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The Functional Renormalisation Group, its Mathematics and Applications to Asymptotic Safety

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Abstract

We present a regularisation scheme for scalar Quantum Field theories that enables a flexible and mathematically consistent formulation of interacting theories in arbitrary dimensions. In contrast to a lattice approach, it retains the smooth features of spacetime and the infinite degrees of freedom such that, in particular, the rotational symmetry can be left unbroken. In this framework, we give a mathematically rigorous derivation of the Wetterich equation as well as sufficient conditions for the passage to the limit of vanishing regularisation.

We also introduce an iterative construction procedure for exact solutions to the Wetterich equation that works by producing higher-order correlation functions form the renormalisation group flow of lower order correlators.

Then a generalisation of Quantum Electrodynamics is considered in the asymptotic safety framework and particular solutions are found that reproduce physical results in a low-energy regime.

Finally, the applicability of the introduced regularisation scheme to the ϕ^4 theory is proved. It follows from an integrability statement that can be thought of as a generalisation of Fernique's theorem on exponential tails of Gaußian measures.

Kurzdarstellung

Wir präsentieren eine Regularisierungsmethode für skalare Quantenfeldtheorien, die eine flexible und konsistente Behandlung wechselwirkender Theorien in beliebig vielen Dimensionen ermöglicht. Im Gegensatz zur Gittermethode bleiben die glatten Eigenschaften der Raumzeit und die unendlich vielen Freiheitsgrade unberührt, sodass u.a die Rotationssymmetrie erhalten werden kann.

Wir stellen auch eine iterative Konstruktionsmethode vor, die es ermöglicht exakte Lösungen der Wetterichgleichung zu erzeugen, indem aus den Flüssen von Korrelationsfunktionen niedriger Ordnung solche höherer Ordnung ermittelt werden.

Anschließend untersuchen wir eine Verallgemeinerung der Quantenelektrodynamik im Kontext der asymptotischen Sicherheit und finden Lösungen, die im Niederenergielimes physikalische Resultate wiedergeben.

Zum Schluss beweisen wir noch, dass die eingeführte Regularisierungsmethode auf ϕ^4 Modelle angewandt werden kann. Diese Eigenschaft folgt aus einem Integrierbarkeitssatz, der als Verallgemeinerung von Ferniques Satz über die exponentielle Integrierbarkeit quadratischer Formen bezüglich Gaußscher Maße betrachtet werden kann.

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Contents

1. Introduction

The most successful models of particle physics are Quantum Field Theories. Over the years numerous experiments have confirmed and several theoretical insights refined the now-called Standard Model of particle physics. It is a Quantum Field Theory (QFT) which models the fundamental building blocks of our universe as matter fields and force carriers. One particular part of the Standard Model is given by Quantum Electrodynamics (QED) which, from a historical perspective, was also the first physical example of an interacting QFT.

However, in four spacetime dimensions the mathematical foundations of interacting QFTs have turned out to be in an unsatisfactory state. One particular reason for this is the known non-convergence of perturbation series which at the same time - in truncations - produce strikingly precise values of physical observables such as the anomalous magnetic moment of the electron. Another reason lies in the possible presence of Landau poles of the renormalisation group (RG) flow of coupling constants leading to the terminology of ultraviolet (UV) incompleteness: A model that maintains its own consistency only for characteristic energies smaller than some energy scale. In particular, QED is widely believed to be such a theory.

Even today, various mathematical and computational difficulties encountered in many relevant QFT models represent unsolved problems. However, over the years, they have led to significant advances in fields ranging from particle physics to pure mathematics. One such result is the renormalisation group in terms of the Wetterich equation which is the thread connecting the different research projects presented in this thesis.

In chapter 2, we give a short review of the path integral quantisation of scalar field theories. Spacetime discretisation and finite volumes are presented as particular examples of regularisation schemes of QFTs. Furthermore, a short discussion about the inevitable dependence of the model parameters on these unphysical regularisations is given, demonstrating the necessity of renormalisation. This is followed by a short digression about the free scalar field which can be modelled in a mathematically satisfactory fashion. Combining these insights, a regularisation scheme is presented that - in contrast to a lattice approach - retains the infinite degrees of freedom of a field theory as well as the physical rotational symmetry. This is achieved in two steps. The first step is analogous to perturbation theory in the sense that the classical action is split into a free part and an interacting part, the former of which can be modelled in a known manner. The second step introduces a regularisation in the form of an operator that takes a rather singular object (a tempered distribution) and returns a more well-behaved object (a Schwartz function). In particular, the latter objects enable a straightforward incorporation of interactions into the path integral.

In chapter 3, Wilson's idea of a scale-dependent action and the resulting recurrence relation is recollected and contrasted with the related approach of the study of the 'effective average action' introduced by Wetterich. Applying the regularisation scheme from chapter 2, a set of axioms for the modified action is proposed from which a mathematically rigorous derivation of the Wetterich equation is given. The axioms cover the established choices of modifications and are general enough to allow many new ones. The derivation is rather technical and requires notions from functional analysis, convex analysis and the theory of Gaußian measures. The main points are the approximation of a 'delta distribution in path space' as shown in theorem 3.1.13 and the proof of the 'quantum equation of motion' as shown in eq. (3.1.43). The result is then precisely the Wetterich equation modulo a field-independent term, where the fields are found to lie in a vector space uniquely determined by the free theory. Because the derivation is given at a fixed regularisation, a natural question is to ask under what circumstances the regularisation can be undone in order to produce the full Quantum Field Theory. While this question is very difficult to answer in general, we find sufficient conditions that guarantee the convergence of a sequence of regulated quantum effective actions to a full effective action. The corresponding proof is based on the observation that there exist topologies on suitable sets of convex functions with respect to which the Legendre-Fenchel transform is a homeomorphism. Consequently, these topologies enable a connection between the convergence of partition functions and the convergence of quantum effective actions. In the last section of this chapter, we discuss the related question of whether the Wetterich equation survives the limit of vanishing regularisation. In this context, we also introduce the concept of asymptotic safety.

In chapter 4, an iterative scheme is presented that enables the construction of exact solutions to the non-regularised Wetterich equation. The idea is to construct higher-order correlation functions from the RG flow of lower order correlators that are consistent with the equation itself as well as possible boundary conditions. It is then applied to a scalar field theory which is afterwards proved to satisfy the unrenormalised boundary condition of a ϕ^4 model.

Chapter 5 deals with a slight generalisation of QED by a Pauli term as was already suggested in [Wei95]. In the framework of asymptotic safety, we find that in our truncation of this model there exist UV-complete RG flows that reproduce known physics in the infrared (IR) regime.

In chapter 6, we prove the applicability of the regularisation scheme presented in section 2.2 to ϕ^4 theory in arbitrary spacetime dimensions. In particular, we prove an integrability theorem for Gaußian measures akin to Fernique's theorem on exponential tails.

Chapter 7 then closes this thesis with a summary of the results and a list of potential directions for further research and applications.

For the convenience of the reader, a collection of relevant mathematical conventions and theorems is compiled in appendix A.

2. The Path Integral Formalism

In 1948 Feynman published his result [Fey05] that the probability amplitude $\langle t_f, q_f | t_i, q_i \rangle$ of a particle at position q_i at time t_i to arrive at q_f at time t_f may be interpreted as a sum over all possible paths connecting these states weighted by a complex phase. In short,

$$\langle t_f, q_f | t_i, q_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \exp\left[iS\left(q\right)\right] \mathcal{D}q\,, \tag{2.0.1}$$

where $S(q)$ denotes the classical action of the path q. The path integral measure $\mathcal{D}q$ is a priori not a measure in the mathematical sense. Instead, it is understood as a formal limit on spaces of discretised paths, i.e

$$\int_{q(t_i)=q_i}^{q(t_f)=q_f} A\left(q\right) \mathcal{D}q = \lim_{\eta\to 0} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^n} A_{\text{disc}}\left(P_x\right) \exp\left[-\frac{1}{2}\eta x^2\right] \mathrm{d}^n x}{\int_{\mathbb{R}^n} \exp\left[-\frac{1}{2}\eta x^2\right] \mathrm{d}^n x} \tag{2.0.2}$$

for functionals A on the space of paths. Here, P_x denotes the discretised path defined by

$$\begin{aligned} t_i &\mapsto x_i \\ t_i + k\frac{t_f - t_i}{n} &\mapsto x_k \qquad k \in \{1, \dots, n-1\} \\ t_f &\mapsto x_f \end{aligned} \tag{2.0.3}$$

and A_{disc} is a suitably discretised version of A e.g. an approximation of a derivative by a finite difference. The exponential factor in the numerator ensures the existence of the integrals while the denominator ascertains that $\mathcal{D}q$ behaves like a probability measure corresponding directly to the unitarity of time-evolution in quantum mechanics. More thorough derivations of the Path integral formalism can be found in introductory textbooks on Quantum Field Theory.

It is straightforward to generalise this procedure to scalar fields by introducing a grid inside a bounded region of spacetime. The boundedness is important because the grid would otherwise contain an infinite number of points. Hence, to define the path integral in QFT one has to take the infinite volume limit as well before sending η to zero as in eq. (2.0.2). More generally, spacetime discretisation and the restriction to finite volumes are examples of so-called regularisations of a QFT and there exist other regularisation schemes that can be chosen instead. In perturbative calculations common choices include the use of momentum cutoffs and dimensional regularisation [BG72; Vel+72] whereas in non-perturbative calculations the spacetime discretisation remains the most common [Sal07]. Summarising this path integral approach, we obtain the statement that Quantum Field Theory should be thought of as the limit of vanishing regularisation of a set of suitably regularised theories.

Having defined QFT in this way one could expect the calculation of probability amplitudes to be a solved problem. However, as it turns out the limits calculated with the above programme are often infinite and thus physically meaningless. The reason for this lies in the classical action S which was taken as a known object that is independent of the regularisation used for the computation. Today, it is known that such a requirement is unphysical in the sense that it may violate the predictions of classical physics at any non-vanishing regularisation. One can see why, by fixing a classical action S of a scalar field parameterised by a finite set $\{g_1, \dots, g_k\}$ of real numbers. These parameters have well-defined meanings in terms of experiments that can be performed in the setting of classical physics. On the stage of QFT, however, the meaning of these parameters changes because the outcome of an experiment is now dictated by a set of probability amplitudes which depend on $g_1, \dots, g_k$ in an inherently non-classical way. An example of this is the Feynman propagator of a scalar field with classical mass m and a self-interaction parameterised by $g \in \mathbb{R}$ which in a perturbative QFT calculation is proportional to

$$\lim_{\epsilon \to 0} \frac{1}{-p_0^2 + \vec{p}^2 + m^2 - \Sigma\left(g, m, p, R\right) - i\epsilon}\,, \tag{2.0.4}$$

where p denotes the momentum, R is taken to parameterise the regularisation and Σ is the self-energy [Wei95, Equation 10.3.15]. Because the pole of the propagator corresponds to the physical mass m_{phys}, we immediately see that there is a discrepancy between m and m_{phys}. In particular, this difference depends on R, that is on the regularisation itself. But since m_{phys} is actually fixed by experiment we can invert the relation in eq. (2.0.4) to write m in terms of g, m_{phys} and R. What we gain from this, is the insight that the parameters that enter the classical action S have to depend on the regularisation itself in order to produce physical results in QFT calculations.

Remark 2.0.1. In the above example, if $g = 0$ the self-energy Σ vanishes and the discrepancy between m and m_{phys} disappears. Such perfect correspondences appear very rarely e.g. in non-interacting theories.

Remark 2.0.2. It is not strictly necessary to demand that a calculated observable $O_{R,g_1,\dots,g_k}$ has the exact value as determined by an experiment. The correct physical requirement is rather that $\lim_{R\to 0} O_{R,g_1,\dots,g_k} = O_{\text{phys}}$, i.e. that the values agree in the limit of vanishing regularisation.

Implementing the dependence on the regularisation in the above way, certain observables are engineered to correspond to physical values while the values of other observables may still remain infinite. Whenever it suffices to know the R-dependence of a finite set of model parameters in order to remove such infinities, a theory is called 'renormalisable' [Wei95].

Remark 2.0.3. With regards to effective field theories and quantum gravity, less restrictive notions of renormalisability may be useful [Wei95].

This immediately raises the question whether there exist simpler criteria for determining the renormalisability of a given model than to check every possible observable by itself. In perturbative calculations a sufficient criterion is given by restricting the mass dimensions of the model parameters [Wei95] and detailed analyses tailored to specific physically relevant models have also been undertaken [Hoo71; Kra98]. The key to perturbative renormalisability is the nature of perturbation theory: To any order m, the collection of n-point functions with $n \leq m$ spans all observables. Hence, at a given order only finitely many quantities have to be examined. Then the inductive principle may be applied to prove renormalisability to arbitrary orders in perturbation theory. In non-perturbative settings the question is much more difficult because one cannot appeal to a similar expansion scheme.

2.1. The Free Scalar Field

As was noted in remark 2.0.1 there exist theories for which one can explicitly compute probability amplitudes. A common example is the case of a free, massive field ϕ in $d \geq 2$ spacetime dimensions with classical action

$$S\left(\phi\right) = -\frac{1}{2}\int_{\mathbb{R}^d} \sum_{\mu,\nu=0}^{d-1} \eta_{\mu\nu}\left(\partial^\mu\phi\right)\left(x\right)\left(\partial^\nu\phi\right)\left(x\right)\mathrm{d}^d x - \frac{m^2}{2}\int_{\mathbb{R}^d}\phi\left(x\right)^2\mathrm{d}^d x \tag{2.1.1}$$

for some $m \in \mathbb{R}$ where $\eta_{\mu\nu}$ denotes the Minkowski metric in the $(-,+,+,\dots)$ signature.

Now, instead of directly applying eq. (2.0.2) we shall mention commonly employed simplifications. In QFT one can use the LSZ formula [LSZ55] to express probability amplitudes in terms of correlation functions (Wightman functions [Jos65]), i.e. functions of the form

$$G_n\left(x_1,\dots,x_n\right) = \int \phi\left(x_1\right)\dots\phi\left(x_n\right)\exp\left[iS\left(\phi\right)\right]\mathcal{D}\phi\,. \tag{2.1.2}$$

Hence, it suffices to know all G_n to know all aspects of a theory. Furthermore, the celebrated Osterwalder-Schrader theorem [OS73] tells us that we can study the Wick-rotated action

$$S_{\text{eucl}}\left(\phi\right) = \frac{1}{2}\int_{\mathbb{R}^d}\sum_{a=1}^{d}\left(\partial_a\phi\right)\left(x\right)\left(\partial_a\phi\right)\left(x\right)\mathrm{d}^d x + \frac{m^2}{2}\int_{\mathbb{R}^d}\phi\left(x\right)^2\mathrm{d}^d x \tag{2.1.3}$$

instead of S by giving conditions for which we can Wick-rotate back the observables via analytic continuation. The benefit is that the corresponding integrals are much more well-behaved for S_{eucl} obliterating the need to use $\eta > 0$. A detailed study of the free, massive scalar field is given in [Sal07]. In particular, on arrives at

$$\int \phi(x)\,\phi(y)\exp\left[-S_{\text{eucl}}(\phi)\right]\mathcal{D}\phi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\exp\left[ip(x-y)\right]}{p^2+m^2}\mathrm{d}^d p\,. \tag{2.1.4}$$

It is clear that one can analytically continue this object and obtain the result of remark 2.0.1.

Apart from being computationally simpler, there is another great benefit of using the Wick-rotated action: For a free, massive scalar field theory, $\exp[-S_{\text{eucl}}]\mathcal{D}\phi$ is actually a bona fide measure enabling the use of a wealth of mathematical tools for its study. More precisely, eq. (2.1.4) along with Wick's theorem [Wic50] or its mathematical counterpart Isserlis's theorem [Iss18] uniquely determine $\exp[-S_{\text{eucl}}]\mathcal{D}\phi$ as the Gaußian measure μ on the space $\mathcal{S}(\mathbb{R})^*_\beta$ of tempered distributions with

$$\int_{\mathcal{S}(\mathbb{R}^d)^*_\beta} T(\phi)\,T(\psi)\,\mathrm{d}\mu(T) = \int_{\mathbb{R}^d}\frac{\hat{\phi}(p)\,\hat{\psi}(-p)}{p^2+m^2}\mathrm{d}^d p \tag{2.1.5}$$

for all Schwartz functions ϕ, ψ on $\mathbb{R}^d$ where the hats denote Fourier transformed quantities defined according to the convention given in definition A.1.1. Note that on the left-hand side, the tempered distribution $T \in \mathcal{S}(\mathbb{R}^d)^*_\beta$ is the integration variable such that formally

$$T(\phi) = \int_{\mathbb{R}^d} T(x_1,\ldots,x_d)\,\phi(x_1,\ldots,x_d)\,\mathrm{d}^d x\,. \tag{2.1.6}$$

Since T as a distribution is in general not defined at every point $(x_1,\ldots,x_d) \in \mathbb{R}^d$ (e.g. the Dirac delta distribution), we shall not make use of the latter notation. Consequently, eq. (2.1.4) has to be understood as a limit with ϕ and ψ tending to Dirac delta distributions δ_x and δ_y at spacetime points x and y respectively. That limit is clearly well-behaved if and only if $x \neq y$. Correspondingly, we define the Schwinger functions

$$S_n(x_1,\ldots,x_n) = \lim_{\phi_k\to\delta_{x_k}}\int T(\phi_1)\ldots T(\phi_n)\,\mathrm{d}\mu(T)\ , \tag{2.1.7}$$

which are known to be analytic except at points with $x_a = x_b$ for some $a \neq b$. The analytic continuations of S_n to the imaginary time axis then precisely correspond to G_n from which physical information may be extracted.

2.2. A Regularisation Scheme for Interacting Scalar Fields

As hinted at in the last section, the Osterwalder-Schrader theorem gives necessary and sufficient conditions (see e.g. [Sim74]) for a set of Schwinger functions S_n to possess analytic continuations to Wightman functions G_n from which a QFT in the sense of the Gårding-Wightman axioms [WG65] can be constructed. With respect to the integral formalism, a modern formulation in terms of a measure on a space of distributions may be found in the book by Glimm and Jaffe [GJ12] and a comprehensive introduction to the Gårding-Wightman axioms is given in the work by Jost [Jos65].[1] The measure introduced in eq. (2.1.5) provides a concrete example of a QFT modelled in these terms.

Thus, for any classical theory - interacting or not - if we find a corresponding measure satisfying the Osterwalder-Schrader axioms we have successfully quantised the theory. It is important to note that the naive ansatz with $S(\phi) = S^{\text{free}}(\phi) + S^{\text{int}}(\phi)$ and the intuitive correspondence

$$\exp\left[-S^{\text{int}}_{eucl}(\phi)\right]\exp\left[-S^{\text{free}}_{eucl}(\phi)\right]\mathcal{D}\phi = \exp\left[-S^{\text{int}}_{eucl}(\phi)\right]\mathrm{d}\mu^{\text{free}}(\phi) \tag{2.2.1}$$

fails. In particular, this happens because μ^{free} lives on a space of distributions while S^{int} is only defined on function spaces.[2] The most common example for this is given by the ϕ^4 interaction with $S^{\text{int}}_{eucl} \sim \int_{\mathbb{R}^d}\phi^4$ which is clearly not well-defined if ϕ is a general tempered distribution.

Hence, one typically regularises the path space $\mathcal{S}(\mathbb{R}^d)^*_\beta$ in some fashion e.g. by restricting to a finite volume and introducing a spacetime grid or enforcing a momentum cutoff, rendering the regularised path space finite-dimensional [RS19, Chapter 8]. The resulting path integrals respectively the corresponding measures are then interpreted as to reflect real physics to a degree where the regularisation is assumed to introduce only negligible effects, e.g

- physics in volumes much smaller than the finite volume introduced by the regularisation,
- physical phenomena with characteristic momentum scales much smaller than the enforced cutoff.[3]

These particular regularisations define a directed subset of $\mathbb{R}\times\mathbb{R}$ where

$$(V, p_{\text{max}}) \leq (V', p'_{\text{max}}) \qquad \Leftrightarrow \qquad V \leq V' \quad \text{and} \quad p_{\text{max}} \leq p'_{\text{max}}\,. \tag{2.2.2}$$

Hence, all correspondingly regularised measures can be collected into a net $\omega_{V,p_{\text{max}}}$ and we may interpret the requirement that they should describe *the same physics* within their respective window of applicability as a kind of gluing instruction. This gluing corresponds precisely to the dependence of the model parameters on the regularisation, i.e. on V and p_{max} in this case. This implies that for ever larger volumes and momentum cutoffs one expects to approach a limit that encompasses *all* physical phenomena. If that limit is indeed a QFT, we may phrase

[1] Strictly speaking, the conditions given by Glimm and Jaffe are stronger and thus not necessary for the existence of a corresponding Gårding-Wightman QFT. For details, see e.g. [Sim74; Frö74].

[2] In fact, μ^{free} is supported on much smaller function subspaces of $\mathcal{S}(\mathbb{R}^d)^*_\beta$ but even on these spaces S^{int} remains ill-defined.

[3] These conditions are debatable in view of the Euclideanised theory not living on physical Minkowski space. In particular, Euclidean volume/momentum do not have exact Minkowski counterparts.

this requirement as the existence of a measure ω satisfying the Osterwalder-Schrader axioms and

$$\lim_{V,p_{\max}\to\infty} \omega_{V,p_{\max}} = \omega \tag{2.2.3}$$

in an appropriate sense. As outlined in the beginning of the chapter, the existence of such a limit generally requires *renormalising* the classical action S, e.g. in the case of having an interaction term $\lambda \int \phi^4$, promoting $\lambda \in \mathbb{R}$ to a function depending on V and $p_{\max}$.

It is quite clear that the choice of regularisation scheme immensely affects concrete calculations. For instance, the finite volume setting is typically implemented as a compactification of $\mathbb{R}^d$ to a torus leading to discrete eigenvalues of the Laplacian, i.e. discrete admissible momenta. The personal belief of the author, however, is that a regularisation scheme maintaining as much smoothness as possible is desirable for computational and analytic methods alike. This is particularly evident in the case of retaining the symmetries of models: By keeping the spacetime intact, e.g. rotational symmetry can be left unbroken in contrast to a lattice approach.

Hence, we shall introduce a regularisation scheme (originally published in [Zie21a]) which works without finite volumes and discretisations, by effectively using smeared-out cutoffs instead. We begin by promoting the classical action S to a family $(S_n)_{n\in\mathbb{N}}$ where n denotes the regularisation parameter and the n-dependence of S_n is just the flow of the model parameters with n. Moreover, the limit $n \to \infty$ will be taken to correspond to the removal of the regularisation.

Definition 2.2.1. From here on we let $(\cdot,\cdot)$ denote the bilinear inner product on the real vector space $L^2(\mathbb{R}^d)$.

1. For each $n \in \mathbb{N}$, split S_n into a free part S_n^{free} and an interacting part S_n^{int} such that $S_n^{\text{free}} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$,

$$\phi \mapsto (\phi, B_n\phi) \tag{2.2.4}$$

for some continuous, bijective, linear operator B_n on $\mathcal{S}(\mathbb{R}^d)$ which is bounded from below in the sense that there is an $\eta_n > 0$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$(\phi, B_n\phi) \geq \eta_n\,(\phi,\phi)\ . \tag{2.2.5}$$

The prototypical example is of course $B_n = m_n^2 - \Delta$ for a free, scalar, massive field theory on $\mathbb{R}^d$. Define the corresponding centred Gaußian probability measure μ_n on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ given by its characteristic function (see appendix A.4)

$$\hat{\mu}_n(\phi) = \exp\left[-\frac{1}{2}\left(\phi, B_n^{-1}\phi\right)\right] \tag{2.2.6}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then, μ_n has full topological support (see appendix A.4) by [Bog98, theorem 3.6.1] and encodes the free theory determined by S_n^{free}.

