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Deal, Thomas J.; Smith, Kevin B.

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Reciprocity relationships in vector acoustics and their application to vector field calculations

Thomas J. Deal[a)]
Naval Undersea Warfare Center Division Newport, Newport, Rhode Island 02841, USA

Kevin B. Smith
Department of Physics, Naval Postgraduate School, Monterey, California 93943, USA

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The reciprocity equation commonly stated in underwater acoustics relates pressure fields and monopole sources. It is often used to predict the pressure measured by a hydrophone for multiple source locations by placing a source at the hydrophone location and calculating the field everywhere for that source. A similar equation that governs the orthogonal components of the particle velocity field is needed to enable this computational method to be used for acoustic vector sensors. This paper derives a general reciprocity equation that accounts for both monopole and dipole sources. This vector-scalar reciprocity equation can be used to calculate individual components of the received vector field by altering the source type used in the propagation calculation. This enables a propagation model to calculate the received vector field components for an arbitrary number of source locations with a single model run for each vector field component instead of requiring one model run for each source location. Application of the vector-scalar reciprocity principle is demonstrated with analytic solutions for a range-independent environment and with numerical solutions for a range-dependent environment using a parabolic equation model. [http://dx.doi.org/10.1121/1.4996458]

[NPC] Pages: 523–529

I. INTRODUCTION

In underwater acoustics, the principle of reciprocity for pressure is commonly used in a form that differs little from that given by Lord Rayleigh.[1] While this form also holds for displacement and velocity potentials, it does not apply to individual components (x, y, and z) of the acoustic vector field.[2] This paper derives a more general form of the reciprocity relationship that accounts for both pressure and particle velocity. This new form is useful for computing the velocity received by a sensor from sources located at multiple ranges and depths with a single propagation model run instead of running the model once for each source location. This derivation follows the general approach of Case[3] using the methods in *Computational Ocean Acoustics*.[4] All signals have an implied time dependence of $e^{-j\omega t}$ that is suppressed for brevity. Boldface indicates vector quantities.

II. RECIPROCITY FOR GENERAL SOURCES

The general form of the linearized inhomogeneous wave equation is[5]

$$\rho(\mathbf{r})\left[\nabla\cdot\frac{\nabla p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right]+k^2p_1(\boldsymbol{r}) = j\omega M_1(\boldsymbol{r}-\boldsymbol{r}_1)+\nabla\cdot\boldsymbol{F}_1(\boldsymbol{r}-\boldsymbol{r}_1), \quad (1)$$

where $p_1(\boldsymbol{r})$ is the pressure at $\boldsymbol{r}$ given a source at $\boldsymbol{r}_1$ that is a superposition of two different types of sources. The first source is an acoustic monopole that produces mass injection rate per unit volume M_1. The second source is an acoustic dipole that produces directional force per unit volume $\boldsymbol{F_1}$. Now consider two solutions to Eq. (1): $p_1(\boldsymbol{r})$ for a source at $\boldsymbol{r_1}$ and $p_2(\boldsymbol{r})$ for a source at $\boldsymbol{r_2}$. Multiply the first equation by p_2, and multiply the second equation by p_1; then subtract the second equation from the first and integrate over the volume of interest,

$$\int_V\left[p_2(\boldsymbol{r})\nabla\cdot\frac{\nabla p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}-p_1(\boldsymbol{r})\nabla\cdot\frac{\nabla p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right]dV = \int_V\left[\frac{p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}j\omega M_1(\boldsymbol{r}-\boldsymbol{r_1})+\frac{p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}\nabla\cdot\boldsymbol{F_1}(\boldsymbol{r}-\boldsymbol{r_1})\right]dV - \int_V\left[\frac{p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}j\omega M_2(\boldsymbol{r}-\boldsymbol{r_2})+\frac{p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}\nabla\cdot\boldsymbol{F_2}(\boldsymbol{r}-\boldsymbol{r_2})\right]dV. \quad (2)$$

Applying Green's Theorem to the left hand side of Eq. (2) converts it to a surface integral that vanishes for the impedance boundary conditions one encounters in underwater acoustics since potentials are proportional to their gradients. Therefore, the integrals on the right-hand side must be equal. This is the form of the general reciprocity relationship for monopole and dipole sources,

$$\int_V\left[\frac{p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}j\omega M_1(\boldsymbol{r}-\boldsymbol{r}_1)+\frac{p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}\nabla\cdot\boldsymbol{F}_1(\boldsymbol{r}-\boldsymbol{r}_1)\right]dV = \int_V\left[\frac{p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}j\omega M_2(\boldsymbol{r}-\boldsymbol{r}_2)+\frac{p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}\nabla\cdot\boldsymbol{F}_2(\boldsymbol{r}-\boldsymbol{r}_2)\right]dV. \quad (3)$$

