The equivalence of linear codes implies semi-linear equivalence

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Abstract

We prove that if two linear codes are equivalent then they are semi-linearly equivalent. We also prove that if two additive MDS codes over a field are equivalent then they are additively equivalent.

1 Introduction

Let F be a finite set. A *code* $\mathcal{A}$ of *length* n is a subset of F^n. The Hamming distance $d(u, v)$ between $u, v \in F^n$ is the number of coordinates in which u and v differ. As in [8], we say that two codes $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if, after some permutation α of the coordinates of the elements of $\mathcal{A}$, there exist permutations σ_i of F such that for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \ldots, \sigma_n(u_n)) \in \mathcal{B}.$$

Let ψ denote the bijection from $\mathcal{A}$ to $\mathcal{B}$ induced by these permutations. Then it is clear that ψ preserves Hamming distance; that is

$$d(u, v) = d(\psi(u), \psi(v))$$

for all $u, v \in \mathcal{A}$.

If F is a finite field and $\mathcal{A}$ and $\mathcal{B}$ are k-dimensional subspaces of F^n then we say that $\mathcal{A}$ and $\mathcal{B}$ are *linear codes*. Two codes are $\mathcal{A}$ and $\mathcal{B}$ are *linearly equivalent* in the above definition of equivalence if, for all $i \in \{1, \ldots, n\}$,

$$\sigma_i(x) = \lambda_i x,$$

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for some non-zero $\lambda_i \in F$. In the case that F has non-trivial automorphisms (fields of non-prime order), for a fixed automorphism β of F, applying the permutation

$$\sigma_i(x) = \lambda_i x^{\beta},$$

to the i-th coordinate of the codewords of $\mathcal{A}$, we get a linear code $\mathcal{B}$ which is equivalent to $\mathcal{A}$. This more general equivalence we will call *semi-linear equivalence*, following [3], [6] and [10]. It is also called *PΓL-equivalence* [2] and is referred to as simply *equivalence* in [4] and [7].

The main result of this note is the following theorem.

Theorem 1. *Two linear codes are equivalent if and only if they are semi-linearly equivalent.*

Based on the above discussion, it is trivial that if two linear codes are semi-linearly equivalent then they are equivalent. Thus the goal here is to prove that if $\mathcal{A}$ and $\mathcal{B}$ are equivalent linear codes then they are semi-linearly equivalent.

Theorem 1 can be compared to Theorem 1.5.10 in [3]. In that theorem they prove that if there is a Hamming distance preserving bijection of F^n which maps subspaces to subspaces then this map is semi-linear, i.e. it is additive and $\sigma(\lambda x) = \lambda^{\beta}\sigma(x)$, for all $\lambda \in F$, where β is an automorphism of F.

Theorem 1 has significant implications when classifying codes. For example, in a classification of codes up to equivalence, as in [1], [8] and [9], one will not find two linear codes in the same equivalence class. Perhaps of more interest is that in a classification of linear codes up to semi-linear equivalence, as in [2] and [5], we can now be sure that two semi-linearly inequivalent codes are also inequivalent.

2 Equivalence implies semi-linear equivalence.

Let $\mathcal{A}$ and $\mathcal{B}$ be k-dimensional linear codes over a finite field $\mathbb{F}_q$ of length n which are equivalent. Since $\alpha(\mathcal{A})$ is also a linear code equivalent to $\mathcal{B}$, we can assume that α, the permutation of the coordinates of the elements of $\mathcal{A}$, is the identity by replacing $\mathcal{A}$ by $\alpha(\mathcal{A})$. Indeed, by re-ordering the coordinates in both codes, if necessary, we can find generator matrices A and B in standard form for $\mathcal{A}$ and $\mathcal{B}$ respectively. Recall that a generator matrix is a $k \times n$ matrix whose row space is the code. By applying elementary row operations, and permuting the columns if necessary, one can always find a generator matrix in standard form, that is a generator matrix in which the initial $k \times k$ sub-matrix is the identity matrix.

As above, we denote by σ_i the permutation in the i-th coordinate which has the property that for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \ldots, \sigma_n(u_n)) \in \mathcal{B}.$$

Lemma 2. *If not all of the columns of A and B are of weight one then, for each $j \in \{1, \ldots, n\}$, the map*

$$\tau_j(x) = \sigma_j(x) - \sigma_j(0)$$

is additive.