2. Fix a sequence $(\mathcal{R}_n)_{n\in\mathbb{N}}$ of linear and continuous operators $\mathcal{S}(\mathbb{R}^d)^*_\beta \to \mathcal{S}(\mathbb{R}^d)$ with dense ranges such that
$$\lim_{n\to\infty} (\mathcal{R}_n T, \phi) = T(\phi) \tag{2.2.7}$$
for all $T \in \mathcal{S}(\mathbb{R}^d)^*_\beta$ and all $\phi \in \mathcal{S}$. A simple example is given by choosing
$$\chi_n(x) = \exp\left[-\frac{1}{2}\frac{\|x\|^2}{n^2K^2}\right], \qquad \xi_n(x) = \left(\frac{n^2\Lambda^2}{2\pi}\right)^{d/2} \exp\left[-\frac{n^2\Lambda^2}{2}\|x\|^2\right] \tag{2.2.8}$$
for some $K, \Lambda > 0$, all $x \in \mathbb{R}^d$ and setting $\mathcal{R}_n T = \chi_n \cdot (\xi_n * T)$ where $*$ denotes convolution of functions. Then, $n\Lambda$ can be viewed as a momentum cutoff and nK as the radius of the ball to which we restrict the theory. The contributions outside these physical windows are not cut off completely, but are heavily suppressed by the exponential decays of χ_n and ξ_n respectively. Furthermore, these particular $\mathcal{R}_n$ commute with $\mathcal{O}(\mathbb{R}^d)$ such that rotational invariance is kept intact.

3. For all $n \in \mathbb{N}$ define the pushforward Borel probability measures $\nu_n = \mu_n \circ \mathcal{R}_n^{-1}$ (see appendix A.4) and
$$\omega_n = \left(\frac{\exp\left[-S_n^{\text{int}}\right]}{\int_{\mathcal{S}(\mathbb{R}^d)} \exp\left[-S_n^{\text{int}}\right] \mathrm{d}\nu_n} \cdot \nu_n\right) \circ \iota^{-1} \tag{2.2.9}$$
on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)^*_\beta$ respectively where ι denotes the continuous inclusion of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)^*_\beta$ given by the inner product $(\cdot,\cdot)$. Note, that the latter is only well-defined if $\exp[-S_n^{\text{int}}] \in L^1(\nu_n)$. Furthermore, by [Bog98, theorem 3.2.3], the density of the range of $\mathcal{R}_n$ and [Bog98, theorem 3.6.1], ν_n has full topological support.

In analogy to eq. (2.2.3), we now have a sequence $(\omega_n)_{n\in\mathbb{N}}$ of measures corresponding to regularised theories and the ultimate goal is to find a limit ω satisfying the Osterwalder-Schrader axioms.

Remark 2.2.2. The use of the $L^2(\mathbb{R}^d)$ inner product in eqs. (2.2.4) and (2.2.9) could be generalised to any other continuous inner product on $\mathcal{S}(\mathbb{R}^d)$. However, in most applications the $L^2(\mathbb{R}^d)$ one suffices and is the simplest to work with such that there is hardly any practical benefit in admitting more general structures.

Remark 2.2.3. To demand that $\exp[-S_n^{\text{int}}] \in L^1(\nu_n)$ is strictly necessary for this regularisation scheme to work. In view of theorem 6.0.1, this is certainly the case for ϕ^4 theory and should be valid for most models in consideration.

3. The Functional Renormalisation Group

As we shall see in section 3.4, a simple sufficient condition for the convergence of ω_n can be phrased in terms of the convergence of the functions

$$Z_n(\phi) = \int_{\mathcal{S}(\mathbb{R}^d)^*_\beta} \exp[T(\phi)] \, \mathrm{d}\omega_n(T) \qquad \phi \in \mathcal{S}\left(\mathbb{R}^d\right) , \tag{3.0.1}$$

i.e. the moment-generating functions (resp. partition functions or generating functionals) (see appendix A.4). It is, however, very difficult to obtain explicit formulae for Z_n.

To see how this problem can be attacked, let us consider a finite volume with a sharp momentum cutoff regularisation and return to the functional integral notation. Then the partition function for a source J reads

$$Z_{V,\Lambda}(J) = \int_{p^2<\Lambda} \exp\left[\int J\phi - S^\Lambda(\phi)\right] \mathcal{D}_V\phi \,, \tag{3.0.2}$$

where S is taken to be a Euclideanised action. Wilson [Wil75] formulated the idea that the partition function does not change, if one integrates out physics above a certain momentum scale first, i.e. for $\Lambda' \leq \Lambda$

$$Z_{V,\Lambda}(J) = \int_{p^2<\Lambda'} \left(\int_{\Lambda'\leq p^2<\Lambda} \exp\left[\int J(\phi+\psi) - S^\Lambda(\phi+\psi)\right] \mathcal{D}_V\phi\right) \mathcal{D}_V\psi \,. \tag{3.0.3}$$

Consequently, there is a modified action $S_J^{\Lambda,\Lambda'}$ with

$$\begin{aligned} S_J^{\Lambda,\Lambda'}(\psi) &= -\ln \int_{\Lambda'\leq p^2<\Lambda} \exp\left[\int J\phi - S^\Lambda(\phi+\psi)\right] \mathcal{D}_V\phi \,, \\ Z_{V,\Lambda}(J) &= \int_{p^2<\Lambda'} \exp\left[\int J\psi - S_J^{\Lambda,\Lambda'}(\psi)\right] \mathcal{D}_V\psi \,. \end{aligned} \tag{3.0.4}$$

Clearly, for all $0 \leq \Lambda'' \leq \Lambda' \leq \Lambda$

$$\begin{aligned}
&Z_{V,\Lambda}(J) = \exp\left[-S_J^{\Lambda,0}(\psi)\right] , \qquad \text{i.e. } S_J^{\Lambda,0} \text{ is actually constant,} \\
&S_J^{\Lambda,\Lambda}(\phi) = S^{\Lambda}(\phi) , \\
&S_J^{\Lambda,\Lambda''}(\psi) = -\ln \int_{\Lambda'' \leq p^2 < \Lambda'} \exp\left[\int J\phi - S_J^{\Lambda,\Lambda'}(\phi+\psi)\right] \mathcal{D}_V \phi ,
\end{aligned} \tag{3.0.5}$$

giving a recurrence relation where the known boundary condition is the classical action S^{Λ} and the other, i.e. $S_J^{\Lambda,0}$ corresponding to the partition function is the one to be computed. This set of equations is commonly referred to as the exact renormalisation group (ERG). The concept was then used by Wegner and Houghton [WH73] and later by Polchinski [Pol84] to turn the recurrence relation into a differential equation. Since then, many results have been obtained by deriving bounds on $Z_{V,\Lambda}$ from the structure of the differential equation and by performing numerical simulations. Some notable recent developments include the proof of the convergence of the operator product expansion [HK12] as well as the proof of the triviality of ϕ^4 theory as obtained with a spacetime discretisation [AD21].

Another path was taken by Wetterich in [Wet93] by considering slightly modified partition functions (we now ignore extra indices coming from a regularisation)

$$Z_k(J) = \int \exp\left[\int J\phi - S(\phi) - \frac{1}{2} F_k(\phi,\phi)\right] \mathcal{D}\phi , \tag{3.0.6}$$

where $k \geq 0$ and F_k is a bilinear operator that essentially (a suggested set of axioms is given in axiom 3.1.3) is proportional to k^2 such that Z_0 is the unaltered partition function of interest. In order to emphasise that $F_k/2$ amounts to a deformation of the classical action, it is commonly written as ΔS_k in existing literature, but we shall not use that notation here. The idea is now to study the k-dependence of the **effective average action**

$$\Gamma_k(\phi) = \sup_J \left[\int J\phi - \ln Z_k(J)\right] - \frac{1}{2} F_k(\phi,\phi) . \tag{3.0.7}$$

The resulting differential equation is dubbed the Wetterich equation or the 'functional renormalisation group' (FRG) and the boundary condition is given by $\lim_{k\to\infty} \Gamma_k = S$. Γ_0 (the quantum effective action) is the object to be computed and it can be used to calculate the Schwinger functions of the model in question. One should note that there is a priori no explicit connection between the regularisation and the scale k as in the Wilsonian approach, where $\Lambda' \leq \Lambda$ are physical quantities.[1]

Remark 3.0.1. It is not uncommon to only admit $k \in [0, \Lambda]$ when using a momentum cutoff

[1] Again, the interpretation of Λ and Λ' as physical momentum scales is debatable because Euclidean momenta do not have direct Minkowski counterparts.

regularisation. Then F_k can be taken as proportional to $k^2\Lambda^2/(\Lambda^2 - k^2)$ instead. A physical interpretation of k (with the usual Euclidean vs. Minkowski reservations) can then be obtained by demanding F_k to give large ($\sim k^2$) contributions to momentum modes below k. Then F_k effectively suppresses quantum fluctuations with momenta below k [Gie12].

The Wetterich equation has been used to derive results in QFT models [GMR20; GZ20], condensed matter physics [SBK05] as well as in hydrodynamics [CDW16] and statistical mechanics [BW13]. It is also routinely used in studies of asymptotic safety scenarios of quantum gravity [RS19; Per17]. For reviews and further applications, see [Dup+20; Met+12; BTW02; FP07; Gie12; Del12; Bra12; Nag14].

3.1. A Rigorous Derivation of the Wetterich Equation

While the Wetterich equation has enjoyed much popularity, until recently, no mathematically rigorous derivation was available. Regarding the achievements made possible by Polchinski's equation this situation was somewhat unfortunate and hence a motivation for publishing [Zie21a]. Here, we shall present the same derivation using the regularisation scheme introduced in section 2.2. Clearly, we need to specify a set of axioms for the operators F_k used in eq. (3.0.6) and we shall also slightly strengthen the requirements on the regularised action S_n^{int}. While the proof is of purely mathematical nature, a discussion in physical terms is given in the next section.

Keeping in mind that all objects in this section shall be taken at a specific regularisation index $n \in \mathbb{N}$, we shall not write these indices explicitly in order to enhance legibility.

Let ν and S^{int} be as in section 2.2.

Axiom 3.1.1. We demand that

- $S^{\text{int}} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ is continuous,
- there is a $q > 1$ such that $\exp[-S^{\text{int}}] \in L^q(\nu)$,
- there is a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)$ and $C > 0$ such that $\exp[-S^{\text{int}}] \leq C \exp[p^2]$.

Remark 3.1.2. The continuity is important for the boundary condition at $k \to \infty$ and it also ensures the strict positivity of $\exp[-S^{\text{int}}]$ as well as its boundedness on compacta. The second condition is only slightly stronger than $\exp[-S^{\text{int}}] \in L^1(\nu)$, which is necessary for the regularisation scheme to work and enables the use of Hölder's inequality.

The author conjectures that the third condition can be relaxed significantly, if not even removed completely. A corresponding discussion is presented in remark 3.2.1.

Let us denote the Cameron-Martin space of ν by $H(\nu) \subset \mathcal{S}(\mathbb{R}^d)$ and the closure of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\nu)$ by $\mathcal{S}(\mathbb{R}^d)^*_\nu$ (see appendix A.4 for details). Furthermore, $R_\nu : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ is the corresponding Hilbert space isomorphism and we shall write

$$\langle K, T \rangle := \int_{\mathcal{S}(\mathbb{R}^d)} K(\psi) \, T(\psi) \, \mathrm{d}\nu(\psi) \tag{3.1.1}$$

for all $T, K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. The spaces $H(\nu)$ and $\mathcal{S}(\mathbb{R}^d)^*_\nu$ will be of central importance in the derivation of the Wetterich equation.

Axiom 3.1.3. We demand that $(F_k)_{k\in\mathbb{R}}$ is a family of symmetric bilinear operators on $\mathcal{S}(\mathbb{R}^d)$ with the following properties:

- *For easy differentiation, allow negative values of k:*
  - $\forall\, \phi \in \mathcal{S}(\mathbb{R}^d), k < 0 : \quad F_k(\phi, \phi) = 0,$
- *For the Dirac delta measure approximation (see theorem 3.1.13):*
  - $$\forall\, \phi \in \mathcal{S}(\mathbb{R}^d), k \geq 0 : \quad 0 \leq F_k(\phi, \phi) \leq k^2 (\phi, \phi), \tag{3.1.2}$$
  - $$\forall\, \phi \in \mathcal{S}(\mathbb{R}^d)\, \exists\, C, K > 0\, \forall\, k \geq K : \quad F_k(\phi, \phi) \geq C k^2 (\phi, \phi), \tag{3.1.3}$$
- *For differentiability in lemmas 3.1.5 and 3.1.6:*
  - F is pointwise continuously k-differentiable, i.e. for all $k \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$

    $$F'_k(\phi, \phi) := \lim_{t \to 0} \frac{F_{k+t}(\phi, \phi) - F_k(\phi, \phi)}{t} \tag{3.1.4}$$

    exists and is jointly continuous in k and ϕ,
  - The above convergence is uniform in ϕ in the sense that there is a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)$ such that for all $k \in \mathbb{R}$ and all $\epsilon > 0$ there exists some $\delta > 0$ as well as a function $o : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{t\to 0} o(t)/t = 0$ and

    $$\left|F_{k+t}(\phi, \phi) - F_k(\phi, \phi) - t F'_k(\phi, \phi)\right| < \epsilon o(t) \, p(\phi)^2 \tag{3.1.5}$$

    for all $t \in (-\delta, \delta)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$,
- *For positivity and interchange of integrations in theorem 3.1.12:*
  - For all $k \in \mathbb{R}$ there is a σ-finite measure space $(X^k, \mathcal{A}^k, m_k)$ (see appendix A.4) and a mapping $U^k : X^k \to (\mathcal{S}(\mathbb{R}^d)^*_\nu)_{\mathbb{C}}$ (see appendix A.2) such that $X^k \to \mathbb{R}, x \mapsto U^k_x(\phi)$ is m_k-measurable for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and

    $$F'_k(\phi, \phi) = \int_{X^k} \left|U^k_x(\phi)\right|^2 \mathrm{d}m_k(x) \ . \tag{3.1.6}$$

While these conditions look very technical, common choices (note the absence of prefactors for separate wave-function renormalisation) like

- $F_k(\phi, \phi) = \int_{\mathbb{R}^d} \left|\hat{\phi}(p)\right|^2 \left(k^2 - \|p\|^2\right) \theta\left(k^2 - \|p\|^2\right) \mathrm{d}p$ a.k.a the Litim regulator,
- $F_k(\phi, \phi) = \int_{\mathbb{R}^d} \left|\hat{\phi}(p)\right|^2 \frac{\|p\|^2}{\exp\left[\frac{\|p\|^2}{k^2}\right]-1} \mathrm{d}p$ a.k.a the exponential regulator,

are included.

Remark 3.1.4. Just as one could generalise to other continuous inner products in eq. (2.2.4), one can generalise the upper and lower F_k bounds in eqs. (3.1.2) and (3.1.3) to other continuous seminorms. But since the established choices of F_k all work with the $L^2(\mathbb{R}^d)$ inner product there is little practical reason to do so.

For brevity, define

$$N_k = \int_S \exp\left[-S^{\text{int}}(\psi) - \frac{1}{2} F_k(\psi, \psi)\right] \mathrm{d}\nu(\psi) \ , \tag{3.1.7}$$

$$f_k(\phi) = \exp\left[-S^{\text{int}}(\phi) - \frac{1}{2} F_k(\phi, \phi)\right] \tag{3.1.8}$$

for all $k \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then we may consider the family $\{f_k/N_k \cdot \nu : k \in \mathbb{R}\}$ of probability measures on $\mathcal{S}(\mathbb{R}^d)$ and in view of the last section the object of interest is $f_0/N_0 \cdot \nu$. By the properties of S^{int} we clearly have that $f_k : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ is strictly positive and there exists a $q \in (1, \infty]$ such that $f_k \in L^q(\nu)$ for all $k \in \mathbb{R}$. Let us now define a family $Z : \mathbb{R} \times \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R}$ of moment-generating functions as

$$Z_k(T) = \frac{1}{N_k} \int_S \exp\left[T(\psi)\right] f_k(\psi) \, \mathrm{d}\nu(\psi) \ . \tag{3.1.9}$$

By employing the Cameron-Martin theorem (see appendix A.4), we have

$$Z_k(T) = \frac{1}{N_k} \exp\left[\frac{1}{2} \langle T, T \rangle\right] \int_S f_k(\psi + R_\nu T) \, \mathrm{d}\nu(\psi) \ . \tag{3.1.10}$$

for all $k \in \mathbb{R}$ and $T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ such that Z is indeed well-defined (everywhere finite). Also, by virtue of [Bog98, Theorem 2.4.8] each $Z_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R}$ is continuous. A straightforward calculation - that we shall omit here - shows that one may differentiate under the integral sign:

Lemma 3.1.5. *Z is continuously Fréchet differentiable and its derivative at (k, J) is given by*

$$D_{k,T} Z = \begin{pmatrix} -\frac{1}{2N_k} \int_S F_k'(\psi, \psi) \exp\left[T(\psi)\right] f_k(\psi) \, \mathrm{d}\nu(\psi) - Z_k(T) \, \partial_k \ln N_k \\ \frac{1}{N_k} \int_S \psi \exp\left[T(\psi)\right] f_k(\psi) \, \mathrm{d}\nu(\psi) \ , \end{pmatrix} \tag{3.1.11}$$

where the term in the second row ($D_T Z_k$) may be understood as a (generalised) Bochner integral in $\mathcal{S}(\mathbb{R}^d)$ [Tho75, theorem 3] and in fact $D_T Z_k \in H(\nu)$.

Let us also define $Y : \mathbb{R} \times \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ with $(k, T) \mapsto D_T Z_k$.

Lemma 3.1.6. Y is continuously Fréchet differentiable and its derivative at (k, J) is given by

$$(D_{k,J}Y)(l,T) = \begin{pmatrix} -\frac{l}{2N_k}\int_{\mathcal{S}} \psi F_k'(\psi,\psi) \exp\left[J(\psi)\right] f_k(\psi)\,\mathrm{d}\nu(\psi) - lD_J Z_k \cdot \partial_k \ln N_k \\ \frac{1}{N_k}\int_{\mathcal{S}} \psi T(\psi) \exp\left[J(\psi)\right] f_k(\psi)\,\mathrm{d}\nu(\psi) \end{pmatrix} \tag{3.1.12}$$

for all $l \in \mathbb{R}, T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. Both integrals may again be understood as generalised Bochner integrals in $\mathcal{S}(\mathbb{R}^d)$ with values in $H(\nu)$.

These properties are inherited by $W : \mathbb{R} \times \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R}, (k, T) \mapsto \ln Z_k(T)$, i.e. W is continuously differentiable, W_k is twice continuously differentiable with $DW_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$. As a matter of fact, DW_k even turns out to be a bijection between $\mathcal{S}(\mathbb{R}^d)^*_\nu$ and $H(\nu)$. The injectivity follows directly from the following positivity property of $D^2 W_k$.

Theorem 3.1.7. For all $k \in \mathbb{R}$ and $J \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ there exists a $C > 0$ such that for all $K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$

$$K\left[\left(D_J^2 W_k\right)(K)\right] \geq C\,\langle K, K\rangle\ . \tag{3.1.13}$$

Proof. By Hölder's inequality,

$$\begin{aligned} K\left[\left(D_J^2 W_k\right)(K)\right] &= \frac{1}{N_k^2 Z_k(J)^2}\int_{\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)} \left[K(\psi)^2 - K(\psi)K(\phi)\right] \\ &\quad \times \exp\left[J(\psi)+J(\phi)\right] f_k(\psi) f_k(\phi)\,\mathrm{d}(\nu\times\nu)(\psi,\phi) \\ &\geq \frac{1}{N_k^2 Z_k(J)^2}\int_{\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)} \left[|K(\psi)K(\phi)| - K(\psi)K(\phi)\right] \\ &\quad \times \exp\left[J(\psi)+J(\phi)\right] f_k(\psi) f_k(\phi)\,\mathrm{d}(\nu\times\nu)(\psi,\phi) \\ &= \frac{1}{N_k^2 Z_k(J)^2}\int_{\mathcal{S}(\mathbb{R}^d)} \left[|K(\psi)| - K(\psi)\right] \exp\left[J(\psi)\right] f_k(\psi)\,\mathrm{d}\nu(\psi) \\ &\quad \times \int_{\mathcal{S}(\mathbb{R}^d)} \left[|K(\psi)| + K(\psi)\right] \exp\left[J(\psi)\right] f_k(\psi)\,\mathrm{d}\nu(\psi) \end{aligned} \tag{3.1.14}$$

which clearly is nonnegative. Let us now suppose that there is a sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^d)^*_\nu$ with $\langle K_n, K_n\rangle = 1$ such that the first integral tends to zero, i.e

$$\lim_{n\to\infty}\int_{\mathcal{S}} \left[|K_n(\psi)| - K_n(\psi)\right] \exp\left[J(\psi)\right] f_k(\psi)\,\mathrm{d}\nu(\psi) = 0\,. \tag{3.1.15}$$

Then, there exists a subsequence $(L_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} |L_n(\psi)| - L_n(\psi) = 0$ for ν-almost every $\psi \in \mathcal{S}(\mathbb{R}^d)$. But since each L_n can in turn be written as the ν-almost everywhere pointwise limit of linear functions, the above implies $\lim_{n\to\infty} L_n(\psi) = 0$ for ν-almost every ψ. One arrives at the same conclusion if one takes the second integral to go to zero instead. By the finiteness of ν we thus have that $L_n \to 0$ in ν-measure as $n \to \infty$. Now, let $\epsilon > 0$ and pick any ν-measurable $A \subset \mathcal{S}(\mathbb{R}^d)$ with $\nu(A) < \epsilon^2/3$. Then, for all $n \in \mathbb{N}$

$$\int_A L_n(\psi)^2 \, \mathrm{d}\nu(\psi) \leq \sqrt{\int_S L_n(\psi)^4 \, \mathrm{d}\nu(\psi)} \sqrt{\nu(A)} < \langle L_n, L_n \rangle \, \epsilon = \epsilon \, . \tag{3.1.16}$$

Thus, Vitali's convergence theorem tells us that L_n goes to zero in $L^2(\nu)$ i.e. in $\mathcal{S}(\mathbb{R}^d)^*_\nu$ for $n \to \infty$ which is a contradiction. □

Corollary 3.1.8. $DW_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ is injective for all $k \in \mathbb{R}$.

Proof. Suppose that $D_J W_k = D_K W_k$ for some $J, K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. Then, by Rolle's theorem, there is a $t \in [0,1]$ such that

$$(J - K)\left[\left(D^2_{tJ+(1-t)K} W_k\right)(J - K)\right] = 0 \, . \tag{3.1.17}$$

By theorem 3.1.7 this can happen only if $J = K$. □

Let us also record the following corollary which will be of paramount importance.

Corollary 3.1.9. For all $k \in \mathbb{R}$ and $J \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ the linear map $D^2_J W_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ is continuously invertible.

Proof. The bilinear form $\mathcal{S}(\mathbb{R}^d)^*_\nu \times \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R}$ given by

$$(K, L) \mapsto K\left[\left(D^2_J W_k\right)(L)\right] \tag{3.1.18}$$

is symmetric which can be seen from writing it out explicitly. By theorem 3.1.7 it is bounded from below. By continuity $R_\nu^{-1} \circ D^2_J W_k$ is thus self-adjoint, continuous and injective and as such has dense range in $\mathcal{S}(\mathbb{R}^d)^*_\nu$. Since it is also bounded from below it is continuously invertible. □

The surjectivity of DW_k is substantially more involved.