III. RECIPROCITY FOR POINT SOURCES

Equation (3) can be further simplified for point sources. Let each monopole source take the form $M(\boldsymbol{r}-\boldsymbol{r_1}) = M_1\delta(\boldsymbol{r}-\boldsymbol{r_1})$, and let each dipole source take the form $\mathbf{F}(\boldsymbol{r}-\boldsymbol{r_1})$

[a)]Electronic mail: thomas.deal@navy.mil

$= \boldsymbol{F_1}\delta(\boldsymbol{r}-\boldsymbol{r_1})$ where $\delta(\cdot)$ is the three-dimensional Dirac delta function. Then the ∇ operation can be moved from **F** to p via integration by parts and the Divergence Theorem, i.e.,

$$\int_V g(\nabla \cdot \boldsymbol{A})dV = -\int_V \boldsymbol{A} \cdot (\nabla g)dV + \int_V \nabla \cdot (g\boldsymbol{A})dV = -\int_V \boldsymbol{A} \cdot (\nabla g)dV + \oint_S (g\boldsymbol{A})dS.$$

When $g = p_2(\boldsymbol{r})/\rho(\boldsymbol{r})$ and $\boldsymbol{A} = \boldsymbol{F_1}\delta(\boldsymbol{r}-\boldsymbol{r_1})$, the surface integral vanishes since the delta function is zero everywhere on the surface, and the volume integral becomes the dot product of the force and the gradient of the pressure. Equation (3) can now be rewritten as

$$\int_V \left[\frac{j\omega M_1\delta(\boldsymbol{r}-\boldsymbol{r}_1)}{\rho(\boldsymbol{r})}p_2(\boldsymbol{r}) - \boldsymbol{F}_1\delta(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\nabla\left(\frac{p_2(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right)\right]dV = \int_V \left[\frac{j\omega M_2\delta(\boldsymbol{r}-\boldsymbol{r}_2)}{\rho(\boldsymbol{r})}p_1(\boldsymbol{r}) - \boldsymbol{F}_2\delta(\boldsymbol{r}-\boldsymbol{r}_2)\cdot\nabla\left(\frac{p_1(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right)\right]dV. \quad (4)$$

Integration over the entire volume extracts only those values where the delta functions are nonzero at the source locations $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, resulting in

$$\frac{j\omega M_1 p_2(\boldsymbol{r}_1)}{\rho(\boldsymbol{r}_1)} - \boldsymbol{F_1} \cdot \nabla\left(\frac{p_2(\boldsymbol{r}_1)}{\rho(\boldsymbol{r}_1)}\right) = \frac{j\omega M_2 p_1(\boldsymbol{r}_2)}{\rho(\boldsymbol{r}_2)} - \boldsymbol{F}_2 \cdot \nabla\left(\frac{p_1(\boldsymbol{r}_2)}{\rho(\boldsymbol{r}_2)}\right). \quad (5)$$

The gradient terms in this equation can be rewritten as

$$\nabla\left(\frac{p(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right) = \nabla\left(\frac{p(\boldsymbol{r})}{\rho_0(1+s(\boldsymbol{r}))}\right), \quad (6)$$

where ρ_0 is the constant equilibrium density and $s(\boldsymbol{r})$ is the condensation. In Ref. 5 the derivation of the wave equation used in Eq. (1) required a linearization of the equation of continuity and Euler's equation. In both of these linearizations, there was an assumption that $s(\boldsymbol{r}) \ll 1$. In many cases, such as in the water column or in a homogeneous layer, that assumption is still valid, so the gradient term can be reduced to

$$\nabla\left(\frac{p(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right) = \frac{\nabla p(\boldsymbol{r})}{\rho_0} = j\omega \boldsymbol{v}(\boldsymbol{r}), \quad (7)$$

since $\nabla p(\boldsymbol{r}) = j\omega\rho_0\boldsymbol{v}(\boldsymbol{r})$. There are situations where this assumption does not hold, such as at boundaries between layers or in an inhomogeneous layer, because the density gradient is not exactly zero. In these cases the full gradient term must be evaluated

$$\nabla\left(\frac{p(\boldsymbol{r})}{\rho(\boldsymbol{r})}\right) = \frac{\nabla p(\boldsymbol{r})}{\rho(\boldsymbol{r})} + \nabla\left(\frac{1}{\rho(\boldsymbol{r})}\right)p(\boldsymbol{r}). \quad (8)$$

Nevertheless, returning to the common case where the condensation is small, the reciprocity relationship for point sources can be stated with both a pressure component and a velocity component as

$$M_1\frac{p_2(\boldsymbol{r}_1)}{\rho(\boldsymbol{r}_1)} - \boldsymbol{F}_1 \cdot \boldsymbol{v}_2(\boldsymbol{r}_1) = M_2\frac{p_1(\boldsymbol{r}_2)}{\rho(\boldsymbol{r}_2)} - \boldsymbol{F}_2 \cdot \boldsymbol{v}_1(\boldsymbol{r}_2). \quad (9)$$

IV. APPLICATIONS

The point source reciprocity relationship has different applications depending on the choice of sources used. Two monopole sources relate pressure to pressure, two dipole sources relate velocity to velocity, and a mix of monopole and dipole sources relates pressure to velocity. In this work, we are interested in the response of a dipole receiver (a vector sensor channel) to a monopole point source. Thus, a mixture of monopole and dipole sources will use this reciprocity relationship to predict the received velocity components from actual monopole sources. Future work will consider the response of dipole receivers due to dipole sources, such as may be encountered at the pressure release surface.