Proof. Let $A = (\alpha_{jr})$ and let $B = (\beta_{jr})$.

We have that for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $b = (b_1, \ldots, b_k) \in \mathbb{F}_q^k$, such that

$$(\sigma_1(a_1), \ldots, \sigma_k(a_k), \sigma_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1} a_j), \ldots) = (b_1, \ldots, b_k, \sum_{j=1}^{k} \beta_{j,k+1} b_j, \ldots). \tag{1}$$

Hence, for $j \in \{1, \ldots, k\}$,

$$b_j = \sigma_j(a_j)$$

and

$$\sum_{j=1}^{k} \beta_{jr} \sigma_j(a_j) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j), \tag{2}$$

for $r \in \{k+1, \ldots, n\}$.

Setting $a_i = 0$ for $i \neq j$ we deduce that

$$\beta_{jr} \sigma_j(a_j) + \sum_{i \neq j} \beta_{ir} \sigma_i(0) = \sigma_r(\alpha_{jr} a_j). \tag{3}$$

Summing over j, and substituting in (2)

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - \sum_{j=1}^{k} \sum_{i \neq j} \beta_{ir} \sigma_i(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j)$$

which implies

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - (k-1) \sum_{i=1}^{k} \beta_{ir} \sigma_i(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j).$$

Again from (2)

$$\sum_{i=1}^{k} \beta_{ir} \sigma_i(0) = \sigma_r(0). \tag{4}$$

Thus,

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - (k-1) \sigma_r(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j). \tag{5}$$

Define
$$\tau_r(x) = \sigma_r(x) - \sigma_r(0).$$
Then (5) becomes
$$\sum_{j=1}^{k} \tau_r(\alpha_{jr}a_j) = \tau_r(\sum_{j=1}^{k} \alpha_{jr}a_j). \tag{6}$$

Firstly, assume that the r-th column of A is not of weight one. Without loss of generality $\alpha_{1r}, \alpha_{2r} \neq 0$, so substituting in (6), $a_1 = \alpha_{1r}^{-1}x$ and $a_2 = \alpha_{2r}^{-1}y$, $a_j = 0$ for $j \in \{3, \dots, k\}$, we have that
$$\tau_r(x + y) = \tau_r(x) + \tau_r(y).$$
We conclude that τ_r is an additive permutation of $\mathbb{F}_q$.

Subtracting (4) from (3), we have that
$$\beta_{jr}(\sigma_j(a_j) - \sigma_j(0)) = \tau_r(\alpha_{1r}a_1)$$
This implies that
$$\tau_j(x) = \sigma_j(x) - \sigma_j(0)$$
is additive too, for $j \in \{1, \dots, k\}$.

Now consider the case that the r-th column of A is of weight one for some $r \in \{k+1, \dots, n\}$. Then, from (2),
$$\sigma_r(\alpha_{ir}a_i) = \sum_{j=1}^{k} \beta_{jr}\sigma_j(a_j)$$
for some $i \in \{1, \dots, k\}$. Since this holds for all $(a_1, \dots, a_k) \in \mathbb{F}_q^k$, we conclude that $\beta_{jr} = 0$ for $j \neq i$ and hence,
$$\sigma_r(\alpha_{ir}a_i) = \beta_{ir}\sigma_i(a_i).$$
This implies $\sigma_r(0) = \beta_{ir}\sigma_i(0)$ and so
$$\tau_r(\alpha_{ir}a_i) = \beta_{ir}\tau_i(a_i).$$
Since τ_i is additive, this implies τ_r is additive. □

The following lemma implies that we can assume that σ_i is additive, by replacing σ_i by τ_i, for each $i \in \{1, \dots, n\}$.

Lemma 3. *If not all of the columns of A and B are of weight one then, for all $u = (u_1, \dots, u_n) \in \mathcal{A}$,*
$$(\tau_1(u_1), \dots, \tau_n(u_n)) \in \mathcal{B}.$$

Proof. By Lemma 2 and (1), for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $b = (b_1, \ldots, b_k) \in \mathbb{F}_q^k$, such that

$$(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1} a_j), \ldots) + v = (b_1, \ldots, b_k)B,$$

where $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$ is defined by $v_j = \sigma_j(0)$.