Theorem 3.1.10. $DW_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ is surjective for all $k \in \mathbb{R}$.

Proof. Let $\phi \in H(\nu)$. Then $J \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ solves the equation $D_J W_k = \phi$ if and only if for all $K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$

$$\int_{\mathcal{S}} K(\psi - \phi) \exp\left[J(\psi)\right] f_k(\psi) \, \mathrm{d}\nu(\psi) = 0 \, . \tag{3.1.19}$$

But since $\phi \in H(\nu)$ we may apply the Cameron-Martin theorem to obtain the equivalent condition

$$\int_{\mathcal{S}} K(\psi) \exp\left[J(\psi) - \left(R_\nu^{-1}\phi\right)(\psi)\right] f_k(\psi + \phi) \, \mathrm{d}\nu(\psi) = 0 \tag{3.1.20}$$

for all $K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. Let us make the ansatz $J = R_\nu^{-1}\phi + H$ for some $H \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. Then the above is true precisely when H minimises the convex function

$$M_\phi : \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R} \qquad T \mapsto \int_{\mathcal{S}} \exp\left[T\left(\psi\right)\right] f_k\left(\psi + \phi\right) \mathrm{d}\nu\left(\psi\right) . \tag{3.1.21}$$

M_ϕ is clearly well-defined and continuous because it admits the representation

$$M_\phi\left(T\right) = \exp\left[\frac{1}{2}\left\langle T, T\right\rangle\right] \int_{\mathcal{S}} f_k\left(\psi + \phi + R_\nu T\right) \mathrm{d}\nu\left(\psi\right) \tag{3.1.22}$$

in analogy to Z_k in eq. (3.1.10). We shall now assume that $(H_n)_{n\in\mathbb{N}}$ is a minimising sequence of M_ϕ and the goal is to show that there is some bounded subsequence.

Since f_k is continuous and $\mathcal{S}(\mathbb{R}^d)$ admits continuous norms, there is a continuous norm p on $\mathcal{S}(\mathbb{R}^d)$ and some $\delta > 0$ such that

$$\forall \psi \in \mathcal{S}(\mathbb{R}^d): \quad p\left(\psi\right) \leq \delta \implies f_k\left(\phi + \psi\right) \geq \frac{1}{2} f_k\left(\phi\right) . \tag{3.1.23}$$

Consider the three mutually exclusive cases

1: $\limsup_{n\to\infty} p\left(R_\nu H_n\right) = 0$,

2: there is a subsequence $(K_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p\left(R_\nu K_n\right) \in (0, \infty)$,

3: $\liminf_{n\to\infty} p\left(R_\nu H_n\right) = \infty$.

1: Suppose there exists a continuous seminorm q on $\mathcal{S}(\mathbb{R}^d)$ such that

$$\limsup_{n\to\infty} \left(p + q\right)\left(R_\nu H_n\right) \neq 0 . \tag{3.1.24}$$

We may then replace the previously used norm p by $p + q$ such that we land either in case 2 or 3. If we have $\limsup_{n\to\infty} \left(p + q\right)\left(R_\nu H_n\right) = 0$ for all continuous seminorms q on $\mathcal{S}(\mathbb{R}^d)$, we clearly have that

$$B := \overline{\left\{R_\nu H_n : n \in \mathbb{N}\right\}} + \left\{\phi\right\} \subset \mathcal{S}(\mathbb{R}^d) \tag{3.1.25}$$

is compact. Also since ν is Radon there is a compact set $C \subseteq \mathcal{S}(\mathbb{R}^d)$ such that $\nu\left(C\right) > 0$. At the same time,

$$M_\phi\left(H_n\right) \geq \exp\left[\frac{1}{2}\left\langle H_n, H_n\right\rangle\right] \int_C f_k\left(\psi + \phi + R_\nu H_n\right) \mathrm{d}\nu\left(\psi\right) . \tag{3.1.26}$$

Now, f_k is continuous such that it attains its infimum on $B + C$ which cannot be zero. Hence,

$$\inf_{\psi\in C} f_k\left(\psi + \phi + R_\nu H_n\right) \geq \inf_{\psi\in B+C} f_k\left(\psi\right) := \alpha > 0 \tag{3.1.27}$$

and

$$M_\phi\left(H_n\right) \geq \alpha \exp\left[\frac{1}{2}\left\langle H_n, H_n\right\rangle\right]\nu\left(C\right) . \tag{3.1.28}$$

But then, H_n cannot be unbounded in $\mathcal{S}(\mathbb{R}^d)^*_\nu$ since it is a minimising sequence of M_ϕ.

For the remaining two cases we shall restrict the integral to a ball of radius $0 < r \leq \delta$ in the norm p. Furthermore, ν_p, the pushforward measure of ν to $\mathcal{S}(\mathbb{R}^d)_p$ via the natural map $\iota_p : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)_p$ (see appendix A.2), is a Gaußian measure on a Banach space and we obtain

$$M_\phi\left(H_n\right) \geq \frac{1}{2} f_k\left(\phi\right) \exp\left[\frac{1}{2}\left\langle H_n, H_n\right\rangle\right]\nu_p\left(\overline{B_r\left(-\iota_p R_\nu H_n\right)}\right) \tag{3.1.29}$$

for all $n \in \mathbb{N}$. Here, $\overline{B_r\left(-\iota_p R_\nu H_n\right)}$ denotes the closed ball of radius r around $-\iota_p R_\nu H_n$ in $\mathcal{S}(\mathbb{R}^d)_p$. Note that the adjoint $\iota_p^* : (\mathcal{S}(\mathbb{R}^d)_p)^* \to \mathcal{S}(\mathbb{R}^d)^*_\beta$ (see appendix A.2) has dense range because ι_p is injective since p is a norm and not just a seminorm (see [SW99, Chapter 4, §4, Corollary 2.3]) and $\mathcal{S}(\mathbb{R}^d)$ is reflexive. Hence, by the continuity of M_ϕ we may assume that $H_n \in \iota_p^*(\mathcal{S}(\mathbb{R}^d)_p)^*$ for all $n \in \mathbb{N}$. Furthermore, letting R_{ν_p} denote the Hilbert isomorphism between the closure of $(\mathcal{S}(\mathbb{R}^d)_p)^*$ in $L^2(\nu_p)$ and $H(\nu_p)$, it is straightforward to verify that $\iota_p R_\nu \iota_p^*$ is equal to the restriction of R_{ν_p} to $(\mathcal{S}(\mathbb{R}^d)_p)^*$.

<u>2</u>: Restrict to a subsequence $(\iota_p^* K_n)_{n\in\mathbb{N}}$ of $(H_n)_{n\in\mathbb{N}}$ with

- $\lim_{n\to\infty} p\left(R_\nu \iota_p^* K_n\right) =: P \in (0,\infty)$,
- $\inf_{n\in\mathbb{N}} p\left(R_\nu \iota_p^* K_n\right) > \frac{1}{2}P$,
- $\sup_{n\in\mathbb{N}} p\left(R_\nu \iota_p^* K_n\right) < 2P$.

Set $\gamma = \min\{\delta, P/2\}$ and in accordance with [KLL94, Corollary 7] (considering $t = 1$ only)

$$r = \frac{\gamma}{4} \qquad \text{and} \qquad \epsilon = 1 - \frac{3}{4}\frac{\gamma}{P} \in (0,1) . \tag{3.1.30}$$

Then, $r < (1-\epsilon)\, p\left(R_\nu \iota_p^* K_n\right)$ for all $n \in \mathbb{N}$. Now, define

$$g_n = -\left(1 - \frac{\epsilon\gamma}{8P}\right) R_{\nu_p} K_n \quad \text{implying} \quad p\left(-R_{\nu_p} K_n - g_n\right) = \frac{1}{8}\epsilon\gamma \frac{p\left(R_\nu \iota_p^* K_n\right)}{P} \leq \frac{1}{4}\epsilon\gamma = \epsilon r \tag{3.1.31}$$

and note that $g_n \in R_{\nu_p}(\mathcal{S}(\mathbb{R}^d)_p)^*$ for all $n \in \mathbb{N}$. Hence, by [KLL94, Corollary 7],

$$\nu_p\left(\overline{B_r\left(-\iota_p R_\nu K_n\right)}\right) \geq \exp\left[-\frac{1}{2}\left(1 - \frac{\epsilon\gamma}{8P}\right)^2 \left\langle \iota_p^* K_n, \iota_p^* K_n\right\rangle\right]\nu_p\left(\overline{B_{(1-\epsilon)r}\left(0\right)}\right) \tag{3.1.32}$$

which combines with eq. (3.1.29) to

$$M_\phi\left(\iota_p^* K_n\right) \geq \frac{1}{2} f_k\left(\phi\right) \exp\left(\frac{1}{2}\left[1 - \left(1 - \frac{\epsilon\gamma}{8P}\right)^2\right]\left\langle \iota_p^* K_n, \iota_p^* K_n\right\rangle\right)\nu_p\left(\overline{B_{(1-\epsilon)r}\left(0\right)}\right) \tag{3.1.33}$$

for all $n \in \mathbb{N}$. Because ν has full topological support, any ball in $\mathcal{S}_p$ has has nonzero measure. Furthermore, $\epsilon\gamma < 8P$ such that $(\iota_p^* K_n)_{n\in\mathbb{N}}$ must be bounded since it is a minimising sequence of M_ϕ.

3: Restrict to a subsequence $(\iota_p^* K_n)_{n\in\mathbb{N}}$ of $(H_n)_{n\in\mathbb{N}}$ with $p\left(R_\nu \iota_p^* K_n\right) > 2\delta$ for all $n \in \mathbb{N}$ and with the same notation as before, set

$$\epsilon = \frac{1}{2}, \qquad r = \delta \qquad \text{and} \qquad g_n = -\left(1 - \frac{\delta}{2p\left(R_{\nu_p} K_n\right)}\right) R_{\nu_p} K_n \tag{3.1.34}$$

for all $n \in \mathbb{N}$. Then, clearly

$$r < (1-\epsilon)\, p\left(R_{\nu_p} K_n\right) \qquad \text{and} \qquad p\left(-R_{\nu_p} K_n - g_n\right) \le \epsilon r\,, \tag{3.1.35}$$

such that

$$\nu_p\left(\overline{B_r\left(-R_\nu \iota_p^* K_n\right)}\right) \ge \exp\left[-\frac{1}{2}\left(1 - \frac{\delta}{2p\left(R_{\nu_p} K_n\right)}\right)^2 \left\langle \iota_p^* K_n, \iota_p^* K_n \right\rangle\right] \nu_p\left(\overline{B_{(1-\epsilon)r}\left(0\right)}\right) \tag{3.1.36}$$

for all $n \in \mathbb{N}$. Now, note that by [Bog98, Theorem 3.2.10(i)], there is a $C > 0$ such that $p(R_\nu K) \le C\sqrt{\langle K, K\rangle}$ for all $K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. Then, since $p(R_{\nu_p} K_n) > \delta/2$ we arrive at

$$\nu_p\left(\overline{B_r\left(-R_\nu \iota_p^* K_n\right)}\right) \ge \exp\left[-\frac{1}{2}\left(1 - \frac{\delta/\left(2C\right)}{\sqrt{\left\langle \iota_p^* K_n, \iota_p^* K_n \right\rangle}}\right)^2 \left\langle \iota_p^* K_n, \iota_p^* K_n \right\rangle\right] \nu_p\left(\overline{B_{(1-\epsilon)r}\left(0\right)}\right)\,, \tag{3.1.37}$$

which combines with eq. (3.1.29) to

$$M_\phi\left(\iota_p^* K_n\right) \ge \frac{1}{2} f_k\left(\phi\right) \exp\left[\frac{\delta\sqrt{\left\langle \iota_p^* K_n, \iota_p^* K_n \right\rangle}}{4C} - \frac{\delta^2}{8C^2}\right] \nu_p\left(\overline{B_{(1-\epsilon)r}\left(0\right)}\right) \tag{3.1.38}$$

for all $n \in \mathbb{N}$. As before, since $(\iota_p^* K_n)_{n\in\mathbb{N}}$ is a minimising sequence of M_ϕ it must be bounded.

Since M_ϕ is continuous and convex, it is also weakly lower semicontinuous and hence attains its minimum by the weak compactness of bounded balls in $\mathcal{S}(\mathbb{R}^d)^*_\nu$. □

So, $DW_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ is a Fréchet differentiable bijection and by corollary 3.1.9, $D_J^2 W_k$ is continuously invertible for all $J \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ and $k \in \mathbb{R}$.

Corollary 3.1.11. The map $(k, \phi) \mapsto (DW_k)^{-1}(\phi)$ is continuously Fréchet differentiable.

Proof. Define $g : \mathbb{R} \times H(\nu) \times \mathcal{S}(\mathbb{R}^d)^*_\nu \to H(\nu)$ with

$$(k, \phi, T) \mapsto D_T W_k - \phi = \frac{D_T Z_k}{Z_k(T)} - \phi \,. \tag{3.1.39}$$

It is continuously Fréchet differentiable by lemma 3.1.6 and for all $k \in \mathbb{R}, \phi \in H(\nu)$ and $K, T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$

$$(D_{k,\phi,T}\, g)(0, 0, K) = \left(D^2_T W_k\right)(K) \,. \tag{3.1.40}$$

Since $D^2_T W_k$ is continuously invertible, g satisfies the conditions of the implicit function theorem. □

Finally, we may define the **effective average action** $\Gamma_k : H(\nu) \to \mathbb{R}$ as

$$\phi \mapsto \sup_{J \in \mathcal{S}(\mathbb{R}^d)^*_\nu} \left[J(\phi) - W_k(J)\right] - \frac{1}{2} F_k(\phi, \phi) \,. \tag{3.1.41}$$

for all $k \in \mathbb{R}$. It is well-defined since the supremum is attained precisely for $J = (DW_k)^{-1}(\phi)$. Hence,

$$\Gamma_k(\phi) = (DW_k)^{-1}(\phi)(\phi) - W_k\left((DW_k)^{-1}(\phi)\right) - \frac{1}{2} F_k(\phi, \phi) \tag{3.1.42}$$

for all $\phi \in H(\nu)$. By the chain rule the above is also Fréchet differentiable with derivative

$$D\Gamma_k : H(\nu) \to \mathcal{S}(\mathbb{R}^d)^*_\nu, \phi \mapsto (DW_k)^{-1}(\phi) - F_k(\phi, \cdot) \,, \tag{3.1.43}$$

where we have used the continuous injection $\mathcal{S}(\mathbb{R}^d)^*_\beta \to \mathcal{S}(\mathbb{R}^d)^*_\nu$. This is precisely the **quantum equation of motion** (see e.g. [Gie12, Equation 22]). But then, we immediately see that we can take another derivative, leading to

$$D^2\Gamma_k : H(\nu) \to \mathcal{L}\left(H(\nu), \mathcal{S}(\mathbb{R}^d)^*_\nu\right) \qquad \phi \mapsto \left(D^2_{(DW_k)^{-1}(\phi)} W_k\right)^{-1} - F_k \,. \tag{3.1.44}$$

Thus, the operators $D^2_\phi \Gamma_k + F_k \in \mathcal{L}(H(\nu), \mathcal{S}(\mathbb{R}^d)^*_\nu)$ are clearly continuously invertible with inverses given by $D^2_{(DW_k)^{-1}(\phi)} W_k$.

A simple calculation entirely analogous to the standard one (see e.g. [Gie12]) now reveals:

Theorem 3.1.12 (The Wetterich equation)**.**

$$\partial_k \Gamma_k(\phi) = \frac{1}{2} \int_{X^k} \overline{U^k_x} \left[\left(D^2_\phi \Gamma_k + F_k\right)^{-1} \left(U^k_x\right)\right] \mathrm{d}m_k(x) + \partial_k \ln N_k \tag{3.1.45}$$

for all $\phi \in H(\nu)$. Here, $\overline{U^k_x} \in (\mathcal{S}(\mathbb{R}^d)^*_\nu)_\mathbb{C}$ denotes the complex conjugate of U^k_x, i.e. with $\overline{U^k_x}(\phi) = \overline{U^k_x(\phi)}$ for all $\phi \in H(\nu)$. Note that $\overline{U^k_x}$ is still complex linear since $H(\nu)$ is a real vector space. Recall that X^k, m_k and U^k_x were defined in eq. (3.1.6).

While this differential equation is in itself already remarkable, its real strength lies in its boundary conditions. The $k \to 0$ limit corresponds to the physical case as outlined in the beginning of this chapter. Before we can derive the boundary condition for $k \to \infty$ we need the following theorem.

Theorem 3.1.13. Let $g : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ be ν-integrable, continuous at zero and $|g| \leq C \exp[p^2]$ for some $C > 0$ and some continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)$. Then

$$\lim_{k\to\infty} \frac{\int_{\mathcal{S}(\mathbb{R}^d)} g(\psi) \exp\left[-\frac{1}{2}F_k(\psi,\psi)\right] \mathrm{d}\nu(\psi)}{\int_{\mathcal{S}(\mathbb{R}^d)} \exp\left[-\frac{1}{2}F_k(\psi,\psi)\right] \mathrm{d}\nu(\psi)} = g(0) \ . \tag{3.1.46}$$

Proof. Let us first prove that the measures

$$\theta_k = \frac{\exp\left[-\frac{1}{2}F_k(\cdot,\cdot)\right]}{\int_{\mathcal{S}(\mathbb{R}^d)} \exp\left[-\frac{1}{2}F_k(\cdot,\cdot)\right] \mathrm{d}\nu(\psi)} \cdot \nu \tag{3.1.47}$$

converge weakly (see appendix A.4) to the Dirac measure δ_0 at the origin as k goes to infinity. To that end, let $A_k : \mathcal{S}(\mathbb{R}^d)^*_\nu \to \mathbb{R}$ denote the moment-generating function of θ_k. By the Cameron-Martin theorem we have that

$$A_k(T) = \exp\left[T(\omega) - \frac{1}{2}\left\langle R_\nu^{-1}\omega, R_\nu^{-1}\omega\right\rangle - \frac{1}{2}F_k(\omega,\omega)\right] A_k\left(T - \left[R_\nu^{-1} + F_k\right]\omega\right) \tag{3.1.48}$$

for any $T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ and $\omega \in H(\nu)$. Here, $F_k\,\omega$ denotes the tempered distribution given by $\phi \to F_k(\omega,\phi)$. Now, note that $R_\nu^{-1} + F_k : H(\nu) \to \mathcal{S}(\mathbb{R}^d)^*_\nu$ is continuously invertible, by the positivity property of F_k given in eq. (3.1.2). Hence, taking $\omega = \left(R_\nu^{-1} + F_k\right)^{-1} T$, we arrive at

$$A_k(T) = \exp\left[\frac{1}{2}T\left(\left(R_\nu^{-1} + F_k\right)^{-1} T\right)\right] A_k(0) = \exp\left[\frac{1}{2}\left\langle T, \left(\mathrm{id} + F_k R_\nu\right)^{-1} T\right\rangle\right] . \tag{3.1.49}$$

By analytic continuation $T \mapsto iT$, we obtain the characteristic functions

$$\hat{\theta}_k(T) = \exp\left[-\frac{1}{2}\left\langle T, \left(\mathrm{id} + F_k R_\nu\right)^{-1} T\right\rangle\right] \tag{3.1.50}$$

for all $T \in \mathcal{S}(\mathbb{R}^d)^*_\beta$. A necessary condition for the sought convergence is that the functions $\hat{\theta}_k$ converge pointwise to 1 as k goes to infinity. Since $F_k(\phi,\phi)$ is increasing with k for all $\phi \in \mathcal{S}$ by eq. (3.1.6), $\left(\mathrm{id} + F_k R_\nu\right)^{-1}$ is a decreasing family of positive operators on $\mathcal{S}(\mathbb{R}^d)^*_\nu$. Fixing, any $T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ we thus have that $\left(\mathrm{id} + F_k R_\nu\right)^{-1} T$ converges in norm to some $K \in \mathcal{S}(\mathbb{R}^d)^*_\nu$ as k tends to infinity. If $K \neq 0$ we obtain

$$\liminf_{k\to\infty} \left\langle K, \frac{\mathrm{id} + F_k R_\nu}{k^2} K\right\rangle = \liminf_{k\to\infty} \frac{1}{k^2} F_k\left(R_\nu K, R_\nu K\right) > 0 \tag{3.1.51}$$

by eq. (3.1.3). At the same time

$$\begin{aligned}\liminf_{k\to\infty}\left\langle K, \frac{\mathrm{id}+F_kR_\nu}{k^2}K\right\rangle &= \liminf_{k\to\infty}\left\langle K, \frac{\mathrm{id}+F_kR_\nu}{k^2}\left[K-(\mathrm{id}+F_kR_\nu)^{-1}T\right]+\frac{T}{k^2}\right\rangle \\ &= \liminf_{k\to\infty}\left\langle \frac{\mathrm{id}+F_kR_\nu}{k^2}K, K-(\mathrm{id}+F_kR_\nu)^{-1}T\right\rangle = 0\,,\end{aligned} \tag{3.1.52}$$

where the last equality follows, since F_kR_ν/k^2 is bounded by eq. (3.1.2). Hence, we have a contradiction and may conclude that $\lim_{k\to\infty}(\mathrm{id}+F_kR_\nu)^{-1}T = 0$ for all $T \in \mathcal{S}(\mathbb{R}^d)^*_\nu$. In particular, we then obtain $\lim_{k\to\infty}\hat{\theta}_k(T) = 1$ for all $T \in \mathcal{S}(\mathbb{R}^d)^*_\beta$. Now, by [Bog98, Corollary 3.8.5], a sufficient criterion for the weak convergence is the uniform tightness (see appendix A.4) of $\{\theta_k : k \in \mathbb{R}\}$. To prove it, observe that

$$\int_{\mathcal{S}(\mathbb{R}^d)} T(\phi)^2\,\mathrm{d}\nu(\phi) = \langle T, T\rangle \geq \left\langle T, (\mathrm{id}+F_kR_\nu)^{-1}T\right\rangle = \int_{\mathcal{S}(\mathbb{R}^d)} T(\phi)^2\,\mathrm{d}\theta_k(\phi) \tag{3.1.53}$$

for all $T \in \mathcal{S}(\mathbb{R}^d)^*_\beta$ and hence, $\theta_k(B) \geq \nu(B)$ for every absolutely convex (see appendix A.2) Borel set $B \subseteq \mathcal{S}(\mathbb{R}^d)$ by [Bog98, Theorem 3.3.6]. Because ν is Radon, for any $\epsilon > 0$ there is a compact set $K \subset \mathcal{S}(\mathbb{R}^d)$ with $\nu(K) > 1-\epsilon$ which, by the completeness of $\mathcal{S}(\mathbb{R}^d)$, may be taken to be absolutely convex. Hence, $\theta_k(K) > 1-\epsilon$ as well and we have proven the weak convergence of θ_k to δ_0.