A. Scalar reciprocity

When only monopole sources of equal amplitude are considered, then $M_1 = M_2$ and $\boldsymbol{F_1} = \boldsymbol{F_2} = 0$, and Eq. (9) reduces to

$$\frac{p_2(\boldsymbol{r}_1)}{\rho(\boldsymbol{r}_1)} = \frac{p_1(\boldsymbol{r}_2)}{\rho(\boldsymbol{r}_2)}. \quad (10)$$

This is the familiar form in Rayleigh[1] and Jensen *et al.*[4] equating the received pressure at point 2 due to a monopole source at point 1 to the received pressure at point 1 due to a monopole source at point 2. This equation enables the computational shortcut commonly used to predict the pressure measured at a fixed location due to a monopole source at any point in the field. In this method, a monopole source is placed at the receiver location and the pressure field is calculated everywhere for that source. This pressure is then scaled by the ratio of densities at source and receiver locations. By reciprocity, this scaled quantity is the pressure that would be measured by a fixed hydrophone for a monopole source located at each point in the field.

This equation cannot be used to predict acoustic velocity since all velocity terms have been eliminated. Although velocity field components due to a fixed source can be calculated by taking the gradient of the pressure field, the equality in Eq. (10) precludes taking the gradient of the left side with respect to $\boldsymbol{r_1}$ while simultaneously taking the gradient of the right side with respect to $\boldsymbol{r_2}$.

B. Vector reciprocity

The vector reciprocity relationship arises when only dipole sources are considered. In this case, $M_1 = M_2 = 0$, and Eq. (9) becomes

$$\boldsymbol{F}_1 \cdot \boldsymbol{v}_2(\boldsymbol{r}_1) = \boldsymbol{F}_2 \cdot \boldsymbol{v}_1(\boldsymbol{r}_2). \quad (11)$$

This is the reciprocal relationship between velocity and force components. When Eq. (11) is expanded in Cartesian coordinates it becomes

$$F_{1x}v_{2x}(\boldsymbol{r}_1) + F_{1y}v_{2y}(\boldsymbol{r}_1) + F_{1z}v_{2z}(\boldsymbol{r}_1) = F_{2x}v_{1x}(\boldsymbol{r}_2) + F_{2y}v_{1y}(\boldsymbol{r}_2) + F_{2z}v_{1z}(\boldsymbol{r}_2). \quad (12)$$

This equation is easier to use when the forces are restricted to a single axis. For example, when both forces act only in the x direction, Eq. (12) reduces to

$$F_{1x}v_{2x}(\boldsymbol{r}_1) = F_{2x}v_{1x}(\boldsymbol{r}_2). \quad (13)$$

When the amplitudes of the force components are equal, $F_{1x} = F_{2x}$, then Eq. (13) states that the x velocity received at point 2 due to a force in the x direction at point 1 is equal to the x velocity received at point 1 due to a force in the x direction at point 2. The same applies to two forces in the y direction and two forces in the z direction.

Equation (12) has a less obvious application when the two forces act in orthogonal directions. For example, when the first force acts in the x direction and the second force acts in the y direction, Eq. (12) becomes

$$F_{1x}v_{2x}(\boldsymbol{r}_1) = F_{2y}v_{1y}(\boldsymbol{r}_2). \quad (14)$$

When the force components are equal, $F_{1x} = F_{2y}$, then Eq. (14) states that the y velocity received at point 2 due to a force in the x direction at point 1 is equal to the x velocity received at point 1 due to a force in the y direction at point 2. This relationship holds for all six pairs of forces in orthogonal directions.

Equation (11) is useful for calculating the received velocity components at a fixed location due to a dipole source at any range and depth in the environment. Suppose the dipole source is oriented along the x axis. First compute the x velocity everywhere in the field for a point force with unit amplitude at the receiver location oriented in the x direction. This quantity is represented by $v_x^{(X)}$, where the subscript x represents the field component that is calculated, and the superscript (X) represents the source orientation used to produce it. By vector reciprocity, this is equal to the x velocity received at the fixed position for a force in the x direction at any source location in the field. Next place a dipole source at the receiver location but orient it in the y direction. Calculate the x velocity everywhere in the field, $v_x^{(Y)}$. This is equal to the y velocity received at the fixed position for a force in the x direction at any source location in the field. Finally, place a dipole source at the receiver location oriented along the z axis. Calculate the x velocity everywhere in the field, $v_x^{(Z)}$. This is equal to the z velocity received at the fixed position for a force in the x direction at any source location in the field. Finally, since the fields were calculated with a unit amplitude source, multiply each quantity by the dipole source amplitude F_x.