Since τ_i is additive, $\tau_i(0) = 0$. Thus, with $a = (0, \ldots, 0)$ the above implies there is a $(b'_1, \ldots, b'_k)$ such that

$$v = (b'_1, \ldots, b'_k)B.$$

Thus

$$(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1} a_j), \ldots) = ((b_1, \ldots, b_k) - (b'_1, \ldots, b'_k))B,$$

and hence for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $c = (c_1, \ldots, c_k) \in \mathbb{F}_q^k$, such that

$$(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1} a_j), \ldots) = (c_1, \ldots, c_k)B.$$

□

We are now in a position to prove the main theorem. Note that an additive permutation of $\mathbb{F}_{p^h}$ is of the form

$$x \mapsto \sum_{i=0}^{h-1} c_i x^{p^i}.$$

This is easily verified since if we consider $\mathbb{F}_{p^h}$ as $\mathbb{F}_p^h$ then an additive map is given by an $h \times h$ matrix over $\mathbb{F}_p$. Thus, there are p^{h^2} in total, which coincides with the number of functions which can be defined as above.

Proof. (of Theorem 1)

Suppose all of the columns of A and B are of weight one. Then, after a suitable permutation of the coordinates, $\mathcal{A}$ is linearly equivalent to a code $\mathcal{A}'$ whose codewords are

$$(\underbrace{a_1, \ldots, a_1}_{m_1}, \underbrace{a_2, \ldots, a_2}_{m_2}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k}). \tag{7}$$

Thus, since $\mathcal{A}'$ and $\mathcal{B}$ are equivalent, applying a suitable permutation of the coordinates of the elements of $\mathcal{B}$ and scaling the columns of B so that its non-zero entries are one, we have that there are permutations θ_i of $\mathbb{F}_q$ and a function f from $\{1,\ldots,n\}$ to $\{1,\ldots,k\}$, such that for all $(b_1,\ldots,b_k)\in\mathbb{F}_q^k$,

$$(\theta_1(b_{f(1)}),\theta_2(b_{f(2)}),\ldots,\theta_n(b_{f(n)}))$$

is equal to (7), for some $(a_1,\ldots,a_k)\in\mathbb{F}_q^k$.

Comparing with (7) we have that, for example, if $m_1 \geqslant 2$,

$$\theta_1(b_{f(1)}) = \theta_2(b_{f(2)})$$

for all $(b_1,\ldots,b_k)\ \in\mathbb{F}_q^k$. This implies $f(1)=f(2)$, and hence $\theta_1=\theta_2$, etc. Thus we conclude that $\mathcal{B}$ is linearly equivalent to a code which has codewords

$$(\underbrace{b_{g(1)},\ldots,b_{g(1)}}_{m_1},\underbrace{b_{g(2)},\ldots,b_{g(2)}}_{m_2},\ldots,\underbrace{b_{g(k)},\ldots,b_{g(k)}}_{m_k}). \tag{8}$$

Hence, $\mathcal{A}$ is linearly equivalent to $\mathcal{B}$.

Suppose not all of the columns of A and B are of weight one.

By Lemma 3, we can assume for $j\in\{1,\ldots,n\}$,

$$\sigma_j(x)=\sum_i c_{ji}x^{p^i}.$$

As in the previous proofs, let $B=(\beta_{jr})$ and let $A=(\alpha_{jr})$.

Substituting in (2), we have that for all $(a_1,\ldots,a_k)\in\mathbb{F}_q^k$ and $r\in\{k+1,\ldots,n\}$,

$$\sum_{j=1}^{k}\sum_{i=0}^{h-1}\beta_{jr}c_{ji}a_j^{p^i}=\sum_i c_{ri}(\sum_{j=1}^{k}\alpha_{jr}a_j)^{p^i}.$$

Therefore, for all $j\in\{1,\ldots,k\}$, $r\in\{k+1,\ldots,n\}$ and $i\in\{0,\ldots,h-1\}$,

$$\beta_{jr}c_{ji}=c_{ri}\alpha_{jr}^{p^i}. \tag{9}$$

Let $t\in\{0,\ldots,h-1\}$ be minimal such that $c_{1t}\neq 0$. We aim to show that we can multiply columns and rows of B by non-zero elements of $\mathbb{F}_q$ in such a way that the matrix B will transform into A^{p^t}. If both α_{jr} and β_{jr} are zero then they will be unaffected by these multiplications and so for this particular entry the previous statement holds. Thus, we assume that α_{jr} and β_{jr} are not both zero for any particular $j\in\{1,\ldots,k\}$ and $r\in\{k+1,\ldots,n\}$.