Let C and p be as stated in the theorem and set $c_k = \theta_k(p^{-1}([0,1]))$. Then

$$\liminf_{k\to\infty} c_k \geq \liminf_{k\to\infty}\theta_k\left(p^{-1}([0,1))\right) \geq 1 \tag{3.1.54}$$

by the Portmanteau theorem (see appendix A.4). Hence,

$$\liminf_{k\to\infty}\alpha_k := \liminf_{k\to\infty}\frac{1}{24}\ln\frac{c_k}{1-c_k} = \infty \tag{3.1.55}$$

and there exists $l \in \mathbb{R}$ such that $c_l \geq 3/4$ and $\alpha_l \geq 2$. Consequently, by Fernique's theorem [Bog98, Theorem 3.2.10 and Theorem 2.8.5], $\int_{\mathcal{S}(\mathbb{R}^d)}\exp[2p^2]\mathrm{d}\theta_l < \infty$ and using [Bog98, Corollary 3.3.7] as well as the monotonically increasing behaviour of $(F_k)_{k\in\mathbb{R}}$,

$$\sup_{k\geq l}\int_{\mathcal{S}(\mathbb{R}^d)}\exp[2p^2]\mathrm{d}\theta_k < \infty\,. \tag{3.1.56}$$

Applying the continuity of g at zero, for every $\epsilon > 0$ there is an open neighbourhood $U \subseteq \mathcal{S}(\mathbb{R}^d)$ of the origin such that $\sup_{\phi\in U}|g(\phi)-g(0)| < \epsilon$. Furthermore,

$$\lim_{k\to\infty}\int_{\mathcal{S}(\mathbb{R}^d)\setminus U}\exp[p^2]\mathrm{d}\theta_k \leq \sqrt{\sup_{k\geq l}\int_{\mathcal{S}(\mathbb{R}^d)}\exp[2p^2]\mathrm{d}\theta_k}\cdot\limsup_{k\to\infty}\sqrt{\theta_k\left(\mathcal{S}\left(\mathbb{R}^d\right)\setminus U\right)} = 0 \tag{3.1.57}$$

by invoking Hölder's inequality and the Portmanteau theorem again. Consequently,

$$\begin{aligned}\limsup_{k\to\infty}\int_{\mathcal{S}(\mathbb{R}^d)}|g(\phi)-g(0)|\ \mathrm{d}\theta_k(\phi)&\leq\limsup_{k\to\infty}\int_U|g(\phi)-g(0)|\ \mathrm{d}\theta_k(\phi)\\&+[C+g(0)]\limsup_{k\to\infty}\int_{\mathcal{S}(\mathbb{R}^d)\setminus U}\exp\left[p(\phi)^2\right]\ \mathrm{d}\theta_k(\phi)\leq\epsilon\end{aligned}\tag{3.1.58}$$

and the claim follows. □

Furthermore, we have the following useful lemmas.

Lemma 3.1.14. Let $k\in\mathbb{R}$, $\phi\in H(\nu)$ and define $X_{k,\phi}:\mathcal{S}(\mathbb{R}^d)^*_\nu\to\mathbb{R}$,

$$T\mapsto\frac{1}{N_k}\int_S\exp\left[T(\psi)-S^{\text{int}}(\psi+\phi)-\frac{1}{2}F_k(\psi,\psi)\right]\mathrm{d}\nu(\psi)\ .\tag{3.1.59}$$

Then, $D_\phi\Gamma_k-R_\nu^{-1}(\phi)$ minimises $X_{k,\phi}$.

Proof. $X_{k,\phi}$ is a moment-generating function and hence convex. Furthermore, by eqs. (3.1.19) and (3.1.43), $D_\phi\Gamma_k-R_\nu^{-1}(\phi)$ extremises $X_{k,\phi}$. □

Lemma 3.1.15. For all $k\in\mathbb{R}$ and $\phi\in H(\nu)$

$$\exp[-\Gamma_k(\phi)]=\exp\left[-\frac{1}{2}R_\nu^{-1}(\phi)(\phi)\right]X_{k,\phi}\left(D_\phi\Gamma_k-R_\nu^{-1}(\phi)\right)\ .\tag{3.1.60}$$

Proof. Let $k\in\mathbb{R}$ and $\phi\in H(\nu)$. Then,

$$\begin{aligned}\exp[-\Gamma_k(\phi)]&=\exp\left[-(DW_k)^{-1}(\phi)(\phi)+W_k\left((DW_k)^{-1}(\phi)\right)+\frac{1}{2}F_k(\phi,\phi)\right]\\&=\frac{1}{N_k}\int_{\mathcal{S}(\mathbb{R}^d)}\exp\Bigg[-S^{\text{int}}(\psi)+(DW_k)^{-1}(\phi)(\psi-\phi)\\&\qquad+\frac{1}{2}F_k(\phi,\phi)-\frac{1}{2}F_k(\psi,\psi)\Bigg]\mathrm{d}\nu(\psi)\\&=\frac{1}{N_k}\int_{\mathcal{S}(\mathbb{R}^d)}\exp\Bigg[-\frac{1}{2}R_\nu^{-1}(\phi)(\phi)-S^{\text{int}}(\psi+\phi)+(DW_k)^{-1}(\phi)(\psi)\\&\qquad-F_k(\psi,\phi)-\frac{1}{2}F_k(\psi,\psi)-R_\nu^{-1}(\phi)(\psi)\Bigg]\mathrm{d}\nu(\psi)\\&=\exp\left[-\frac{1}{2}R_\nu^{-1}(\phi)(\phi)\right]X_{k,\phi}\left(D_\phi\Gamma_k-R_\nu^{-1}(\phi)\right)\ .\end{aligned}\tag{3.1.61}$$

□

Collecting all of the above, we can finally calculate the limit for $k\to\infty$.

Theorem 3.1.16. For all $\phi \in H(\nu)$,

$$\lim_{k\to\infty} \Gamma_k(\phi) = \frac{1}{2} R_\nu^{-1}(\phi)(\phi) + S^{\text{int}}(\phi) - S^{\text{int}}(0) \ . \tag{3.1.62}$$

Note that, most often $S^{\text{int}}(0) = 0$ such that this term may be seen as immaterial at this point.

Proof. Let $k \in \mathbb{R}$ and $\phi \in H(\nu)$. Since $D_\phi \Gamma_k - R_\nu^{-1}(\phi)$ minimises $X_{k,\phi}$, we have

$$\begin{aligned} \exp[-\Gamma_k(\phi)] &\leq \exp\left[-\frac{1}{2} R_\nu^{-1}(\phi)(\phi)\right] X_{k,\phi}(0) \\ &= \frac{1}{N_k} \int_S \exp\left[-\frac{1}{2} R_\nu^{-1}(\phi)(\phi) - S^{\text{int}}(\psi+\phi) - \frac{1}{2} F_k(\psi,\psi)\right] \mathrm{d}\nu(\psi) \ . \end{aligned} \tag{3.1.63}$$

Because $\exp[-S^{\text{int}}(\phi+\psi)] \leq C_\phi \exp[p(\psi)^2]$ for some $C_\phi > 0$ and some continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)$, we may now apply theorem 3.1.13 leading to

$$\limsup_{k\to\infty} \exp[-\Gamma_k(\phi)] \leq \exp\left[-\frac{1}{2} R_\nu^{-1}(\phi)(\phi) - S^{\text{int}}(\phi) + S^{\text{int}}(0)\right] \ . \tag{3.1.64}$$

For the converse inequality, let $n \in \mathbb{N}$ and pick a balanced neighbourhood U_n of zero in $\mathcal{S}(\mathbb{R}^d)$ such that

$$\forall \psi \in U_n : \quad \exp\left[-S^{\text{int}}(\phi+\psi)\right] \geq \frac{n}{n+1} \exp\left[-S^{\text{int}}(\phi)\right] \ . \tag{3.1.65}$$

Then,

$$\inf_{T\in\mathcal{S}(\mathbb{R}^d)^*_\nu} X_{k,\phi}(T) \geq \frac{n}{n+1} \exp\left[-S^{\text{int}}(\phi)\right] \inf_{T\in\mathcal{S}(\mathbb{R}^d)^*_\nu} \frac{1}{N_k} \int_{U_n} \exp\left[T(\psi) - \frac{1}{2} F_k(\psi,\psi)\right] \mathrm{d}\nu(\psi) \ . \tag{3.1.66}$$

Now, note that the above integral is invariant under the change of $T \mapsto -T$ since ν is centred. Furthermore, it is clearly a convex function of T. Hence, the infimum is attained at $T = 0$ and

$$\inf_{T\in\mathcal{S}(\mathbb{R}^d)^*_\nu} X_{k,\phi}(T) \geq \frac{n}{n+1} \exp\left[-S^{\text{int}}(\phi)\right] \frac{1}{N_k} \int_{U_n} \exp\left[-\frac{1}{2} F_k(\psi,\psi)\right] \mathrm{d}\nu(\psi) \ . \tag{3.1.67}$$

But then

$$\liminf_{k\to\infty} \exp[-\Gamma_k(\phi)] \geq \frac{n}{n+1} \exp\left[-\frac{1}{2} R_\nu^{-1}(\phi)(\phi) - S^{\text{int}}(\phi) + S^{\text{int}}(0)\right] \tag{3.1.68}$$

by theorem 3.1.13 and since $n \in \mathbb{N}$ was arbitrary, the result follows. □

Most commonly, the Wetterich equation, FRG or FRGE (functional renormalisation group (equation)) is used without regard to domains and often without taking the effect of regularising operators such as $\mathcal{R}$ into account. Recall that $\mathcal{R}$ (with an extra regularisation index n that we do not write out explicitly in this section) is given by the regularisation scheme introduced in section 2.2. However, we now have the tools to formulate the standard procedure in a rigorous fashion. The standard choice for F_k is as a multiplication operator in Fourier space. Hence, let us choose F_k such that $X^k = \mathbb{R}^d$ for all $k \in \mathbb{R}$, $\mathcal{A}^k$ is the corresponding Borel sigma algebra and

$$\begin{aligned} U_p^k = \mathcal{F}_p := \left[\phi \mapsto \hat{\phi}(p)\right] , \quad & R : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, (k,p) \mapsto R_k(p) , \\ F_k(\phi,\phi) = \left\langle \hat{\phi}, R_k \cdot \hat{\phi} \right\rangle_{L^2(\mathbb{R}^d)_\mathbb{C}} , \quad & m_k = \partial_k R_k \cdot L_d , \end{aligned} \tag{3.1.69}$$

where L_d is the Lebesgue measure on $\mathbb{R}^d$, $\partial_k R_k$ is taken to exist L_d-almost everywhere and R is regular enough for eq. (3.1.6) to hold. Let us fix some orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$. With the continuous injection of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)^*_\beta$ via the $L^2(\mathbb{R}^d)$ inner product, we then have

$$\begin{aligned} \mathrm{Tr}\left(D_\phi^2\Gamma_k + F_k\right)^{-1} &= \sum_{n=1}^{\infty} \left(D^2_{(DW_k)^{-1}(\phi)} W_k\right)(e_n, e_n) \\ &\leq \frac{2/N_k}{Z_k\left((DW_k)^{-1}(\phi)\right)} \int_{\mathcal{S}(\mathbb{R})} \|\psi\|^2_{L^2(\mathbb{R}^d)} \exp\left[(DW_k)^{-1}(\phi)(\psi)\right] f_k(\psi)\, \mathrm{d}\nu(\phi) < \infty \end{aligned} \tag{3.1.70}$$

by Hölder's inequality and Fernique's theorem [Bog98, Theorem 3.2.10]. In particular, $(D_\phi^2\Gamma_k + F_k)^{-1}$ induces a trace-class operator on $L^2(\mathbb{R}^d)$. Moreover, by passing to $L^2(\mathbb{R}^d)_\mathbb{C}$,

$$\mathrm{Tr}\left(D_\phi^2\Gamma_k + F_k\right)^{-1} = \int_{\mathbb{R}^d} \mathcal{F}_{-p}\left[\left(D_\phi^2\Gamma_k + F_k\right)^{-1}\mathcal{F}_p\right] \mathrm{d}p < \infty . \tag{3.1.71}$$

Remark 3.1.17. The same reasoning also applies to R_ν with

$$\mathrm{Tr}R_\nu = \int_{\mathbb{R}^d} \delta_x\left[R_\nu \delta_x\right] \mathrm{d}x < \infty , \tag{3.1.72}$$

where δ_x denotes the Dirac distribution at the point $x \in \mathbb{R}^d$. Consequently, $R_\nu \neq 0$ cannot be translation equivariant because in that case the above integrand would be constant and the integral infinite.

Letting $M(f)$ denote the multiplication operator on $L^2(\mathbb{R}^d)_\mathbb{C}$ by a function f, it is now clear that

$$\partial_k\Gamma_k(\phi) = \frac{1}{2}\mathrm{Tr}\left\{M(\partial_k R_k)\left[\mathcal{F}\circ D_\phi^2\Gamma_k \circ \mathcal{F}^{-1} + M(R_k)\right]^{-1}\right\} + \partial_k \ln N_k . \tag{3.1.73}$$

Apart from the $\partial_k \ln N_k$ term this is exactly the FRGE in physicists' notation. Making a simple subtraction even removes $\partial_k \ln N_k$ completely.

Theorem 3.1.18 (The Wetterich equation (second formulation)).

$$\partial_k \bar{\Gamma}_k(\phi) = \frac{1}{2}\mathrm{Tr}\Bigg\{ M(\partial_k R_k) \Bigg(\left[\mathcal{F} \circ D^2_\phi \bar{\Gamma}_k \circ \mathcal{F}^{-1} + M(R_k)\right]^{-1} - \left[\mathcal{F} \circ D^2_0 \bar{\Gamma}_k \circ \mathcal{F}^{-1} + M(R_k)\right]^{-1} \Bigg)\Bigg\} \tag{3.1.74}$$

for all $\phi \in H(\nu)$ where $\bar{\Gamma}_k(\phi) = \Gamma_k(\phi) - \Gamma_k(0)$. Furthermore,

$$\Gamma_k(0) = -\inf_{\phi \in H(\nu)} \bar{\Gamma}_k(\phi) \quad \text{and} \quad \lim_{k\to\infty} \bar{\Gamma}_k(\phi) = \frac{1}{2} R_\nu^{-1}(\phi)(\phi) + S^{\text{int}}(\phi) - S^{\text{int}}(0) \ . \tag{3.1.75}$$

Proof. The differential equation and the $k \to \infty$ limit is clear. The expression for $\Gamma_k(0)$ follows from $W_k(0) = 0$ since $f_k/N_k \cdot \nu$ is a probability measure. □

3.2. Physical Interpretation

The subtraction of the $\phi = 0$ contribution of the full propagator in theorem 3.1.18 is quite remarkable because it is precisely what is done under the hood in concrete calculations involving the 'non-regularised' Wetterich equation. Furthermore, it motivates the common practise of expanding both sides in powers of ϕ and solving the equation order by order. In fact, it is easy to show that under rather mild conditions one may Gâteaux differentiate under the trace owing to the fact that Z_k is actually Fréchet-C^∞. Hence, solving order by order is justified but whether the resulting solution is analytic is not clear from these equations. Resulting expressions including combinatorics may be extracted from [Zie21b].

The achievement of theorem 3.1.18 is the rigorous derivation of the Wetterich equation and the exposure of correct domains and boundary conditions of $\bar{\Gamma}_k$. Notably, the $k \to \infty$ limit depends on $\mathcal{R}$ since R_ν^{-1} can be seen as the continuous extension of $(\mathcal{R}^*)^{-1} \circ B \circ \mathcal{R}^{-1}$ on a suitable domain, where $\mathcal{R}^*$ is the adjoint of $\mathcal{R}$ as defined in appendix A.2. In the event where the $n \to \infty$ limit (the limit of vanishing regularisation) can be taken without introducing divergences one obtains

$$\lim_{n\to\infty} \lim_{k\to\infty} \bar{\Gamma}_k(\phi) = (\phi, B_\infty \phi) + S^{\text{int}}_\infty(\phi) - S^{\text{int}}_\infty(0) \ , \tag{3.2.1}$$

in accordance with the common assumption that taking $k \to \infty$ should recover the 'classical limit'. Of course, B_∞ and S^{int}_∞ do not generally exist and one has to conclude that - at least in the presented scheme - the classical limit for finite $n \in \mathbb{N}$ is modified by the regularisation. An unfortunate consequence is that it is impossible to obtain a translation-equivariant R_ν as was noted in remark 3.1.17. Hence, while the differential equation for $\bar{\Gamma}_k$ respects translation invariance the boundary condition at $k \to \infty$ does not, which may pose a severe difficulty in concrete calculations. On the bright side, the fact that the involved operators are trace class and self-adjoint on $L^2(\mathbb{R}^d)_\mathbb{C}$ implies that they have a complete basis of eigenvectors. It seems likely that this fact can be exploited in numerical calculations. All in all, the Wetterich equation is shown to have a rigorous foundation including the contributions arising from the regularisation.

Remark 3.2.1. The requirement that $|g| \leq C \exp[p^2]$ was necessary in the proof of theorem 3.1.13 to ensure that $\lim_{k\to\infty} \int_{\mathcal{S}(\mathbb{R}^d)\setminus U} g \, \mathrm{d}\nu = 0$ for all neighbourhoods U of zero. The most general condition yielding this, is that the family $\{(1 - I_U) g \cdot \mathrm{d}\theta_k/\mathrm{d}\nu : k \text{ large enough}\}$ is uniformly integrable with respect to ν, where I_U is the indicator function of U, for all open neighbourhoods U of zero. Because in our applications $g \in L^a(\nu)$ for some $a > 1$, a sufficient condition for this to be true would be that

$$\lim_{k\to\infty} \int_{\mathcal{S}(\mathbb{R}^d)\setminus U} \left(\frac{\mathrm{d}\theta_k}{\mathrm{d}\nu}\right)^b \mathrm{d}\nu = 0 \tag{3.2.2}$$

for all $b > 1$. Unfortunately, the author has not been able to prove such a statement from the given axioms, while he does believe it to be true.

From a physical perspective, the boundedness by $C \exp[p^2]$ covers the case of the ϕ^4 model for $d = 4$. In that setting, it is known that a negative mass counterterm is needed that diverges with vanishing regularisation. The resulting term in S^{int} is precisely of the form $-\Delta m^2 \|\cdot\|^2_{L^2(\mathbb{R}^d)}$ which exactly corresponds to the boundedness by $C \exp[p^2]$. That this point is important and also raises the question of the integrability of functions of the form

$$\phi \to \exp\left[-\lambda \int_{\mathbb{R}^d} \phi^4 + \Delta m^2 \int_{\mathbb{R}^d} \phi^2\right] , \tag{3.2.3}$$

was pointed out in private communications with Christoph Kopper. It was answered positively in [HJZ22] and is proven in a more general setting in chapter 6.

Remark 3.2.2. It is clear that the mathematically rigorous derivation as given in section 3.1 does not apply to models with

- fermionic fields,
- multiple fields,
- fields with gauge symmetries.

However, it seems very likely to the author that a similar treatment can be given to coupled fermions and bosons validating the common formal derivation of the Wetterich equation. Some possibles issues that might arise are discussed in chapter 7. Another new feature in the case of fermions is the promotion of the trace operation in theorem 3.1.18 to a supertrace operation taking the parity (in terms of a superalgebra) of the resulting objects into account [Gie12].

A treatment of gauge fields in the presented framework should not present any new obstacles. In particular, after the addition of a gauge fixing term to the free part of the action, a Gaußian measure modelling a free, gauge-fixed version of the theory exists. One would also need to model the corresponding ghost fields and the usual reservations regarding Gribov ambiguities [Gri78] still apply.

3.3. A Theorem on Lower Semicontinuous Envelopes

In the context of the Wetterich equation, a major role is played by the effective average action. It is formally defined in eq. (3.0.7) via the Legendre-Fenchel transform of the logarithm of the partition function which is convex and lower semicontinuous by lemma A.4.6. Hence, it is clear that the study of the effective average action necessitates some features of convex analysis.

In this section we introduce the concepts of supercoercivity and lower semicontinuous envelopes and prove some simple results that the author believes to be novel.

Definition 3.3.1. Let X be a locally convex space and $f : X \to \bar{\mathbb{R}}$ (see appendix A.3) a convex function with

$$\sup_{x \in X} \left[p\left(x\right) - f\left(x\right)\right] < \infty \tag{3.3.1}$$

for all continuous seminorms p on X. Then f is **supercoercive**.

Definition 3.3.2. Let X be a locally convex space, Y a normed space, $\iota : X \to Y$ linear and continuous withe dense range and $f : X \to \bar{\mathbb{R}}$ a convex and lower semicontinuous function. Then the **lower semicontinuous envelope** $LSC(f, \iota) : Y \to \bar{\mathbb{R}}$ of f with respect to ι is given by

$$LSC(f, \iota)\left(x\right) = \inf \left\{ \liminf_{n \to \infty} f\left(x_n\right) \,\middle|\, (x_n)_{n \in \mathbb{N}} \text{ in } X \text{ with } \lim_{n \to \infty} \left\| \iota\left(x_n\right) - x \right\|_Y = 0 \right\} \tag{3.3.2}$$

Remark 3.3.3. This definition is a generalisation of the one that is commonly used in literature, see e.g. [Zal02]. It is easy to see that $LSC(f, \iota)$ is lower semicontinuous. Furthermore, if ι is injective, $LSC(f, \iota)$ is the largest lower semicontinuous function that is not greater than $f \circ \iota^{-1}$ on $\iota(X)$.

The following lemma shows that our definition of lower semicontinuous envelopes plays well with Legendre-Fenchel conjugation.

Lemma 3.3.4. Let X be a Hausdorff reflexive space, p a continuous seminorm on X^*_β, $\iota_p : X^*_\beta \to (X^*_\beta)_p = Y$ the natural map and $f : X \to \bar{\mathbb{R}}$ a proper convex and lower semicontinuous function. Let f^p denote the restriction of f to the Banach space Y^* considered as a subspace of X. Then $LSC(f^c, \iota_p)^c = f^p$ and if f^p is proper, $LSC(f^c, \iota_p) = (f^p)^c|_Y$.

Remark 3.3.5. The adjoint $\iota_p^* : Y^* \to X$ is injective by [SW99, Chapter 4, §4, Corollary 2.3].

Proof. By Fenchel-Moreau it suffices to show that $LSC(f^c, \iota_p)^c = f^p$. Plugging in the definition of $LSC(f^c, \iota_p)^c$, for every $x \in Y^*$ there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ in Y and a sequence $(\psi_n)_{n\in\mathbb{N}}$ in X_β^* with $\lim_{n\to\infty} p(\phi_n - \iota_p\psi_n) = 0$ such that

$$LSC(f^c, \iota_p)^c\,(x) = \lim_{n\to\infty} \left[x\,(\phi_n) - f^c\,(\psi_n)\right] . \tag{3.3.3}$$

But since $x \in Y^*$, we have

$$\lim_{n\to\infty} \left|x\,(\phi_n - \iota_p\psi_n)\right| \leq \lim_{n\to\infty} Cp\,(\phi_n - \iota_p\psi_n) = 0 \tag{3.3.4}$$

for some $C > 0$. Consequently,

$$LSC(f^c, \iota_p)^c\,(x) = \lim_{n\to\infty} \left[x\,(\iota_p\psi_n) - f^c\,(\psi_n)\right] \leq f^{cc}\left(\iota_p^* x\right) = f^p\,(x) . \tag{3.3.5}$$

For the converse inequality, note that

$$\begin{aligned} LSC\,(f^c, \iota_p)^c\,(x) &\geq \sup_{\phi\in X_\beta^*} \left[x\,(\iota_p\phi) - LSC(f^c, \iota_p)\,(\iota_p\phi)\right] \qquad (3.3.6)\\ &\geq \sup_{\phi\in X_\beta^*} \left[x\,(\iota_p\phi) - f^c\,(\phi)\right] = f^{cc}\left(\iota_p^* x\right) = f^p\,(x) . \end{aligned}$$

□

While this lemma demonstrates a useful property, it may in general be difficult to actually calculate the lower semicontinuous envelope of a given function. In the supercoercive case, however, we obtain a particularly simple expression.