Since any force can be decomposed into a sum of components in each orthogonal direction, F_x, F_y, and F_z, this process can be extended to arbitrary source orientations via superposition. The reciprocal velocity field must be calculated for each orthogonal source direction. This can be accomplished with three propagation model runs provided all three velocity components are saved for each model run, e.g., $v_x^{(X)}$, $v_y^{(X)}$, and $v_z^{(X)}$ for the source in the x direction. This produces three x velocities ($v_x^{(X)}$, $v_x^{(Y)}$, $v_x^{(Z)}$), three y velocities ($v_y^{(X)}$, $v_y^{(Y)}$, $v_y^{(Z)}$), and three z velocities ($v_z^{(X)}$, $v_z^{(Y)}$, $v_z^{(Z)}$). The total x velocity is the inner product of the three x velocities with the three force components, the total y velocity is the inner product of the three y velocities with the three force components, and the total z velocity is the inner product of the three z velocities with the three force components, i.e.,

$$\begin{aligned} v_x &= F_x v_x^{(X)} + F_y v_x^{(Y)} + F_z v_x^{(Z)}, \\ v_y &= F_x v_y^{(X)} + F_y v_y^{(Y)} + F_z v_y^{(Z)}, \\ v_z &= F_x v_z^{(X)} + F_y v_z^{(Y)} + F_z v_z^{(Z)}. \end{aligned} \quad (15)$$

C. Vector-scalar reciprocity

The final reciprocal relationship arises when $M_1 = 0$ and $\boldsymbol{F_2} = 0$. This reduces Eq. (9) to

$$-\boldsymbol{F_1} \cdot \boldsymbol{v}_2(\boldsymbol{r}_1) = M_2 \frac{p_1(\boldsymbol{r}_2)}{\rho(\boldsymbol{r}_2)}. \quad (16)$$

This form of the reciprocity relationship for point sources permits the computational shortcut commonly used for received pressure calculations to be used for received velocity calculations. Applying this method to Eq. (16) is a two-step process. First, a dipole source is placed in a particular orientation (aligned with the x, y, or z axis) at the receiver location and the pressure field is calculated everywhere for that source. The pressure at each point in the field is then divided by the density and negated to account for the direction reversal. Provided the sources have equivalent magnitudes, $|M_2| = ||\boldsymbol{F_1}||$, the result is the amplitude of the velocity component in the same direction that the dipole was oriented in the first step. To obtain the other received velocity components, these steps are repeated for other orientations of the reciprocal source dipole.

D. Vector fields for fixed receivers

When combined with a propagation model that calculates pressure and velocity components of the vector field for a fixed source, Eqs. (10), (11), and (16) enable the reciprocal computation of the vector field for a fixed point receiver due to a monopole or dipole source. Let the model calculate the received pressure, radial velocity, and vertical velocity for a reciprocal monopole source at the fixed receiver location, $p^{(M)}$, $v_r^{(M)}$, and $v_z^{(M)}$. Likewise, the model outputs for a reciprocal horizontal dipole source are $p^{(H)}$, $v_r^{(H)}$, and $v_z^{(H)}$, and the model outputs for a reciprocal vertical dipole source are $p^{(V)}$, $v_r^{(V)}$, and $v_z^{(V)}$. Using these model outputs properly scaled for units of Pascals and meters per second, the received pressure and velocity at the fixed receiver due to a monopole source are

$$\begin{aligned} p &= p^{(M)}, \\ v_r &= -\left(\frac{1}{\rho_0 c}\right) p^{(H)}, \\ v_z &= -\left(\frac{1}{\rho_0 c}\right) p^{(V)}, \end{aligned} \quad (17)$$

the received pressure and velocity at the fixed receiver due to a horizontal dipole source are

$$p = -(\rho_0 c) v_r^{(M)},$$
$$v_r = v_r^{(H)},$$
$$v_z = v_r^{(V)}, \tag{18}$$

and the received pressure and velocity at the fixed receiver due to a vertical dipole source are

$$p = -(\rho_0 c) v_z^{(M)},$$
$$v_r = v_z^{(H)},$$
$$v_z = v_z^{(V)}. \tag{19}$$