If $\beta_{jr}=0$ then, by the previous paragraph, we can assume $\alpha_{jr}\neq 0$. Hence, $c_{ri}=0$ for all $i\in\{0,\ldots,h-1\}$ which implies $\sigma_r=0$, a contradiction.

If $\alpha_{jr}=0$ then, by the previous paragraph, we can assume $\beta_{jr}\neq 0$. Hence, $c_{ji}=0$ for all $i\in\{0,\ldots,h-1\}$ which implies $\sigma_j=0$, a contradiction.

Thus, we can assume c_{1t}, α_{1r} and β_{1r} are not zero. Thus, (9) implies $c_{rt} \neq 0$ for all $r \in \{k+1, \dots, n\}$, and so

$$\frac{\beta_{jr} c_{jt}}{\beta_{1r} c_{1t}} = \frac{\alpha_{jr}^{p^t}}{\alpha_{1r}^{p^t}}.$$

If $c_{jt} = 0$ for some j then $\alpha_{jr} = 0$, which we have already ruled out.

Thus, we can divide the r-th column of B by β_{1r} and multiply the j-th row of this generator matrix by

$$\frac{c_{jt}}{c_{1t}}$$

(which does not change the code) and we get a generator matrix whose j-th coordinate of the r-th column is

$$\frac{\alpha_{jr}^{p^t}}{\alpha_{1r}^{p^t}}$$

Now, multiplying the r-th column of the generator matrix by $\alpha_{1r}^{p^t}$, we get a generator matrix of a code linearly equivalent to $\mathcal{B}$ whose r-th column is the r-th column of A^{p^t}.

Therefore, we conclude that $\mathcal{A}$ and $\mathcal{B}$ are semi-linearly equivalent. $\square$

3 Additive MDS codes

The motivation for this article stems from the discussion after Theorem 3.4 in [2]. In that discussion the following question is asked. “If two additive codes $\mathcal{A}$ and $\mathcal{B}$ are equivalent then are they necessarily PΓL-equivalent? Equivalently, if two additive codes $\mathcal{A}$ and $\mathcal{B}$ are PΓL-inequivalent then is it true that they are inequivalent?” They also pose the same question for linear codes.

We have proved in Theorem 1 that it is true for linear codes. Here, we answer the question in the affirmative for additive MDS codes. Recall that an MDS code is a code attaining the Singleton bound. That is, C is a MDS code of length n over F of minimum distance d if and only if $|C| = |F|^{n-d+1}$.

An additive code over $\mathbb{F}_q$ is linear over the prime subfield $\mathbb{F}_p$, where $q = p^h$. Two additive codes $\mathcal{A}$ and $\mathcal{B}$ are *additively equivalent* if, after some permutation of the coordinates of the elements of $\mathcal{A}$, there exist permutations

$$\sigma_j = \sum_{i=0}^{h-1} c_{ij} x^{p^i}.$$

such that for all $u = (u_1, \dots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \dots, \sigma_n(u_n)) \in \mathcal{B}.$$

An additive MDS code of size q^k is the row space over $\mathbb{F}_p$ of a $kh \times n$ matrix, whose elements are from $\mathbb{F}_q$. We can consider the elements of $\mathbb{F}_q$ as row vectors of $\mathbb{F}_p^h$, so that the generator matrix becomes a $kh \times nh$ matrix over $\mathbb{F}_p$. Suppose that the initial $kh \times kh$ sub-matrix M of this matrix is singular. Therefore there is a $\mathbb{F}_p$-linear combination of the rows whose initial kh coordinates are zero. This implies that there is a codeword whose initial k coordinates are zero. Since the zero vector of $\mathbb{F}_q^n$ is also a codeword, this contradicts the fact that the minimum distance of the code is $n-k+1$. Hence, M is non-singular. Thus, left multiplying the generator matrix by M^{-1}, we obtain a generator matrix whose initial $kh \times kh$ sub-matrix is the identity matrix. This we can write as