Lemma 3.3.6. Let X be a Hausdorff reflexive space and $f : X \to \bar{\mathbb{R}}$ a convex, lower semicontinuous and supercoercive function. For any continuous seminorm p, let $\iota_p : X \to X_p$ denote the natural map. Then the lower semicontinuous envelope g of f with respect to ι_p takes the form

$$g\,(x) = \begin{cases} \inf\left\{f\,(y) : y \in \iota_p^{-1}\,(\{x\})\right\} & x \in \iota_p\,,(X) \\ \infty & \text{otherwise} \end{cases} \tag{3.3.7}$$

and is supercoercive.

Proof. It is immediately clear that $LSC(f, \iota_p)(x) \leq g(x)$ for all $x \in X_p$. Suppose that $g(x) > LSC(f, \iota_p)(x)$ for some $x \in X_p$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} p(x - \iota_p x_n) = 0$ such that $\lim_{n\to\infty} f(x_n) < g(x)$. If x_n is bounded in X there is a subnet (y_α) that is weakly converging to some $y \in X$ with $x = \iota_p(y)$. But f is weakly lower semicontinuous by

[Zal02, Theorem 2.2.1] such that

$$f(y) \leq \lim_{\alpha} f(y_\alpha) = \lim_{n\to\infty} f(x_n) < g(x) , \tag{3.3.8}$$

which contradicts the definition of g.

If x_n is unbounded in X there is some subsequence $(y_n)_{n\in\mathbb{N}}$ and a continuous seminorm q on X with $\lim_{n\to\infty} q(y_n) = \infty$. By the supercoercivity of f, we must then have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = \infty$ which is again a contradiction.

For the supercoercivity, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\iota_p(X)$. Then for every x_n there is also some $y_n \in \iota_p^{-1}(\{x_n\})$ with

$$|f(y_n) - g(x_n)| < \frac{1}{n} . \tag{3.3.9}$$

By the definition of ι_p, $p(x_n) = p(y_n)$ such that

$$\limsup_{n\to\infty} [p(x_n) - g(x_n)] \leq \limsup_{n\to\infty} \left[p(y_n) - f(y_n) + \frac{1}{n}\right] < \infty \tag{3.3.10}$$

by the supercoercivity of f. Because the sequence (x_n) was arbitrary, g is supercoercive. □

3.4. A Renormalisation Theorem

In our case, the measures $(\omega_n)_{n\in\mathbb{N}}$ encoding the regularised QFTs are Borel measures on the space $\mathcal{S}(\mathbb{R}^d)^*_\beta$ of tempered distributions equipped with its strong dual topology. Such measures enjoy strong regularity and convergence properties that we shall exploit.

Remark 3.4.1. The spaces $\mathcal{S}(\mathbb{R}^d)^*_\beta$ are **Radon spaces** [Sch73], i.e. every Borel measure μ on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ is a Radon measure.

Theorem 3.4.2 (Lévy continuity theorem [Col66])**.** Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that their characteristic functions converge pointwise to a function that is continuous at zero. Then there is a Borel measure ω on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that ω_n converges weakly to μ.

In concrete physical calculations one rarely has any control over the characteristic functions of the regularised theories. Instead, one typically calculates regularised moments as in eq. (2.1.5) and attempts to study their respective limits of vanishing regularisation. This can obviously work only, if

- all regularised moments (smeared Schwinger functions) are finite,
- the regularised moments converge in the limit of vanishing regularisation.

If one of these conditions is not satisfied, the physical wellposedness and/or tractability of the given theory should be questioned.

Mathematically, there is a simple condition ensuring that both of the above are true for sufficiently small regularisations, namely that the moment-generating functions converge. In view of the Osterwalder-Schrader theorem this is rather natural, because the final measure should have a continuous and everywhere finite moment-generating function in order to be physically meaningful [GJ12]. Hence it is natural to demand that the regularised moment-generating functions are pointwise eventually bounded. While somewhat abstract, this condition is well-motivated by the above discussion and turns out to enable strong statements about the convergence of the regularised theories.

In view of the specific application of the Wetterich equation, we shall ultimately give a convergence theorem in terms of the convergence of a sequence of regularised quantum effective actions. Recall that the effective action is obtained through Legendre-Fenchel conjugation of the logarithm of the moment-generating function. Hence, we will need to find the connection between weak convergence of measures and a corresponding notion for moment-generating functions as well as a bridge to Legendre-Fenchel conjugates. This was originally done by the author in [Zie21a].

The convergence of convex conjugate functions was originally studied on reflexive Banach spaces by Mosco who introduced the now-called Mosco convergence [Mos71]. It was later generalised to locally convex spaces by Beer and Borwein [BB90], de Acosta [Aco88] and Zabell [Zab92]. Following the latter, we give the following definition:[2]

Definition 3.4.3. Let X be a Banach space or a reflexive, locally convex space and $(f_n)_{n=0}^{\infty} : X \to \bar{\mathbb{R}}$ be a sequence of extended real-valued functions on X. Then

- f_n **(M1)-converges** to f_0 if for every $x \in X$ there is some sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x such that
$$\limsup_{n\to\infty} f_n(x_n) \leq f_0(x) .$$
- f_n **(K2)-converges** to f_0 if for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x
$$\liminf_{n\to\infty} f_n(x_n) \geq f_0(x) .$$
- f_n **(M2)-converges** to f_0 if for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ that converges weakly to x
$$\liminf_{n\to\infty} f_n(x_n) \geq f_0(x) .$$

[2] In [Zab92] Zabell gives the definition of Mosco convergence in terms of Mackey convergence. On Banach spaces and reflexive locally convex spaces, the notions of norm convergence and Mackey convergence coincide [SW99, Chapter 4, Theorem 3.4] and since we shall not work on more general spaces, the given definitions suffice.

If f_n (M1)- and (K2)-converges to f_0, we shall say that f_n **epi-converges** to f_0 or converges to f_0 in the **Painlevé-Kuratowski** sense [Bee93, Theorem 5.3.5]. If f_n (M1)- and (M2)-converges to f_0, we shall say that f_n **Mosco converges** to f_0.

Remark 3.4.4. Clearly, Mosco convergence implies epi-convergence.

Using Mosco convergence it is possible to express a continuity theorem of the Legendre-Fenchel conjugation. As Zabell proved in [Zab92] however, the Legendre-Fenchel conjugation is not a homeomorphism with respect to Mosco convergence. A stronger notion offering this feature is given by the so-called **Attouch-Wets convergence** which we shall exploit in our final theorem. Its precise formulation is somewhat complicated such that we refer to [BV95] for a definition. In fact, we will not need to prove Attouch-Wets convergence from first principles, but only indirectly through the theorems presented in [BV95] such that a lack of definition appears tolerable to the author. Another great achievement of the Attouch-Wets convergence, is its compatibility with pointwise convergence which was also worked out in [BV95].

Let us begin with the following simple lemma which trivially follows from the metrisability of $\mathcal{S}(\mathbb{R}^d)$.

Lemma 3.4.5. Let $(\phi_n)_{n\in\mathbb{N}}$ be a null sequence in $\mathcal{S}(\mathbb{R}^d)$. Then, there exists a monotonically increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that

$$\lim_{n\to\infty} t_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} t_n\phi_n = 0\,. \tag{3.4.1}$$

For the remainder of the section, we will use the following abbreviation.

Definition 3.4.6. For a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, let $\iota_p : \mathcal{S}(\mathbb{R}^d)^*_\beta \to \mathcal{S}(\mathbb{R}^d)^*_p$ denote the natural map to the corresponding completion (see appendix A.2). Furthermore, for any function f on $\mathcal{S}(\mathbb{R}^d)$ let f^p denote its restriction to $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$.

Remark 3.4.7. f^p is well-defined because ι_p has dense range, whence its adjoint map $\iota_p^* : (\mathcal{S}(\mathbb{R}^d)^*_p)^* \to \mathcal{S}(\mathbb{R}^d)$ is injective [SW99, Chapter 4, Corollary 2.3].

We may now formulate a sufficient condition for the weak convergence of a sequence of measures.

Theorem 3.4.8. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ and $(Z_n)_{n\in\mathbb{N}}$ the corresponding moment-generating functions. Suppose that $\limsup_{n\to\infty} Z_n(\phi) < \infty$ for all $\phi \in \mathcal{S}$. Then, ω_n converges weakly to another Borel probability measure ω if and only if

- Z_n Mosco converges to a convex, lower semicontinuous function $Z : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ and
- for all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ the restrictions Z_n^p Mosco converge to the corresponding restriction Z^p.

Moreover, in the affirmative case Z is continuous and the moment-generating function of ω.

Proof. Suppose $\omega_n \to \omega$ weakly and fix some $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then, we clearly have pointwise convergence of the characteristic functions $\widehat{\phi_*\omega_n}$ of the one-dimensional pushforward measures to $\widehat{\phi_*\omega}$. Because $Z_n(\phi)$ and $Z_n(-\phi)$ are eventually finite it is known that $\widehat{\phi_*\omega_n}$ has an analytic continuation $\overline{\phi_*\omega_n}$ to $\{z \in \mathbb{C} : |z| \leq 1\}$ satisfying [LC70]

$$\max\left\{Z_n(\phi), Z_n(-\phi)\right\} = \sup_{|z|\leq 1} \overline{\phi_*\omega_n}(z\phi) \tag{3.4.2}$$

where z is now a complex variable. Hence the family $\left\{\overline{\phi_*\omega_n} : n \in \mathbb{N}\right\}$ is eventually uniformly bounded within the unit ball of $\mathbb{C}$. By the Vitali-Porter theorem [Sch93], pointwise convergence on the real axis implies pointwise convergence on the imaginary axis. Since this is true for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, we obtain pointwise convergence of Z_n to some real-valued function Z which is convex and continuous by lemma A.3.4. This clearly implies that Z_n (M1)-converges to Z. Moreover, by lemma A.4.15, the moment-generating function M of ω is bounded by Z and is, in particular, finite everywhere. But then the analytic continuations of $t \to M(t\phi)$ and $t \to Z(t\phi)$ to the imaginary axis must both equal the characteristic function $\widehat{\phi_*\omega}$. Consequently, $M = Z$. Regarding the (M2)-convergence, it is straightforward to see that the methods used for proving [FKZ14, theorem 1.1] apply also in our case such that for every weakly convergent sequence $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} Z(\phi) &= \int_{\mathcal{S}(\mathbb{R}^d)^*_\beta} \exp\left[\phi(T)\right] \mathrm{d}\omega(T) = \int_{\mathcal{S}(\mathbb{R}^d)^*_\beta} \liminf_{n\to\infty, T'\to T} \exp\left[\phi_n(T')\right] \mathrm{d}\omega(T) \\ &\leq \liminf_{n\to\infty} \int_{\mathcal{S}(\mathbb{R}^d)^*_\beta} \exp\left[\phi_n(T)\right] \mathrm{d}\omega_n(T) = \liminf_{n\to\infty} Z_n(\phi_n)\ , \end{aligned} \tag{3.4.3}$$

where $T' \to T$ is considered in the topology of $\mathcal{S}(\mathbb{R}^d)^*_\beta$. The second equality follows from the boundedness of $(\phi_n)_{n\in\mathbb{N}}$ by the definition of the strong dual topology. Hence, Z_n (M2)-converges to Z. Now, let p be a continuous seminorm on $\mathcal{S}(\mathbb{R}^d)^*_\beta$. Clearly Z_n^p (M2)-converges to Z^p since every weakly convergent sequence in $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ is also weakly convergent in $\mathcal{S}(\mathbb{R}^d)$. Finally, the pointwise convergence ensures (M1)-convergence of Z_n^p to Z^p.

Conversely, assume that Z_n Mosco converges to Z and that the same holds for the corresponding restrictions as above. For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ fix a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that $\phi \in (\mathcal{S}(\mathbb{R}^d)^*_p)^*$. Then, by assumption, Z_n^p Mosco converges to Z^p and, in particular, also converges in the Painlevé-Kuratowski sense. Furthermore, because all Z_n are lower semicontinuous (see lemma A.4.6), so are all Z_n^p. Since $\limsup_{n\to\infty} Z_n^p(\psi) < \infty$ for all $\psi \in (\mathcal{S}(\mathbb{R}^d)^*_p)^*$, we have that Z_n^p converges pointwise to Z^p by [BV95, Corollary 2.3.]. Since, $\phi \in \mathcal{S}(\mathbb{R}^d)$ was arbitrary, we obtain $Z_n \to Z$ pointwise and the continuity of Z by lemma A.3.4.

In analogy to the first part of the proof, we may now use the pointwise convergence of

$\overline{\phi_* \omega_n}$ along the imaginary axis, the bound given in eq. (3.4.2) and the Vitali-Porter theorem to conclude that $\hat{\omega}_n$ converges pointwise to some function $c : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. By theorem 3.4.2, it just remains to show that c is continuous at zero. By the continuity of Z, there is some balanced neighbourhood $U \subseteq S$ of the origin such that

$$\sup_{\phi \in U} |Z(\phi) - Z(0)| \leq 1 \qquad \text{and hence} \qquad 0 \leq \sup_{\phi \in U} Z(\phi) \leq 2\,. \tag{3.4.4}$$

But then, for all $t \in (0, 1)$,

$$\begin{aligned} \sup_{\phi \in tU} |c(\phi) - c(0)| = \sup_{\phi \in U} \lim_{n\to\infty} |\hat{\omega}_n(t\phi) - \hat{\omega}_n(0)| &\leq \sup_{\phi \in U} \limsup_{n\to\infty} \int_{\mathcal{S}'_\beta} t\,|T(\phi)|\, \mathrm{d}\omega_n(T) \\ &\leq \frac{t}{2} \sup_{\phi \in U} \limsup_{n\to\infty} [Z_n(\phi) + Z_n(-\phi)] \leq 2t\,. \end{aligned} \tag{3.4.5}$$

Since every null net $(\phi_\alpha)_{\alpha \in I}$ in $\mathcal{S}(\mathbb{R}^d)$ is eventually in tU for all $t \in (0, 1)$, the continuity of c at the origin follows. □

A simple corollary to this theorem, is that we may in fact get rid of the Mosco convergence in $\mathcal{S}(\mathbb{R}^d)$.

Corollary 3.4.9. Let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence of Borel probability measures on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ and $(Z_n)_{n \in \mathbb{N}}$ the corresponding moment-generating functions. Suppose that $\limsup_{n\to\infty} Z_n(\phi) < \infty$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then, ω_n converges weakly to another Borel probability measure ω if and only if there exists a lower semicontinuous, convex function $Z : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ such that for all continuous norms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, the restrictions Z_n^p Mosco converge to Z^p.

Moreover, in the affirmative case Z is continuous and the moment-generating function of ω.

Proof. By theorem 3.4.8, we just need to prove that the Mosco convergence of Z_n^p implies that of Z_n. Since for every ϕ there is a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that $\phi \in (\mathcal{S}(\mathbb{R}^d)^*_p)^*$, we clearly have that Z_n (M1)-converges to Z.

For the (M2)-convergence, note that for any weakly convergent sequence $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^d)$, the set

$$B = \{\phi_n : n \in \mathbb{N}\} \cup \{\phi\} \subset \mathcal{S}(\mathbb{R}^d) \tag{3.4.6}$$

is bounded and induces a continuous seminorm q on $\mathcal{S}(\mathbb{R}^d)^*_\beta$. Since, $\mathcal{S}(\mathbb{R}^d)^*_\beta$ is nuclear (see definition A.2.24), q is majorised by some continuous Hilbert norm p. Hence, B is a bounded subset of the reflexive Banach space $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ which is separable because $\mathcal{S}(\mathbb{R}^d)^*_p$ is. Consequently, every subsequence of $(\phi_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence in $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$. By assumption, it follows that ϕ_n converges weakly to ϕ in $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ and the result follows from the (M2)-convergence of Z_n^p. □

The above corollary may appear rather inelegant and in fact we can do a lot better by using the following lemma.

Lemma 3.4.10. Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of proper convex and lower semicontinuous functions from $\mathcal{S}(\mathbb{R}^d)$ to $\bar{\mathbb{R}}$. Given another function $Z : \mathcal{S}(\mathbb{R}^d) \to \bar{\mathbb{R}}$, the following are equivalent:

(i) For all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, the restrictions Z^p_n converge uniformly on compact sets to Z^p.

(ii) For all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, the restrictions Z^p_n converge uniformly on bounded sets to Z^p.

Proof. Let p be a continuous seminorm on $\mathcal{S}(\mathbb{R}^d)^*_\beta$. By nuclearity, there is a continuous seminorm $q > p$ such that the natural map $\mathcal{S}(\mathbb{R}^d)^*_q \to \mathcal{S}(\mathbb{R}^d)^*_p$ has dense range and is nuclear, thus in particular is compact. But then its adjoint map is compact and injective such that every bounded subset of $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ is mapped injectively to a compact subset of $(\mathcal{S}(\mathbb{R}^d)^*_q)^*$ on which we have uniform convergence.

The converse is clear, since every compact set is bounded. □

The uniform convergence on bounded sets enables the following corollary.

Corollary 3.4.11. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ and $(Z_n)_{n\in\mathbb{N}}$ the corresponding moment-generating functions. Then the following are equivalent:

(i) $\limsup_{n\to\infty} Z_n(\phi) < \infty$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and ω_n converges weakly to another Borel probability measure ω.

(ii) There exists a convex and continuous function $Z : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ such that for all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, the restrictions Z^p_n Attouch-Wets converge to Z^p.

(iii) Z_n converges to some function $Z : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ uniformly on bounded sets.

Moreover, in this case Z is the moment-generating function of ω.

Proof. $(i) \implies (ii)$: By corollary 3.4.9, for all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*$, Z^p_n Mosco-converge to Z^p for some convex and continuous function $Z : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$. In particular, Z^p_n also converge to Z^p in the Painlevé-Kuratowski sense which by [BV95, Corollary 2.3] implies the uniform convergence on compact subsets of $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$. Applying lemma 3.4.10 we also obtain uniform convergence on bounded subsets of $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$. Furthermore, every bounded subset of $\mathcal{S}(\mathbb{R}^d)$ is precompact, such that Z^p is bounded on bounded subsets of $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ which implies the Attouch-Wets convergence by [BV95, Lemma 1.4].

$(ii) \implies (iii)$: Every bounded subset $B \subset \mathcal{S}(\mathbb{R}^d)$ induces a continuous seminorm p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that B is a bounded subset of $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$. Furthermore, Z is bounded on B by the precompactness of B in $\mathcal{S}(\mathbb{R}^d)$. Hence, [BV95, Corollary 2.2.] implies the uniform convergence on B.

$(iii) \implies (i)$: Let p be a continuous seminorm on $\mathcal{S}(\mathbb{R}^d)^*_\beta$. By the pointwise convergence, we clearly have that Z_n^p (M1)-converges to Z^p while the continuity of Z^p follows from lemma A.3.4. For the (M2)-convergence, note that any sequence $(\phi_n)_{n\in\mathbb{N}}$ in $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$ converging weakly to some $\phi \in (\mathcal{S}(\mathbb{R}^d)^*_p)^*$ is bounded and consequently also bounded in $\mathcal{S}(\mathbb{R}^d)$. Also, by assumption, Z_n^p converges to Z^p uniformly on bounded sets. Now, keeping in mind that Z^p is also convex and continuous and thus weakly lower semicontinuous by [Zal02, Theorem 2.2.1], we obtain

$$\liminf_{n\to\infty} Z_n^p(\phi_n) \geq \liminf_{n\to\infty} [Z_n^p(\phi_n) - Z^p(\phi_n)] + \liminf_{n\to\infty} Z^p(\phi_n) \geq Z^p(\phi) \; . \tag{3.4.7}$$

Thus Z_n^p Mosco-converges to Z^p and corollary 3.4.9 applies. □

It would now be most tempting to conclude that there is some sort of Attouch-Wets topology on $\mathcal{S}(\mathbb{R}^d)$ with respect to which Z_n converges to Z. However, to the knowledge of the author, there have not been many studies of generalisations of Attouch-Wets convergence to non-normed spaces. Consequently, no such result is available at the time of this writing. Before coming to the final theorem stating the weak convergence of measures in terms of Attouch-Wets convergence of conjugate functions, we need another short lemma.

Lemma 3.4.12. Let $f : \mathcal{S}(\mathbb{R}^d) \to \bar{\mathbb{R}}$ be proper convex and lower semicontinuous. Then f is a continuous function $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ if and only if its convex conjugate (Legendre-Fenchel transform) f^c is supercoercive.

Proof. Let $B \subseteq \mathcal{S}(\mathbb{R}^d)$ be balanced, bounded and hence precompact. Hence, by Fenchel-Moreau,

$$\sup_{\phi\in B} f(\phi) = \sup_{\phi\in B, T\in\mathcal{S}(\mathbb{R}^d)^*_\beta} [T(\phi) - f^c(T)] = \sup_{T\in\mathcal{S}(\mathbb{R}^d)^*_\beta} [p_B(T) - f^c(T)] \; , \tag{3.4.8}$$

where p_B is the continuous seminorm induced by B.

⇒: Since all continuous seminorms p are bounded by some p_B the implication is clear.

⇐: Equation (3.4.8) shows that f is bounded on bounded subsets of $\mathcal{S}(\mathbb{R}^d)$. Hence, it is finite everywhere and theorem A.3.3 applies. □

With this lemma at hand, we arrive at the final theorem of this section with immediate relevance for the Wetterich equation.

Theorem 3.4.13. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, $(Z_n)_{n\in\mathbb{N}}$ the corresponding moment-generating functions and set $W_n = \ln \circ Z_n$. Then the following are equivalent:

(i) $\limsup_{n\to\infty} W_n(\phi) < \infty$ for all $\phi \in \mathcal{S}$ and ω_n converges weakly to another Borel probability measure ω.

(ii) There exists a proper convex, lower semicontinuous and supercoercive function $\Gamma : \mathcal{S}(\mathbb{R}^d)^*_\beta \to \bar{\mathbb{R}}$ such that for all continuous seminorms p on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ the lower semicontinuous envelopes $LSC(W_n^c, \iota_p)$ Attouch-Wets converge to $LSC(\Gamma, \iota_p)$, where $\iota_p : \mathcal{S}(\mathbb{R}^d)^*_\beta \to \mathcal{S}(\mathbb{R}^d)^*_p$ denotes the natural map.

Proof. Let us first note that all W_n are convex because all Z_n are logarithmically convex. Furthermore, all ω_n are probability measures such that Z_n does not attain the value 0 and W_n does not attain the value $-\infty$. Moreover, all W_n are lower semicontinuous by the monotony of the logarithm and, by definition, $W_n(0) = 0$. Thus, letting p denote some continuous seminorm on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, the restrictions W_n^p are also proper convex and lower semicontinuous functions. Recalling that the Legendre-Fenchel transform is a bijection between proper convex and lower semicontinuous functions on $\mathcal{S}(\mathbb{R}^d)^*_p$ and $(\mathcal{S}(\mathbb{R}^d)^*_p)^*$, there exist proper convex and lower semicontinuous functions $\Gamma_{n,p} : \mathcal{S}(\mathbb{R}^d)^*_p \to \mathbb{R}$ such that $\Gamma_{n,p}^c = W_n^p$ for all $n \in \mathbb{N}$. By Fenchel-Moreau $\Gamma_{n,p}$ is equal to the restriction of $(W_n^p)^c$ to $\mathcal{S}(\mathbb{R}^d)^*_p$. Furthermore, $\limsup_{n\to\infty} W_n(\phi) < \infty$ implies $\limsup_{n\to\infty} Z_n(\phi) < \infty$.