A dipole with arbitrary orientation within the plane of calculation can be parameterized by its rotation from vertical by an angle φ. For a unit magnitude source, the horizontal and vertical components of this dipole are equal to $\sin(\varphi)$ and $\cos(\varphi)$, respectively. The received pressure and velocity at the fixed receiver due to an arbitrarily-rotated source are

$$p = -(\rho_0 c) v_r^{(M)} \sin(\varphi) - (\rho_0 c) v_z^{(M)} \cos(\varphi),$$
$$v_r = v_r^{(H)} \sin(\varphi) + v_z^{(H)} \cos(\varphi),$$
$$v_z = v_r^{(V)} \sin(\varphi) + v_z^{(V)} \cos(\varphi), \tag{20}$$

of which the horizontal and vertical dipole are special cases for $\varphi = \pi/2$ and $\varphi = 0$. These results are for a model that calculates field components in a single plane with one vertical axis z and one horizontal axis r, but it immediately extends to a model that calculates a full three-dimensional field. In that case a dipole source can also be arbitrarily rotated by an angle ϑ in the horizontal plane, so the full direction cosines needed to combine model outputs are $\cos(\vartheta)\sin(\varphi)$, $\sin(\vartheta)\sin(\varphi)$, and $\cos(\varphi)$.

V. RANGE-INDEPENDENT ENVIRONMENT

The range-independent waveguide produces a closed-form solution that demonstrates the vector-scalar reciprocity relationship. In this example, density is constant in the layer, so $\rho(\boldsymbol{r}_1) = \rho(\boldsymbol{r}_2) = \rho_0$. Upon calculating the displacement potential, $\psi(\boldsymbol{r}_1)$, the pressure and velocity are readily calculated as $p(\boldsymbol{r}_1) = \rho_0 \omega^2 \psi(\boldsymbol{r}_1)$ and $\boldsymbol{v}(\boldsymbol{r}_1) = -j\omega\nabla\psi(\boldsymbol{r}_1)$. In cylindrical coordinates $[\boldsymbol{r}_1 = (r_1, z_1, \theta_1)]$, the received pressure, radial velocity, and vertical velocity at point $\boldsymbol{r}_1$ from a monopole source M_2 at point $\boldsymbol{r}_2$ are[6,7]

$$p(\boldsymbol{r}_1) = \frac{\omega M_2}{4} \sum_{n=0}^{\infty} \Psi_n(z_2)\Psi_n(z_1) H_0^{(1)}(k_{rn} r), \tag{21}$$

$$v_r(\boldsymbol{r}_1) = -\frac{jM_2}{4\rho_0} \sum_{n=0}^{\infty} \Psi_n(z_2)\Psi_n(z_1) k_{rn} H_1^{(1)}(k_{rn} r), \tag{22}$$

$$v_z(\boldsymbol{r}_1) = \frac{jM_2}{4\rho_0} \sum_{n=0}^{\infty} \Psi_n(z_2)\Psi_n'(z_1) H_0^{(1)}(k_{rn} r), \tag{23}$$

where Ψ_n is the depth-dependent mode shape function for the nth mode with the prime indicating differentiation with respect to its argument, k_{rn} is the horizontal wavenumber for the nth mode, and $r = |r_1 - r_2|$ is the absolute horizontal distance between the source and receiver.

Haug *et al.*[8] developed an expression for the pressure generated by a dipole source by expanding the field around the source in a Taylor series. For a dipole of source strength $F_1 = ||\boldsymbol{F}_1||$ at point $\boldsymbol{r}_1$ rotated φ from the vertical axis and ϑ from the radial axis, it produces

$$p(\boldsymbol{r}_2) = \frac{F_1}{4j} \sum_{n=0}^{\infty} [-\Psi_n(z_2)\Psi_n(z_1) k_{rn} H_1^{(1)}(k_{rn} r) \times \sin(\varphi)\cos(\vartheta - \theta) + \Psi_n(z_2)\Psi_n'(z_1) H_0^{(1)}(k_{rn} r)\cos(\varphi)]. \tag{24}$$

Here $r = |r_2 - r_1|$ is the absolute horizontal distance between the source and receiver, which is the same as the distance r in Eqs. (21)–(23).

When the dipole is oriented horizontally, $\sin(\varphi) = 1$ and $\cos(\varphi) = 0$, the first bracketed term in Eq. (24) determines the pressure. When the dipole is aligned with the radial axis, $\cos(\vartheta - \theta) = 1$, this term is identical to the radial velocity in Eq. (22). Similarly, when the source dipole is oriented vertically, $\cos(\varphi) = 1$ and $\sin(\varphi) = 0$, the pressure field is determined entirely by the second bracketed term in Eq. (24). This equals the received vertical velocity in Eq. (23). With the proper source scaling factor, $F_1 = -M_2/\rho_0$, the reciprocal relationship holds. Therefore, the quantity in the brackets is the dot product of the dipole source with the received velocity due to a monopole source, as predicted in Eq. (16).

VI. RANGE-DEPENDENT ENVIRONMENT

A. Modeling range-dependent vector fields

Extending these results to range-dependent environments requires a propagation model capable of computing pressure and velocity fields for monopole and dipole sources. The split-step Fourier parabolic equation (PE) program MMPE (Ref. 9) produces pressure and velocity transmission loss (TL) fields for monopole sources and arrays of monopole sources. Generating fields for dipole sources is accomplished by modifying the PE starter field. Vecherin *et al.*[10] prescribes an equivalent source method that decomposes vertical and horizontal dipole sources into vertical monopole arrays with element amplitude scaling such that the array source strength is equal to that of a single monopole.