$$\begin{pmatrix} I_h & O_h & O_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,1} & \alpha_{r+1,1} & \dots \\ O_h & I_h & O_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,2} & \alpha_{r+1,2} & \dots \\ O_h & O_h & I_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,3} & \alpha_{r+1,3} & \dots \\ O_h & O_h & O_h & I_h & \ddots & \dots & O_h & O_h & \alpha_{r,4} & \alpha_{r+1,4} & \dots \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \vdots & \\ O_h & O_h & O_h & O_h & \dots & \dots & I_h & O_h & \alpha_{r,k-1} & \alpha_{r+1,k-1} & \dots \\ O_h & O_h & O_h & O_h & \dots & \dots & O_h & I_h & \alpha_{r,k} & \alpha_{r+1,k} & \dots \end{pmatrix}, \tag{10}$$

where I_h is the $h \times h$ identity matrix, O_h is the $h \times h$ zero matrix and $\alpha_{i,j}$ are $h \times h$ matrices over $\mathbb{F}_p$.

Theorem 4. *Two additive MDS codes over a field are equivalent if and only if they are additively equivalent.*

Proof. If two codes are additively equivalent then they are equivalent.

Suppose that the additive MDS codes $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Then, according to (10), we can find generator matrices of the form $A = (\alpha_{jr})$ and $B = (\beta_{jr})$ for $\mathcal{A}$ and $\mathcal{B}$ respectively, where α_{jr} and β_{jr} are $h \times h$ matrices. We can then mimic the proof of Lemma 2 and Lemma 3, replacing a_i and b_i by row vectors in $\mathbb{F}_p^h$. The two proofs follows in exactly the same way as for linear codes. Since the codes are MDS we do not have the case in which all the columns of A or B are of weight one. The conclusion is then that σ_j can be taken to be additive for all $j \in \{1, \dots, n\}$. Thus, we conclude that $\mathcal{A}$ and $\mathcal{B}$ are additively equivalent. □

Example 5. *In [2, Table 3], the second row shows that there are 3 additive MDS $(8, 9^3, 6)_9$ codes, i.e. MDS codes of length 8 and minimum distance 6. Two of these codes, C_1 and C_2 are linear MDS codes and have generator matrices*

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & e & e^6 & e^6 & e \\ 0 & 1 & 0 & e & 1 & e & e^6 & e^6 \\ 0 & 0 & 1 & e^6 & e & 1 & e & e^6 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & e^5 & e^7 & e^7 & e^5 \\ 0 & 1 & 0 & e & 1 & e^5 & e^7 & e^7 \\ 0 & 0 & 1 & e^7 & e^5 & 1 & e^5 & e^7 \end{pmatrix}$$

respectively. Note that $e \in \mathbb{F}_9$ *is a primitive element such that* $e^2 = e + 1$*. It is explained in [2] that* C_2 *is a truncated Reed- Solomon code, while* C_1 *is not, implying that they are not semi-linearly equivalent. Additionally, one may show that* C_1 *and* C_2 *are not semi-linearly equivalent by confirming that the columns of* G_2 *are contained in the conic*

$$x_1x_2 + e^3x_1x_3 + x_2x_3,$$

while the columns of G_1 *are not contained in any conic.*

The last of the three additive MDS $(8, 9^3, 6)_9$ *codes,* C_3*, is not semi-linearly equivalent to a linear code. It has generator matrix*

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 2 & 0 & 0 & 2 \end{array}\right).$$

We now know, by Theorem 1, that C_1 *and* C_2 *are not equivalent and, by Theorem 4, these codes are also not equivalent to* C_3*.*

MDS codes have been classified (up to equivalence) over alphabets of size at most 8*, see [1], [8] and [9]. If and when MDS codes over alphabets of size* 9 *are classified, there will be at least three equivalence classes of codes of length* 8*, two of which will contain linear codes and one of which will contain an additive code. We conjecture that there will be no further equivalence class.*

4 Comments

It is unclear to us if Theorem 4 will apply to all additive codes. The principal problem in proving such a statement is that one cannot assume a generator matrix in standard form. It would be interesting to have a counterexample. In the proof of Theorem 4, we view additive MDS codes over $\mathbb{F}_q$, $q = p^h$, as linear codes over the vector space $\mathbb{F}_p^h$, i.e. $1 \times h$ matrices over $\mathbb{F}_p$. It is possible that one can extend Theorem 4 to codes over general matrices. Indeed it would be interesting to study MDS codes over matrices. In other words, the alphabet itself is taken to be a ring of matrices. The smallest such alphabet which is not equivalent to a linear or additive code over $\mathbb{F}_q$ would be over 2×2 matrices over $\mathbb{F}_2$; so the alphabet would have size 16.

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