$\Rightarrow$: By corollary 3.4.11, Z_n converges uniformly on bounded sets to the moment-generating function Z of ω which is also continuous and never zero because ω is a probability measure. On bounded sets Z is always bounded away from zero by the precompactness of such sets. Consequently, W is continuous, W_n also converges uniformly to W on bounded sets and the same is true for the restrictions, i.e. W_n^p converges uniformly to W^p on bounded sets. Applying [BV95, Lemma 1.4], we have that W_n^p Attouch-Wets converges to W^p.

Recalling that the Legendre-Fenchel transform is a homeomorphism with respect to Attouch-Wets convergence [Bac86], it is clear that $\Gamma_{n,p}$ Attouch-Wets converges to the restriction of W_p^c to $\mathcal{S}(\mathbb{R}^d)^*_p$. Now, lemma 3.3.4 shows that W_p^c and $LSC(W^c, \iota_p)$ agree on $\mathcal{S}(\mathbb{R}^d)^*_p$. That W^c is supercoercive is clear from lemma 3.4.12.

$\Leftarrow$: By the homeomorphism property of the Legendre-Fenchel transform and lemma 3.3.4, W_n^p Attouch-Wets converges to $(\Gamma^c)^p$. Furthermore, Γ^c is continuous by lemma 3.4.12 such that $(\Gamma^c)^p$ is bounded on bounded sets by the precompactness of bounded sets in $\mathcal{S}(\mathbb{R}^d)$. Hence, W_n^p actually converges to Γ^c uniformly on bounded sets [BV95, Corollary 2.2]. Because every bounded set B in $\mathcal{S}(\mathbb{R}^d)$ induces a continuous seminorm q on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ such that B is also bounded in $(\mathcal{S}(\mathbb{R}^d)^*_q)^*$, it follows that W_n converges to Γ^c uniformly on bounded sets. Finally, since bounded sets in $\mathcal{S}(\mathbb{R}^d)$ are precompact, Γ^c is bounded on bounded sets such that Z_n converges to $\exp[\Gamma^c]$ uniformly on bounded sets. Hence, corollary 3.4.11 applies. □

Remark 3.4.14. While the convergence criterion for the dual functions W_n^c is somewhat complicated, the concrete form of $LSC(\Gamma, \iota_p)$ is rather simple and may be extracted from lemma 3.3.6

due to the supercoercivity of Γ. In the case where p is a continuous norm on $\mathcal{S}(\mathbb{R}^d)^*_\beta$, $LSC(\Gamma, \iota_p)$ is in fact simply given by Γ on $\mathcal{S}(\mathbb{R}^d)^*_\beta$ considered as a subspace of $\mathcal{S}(\mathbb{R}^d)^*_p$ and ∞ everywhere else. Likewise, if the regularisation happens to be such that all W_n are continuous, the resulting supercoercivity also gives simple expressions for W^c_n. As seen in section 3.1, this is indeed the case for the presented regularisation scheme. Even in these cases, it appears to the author that there is no straightforward simplification of the dual convergence criterion. The reason is that Attouch-Wets convergence in the spaces $\mathcal{S}(\mathbb{R}^d)^*_p$ does not provide enough uniformity for a sequential statement such as (M1)-convergence in $\mathcal{S}(\mathbb{R}^d)^*_\beta$ because the latter is not a Fréchet-Urysohn space.

The objects W^c_n in the above theorem correspond to a set of quantum effective actions of regularised theories and are precisely those objects which may be computed from the Wetterich equation. If they are ordered such that any regularisation vanishes in the $n \to \infty$ limit, theorem 3.4.13 states necessary and sufficient conditions for their convergence to a full theory in a physically meaningful manner according to the discussion in the beginning of this section.

3.5. The Non-regularised Limit and Asymptotic Safety

In literature, the Wetterich equation is usually treated as a statement that can be derived directly from the path integral provided a suitable UV regularisation has been performed. In practise, one often dismisses such regularisations altogether on the ground that F_k resp. R_k should suffice to make the theory well-defined. But in these cases we do not know what the path integral means, because we can only define it at a given regularisation. Consequently, it makes little sense a priori to speak of a non-regularised Wetterich equation. It is, however, rather natural to ask if the Wetterich equation survives the limit $n \to \infty$ of vanishing regularisation in some sense. In particular, supposing that a limit measure ω_∞ exists such that theorem 3.4.13 can be applied, there is a sequence of quantum effective actions $(\bar{\Gamma}^n_0)_{n\in\mathbb{N}}$ converging to some function $\bar{\Gamma}^\infty_0$ in the corresponding sense. If we now have chosen F_k to be independent on n, each of these effective actions $\bar{\Gamma}^n_0$ has an extension to a trajectory $(\bar{\Gamma}^n_k)_{k\in\mathbb{R}}$ traced out by the same differential equation such that it appears natural to ask whether there also is a corresponding trajectory $(\bar{\Gamma}^\infty_k)_{k\in\mathbb{R}}$ for the limit object.

An obvious question that arises is what the correct domain for $\bar{\Gamma}^\infty_0$ resp. $\bar{\Gamma}^\infty_k$ should be. It needs to be such that the right-hand side of the Wetterich equation is well-defined and, in particular, we need a way to take derivatives and to ensure that $D^2_\phi\bar{\Gamma}^n_k + F_k$ is invertible. Another pressing question is analogous to the renormalisation group equation in perturbative renormalisation schemes: With the example for $\mathcal{R}_n$ given in section 2.2, is $\bar{\Gamma}^\infty_0$ independent of Λ and K? If not, there is a residual regularisation dependence which would disqualify $\bar{\Gamma}^\infty_0$ as a physical theory.

If we assume such problems to be solved, we should expect a trajectory $(\bar{\Gamma}_k^\infty)_{k\in\mathbb{R}}$ satisfying the Wetterich equation. Then the necessity of renormalisation tells us that the boundary condition at $k \to \infty$ has to contain infinities. Hence, the next best classification of the boundary conditions is by their asymptotic behaviour. Let us parameterise the space of all admissible $\bar{\Gamma}_k^\infty$ of theories with FRG scale k by coupling constants $(\bar{g}_1, \dots)$ (in an expansion scheme, this can always be done with countably many coupling constants) with

$$\bar{g}_j(k) = k^{[\bar{g}_j]} g_j(k) \tag{3.5.1}$$

for all $j \in \{1, 2, \dots\}$ where $[\cdot]$ denotes the mass dimension and $[k] = 1$. Then all $g \in \mathbf{g} = (g_1, \dots)$ are dimensionless and the Wetterich equation (at least in an expansion scheme) is equivalent to an autonomous system of ordinary differential equations

$$k\partial_k g_j = f_j(\mathbf{g}) , \tag{3.5.2}$$

for some functions f_j. This follows directly from power counting, because k is the only dimensionful quantity. Let us call a tuple $\mathbf{g}^*$ of dimensionless coupling constants a **fixed point** of the renormalisation group flow if $f_j(\mathbf{g}^*) = 0$ [Wei76; Wei79].

The **Asymptotic Safety** hypothesis now states that any physical trajectory should run into such a fixed point according to the above definition in the limit of k going to infinity, i.e

$$\lim_{k\to\infty} g_j(k) = g_j^*(k) . \tag{3.5.3}$$

As in the theory of differentiable dynamics one may characterise the tangent vectors at a fixed point in the space spanned by all coupling constants into UV-attractive, UV-stationary and UV-repulsive by their respective $k \to \infty$ behaviour. The asymptotic safety postulate now amounts to the statement that any physical trajectory has to lie in the submanifold (**UV critical surface**) traced out by the UV-attractive (and possibly UV-stationary depending on higher-order corrections) directions. Thus, knowing the fixed points and the classification of the corresponding tangent vectors the actual physics at $k \to 0$ is strongly constrained, whence one speaks of the enhanced predictivity of asymptotic safety. Whether physically relevant QFTs are asymptotically safe is a focus of current research. For further details and reviews of asymptotic safety, see [Wei79; Per17; RS19; BGS11; Bra12; Nag14].

In practice one often restricts oneself to a finite set of coupling constants which corresponds to truncations of the flow equations with respect to some expansion scheme. Then the stability of fixed points, the dimensionality of the UV critical surface as well as the qualitative features of flows of the coupling constants with respect to successive inclusion of more parameters can be used as heuristics towards validating the asymptotic safety conjecture.

4. A Prototypical Exact Solution to the Wetterich Equation

This chapter is derived from [Zie21b].

In QFT and related fields one rarely has access to exact expressions for quantities of interest. Instead, one generally resorts to approximation schemes such as truncations of power series or lattice discretisations. But the use of such approximations raises the question of their respective reliability. In terms of observables, one is interested in quantitative bounds on deviations from exact values. However, the necessity of renormalisation turns the analysis of such deviations into a complicated task. They are commonly studied by investigating artificial regulator dependencies, the apparent convergence of truncation schemes or by purely qualitative methods such as apparent stability of features like fixed points or phase transitions. Nonetheless, it usually remains very difficult and often practically impossible to provide quantitative bounds on absolute errors and hence to explicitly specify the region of applicability of any given approximation procedure.

There are some notable exceptional cases in which exact results have been obtained such as the Schwinger model [Sch62], the Thirring model [Thi58] and lattice ϕ^4_3 and $\phi^4_{d>4}$ theories [BFS83; Aiz81]. Further exact results in QFT models [GMR20], condensed matter physics [SBK05] as well as in hydrodynamics [CDW16] and statistical mechanics [BW13] have been obtained through the use of the functional renormalisation group which is also at the core of this chapter. It constitutes a renormalisation scheme of the path integral quantisation and leads to well known expressions for the renormalisation group flow. These include the Wegner-Houghton [WH73], the Polchinski [Pol84] and the Wetterich [Wet93] equations, the latter being in the focus here. In particular, it is also routinely used in studies of asymptotic safety scenarios of quantum gravity [RS19; Per17]. For reviews and further applications, see [Dup+20; Met+12; BTW02; FP07; Gie12; Del12; Bra12; Nag14].

The expansion of the Wetterich equation in powers of quantum fields corresponds to an expansion in one-particle irreducible vertices. It constitutes a countably infinite tower of non-linear ordinary differential equations encoding the renormalisation group flow of correlation functions of the QFT at hand. As will be demonstrated, it is possible to bootstrap formally (in the sense of not necessarily analytic) exact solutions to these equations by providing a well-behaved, consistent set of low-order correlation functions and giving an explicit construction procedure for the higher-order ones. In this chapter the above method is employed to construct exact solutions to the Wetterich equation for QFTs on Euclidean spacetimes of dimensions $d > 2$ that satisfy the naive boundary conditions of massive and interacting real scalar ϕ^4 theories in the classical limit. This boundary condition corresponds to strictly finite renormalisations of all coupling constants. Consequently, the results do not agree with the rigorously known results for the ϕ^4_3 theory in the Ising universality class. In particular, the constructed solutions are shown to correspond to generalised free Quantum Field Theories.

Nonetheless, the author believes that exact solutions may provide good grounds for further research on the functional renormalisation group and its applications. Through their constructive nature the solutions given in this paper may also be able to open the door to more rigorous error estimates because the knowledge of bounds on lower-order correlators may be employed to produce bounds on higher-order ones.

4.1. The UV Unregularized Functional Renormalisation Group

Let us start with the Euclidean path integral quantisation of a classical action S_Λ for a real scalar field at an ultraviolet regularisation scale $\Lambda > 0$. Then (c.f eq. (3.1.61))

$$\exp\left[-\Gamma\left(\phi\right)\right] = \int \exp\left[-S_\Lambda\left(\phi+\psi\right) + \left(D_\phi \Gamma\right)\left(\psi\right)\right] \mathcal{D}_\Lambda \psi \,, \tag{4.1.1}$$

where $\mathcal{D}_\Lambda$ denotes the regularised path integral measure and Γ is the effective action. For clarity and brevity we shall use Fréchet derivatives instead of functional derivatives throughout this work which are related by

$$\left(D_\phi \Gamma\right)\left(\psi\right) = \int_{\mathbb{R}^d} \frac{\delta \Gamma\left(\phi\right)}{\delta \phi\left(x\right)} \psi\left(x\right) \mathrm{d}x \tag{4.1.2}$$

for suitable test functions ψ. Introducing the effective average action $\Gamma_{k,\Lambda}$, one obtains[RW94; MR09]

$$\exp\left[-\Gamma_{k,\Lambda}\left(\phi\right)\right] = \int \exp\Big[-S_\Lambda\left(\phi+\psi\right) + \left(D_\phi \Gamma_{k,\Lambda}\right)\left(\psi\right) - \frac{1}{2}\left(\psi, R_k \psi\right)\Big] \mathcal{D}_\Lambda \psi \,, \tag{4.1.3}$$

where R_k is a suitable scale-dependent regulator and $(\cdot, \cdot)$ denotes the standard inner product on $L^2(\mathbb{R}^d)_\mathbb{C}$. In particular, for $k \to 0$ the regulator R_k should vanish such that $\lim_{k\to 0} \Gamma_{k,\Lambda}$ reproduces the ordinary effective action Γ. On the other extreme R_k should diverge when $k \to \Lambda$ causing it to act as a delta functional with respect to the path integral ensuring $\lim_{k\to\Lambda} \Gamma_{k,\Lambda} \approx S_\Lambda$ [Wet93], although it is known that this correspondence involves a reconstruction problem

[MR09]. Through the standard derivations one also obtains the Wetterich equation [Wet93; Ell94; Mor94]

$$\partial_k \Gamma_{k,\Lambda}(\phi) = \frac{1}{2} \operatorname{Tr}_\Lambda \left[(\partial_k R_k) \left(\Gamma^{(2)}_{k,\Lambda}\Big|_\phi + R_k \right)^{-1} \right] , \tag{4.1.4}$$

where $\Gamma^{(2)}_{k,\Lambda}|_\phi$ denotes the second derivative of $\Gamma_{k,\Lambda}$ at ϕ interpreted as an operator.[1]

Remark 4.1.1. Compared to section 3.1, we employ a somewhat different notation in this chapter that is closer to the one that is typically used in related literature. Furthermore, the lack of the field-independent term compared to theorem 3.1.18 does not cause any harm, as we shall perform an expansion in powers of ϕ and ignore the constant term.

As in remark 3.0.1, we shall assume that $R_k \sim k^2$ for modes with small momentum. Consequently, R_k acts as an infrared regulator in the sense that it contributes a mass term to these modes. Hence loop integrals will be infrared finite because of R_k and UV finite by the UV regularisation scale Λ. It is, however, possible, to choose R_k such that the Wetterich equation as in theorem 3.1.18 is well-defined in the limit $\Lambda \to \infty$ by demanding that $\partial_k R_k$ decays fast enough. A common choice that works is the exponential regulator as given in section 3.1. In view of chapter 2, it is clear that this limit will generally entail a divergence of the coupling constants in the action S_Λ making this procedure problematic. A feasible solution to this issue is given by the analysis of fixed points of the renormalisation group flow which is explained in detail in chapter 5. Nonetheless, in this chapter we shall assume that the limit $\lim_{\Lambda\to\infty} S_\Lambda$ exists because it enables us to prove the existence as well as a construction of an exact solution to the Λ-free [MR09] Wetterich equation.

While the resulting solution may be unphysical it is to the knowledge of the author the only (nontrivial) full solution for higher-dimensional interacting QFTs that has been constructed at the time of this writing (See e.g. [TL96; Kno21] for exact equations/solutions in the limit of large flavour numbers). Furthermore, there is the hope that, once one exact solution can be found, others might follow by suitable generalisations.

Let us refer to the Λ-free effective average action as $\Gamma_k = \lim_{\Lambda\to\infty} \Gamma_{k,\Lambda}$ to which we shall devote our attention throughout this chapter. Expanding the right hand side of the Λ-free Wetterich equation in powers of a real scalar field ϕ gives us

$$\partial_k \Gamma_k(\phi) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \operatorname{Tr} \left[(\partial_k R_k) \left(D_0^n A \right) \left(\phi^{\otimes n} \right) \right] , \tag{4.1.5}$$

[1] i.e. for all suitable test functions ψ_1, ψ_2 on $\mathbb{R}^d$ we have

$$\left(\psi_1^*, \Gamma^{(2)}_k \Big|_\phi \psi_2 \right) = \int_{\mathbb{R}^d} \psi_1 \, \Gamma^{(2)}_{k,\Lambda}\Big|_\phi \psi_2 = \left(D^2_\phi \Gamma_k \right)(\psi_1, \psi_2) \, .$$

where $A(\phi) = (\Gamma_k^{(2)}|_\phi + R_k)^{-1}$ and the $n = 0$ term is dropped because it does not contribute to observables. We have also assumed that the sum may be taken out of the trace corresponding to an interchange of limits. $\phi^{\otimes n}$ denotes the tensor product $\phi \otimes ... \otimes \phi$ with a total of n factors.

Expanding the left hand side in powers of ϕ and comparing the coefficients leads to

$$\partial_k D_0^n \Gamma_k \left(\phi^{\otimes n}\right) = \frac{1}{2} \operatorname{Tr} \left[(\partial_k R_k) \left(D_0^n A\right) \left(\phi^{\otimes n}\right) \right] . \tag{4.1.6}$$

We now wish to find an explicit expression for $D_0^n A$ which may be achieved inductively by noting that

$$(D_\phi A)(\psi) = -A(\phi) \circ \left(D_\phi \Gamma_k^{(2)} \right)(\psi) \circ A(\phi) \quad \text{or shorter} \quad DA = -A \circ D\Gamma_k^{(2)} \circ A\,. \tag{4.1.7}$$

An educated guess produces the induction hypothesis

$$D^n A = \sum_{c \in \mathcal{C}(n)} (-1)^{\#c} \frac{n!}{c!} A \circ \prod_{l=1}^{\#c} \left[D^{c_l} \Gamma_k^{(2)} \circ A \right] , \tag{4.1.8}$$

where $\mathcal{C}(n)$ denotes the set of all multi-indices with positive entries that are combinations[2] of the natural number n, e.g

$$\mathcal{C}(3) = \{(1,1,1),(1,2),(2,1),(3)\} . \tag{4.1.9}$$

In eq. (4.1.8), $\#c$ is the length of such a multi-index and

$$c! = \prod_{l=1}^{\#c} (c_l!) \,, \qquad |c| = \sum_{l=1}^{\#c} c_l = n \tag{4.1.10}$$

for all $n \in \mathbb{N}$ and any $c \in \mathcal{C}(n)$. The inductive proof of eq. (4.1.8) is given in section 4.7.1. Inserting this result into eq. (4.1.6) then yields

$$\partial_k D_0^n \Gamma_k \left(\phi^{\otimes n}\right) = \frac{1}{2} \sum_{c \in \mathcal{C}(n)} (-1)^{\#c} \frac{n!}{c!} \operatorname{Tr} \left\{ (\partial_k R_k) A(0) \prod_{l=1}^{\#c} \left[\left(D_0^{c_l} \Gamma_k^{(2)} \right) \left(\phi^{\otimes c_l} \right) A(0) \right] \right\} . \tag{4.1.11}$$

Equation (4.1.11) expresses all possible one-loop diagrams generated by an arbitrary action Γ_k contributing to the renormalisation group flow of a given correlation function. As is common practice, we shall work with them explicitly in the Fourier picture. Restricting ourselves to

[2] Partitions including permutations

translation-invariant QFTs[3], for every $n \in \mathbb{N}$ there is a (k-dependent) function κ_n[4] such that

$$\begin{aligned}\left(D_0^n \Gamma_k\right)\left(\phi_1 \otimes \ldots \otimes \phi_n\right) = (2\pi)^{\frac{d}{2}(2-n)} \int_{\left(\mathbb{R}^d\right)^{n-1}} \kappa_n\left(p_1, \ldots, p_{n-1}; k\right) \\ \times \hat{\phi}_1\left(p_1\right) \ldots \hat{\phi}_{n-1}\left(p_{n-1}\right) \hat{\phi}_n\left(-\left[p_1 + \ldots + p_{n-1}\right]\right) \mathrm{d}p_1 \ldots \mathrm{d}p_{n-1}\end{aligned} \tag{4.1.12}$$

for all test functions $\phi_1, .., \phi_n$. These κ_n are precisely the commonly considered one-particle irreducible n-point functions in Fourier space stripped of their delta-functions:

$$\Gamma_k^{(n)}\left(p_1, \ldots, p_n\right) = \kappa_n\left(p_1, \ldots, p_{n-1}\right) \delta\left(p_1 + \ldots + p_n\right) . \tag{4.1.13}$$

Consequently, any such $D_0^n \Gamma_k$ is translation invariant in the sense that

$$\left(D_0^n \Gamma_k\right)\left(T\phi_1 \otimes \ldots \otimes T\phi_n\right) = \left(D_0^n \Gamma_k\right)\left(\phi_1 \otimes \ldots \otimes \phi_n\right) \tag{4.1.14}$$

for all translations T of $\mathbb{R}^d$ by the properties of the Fourier transform. Furthermore, such a $D_0^n \Gamma_k$ is obviously $\mathcal{O}(d)$-invariant whenever the corresponding κ_n is.[5] To simplify equations from this point on, any k-dependence will be notationally suppressed whenever it does not lead to ambiguities. Since Fréchet derivatives are invariant under permutations there are corresponding symmetries of the κ_n: For all $\sigma \in \mathrm{Sym}_{n-1}$

$$\kappa_n\left(p_{\sigma(1)}, \ldots, p_{\sigma(n-1)}\right) = \kappa_n\left(p_1, \ldots, p_{n-1}\right) \tag{4.1.15}$$

and also

$$\kappa_n\left(p_1, \ldots, p_{n-1}\right) = \kappa_n\left(-\left[p_1 + \ldots + p_{n-1}\right], p_2, \ldots, p_{n-1}\right) \tag{4.1.16}$$

for all $p_1, \ldots, p_{n-1} \in \mathbb{R}^d$. We shall refer to functions f satisfying these symmetries as Sym^*_{n-1} symmetric.[6] It remains to phrase eq. (4.1.11) in terms of the correlation functions κ_n. While the left-hand side is simple, let us take a look at the right-hand side first: If the expression within the trace is viewed as an integral operator the trace can be evaluated by integration along the diagonal. From the definition of the κ_n we already know the integral form of the derivatives

[3] We work in the $\Lambda \to \infty$ limit. Hence this does not contradict the discussion in section 3.2.

[4] The prefactors $(2\pi)^{\frac{d}{2}(2-n)}$ are chosen such that they vanish in position space.

[5] The action of $\mathcal{O}(d)$ on a function $g : (\mathbb{R}^d)^n \to \mathbb{R}$ is the standard one, defined as

$$(Og)(p_1, \ldots, p_n) = g(O^{-1}p_1, \ldots, O^{-1}p_n)$$

for all $O \in \mathcal{O}(d)$ and $p_1, \ldots, p_n \in \mathbb{R}^d$.