A vertical dipole is a special case of a vertical array with two elements 180° out of phase with each other and separated by a distance $2d_z$ much smaller than a wavelength ($kd_z \leq 0.1$). The desired shape of the acoustic field for this source is $\cos(\phi)$ for angles ϕ measured from the vertical axis. The horizontal dipole has a desired shape of $\sin(\phi)$, which Vecherin *et al.* represents as a sum of cosines for angles close to $\pi/2$,

$$\sin(\phi) \approx 1 - \frac{1}{2}\cos^2(\phi) - \frac{1}{8}\cos^4(\phi), \tag{25}$$

which is valid for wide-angle PE calculations. Each $\cos^n(\phi)$ term in Eq. (25) can be represented by an array of $(n+1)$

elements vertically spaced by $2d_z$. Adding these arrays together produces the desired PE starter field. Although these amplitude weights and element spacings can be implemented with MMPE's existing vertical array source, the utility of dedicated horizontal and vertical dipole sources led to a direct implementation of the starter field in the vertical wavenumber domain such that the user need only specify the source depth and frequency.

B. Numerical example

The shelf break test case from the SWAM'99 workshop[11] provides varying bathymetry and sound speed that is useful for demonstrating vector-scalar reciprocity in an arbitrary environment, as shown in Fig. 1. The water has constant density of 1.0 g/cm^3 and no attenuation, and the bottom is homogeneous with sound speed 1700 m/s, density 1.5 g/cm^3, compressional attenuation 0.1 dB/λ, and no shear. The range-dependent bathymetry has depth

$$z_b = \begin{cases} 100, & r \le 5000 \\ \dfrac{1}{50}r, & 5000 < r \le 20\,000, \end{cases} \tag{26}$$

where z_b and r are in meters. For depths $z < z_b$, the water sound speed in m/s is defined by

$$c(r,z) = 1450 + 4.6T(r,z) - 0.055T^2(r,z) + 0.016z, \tag{27}$$

where $T(r, z)$ is the temperature in degrees Celsius defined by

$$T(r,z) = 5 + T_0(r)\left(\frac{\sinh\left(\dfrac{\pi(400-z)}{200}\right)}{265}\right)^2, \tag{28}$$

with the surface temperature in degrees Celsius defined by

$$T_0(r) = 15 - 5\left(1 + \tanh\left(10 - \frac{r}{1000}\right)\right). \tag{29}$$

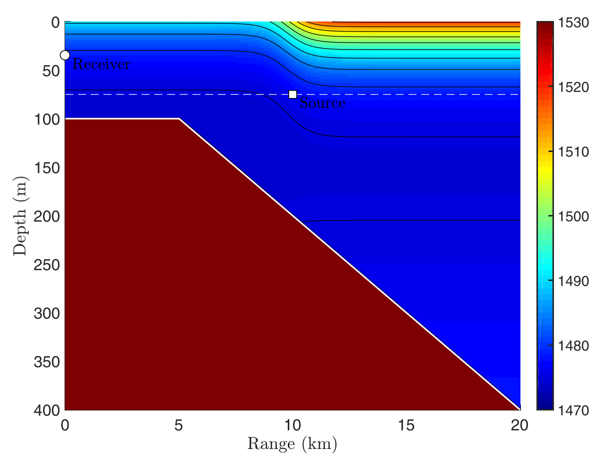

FIG. 1. (Color online) Range-dependent geometry showing fixed receiver location, variable-range source depth, bathymetry, and sound speed profile. Sound speed contours are drawn every 5 m/s.

The receiver is located on the shelf in the water column at a depth of 35 m and range of 0 m. The goal is to calculate the vector field received by that sensor for a 25 Hz monopole source located at a depth of 75 m for ranges from 0 to 20 km.

C. Results

We define pressure TL in decibels as $20\log_{10}|p(r,z)/p_0|$, where p_0 is the root-mean-square (rms) pressure amplitude in Pa 1 m from the source. This source produces an rms particle velocity amplitude of $v_0 = p_0/(\rho_0 c)$ m/s 1 m from the source, so we define horizontal and vertical velocity transmission losses as $20\log_{10}|v_r(r,z)/v_0|$ and $20\log_{10}|v_z(r,z)/v_0|$. This produces plots of comparable amplitude for pressure and velocity TL despite the large difference in their values when expressed in SI units of Pa and m/s. It also incorporates the characteristic acoustic impedance $\rho_0 c$, evaluated at the receiver location, required by Eqs. (17)–(19) for computing received vector fields.