[6] 'Sym' standing for the symmetric (permutation) group and '$*$' for the involution given by

$$\left(p_1, \ldots, p_{n-1}\right) \mapsto \left(-\left[p_1 + \ldots + p_{n-1}\right], p_2, \ldots, p_{n-1}\right) .$$

The full group Sym^*_{n-1} of symmetries is isomorphic to Sym_n but the underlying action is a non-standard one on $(n-1)$-tuples, hence the alternative naming.

of Γ_k and it only remains to express R_k appropriately. It is common practice to define R_k in momentum space as a family of multiplication operators parameterised by k, i.e

$$\left[\mathcal{F}R_k\mathcal{F}^{-1}\phi\right](p) = \bar{r}(p;k)\,\phi(p) \tag{4.1.17}$$

for some $\bar{r} : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$.[7] The role of $\bar{r}$ is to contribute a 'momentum-dependent mass' that protects against IR singularities and at the same time screens UV divergences at any finite scale $k > 0$ by a rapid decay for large momenta. Thus, $\bar{r}$ and κ_2 have to be treated on similar footings so that we have to demand $\bar{r}(q) = \bar{r}(-q)$ for all $q \in \mathbb{R}^d$ in accordance with the Sym_1^* symmetry of κ_2. Choosing a Sym_1^*-violating regulator would generate further symmetry-breaking terms leading to undesirable contributions that are not translation invariant. The trace in eq. (4.1.11) then becomes

$$\begin{aligned}\mathrm{Tr}\{...\} = (2\pi)^{-\frac{|c|d}{2}} \int_{\left(\mathbb{R}^d\right)^{|c|-1}} &\lambda_c\left(p_1, ..., p_{|c|-1}\right) \\ &\times \hat{\phi}(p_1)\,...\hat{\phi}\left(p_{|c|-1}\right)\hat{\phi}\left(-\left[p_1 + ... + p_{|c|-1}\right]\right)\mathrm{d}p_1...\mathrm{d}p_{|c|-1}\,,\end{aligned} \tag{4.1.18}$$

where

$$\begin{aligned}\lambda_c\left(p_1^1, ..., p_{c_1}^1, ..., p_{c_{\#c}-1}^{\#c}\right) = \int_{\mathbb{R}^d} &\frac{(\partial_k\bar{r})(q)}{\left[\kappa_2(q) + \bar{r}(q)\right]^2}\kappa_{2+c_{\#c}}\left(p_1^{\#c}, ..., p_{c_{\#c}-1}^{\#c}, -\sum_{a=1}^{\#c}\sum_{b=1}^{c_{\#c}-1} p_b^a, q\right) \\ &\times \prod_{l=1}^{\#c-1}\frac{\kappa_{2+c_l}\left(p_1^l, ..., p_{c_l}^l, q - \sum_{a=1}^{l}\sum_{b=1}^{c_l} p_b^a\right)}{(\kappa_2 + \bar{r})\left(q - \sum_{a=1}^{l}\sum_{b=1}^{c_l} p_b^a\right)}\mathrm{d}q\,.\end{aligned} \tag{4.1.19}$$

This represents an integral over an arbitrary one-loop diagram containing all possible vertices in closed form.

Let us now collect the above and rewrite eq. (4.1.11) as

$$\begin{aligned}0 = \int_{\left(\mathbb{R}^d\right)^{n-1}} &\left[(2\pi)^d\,(\partial_k\kappa_n)(p_1, ..., p_{n-1}) - \frac{1}{2}\sum_{c\in\mathcal{C}(n)}(-1)^{\#c}\frac{n!}{c!}\lambda_c(p_1, ..., p_{n-1})\right] \\ &\times \hat{\phi}(-[p_1 + ... + p_{n-1}])\,\hat{\phi}(p_1)\,...\hat{\phi}(p_{n-1})\,\mathrm{d}p_1...\mathrm{d}p_{n-1}\,.\end{aligned} \tag{4.1.20}$$

Before the fundamental lemma of the calculus of variations may be invoked here, we need to polarise this equation, allowing for arbitrary test functions of the form $\phi_1 \otimes ... \otimes \phi_n$ in-

[7]We use $\bar{r}$ to avoid confusion with the commonly used shape function r defined by

$$\bar{r}(p) = p^2 r\left(\frac{p^2}{k^2}\right)\,.$$

stead of purely diagonal ones $\phi^{\otimes n}$. However, this polarisation will leave the κ_n part invariant (after proper substitutions of the integral variables) by eq. (4.1.12) due to its Sym^*_{n-1} symmetry. Therefore, such a polarisation is exactly the same as a Sym^*_{n-1} symmetrisation. Hence, simply defining

$$\bar{\lambda}_c\left(p_1, \ldots, p_{n-1}\right) = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} \lambda_c\left(p_{\sigma(1)}, \ldots, p_{\sigma(n-1)}\right) , \tag{4.1.21}$$

where we set $p_n = -[p_1 + \ldots + p_{n-1}]$, sidesteps the explicit polarisation. Invoking the fundamental lemma of the calculus of variations then leads to

$$\partial_k \kappa_n = \frac{1}{2\left(2\pi\right)^d} \sum_{c \in \mathcal{C}(n)} \left(-1\right)^{\#c} \frac{n!}{c!} \bar{\lambda}_c \, . \tag{4.1.22}$$

This is an equivalent formulation of eq. (4.1.11) and will be referred to as the flow equation of the correlation function κ_n. While these equations for arbitrary $n \in \mathbb{N}$ are certainly implied by the Λ-free form of eq. (4.1.4) if

- Γ_k is analytic,
- $\phi \mapsto \left(\Gamma_k^{(2)}\Big|_\phi + R_k \right)^{-1}$ is analytic,
- the sum in eq. (4.1.5) may be pulled out of the trace,

the converse is not necessarily true: A given solution might not correspond to an analytic Γ_k, that is the formal series

$$\sum_{n=1}^{\infty} \frac{\left(2\pi\right)^{\frac{d}{2}(2-n)}}{n!} \int_{\left(\mathbb{R}^d\right)^{n-1}} \kappa_n\left(p_1, \ldots, p_{n-1}\right) \hat{\phi}\left(-\left[p_1 + \ldots + p_{n-1}\right]\right) \hat{\phi}\left(p_1\right) \ldots \hat{\phi}\left(p_{n-1}\right) \mathrm{d}p_1 \ldots \mathrm{d}p_{n-1} \tag{4.1.23}$$

might diverge for some non-zero test function ϕ. Nonetheless, in the study of differential equations a lot of insight is often gained by an initial broadening of the space of admissible solutions and in this spirit, one might even expect such formal solutions to be very important for the general study of the Wetterich equation.

One further remark is in order at this point: Upon solving eq. (4.1.4) it is not clear whether there always exists a corresponding S_Λ satisfying eq. (4.1.3) amounting to the reconstruction problem [RS19]. Especially, a possible non-uniqueness of solutions to eq. (4.1.4) casts doubts on a positive conjecture. The situation is made even less clear by studying solutions to the Λ-free version of the Wetterich equation due to the difficulty of non-regularised path integrals.

4.2. A Constructive Solution for the Correlation Functions

A full solution to the flow eq. (4.1.22) with $\kappa_m \neq 0$ for some $m \in \mathbb{N}_{\geq 3}$ of course seems rather difficult to find due to the non-linear structure of the $\bar{\lambda}_c$ terms. This is the reason why one in practise usually truncates the equations at a finite $n \in \mathbb{N}$. There are, however, precisely three terms on the right hand side of eq. (4.1.22) for $n \in \mathbb{N}_{\geq 3}$ revealing a somewhat linearish structure, namely

$$\begin{aligned} c &= (n) && \Rightarrow \bar{\lambda}_c \text{ depends linearly on } \kappa_{n+2}\,, \\ c &= (n-1,1) && \Rightarrow \bar{\lambda}_c \text{ depends linearly on } \kappa_{n+1}\,, \\ c &= (1,n-1) && \Rightarrow \bar{\lambda}_c \text{ depends linearly on } \kappa_{n+1}\,. \end{aligned} \tag{4.2.1}$$

Phrased differently, for all $n \in \mathbb{N}_{\geq 3}$ there exist linear operators I_n implicitly depending on $\{\kappa_2, \bar{r}\}$ and J_n implicitly depending on $\{\kappa_2, \kappa_3, \bar{r}\}$ such that

$$I_n \kappa_{n+2} = -2\,(2\pi)^d\,\partial_k \kappa_n + n J_n \kappa_{n+1} + \sum_{c \in \mathcal{C}(n) \setminus \{(n),(n-1,1),(1,n-1)\}} (-1)^{\#c}\,\frac{n!}{c!}\bar{\lambda}_c\,. \tag{4.2.2}$$

The significance of this equation lies in the fact, that the right-hand side depends only on $\{\kappa_2, ..., \kappa_{n+1}, \bar{r}\}$. Suppose now that all I_n possess right inverses ρ_n, i.e. mappings such that $I_n \circ \rho_n = \mathrm{id}$. Then, setting

$$\kappa_{n+2} = \rho_n \left[-2\,(2\pi)^d\,\partial_k \kappa_n + n J_n \kappa_{n+1} + \sum_{c \in \mathcal{C}(n) \setminus \{(n),(n-1,1),(1,n-1)\}} (-1)^{\#c}\,\frac{n!}{c!}\bar{\lambda}_c \right] \tag{4.2.3}$$

will evidently solve eq. (4.2.2). This fact suggests the following approach for solving the flow equation for the correlators:

1. For some $N \in \mathbb{N}$ find $\kappa_1, ..., \kappa_{N+1}$ satisfying eq. (4.1.22) for all $n \in \mathbb{N}_{<N}$.

2. Find a right inverse ρ_N of I_N.

3. Construct κ_{N+2} as in eq. (4.2.3).

4. Increase N by 1 and go back to step 2.

This iterative construction will produce κ_n for all $n \in \mathbb{N}$ and they will satisfy their respective flow equation. Evidently, this construction depends crucially on the initial $\kappa_1, ..., \kappa_{N+1}$ which have to be given as input for all values of momenta and the scale k. This input may be pictured as a different kind of boundary condition to the Wetterich equation: Instead of specifying the classical theory at $k \to \infty$ one specifies the full renormalisation group flow of a finite set of one-particle irreducible correlators which is exemplified in fig. 4.1. A prototypical input might be given by a propagator κ_2 obtained through some other well-developed method like a

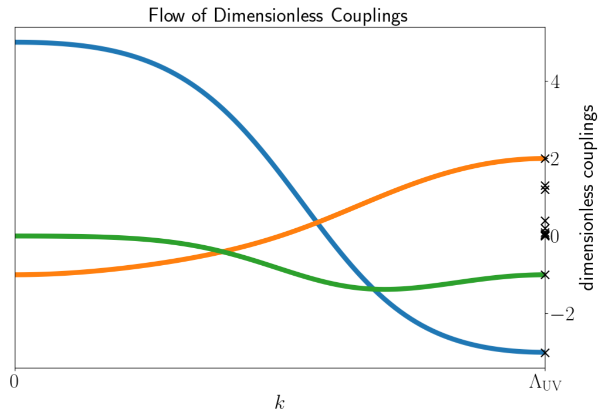

Figure 4.1.: A fictitious renormalisation group flow from a possibly infinite UV scale Λ_{UV} to 0. In the presented approach initial conditions correspond to the exemplified flow of three dimensionless couplings for all values of k. In the traditional approach the initial conditions are given by the values of infinitely many couplings at Λ_{UV} exemplified by the crosses on the right axis.

derivative expansion or even a numerical lattice computation. In case of a $\mathbb{Z}_2$-symmetric theory such input in fact provides a full starting point for the presented scheme since κ_1 and κ_3 vanish.

As is clear from this discussion, there is a certain amount of choice involved. Furthermore, in every iteration there may be several right inverses to choose from because the kernel of any I_n might be non-empty. Hence, this procedure is quite different from the usual approach of giving specific boundary conditions at some scale k or at $k \to \infty$. In fact, it shall be demonstrated that imposing the naive boundary condition of a real scalar ϕ^4 theory in the UV limit $k \to \infty$ does not guarantee the uniqueness of solutions to eq. (4.1.22). However, before diving into the specifics of ϕ^4 theory, we shall give explicit expressions for the I_n and particularly simple choices of linear right inverses ρ_n. For brevity, define

$$K(q) = \frac{(\partial_k \bar{r})(q)}{[\kappa_2(q) + \bar{r}(q)]^2}\,, \tag{4.2.4}$$

allowing to write

$$\lambda_{(n)}(p_1, \ldots, p_{n-1}) = \int_{\mathbb{R}^d} K(q)\, \kappa_{n+2}(p_1, \ldots, p_{n-1}, -[p_1 + \ldots + p_{n-1}], q)\, \mathrm{d}q\,. \tag{4.2.5}$$

By the Sym^*_{n+1} symmetry of κ_{n+2}, this is Sym^*_{n-1} symmetric such that $\bar{\lambda}_{(n)} = \lambda_{(n)}$ and

$$\bar{\lambda}_{(n)}(p_1, \ldots, p_{n-1}) = \int_{\mathbb{R}^d} K(q)\, \kappa_{n+2}(p_1, \ldots, p_{n-1}, -q, q)\, \mathrm{d}q\,. \tag{4.2.6}$$

Thus, one may write

$$(I_n f)(p_1, ..., p_{n-1}) = \int_{\mathbb{R}^d} K(q) f(p_1, ..., p_{n-1}, -q, q) \, \mathrm{d}q \tag{4.2.7}$$

for all functions $f : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ where the integral exists. The reason for allowing arbitrary functions f and not just Sym^*_{n+1}-symmetric ones is to facilitate the proof given in section 4.7.2 that the yet to be defined ρ_n are indeed right inverses of the corresponding I_n. An obvious choice of linear right inverse of the above I_n is given by

$$(\bar{\rho}_n g)(p_1, ..., p_{n+1}) = \frac{g(p_1, ..., p_{n-1})}{\int_{\mathbb{R}^d} K} . \tag{4.2.8}$$

However, in general such $\bar{\rho}_n g$ will not be Sym^*_{n+1} symmetric whenever g is Sym^*_{n-1} symmetric. This is unacceptable here, as it would generate terms that are not momentum conserving. Taking $\bar{\rho}_n$ as an ansatz and successively eliminating all Sym^*_{n+1}-violating terms generated by the action of Sym^*_{n+1} on functions of the form $\bar{\rho}_n g$ where g is taken to be Sym^*_{n-1} symmetric leads to the better choice

$$\begin{aligned} (\rho_n g)(p_1, ..., p_{n+1}) = & \sum_{J \subseteq \{0,...,n+1\}} \sum_{l=0}^{\left\lfloor \frac{n-1-\#J}{2} \right\rfloor} \frac{\alpha^n_{\#J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-l}} \\ & \times \int_{\left(\mathbb{R}^d\right)^{n-1-\#J-l}} g(p_J, -s_1, s_1, \ldots, -s_l, s_l, t_1, ..., t_{n-1-\#J-2l}) \\ & \times K(s_1) ... K(s_l) K(t_1) ... K(t_{n-1-\#J-2l}) \, \mathrm{d}s ... \mathrm{d}t ... , \end{aligned} \tag{4.2.9}$$

with

$$\alpha^n_{a,b} = \frac{(-1)^{n-1-a-b}}{n} 2^{n-1-a-2b} \binom{n-1-a-b}{b} . \tag{4.2.10}$$

In the above expression we have defined $p_0 = -[p_1 + ... + p_{n+1}]$ and introduced the shorthand notation $p_J := p_{J_1}, ..., p_{J_{\#J}}$. Note that the particular order of the corresponding momenta p in the above expression does not matter since g is presumed symmetric. Hence, we do not need another sum over all permutations of index sets J. For a proof that ρ_n is indeed a right inverse of I_n when restricted to Sym^*_{n-1}-symmetric functions, see section 4.7.2.

It is obvious that ρ_n is a linear operator and thus a particularly simple choice of right inverse of κ_n. Furthermore, it preserves $\mathcal{O}(d)$-invariance provided K itself is $\mathcal{O}(d)$-invariant. In our naive approach to ϕ^4 theory, we shall consider a two-point function that does not scale with k and approximates the free propagator

$$\kappa_{2,\text{free}}(p) = m^2 + \|p\|^2 \tag{4.2.11}$$

for some mass m. Hence, any k scaling of K comes from the choice of a regulator. Furthermore, common regulators scale like k^2 at small momenta leading to an overall k scaling of K as k^{-3}. A simple power counting in eq. (4.2.9) then reveals that ρ_n scales like k^{3-d}. This fact is remarkable, as it indicates that in $d > 3$ dimensions the correlators constructed through ρ_n are strongly suppressed for large k. This simplifies the control of the 'classical limit' $k \to \infty$, as one usually considers only a finite set of non-zero correlation functions in this limit. The small k behaviour is precisely the opposite. Here ρ_n grows arbitrarily large, possibly leading to IR divergences.

As mentioned before, the choice of a right inverse is not necessarily unique which can be seen explicitly in the case of $n = 2$. With the previous construction, we have

$$\begin{aligned}(\rho_2 g)(p,q,r) = &\frac{1}{2\int_{\mathbb{R}^d} K}\left[g(p) + g(q) + g(r) + g(-p-q-r)\right] \\ &- \frac{1}{\left(\int_{\mathbb{R}^d} K\right)^2}\int_{\mathbb{R}^d} g(t)\,K(t)\,\mathrm{d}t\,,\end{aligned} \tag{4.2.12}$$

which satisfies $(I_2 \circ \rho_2)\, g = g$ whenever g is Sym_1^* symmetric. However, there also exists a suitable non-linear right inverse ρ_2' given by

$$\begin{aligned}(\rho_2' g)(p,q,r) = &\frac{1}{8\int_{\mathbb{R}^d} gK}\Big(\left[g(p) + g(q) + g(r) + g(-p-q-r)\right]^2 \\ &- 2\left[g(p)^2 + g(q)^2 + g(r)^2 + g(-p-q-r)^2\right]\Big)\,.\end{aligned} \tag{4.2.13}$$

Hence, the operator I_2 indeed has a non-trivial kernel, since $I_2 \circ (\rho_2 - \rho_2') = 0$. Thus, there is a certain degree of freedom involved in the choice of a right inverse to I_2. In particular this choice may be used to construct higher correlators that satisfy certain constraints such as boundary conditions (e.g. at $k \to 0$ or $k \to \infty$) or decay properties like those produced in [FP07].

4.3. Solving the Flow Equations

We shall consider a real scalar QFT in d Euclidean dimensions without spontaneous symmetry breaking with the 'classical limit'

$$\lim_{k\to\infty} \kappa_2(p) = \kappa_{2,\text{free}}(p) = m^2 + \|p\|^2 \qquad \text{and} \qquad \lim_{k\to\infty} \kappa_4(p,q,r) = \frac{\lambda}{|m|^{d-4}} \tag{4.3.1}$$

for some $m \in \mathbb{R}$, $\lambda > 0$ and $\lim_{k\to\infty} \kappa_n = 0$ for all $n \in \mathbb{N} \setminus \{2,4\}$.[8] where the limits should be understood in a distributional sense.[9] In particular, for $k \to \infty$ all correlation functions of odd

[8]The κ_4 limit has been chosen such that λ is dimensionless.

[9]Technically speaking, κ_n is a distribution on $\mathbb{R}^{(n-1)d}$ and the k limits should be understood as pointwise convergence.

order vanish. We shall now set $N = 3$ and proceed as outlined in the preceding section. The reason for setting $N = 3$ is of course to be able to satisfy the boundary condition for $\kappa_{N+1} = \kappa_4$ for $k \to \infty$. We thus choose the ansatz

$$\kappa_4 (p, q, r; k) = \frac{\lambda}{|m|^{d-4}} \exp \left[- \frac{\|p\|^d + \|q\|^d + \|r\|^d + \|p+q+r\|^d + |m|^d}{k \, |m|^{d-1}} \right] , \tag{4.3.2}$$

which is obviously Sym_3^* and $\mathcal{O}(d)$ invariant and satisfies eq. (4.3.1). The rationale for choosing this particular form for κ_4 is to keep the upcoming integrals as simple as possible and to ensure a rapid decrease of κ_4 and its k derivatives for $k \to 0$. The latter is paramount for controlling the divergent k behaviour of ρ_n in this limit. At the same time, all higher correlators as generated by the ρ_n will vanish in the UV due to the very same k-scaling. The most natural choice for the lower odd correlators is

$$\kappa_3 = 0 \qquad \text{and} \qquad \kappa_1 = 0 \, , \tag{4.3.3}$$

which alongside the given construction procedure guarantees the vanishing of all odd correlators because

- for all odd $n \in \mathbb{N}$ any $c \in \mathcal{C}(n)$ contains an odd entry,
- the chosen ρ_n are linear.

This implements the standard $\mathbb{Z}_2$ symmetry such that only even correlators have to be dealt with. Equation (4.2.3) then simplifies to

$$\kappa_{2n+2} = \rho_{2n} \left[- 2 \, (2\pi)^d \, \partial_k \kappa_{2n} + \sum_{c \in \bar{\mathcal{C}}(2n) \setminus \{(2n)\}} (-1)^{\#c} \, \frac{(2n)!}{c!} \bar{\lambda}_c \right] , \tag{4.3.4}$$

where $\bar{\mathcal{C}}(n) \subset \mathcal{C}(n)$ denotes the set of combinations with even entries. The next step is now to find κ_2, since the flow equation for κ_1 is trivially satisfied. Equation (4.1.22) for $n = 2$ reads

$$\partial_k \kappa_2 (p) = - \frac{1}{2 \, (2\pi)^d} \left(I_2 \kappa_4 \right) (p) = - \frac{1}{2 \, (2\pi)^d} \int_{\mathbb{R}^d} K(q) \, \kappa_4 (p, -q, q) \, \mathrm{d}q \, , \tag{4.3.5}$$

which in general cannot be expected to have a solution that can be put in closed form due to the dependence of K on $\bar{r}$ and κ_2. One may, however, show that the differential equation may be solved iteratively as is done in section 4.7.3. The initial ansatz is chosen to be the free propagator $\kappa_{2,\text{free}}$ and the regulator is chosen as

$$\bar{r} (p; k) = \frac{\|p\|^2}{\exp \left[\frac{\|p\|^2}{k^2} \right] - 1} , \tag{4.3.6}$$

both of which are $\mathcal{O}(d)$-invariant. It is then demonstrated that whenever

$$0 \leq \lambda < \left(\sqrt{3}-1\right) d\, 2^{d+1} \pi^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) , \tag{4.3.7}$$

a bounded, $\mathcal{O}(d)$-invariant and smooth (in its momentum argument as well as in k) solution satisfying the boundary condition (4.3.1) exists and is approached by the iterative scheme. Note that the upper bound for λ does not denote a critical coupling, it merely ensures that rather straightforward estimates may be applied. We shall henceforth assume λ to be bounded as in eq. (4.3.7). At the core of the proof lies the inequality

$$\frac{1}{m^2 + \|p\|^2 + \bar{r}\,(p)} \leq \frac{1}{m^2 + k^2} , \tag{4.3.8}$$

leading to the existence of a $\kappa_2 > m^2$ satisfying

$$\begin{aligned} \kappa_2\,(p;k) &= \kappa_{2,\text{free}}\,(p;k) \\ &\quad + \frac{1}{2}\,(2\pi)^{-d} \int_k^\infty \int_{\mathbb{R}^d} \frac{\partial_{k'} r\,(q;k')}{\left[\kappa_2\,(q;k') + \bar{r}\,(q;k')\right]^2} \kappa_4\,(p,-q,q;k')\,\mathrm{d}q\,\mathrm{d}k' . \end{aligned} \tag{4.3.9}$$

The iterative construction procedure of κ_2 also guarantees the existence of the IR limit $k \to 0$. Note however, that in this limit κ_2 does not correspond to the free propagator. Once we know κ_2, constructing the higher-order correlation functions is straightforward employing eq. (4.3.4). Their respective $\mathcal{O}(d)$-invariance follows from that of K. It remains to discuss the behaviour of the correlators in the IR limit $k \to 0$ and the UV limit $k \to \infty$ respectively: Obviously κ_4 vanishes in the limit of $k \to 0$. As is proved in section 4.7.4 for all $n \in \mathbb{N}_{\geq 2}$ there are constants $B^{0,1}_{2n} > 0$ such that

$$\|\kappa_{2n}\|_{L^\infty} \leq B^{0,1}_{2n} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}\, k}{(k+|m|)^{(n-2)(1+\Delta)+1}} \tag{4.3.10}$$

for

$$\Delta = \begin{cases} 1 & d \geq 4 \\ d-3 & d < 4 \end{cases} . \tag{4.3.11}$$

These equations establish the central result of this work: For $d > 2$ all higher correlators vanish in both limits $k \to 0$ and $k \to \infty$. Thus, the IR limit is a non-interacting theory with a non-trivially momentum dependent propagator κ_2 - a generalised free theory. It may also be possible that the given solutions generalise to $d = 2$, since the proofs only make use of the property that the UV behaviour of $|\partial_k^l \kappa_4|$ is bounded by $\sim k^{-l}$. It is, however, even bounded by $\sim k^{-l-1}$ whenever $l \in \mathbb{N}$ which should guarantee the correct UV limits, while eq. (4.3.10) still ensures trivial IR limits. A formal argument showing this has not yet been produced.