The baseline for model comparison is a set of 100 MMPE runs with a monopole source at the source depth and discrete range intervals of 200 m. At each range step, the sound speed profiles and bathymetry are reversed so the computation begins at the source location. The pressure, radial velocity, and vertical velocity TL output of each run for the receiver location are saved in an array. These are the values expected from the reciprocal calculation method.

Figure 2 shows the comparison between radial velocity TL calculations. The vector-scalar reciprocal method using a horizontal dipole source is the solid black line, and it matches the discrete range baseline indicated by plus signs. The magnitude errors are within 5%, and the phase errors are generally within 5°, as shown in Fig. 3. The ranges where the phase error increases correspond to nulls in the radial velocity field where the magnitude is small enough that the phase errors have little effect.

The dashed red line in Fig. 2 is the radial velocity TL measured at the source depth for a monopole source at the

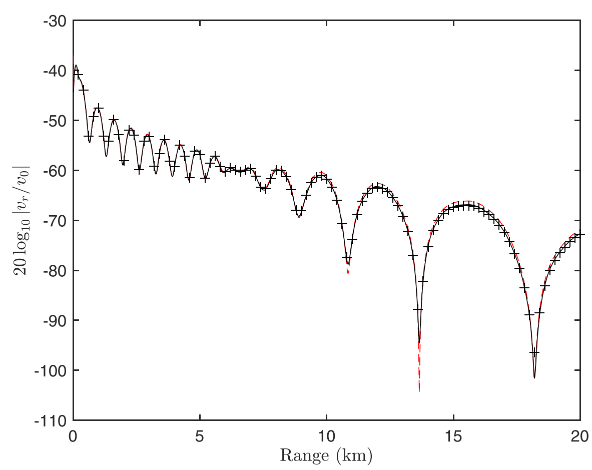

FIG. 2. (Color online) Radial velocity TL magnitude comparison. Plus signs are baseline radial velocity TL measured at the receiver location for a monopole source at the source depth. Dashed red line is radial velocity TL measured at the source depth for a monopole source at the receiver location. Solid black line is pressure TL measured at the source depth for a horizontal dipole source at the receiver location.

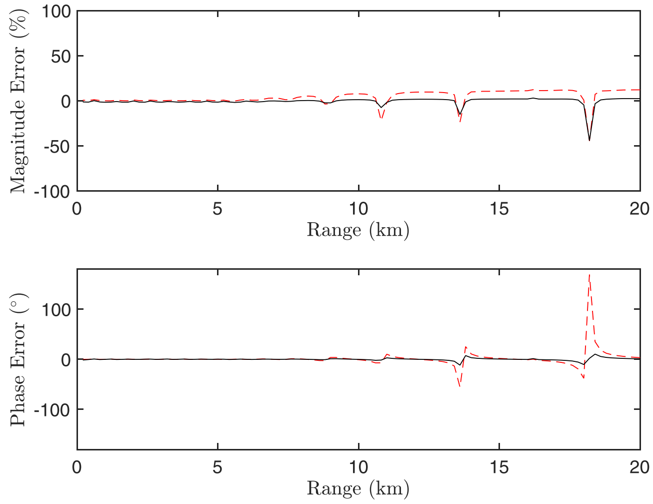

FIG. 3. (Color online) Radial velocity TL calculation error comparing 100 model runs for a moving monopole source with a single model run for a fixed monopole source and a single model run for a fixed horizontal dipole source. Dashed red line is the error for the radial velocity TL measured at the source depth for a monopole source at the receiver location. Solid black line is the error for the pressure TL measured at the source depth for a horizontal dipole source at the receiver location.

receiver location, which is the typical approach used in modeling that assumes a reciprocal solution. While it has good agreement with the baseline for short ranges, its error grows with range, which is more clearly shown in Fig. 3. This is to be expected because the environment is nearly range-independent for the first 5 km, and as shown in Eq. (22) the radial velocity is the same when source and receiver positions are exchanged in range-independent environments. As the environment begins to vary in range, however, the radial velocity TL for a fixed monopole source begins to deviate from the baseline, and the pressure TL for a fixed horizontal dipole source becomes a better match for the radial velocity TL field.

The comparison between vertical velocity TL calculations is shown in Fig. 4, which highlights the advantage of the vector-scalar reciprocity relationship more clearly. The solid black line for vertical velocity TL using the vector-scalar reciprocal method matches well with the baseline velocity TL indicated by plus signs. The magnitude and phase errors shown in Fig. 5 are comparable to those for reciprocal radial velocity TL and, again, ranges with larger magnitude and phase errors correspond to nulls in the vertical velocity field and will therefore have less effect on the overall field. In contrast, the dashed red line in Figs. 4 and 5 is the vertical velocity TL measured at the source depth for a monopole source at the receiver location. This produces a very poor match to the baseline at any range.

It should not be surprising that misapplying the scalar reciprocity equation to radial velocity produces an approximately correct solution for an axisymmetric environment. Since the wavefronts are expanding outward in the radial direction, radial intensity is primarily active and radial velocity is in phase with pressure in the far field. This relationship does not hold for environments that produce horizontal reflection and refraction of acoustic waves, and errors in the approximate solution will be greater for those environments. The complicated phase relationship between pressure and vertical velocity in even the simplest stratified environments is the reason for the large errors when attempting to use the scalar reciprocity equation to calculate received vertical velocity.

VII. CONCLUSION

The scalar reciprocity equation that governs the relationship between monopole sources and pressure fields is readily extendable to include horizontal and vertical dipole sources and the velocity components of acoustic vector fields. Using the vector and vector-scalar reciprocal methods makes it computationally efficient to calculate received vector fields for monopole and dipole point sources, completing an entire

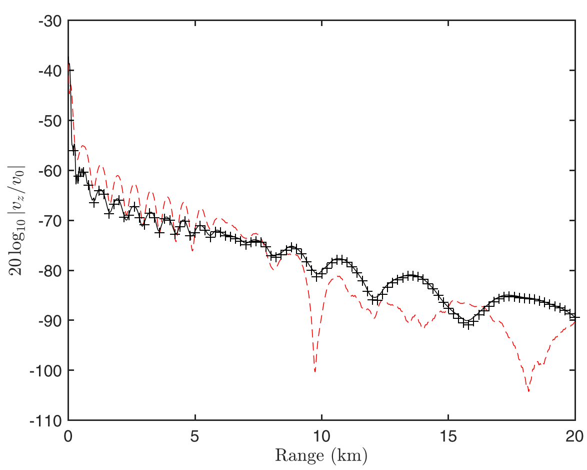

FIG. 4. (Color online) Vertical velocity TL magnitude comparison. Plus signs are baseline vertical velocity TL measured at the receiver location for a monopole source at the source depth. Dashed red line is vertical velocity TL measured at the source depth for a monopole source at the receiver location. Solid black line is pressure TL measured at the source depth for a vertical dipole source at the receiver location.

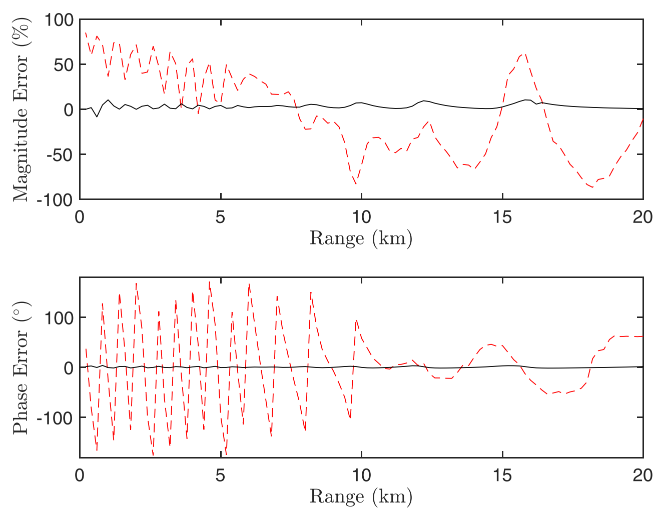

FIG. 5. (Color online) Vertical velocity TL calculation error comparing 100 model runs for a moving monopole source with a single model run for a fixed monopole source and a single model run for a fixed vertical dipole source. Dashed red line is the error for the vertical velocity TL measured at the source depth for a monopole source at the receiver location. Solid black line is the error for the pressure TL measured at the source depth for a vertical dipole source at the receiver location.

field calculation with no more than four propagation model runs (one for pressure and each orthogonal velocity direction) rather than having to run the propagation model once for every point in the field. This is particularly useful for computationally-intensive range-dependent PE and finite element propagation models. This is also useful for matched-field and time-reversal processing involving vector sensors. The simplified forms of the vector and vector-scalar reciprocity equations derived here depend on constant density, which is a reasonable assumption when the source and receiver are located in the water column.

The vector reciprocity equation relating vector fields to dipole sources was not demonstrated, but one application for it is predicting the received vector field created by wind-driven surface noise. This noise source is often represented by a large number of vertical dipoles evenly distributed on a horizontal layer less than a wavelength below the surface. Propagating the field from each source to a fixed receiver would be computationally intensive due to the large number of sources required. By invoking the vector reciprocity principle, however, one propagation model run with a dipole source at the receiver location could predict the velocity received from the surface noise sources at all ranges.

The reciprocity equation derivation can also be repeated with higher order sources such as quadrupoles. Like dipoles, quadrupoles tend to be inefficient radiators and are not commonly used in underwater acoustics. However, their inclusion in a reciprocal equation could provide some mathematical efficiency for calculating other quantities of interest such as field circularity or higher-order moments. For any source order, proper application of the reciprocity principle requires careful attention to the relationship between source type and received quantity. When this is done, the vector and vector-scalar reciprocity equations provide many computational advantages for calculating received vector fields.

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