In the definition of κ_4 in eq. (4.3.2), note that the argument in the exponential can be multiplied by any positive real number and still all estimates hold analogously with modified constants. Furthermore, the boundary conditions at $k \to \infty$ remain satisfied and all higher correlators vanish at $k = 0$ upon such a modification of κ_4. At the same time, the IR limit of κ_2 will in general be different. Such ansatzes do not correspond to a rescaling of k since the k dependence of $\bar{r}$ remains unaltered. Instead, they lead to different flows solving the flow equations for the correlators.

4.4. The Flow of the Dimensionless Potential

It is possible to extract the quantum potential from the correlators by examining their behaviour at zero momentum. Of particular interest is the flow of the dimensionless potential v given by

$$v(s) := \sum_{n=1}^{\infty} \frac{\kappa_{2n}(0, ..., 0)}{k^{2+(2-d)(n-1)}} \frac{s^{2n}}{(2n)!} . \tag{4.4.1}$$

It is appropriate to analyse its dimensionless flow, i.e. $k\partial_k v$ which we shall examine in the limits $k \to 0$ and $k \to \infty$. The κ_2 contribution is determined by eq. (4.3.9) where the second term on the right-hand side is non-negative for all $p \in \mathbb{R}^d$. Hence,

$$\lim_{k\to 0} \frac{\kappa_2(0)}{k^2} \geq \lim_{k\to 0} \frac{\kappa_{2,\text{free}}(0)}{k^2} = \infty , \tag{4.4.2}$$

so that the resulting two-point correlator contains a gap that is bounded from below by the bare gap. Furthermore,

$$\begin{aligned} \lim_{k\to 0} k\partial_k \frac{\kappa_2(0)}{k^2} &\leq \lim_{k\to 0} \left[\frac{\|\partial_k \kappa_2\|_{L^\infty}}{k} - 2\frac{\kappa_{2,\text{free}}(0)}{k^2} \right] \\ &\leq \lim_{k\to 0} \left[\frac{(2\pi)^{-d}}{2} R_1 A_4^0 \frac{|m|^3 k}{(k^2+m^2)^2} - 2\frac{\kappa_{2,\text{free}}(0)}{k^2} \right] = -\infty \end{aligned} \tag{4.4.3}$$

for constants $R_1, A_4^0 \geq 0$, where the $\|\partial_k \kappa_2\|_{L^\infty}$ estimate is taken from eq. (4.7.47) in section 4.7.4. Thus, the contribution of the propagator to the dimensionless potential diverges in the limit of $k \to 0$ which may be expected, since m is taken to not scale with k. The UV limits become

$$\lim_{k\to \infty} \frac{\kappa_2(0)}{k^2} = \lim_{k\to \infty} \frac{m^2}{k^2} = 0 \tag{4.4.4}$$

and

$$\begin{aligned} \lim_{k\to \infty} k\partial_k \frac{\kappa_2(0)}{k^2} &\leq \lim_{k\to \infty} \left[\frac{\|\partial_k \kappa_2\|_{L^\infty}}{k} - 2\frac{m^2}{k^2} \right] \\ &\leq \lim_{k\to \infty} \left[\frac{(2\pi)^{-d}}{2} R_1 A_4^0 \frac{|m|^3 k}{(k^2+m^2)^2} - 2\frac{m^2}{k^2} \right] = 0 . \end{aligned} \tag{4.4.5}$$

Thus in the limit of $k \to \infty$ the corresponding contribution to v vanishes and the solution lives in the deep-Euclidean region. For the contributions from the higher correlators, we use theorem 4.7.11 from section 4.7.4 to produce the estimates

$$\|\kappa_{2n}\|_{L^\infty} \leq B_{2n}^{0,x} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}\, k^x}{(k+|m|)^{(n-2)(1+\Delta)+x}}\,, \tag{4.4.6}$$

$$\|\partial_k \kappa_{2n}\|_{L^\infty} \leq B_{2n}^{1,x} \frac{|m|^{2+(2-d)(n-1)+(n-2)(1+\Delta)}\, k^x}{(k+|m|)^{(n-2)(1+\Delta)+1+x}}\,, \tag{4.4.7}$$

with constants $B_{2n}^{0,x}, B_{2n}^{1,x} \geq 0$ for all $x \in \mathbb{N}$ and $n \in \mathbb{N}_{\geq 2}$. Hence, for all such n,

$$\left| k\partial_k \frac{\kappa_{2n}(0,\ldots,0)}{k^{2+(2-d)(n-1)}} \right| \leq \left| \frac{\partial_k \kappa_{2n}(0,\ldots,0)}{k^{1+(2-d)(n-1)}} \right| + \left| \frac{2+(2-d)(n-1)}{k^{2+(2-d)(n-1)}} \kappa_{2n}(0,\ldots,0) \right|\,. \tag{4.4.8}$$

With the previous inequalities, we then obtain

$$\begin{aligned} \lim_{k\to 0} \left| \frac{\kappa_{2n}(0,\ldots,0)}{k^{2+(2-d)(n-1)}} \right| &\leq B_{2n}^{0,\max\{1,3+(2-d)(n-1)\}} \\ &\quad \times \lim_{k\to 0} \left(\frac{|m|}{k} \right)^{2+(2-d)(n-1)-\max\{1,3+(2-d)(n-1)\}} = 0\,. \end{aligned} \tag{4.4.9}$$

Likewise

$$\begin{aligned} \lim_{k\to 0} \left| \frac{\partial_k \kappa_{2n}(0,\ldots,0)}{k^{1+(2-d)(n-1)}} \right| &\leq B_{2n}^{1,\max\{1,2+(2-d)(n-1)\}} \\ &\quad \times \lim_{k\to 0} \left(\frac{|m|}{k} \right)^{1+(2-d)(n-1)-\max\{1,2+(2-d)(n-1)\}} = 0\,, \end{aligned} \tag{4.4.10}$$

so that v and $k\partial_k v$ in the limit of small k are fully determined by the κ_2 contributions. For large k the estimates

$$\left| \frac{\kappa_{2n}(0,\ldots,0)}{k^{2+(2-d)(n-1)}} \right| \leq B_{2n}^{0,1} \left(\frac{|m|}{k} \right)^{(3-d+\Delta)(n-2)+4-d}\,, \tag{4.4.11}$$

$$\left| \frac{\partial_k \kappa_{2n}(0,\ldots,0)}{k^{1+(2-d)(n-1)}} \right| \leq B_{2n}^{1,1} \left(\frac{|m|}{k} \right)^{(3-d+\Delta)(n-2)+4-d} \tag{4.4.12}$$

produce meaningful bounds whenever $d \leq 4$:

$$\lim_{k\to\infty} \left| \frac{\kappa_{2n}(0,...,0)}{k^{2+(2-d)(n-1)}} \right| \leq \begin{cases} 0 & d \leq 4 \\ B_{2n}^{0,1} & d = 4 \\ \infty & \text{otherwise}, \end{cases} \tag{4.4.13}$$

$$\lim_{k\to\infty} \left| \frac{\partial_k \kappa_{2n}(0,...,0)}{k^{1+(2-d)(n-1)}} \right| \leq \begin{cases} 0 & d \leq 4 \\ B_{2n}^{1,1} & d = 4 \\ \infty & \text{otherwise}. \end{cases} \tag{4.4.14}$$

Thus,

$$\lim_{k\to\infty} |v(s)| \leq \begin{cases} 0 & d < 4 \\ \sum_{n=2}^{\infty} B_{2n}^{0,1} \frac{s^{2n}}{(2n)!} & d = 4 \end{cases} \tag{4.4.15}$$

and

$$\lim_{k\to\infty} |k\partial_k v(s)| \leq \begin{cases} 0 & d < 4 \\ \sum_{n=2}^{\infty} Y_{2n} \frac{s^{2n}}{(2n)!} & d = 4 \end{cases} \tag{4.4.16}$$

for $Y_{2n} = B_{2n}^{1,1} + |2 + (2-d)(n-1)| B_{2n}^{0,1}$. In particular no definite statement is obtained by these methods for $d > 4$. However, the κ_4 contribution to v may be calculated explicitly:

$$k\partial_k \frac{\kappa_4(0,0,0)}{k^{4-d}} = \lambda \exp\left[-\frac{|m|}{k}\right] \left((4-d) \left(\frac{|m|}{k}\right)^{4-d} + \left(\frac{|m|}{k}\right)^{5-d} \right). \tag{4.4.17}$$

Hence,

$$\lim_{k\to\infty} k\partial_k \frac{\kappa_4(0,0,0)}{k^{4-d}} = \begin{cases} 0 & d \leq 4 \\ -\infty & \text{otherwise}. \end{cases} \tag{4.4.18}$$

so that for $d = 4$ the beta function of the quartic term of the dimensionless potential vanishes in the limit of $k \to \infty$.

In $d < 4$ we see that the dimensionless potential as well as its flow vanishes in the large k limit owing to the fact that κ_2 and κ_4 are bounded and have positive mass dimensions. Hence, in comparison with the Wilson-Fisher fixed point the constructed solution features a completely different phenomenology.

In the case of $d = 4$ the dimensionless potential is obviously non-vanishing while the fate of its flow in the limit of $k \to \infty$ is unclear. A numerical analysis showed that the coefficients Y_{2n} with $d = 4$ grow so fast that a zero radius of convergence is probable. Thus, we do not obtain a useful estimate of $\lim_{k\to\infty} |k\partial_k v(s)|$ in this case.

4.5. Possible Applications

The iterative scheme presented in section 4.2 provides a systematic approach to producing exact solutions to the expanded Wetterich equation. It may be straightforwardly adapted to fermionic fields as well as to models with multiple fields with few modifications. As such the approach is extremely general and can be applied in many situations. Obvious candidates are theories in which approximation schemes have produced a set of one-particle irreducible correlation functions such as propagators or flows of lower-order vertices. Such approximations may e.g. have been produced by expansion schemes or lattice computations and are not limited to analytic input but can just as well be numerical. In the case where such quantities have only been calculated at $k = 0$ without a regulator, a renormalisation group flow may be imposed through a suitable interpolation between the given result and some initial conditions along with a regulator. The approach is then to construct operators ρ_n that are compatible with boundary or regularity conditions and study features such as relevance and irrelevance of resulting higher order-operators along the renormalisation group flow. Likewise, proposed flows of lower-order correlators may be scrutinised and possibly dismissed if the flow of the higher-order correlators proves to be singular or fails to have correct asymptotics.

This constitutes a new bootstrap strategy to explore exact properties of the theory space using the functional renormalisation group.

4.6. Discussion

It has been demonstrated that a Euclidean invariant exact solution to eq. (4.1.22) satisfying the boundary conditions (4.3.1) exists and may be constructed as outlined in section 4.3. Furthermore, explicit bounds on the flow as given by eq. (4.3.10) and more generally by the methods applied in sections 4.7.3 and 4.7.4 may be utilised to approximate the flow of any given correlation function to arbitrary precision. By construction, the mass and the quartic coupling only undergo finite renormalisations during the flow from $k \to \infty$ to $k \to 0$. Thus, the theory in the latter limit does not correspond to the result for ϕ_3^4 in the Ising universality class which requires infinite renormalisations. This raises the question of how to determine the physically correct boundary conditions in the large k limit which is of course intimately connected with the physically appropriate choice of classical action S_Λ. Conversely, one may ask how a given renormalisation group flow determines S_Λ which precisely amounts to the reconstruction problem [MR09]. With S_Λ being unknown in this case, it is unclear whether $\lim_{k\to 0} \Gamma_k$ is independent of the choice of renormalisation scheme. In particular it was demonstrated that the flow was not uniquely determined by $\lim_{k\to\infty} \Gamma_k$. Hence, it may be expected that there is a yet to be uncovered connection between exact solutions to the flow equations and a possibly unique physical one.

The given solution was obtained through a very straightforward construction procedure that essentially enables the extrapolation of higher-order correlation functions from a set of lower-order ones. Though these extrapolations should not be expected to be unique, one may hope that their asymptotic behaviour for small and large values of k are strongly constrained. Such constraints can then reveal lots of structure of the higher correlators. In particular, the construction principle may be extended to models with multiple scalar fields as well as fermions without gauge symmetries. In the presence of gauge symmetries, the right inverses ρ_n would have to be chosen such that the symmetry constraints of the Ward identities are satisfied. Applying similar choices of ρ_n operators to systems truncated at finite $n \in \mathbb{N}$ may then give hints for or against the applicability of the truncations in use and possibly even enable the explicit calculation of uncertainties.

4.7. Mathematical Proofs

4.7.1. Proof of the Derivative Identity for the Propagator

Before stepping into the induction proof, note that eq. (4.1.8) corresponds to eq. (4.1.7) for $n = 1$. In order to further shorten notation, let us write

$$\langle c \rangle = \langle c_1, ..., c_l \rangle = A \circ \prod_{l=1}^{\#c} \left(D^{c_l} \Gamma_k^{(2)} \circ A \right) \tag{4.7.1}$$

for all $l \in \mathbb{N}$ and any multi-index $c \in \mathbb{N}^l$. Now define two operations on such multi-indices:

$$s_j : \mathbb{N}^l \to \mathbb{N}^l, \qquad (n_1, ..., n_l) \mapsto (n_1, ..., n_{j-1}, 1 + n_j, n_{j+1}, ..., n_l) \qquad j \in \mathbb{N}_{\leq l} \tag{4.7.2}$$

$$t_j : \mathbb{N}^l \to \mathbb{N}^{l+1}, \qquad (n_1, ..., n_l) \mapsto (n_1, ..., n_{j-1}, 1, n_j, ..., n_l) \qquad j \in \mathbb{N}_{\leq l+1} \tag{4.7.3}$$

For the inductive principle, assume the validity of eq. (4.1.8) for a fixed $n \in \mathbb{N}$. Then,

$$D^{n+1} A = \sum_{c \in \mathcal{C}(n)} (-1)^{1+\#c} \frac{n!}{c!} \sum_{j=1}^{1+\#c} \langle t_j (c) \rangle + \sum_{c \in \mathcal{C}(n)} (-1)^{\#c} \frac{n!}{c!} \sum_{j=1}^{\#c} \langle s_j (c) \rangle \ . \tag{4.7.4}$$

It is apparent that s_j and t_j are both injective maps from $\mathcal{C}(n)$ to $\mathcal{C}(n+1)$ for all possible j. Thus, we may equally well sum over $\mathcal{C}(n+1)$ instead of $\mathcal{C}(n)$ giving

$$D^{n+1} A = \sum_{c \in \mathcal{C}(n+1)} (-1)^{\#c} \frac{n!}{c!} \sum_{\substack{j=1 \\ c_j = 1}}^{\#c} \langle c \rangle \ = + \sum_{c \in \mathcal{C}(n+1)} (-1)^{\#c} \frac{n!}{c!} \sum_{\substack{j=1 \\ c_j \neq 1}}^{\#c} c_j \langle c \rangle \ , \tag{4.7.5}$$

where it is now obvious that

$$D^{n+1}A = \sum_{c\in\mathcal{C}(n+1)} (-1)^{\#c} \frac{n!}{c!} \sum_{j=1}^{\#c} c_j \langle c\rangle = \sum_{c\in\mathcal{C}(n+1)} (-1)^{\#c} \frac{(n+1)!}{c!} \langle c\rangle \ . \tag{4.7.6}$$

This proves eq. (4.1.8).

4.7.2. Proof that $I_n \circ \rho_n = \mathrm{id}$

Let us fix a real Sym^*_{n-1}-symmetric function g on $(\mathbb{R}^d)^{n-1}$ and compute $I_n\rho_n g$. In order to facilitate the proof, let us split $\rho_n g$ into the following parts defined by restricting the sum over J in eq. (4.2.9):

- $\rho_n^1 g$ where J contains no index $\geq n$,
- $\rho_n^2 g$ where J contains precisely one index $\geq n$,
- $\rho_n^3 g$ where J contains precisely two indices $\geq n$.

Then, $I_n\rho_n g = I_n\rho_n^1 g + I_n\rho_n^2 g + I_n\rho_n^3 g$ by the linearity of I_n.[10] Hence, it suffices to analyse the three parts individually: The first part becomes

$$\begin{aligned}\left(I_n\rho_n^1 g\right)(p_1,\dots,p_{n-1}) = &\sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=0}^{\left\lfloor\frac{n-1-\#J}{2}\right\rfloor} \frac{\alpha^n_{\#J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-\#J-l}} \\ &\times \int_{\left(\mathbb{R}^d\right)^{n-1-\#J-l}} \int_{\mathbb{R}^d} K(q)\, g\left(p_J, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-1-\#J-2l}\right) \\ &\times K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-1-\#J-2l}\right) \mathrm{d}q\, \mathrm{d}s_{\dots}\mathrm{d}t_{\dots}\end{aligned} \tag{4.7.7}$$

where p_J contains neither q nor $-q$. Thus, we may evaluate the q integral and obtain

$$\begin{aligned}\left(I_n\rho_n^1 g\right)(p_1,\dots,p_{n-1}) = &\sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=0}^{\left\lfloor\frac{n-1-\#J}{2}\right\rfloor} \frac{\alpha^n_{\#J,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-1-\#J-l}} \\ &\times \int_{\left(\mathbb{R}^d\right)^{n-1-\#J-l}} g\left(p_J, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-1-\#J-2l}\right) \\ &\times K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-1-\#J-2l}\right) \mathrm{d}s_{\dots}\mathrm{d}t_{\dots}\ .\end{aligned} \tag{4.7.8}$$

In the second part p_J contains either q or $-q$. But $K(q) = K(-q)$ since $\bar{r}(q) = \bar{r}(-q)$, such that contributions are identical. Removing the index n or $n+1$ respectively from J and

[10] This splitting makes sense, because I_n is also defined for non-Sym_{n+1}-symmetric functions.

inserting q explicitly then leads to

$$\begin{aligned}\left(I_n\rho_n^2 g\right)(p_1,\dots,p_{n-1}) = 2 \sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=0}^{\left\lfloor\frac{n-2-\#J}{2}\right\rfloor} \frac{\alpha^n_{\#J+1,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-1-\#J-l}} \\ \times \int_{\left(\mathbb{R}^d\right)^{n-2-\#J-l}} \int_{\mathbb{R}^d} K(q)\, g\left(p_J, q, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-2-\#J-2l}\right) \\ \times K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-2-\#J-2l}\right) \mathrm{d}q\, \mathrm{d}s\dots \mathrm{d}t\dots\,,\end{aligned} \tag{4.7.9}$$

where the factor of 2 comes from the two possibilities of picking either n or $n+1$. Relabelling q to $t_{n-1-\#J-2l}$ simplifies this part to

$$\begin{aligned}\left(I_n\rho_n^2 g\right)(p_1,\dots,p_{n-1}) = \sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=0}^{\left\lfloor\frac{n-2-\#J}{2}\right\rfloor} \frac{2\alpha^n_{\#J+1,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-1-\#J-l}} \\ \times \int_{\left(\mathbb{R}^d\right)^{n-1-\#J-l}} g\left(p_J, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-1-\#J-2l}\right) \\ \times K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-1-\#J-2l}\right) \mathrm{d}s\dots \mathrm{d}t\dots\end{aligned} \tag{4.7.10}$$

where the similarity to eq. (4.7.8) is immediate. In the third part J contains both n and $n+1$ corresponding to p_J containing both q and $-q$. Removing these indices from J, one obtains

$$\begin{aligned}\left(I_n\rho_n^3 g\right)(p_1,\dots,p_{n-1}) = \sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=0}^{\left\lfloor\frac{n-3-\#J}{2}\right\rfloor} \frac{\alpha^n_{\#J+2,l}}{\left(\int_{\mathbb{R}^d} K\right)^{n-2-\#J-l}} \\ \times \int_{\left(\mathbb{R}^d\right)^{n-3-\#J-l}} \int_{\mathbb{R}^d} g\left(p_J, -q, q, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-3-\#J-2l}\right) \\ \times K(q)\, K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-3-\#J-2l}\right) \mathrm{d}q\, \mathrm{d}s\dots \mathrm{d}t\dots\,.\end{aligned} \tag{4.7.11}$$

Relabelling q to s_{l+1} and shifting the index l by 1 leads to

$$\begin{aligned}\left(I_n\rho_n^3 g\right)(p_1,\dots,p_{n-1}) = \sum_{J\subseteq\{0,\dots,n-1\}} \sum_{l=1}^{\left\lfloor\frac{n-1-\#J}{2}\right\rfloor} \frac{\alpha^n_{\#J+2,l-1}}{\left(\int_{\mathbb{R}^d} K\right)^{n-1-\#J-l}} \\ \times \int_{\left(\mathbb{R}^d\right)^{n-1-\#J-l}} g\left(p_J, -s_1, s_1, \dots, -s_l, s_l, t_1, \dots, t_{n-1-\#J-2l}\right) \\ \times K(s_1)\dots K(s_l)\, K(t_1)\dots K\left(t_{n-1-\#J-2l}\right) \mathrm{d}s\dots \mathrm{d}t\dots\,.\end{aligned} \tag{4.7.12}$$

It is now straightforward to add up the parts in eqs. (4.7.8), (4.7.10) and (4.7.12). Furthermore, the coefficients $\alpha^n_{a,b}$ may be determined by demanding $I_n \rho_n g = g$ translating to

$$\begin{aligned} & & n\,\alpha^n_{n-1,0} &= 1\,, \\ \forall a \in \{0,\dots,n-4\}\,, b \in \left\{1,\dots,\left\lfloor \frac{n-2-a}{2} \right\rfloor\right\} &: & \alpha^n_{a,b} + 2\alpha^n_{a+1,b} + \alpha^n_{a+2,b-1} &= 0\,, \\ \forall a \in \{0,\dots,n-3\}\,, n-a \text{ odd} &: & \alpha^n_{a,(n-1-a)/2} + \alpha^n_{a+2,(n-3-a)/2} &= 0\,, \\ \forall a \in \{0,\dots,n-2\} &: & \alpha^n_{a,0} + 2\alpha^n_{a+1,0} &= 0\,. \end{aligned}$$

Here the first factor of n comes from the n different subsets of $\{0,\dots,n-1\}$ of length $n-1$. All these subsets give the same contribution to $I_n\rho_n g$ due to the Sym^*_{n-1} symmetry of g. As may easily be verified, eq. (4.2.10) solves these recursion relations. Furthermore, this solution is unique because all $\alpha^n_{a,b}$ for $n \in \mathbb{N}$ and $a, b \in \mathbb{N}_0$ with $a + 2b \leq n-1$ are uniquely determined by the values of $\alpha^n_{n-1,0}$.

4.7.3. Existence Proof of κ_2

Let $\kappa^1_2(p;k) = m^2 + \|p\|^2$ and for any $n \in \mathbb{N}$, define

$$\kappa^{n+1}_2(p;k) = \kappa^1_2(p) + \frac{1}{2(2\pi)^d} \int_k^\infty \int_{\mathbb{R}^d} \frac{\partial_{k'} r(q,k')}{\left[\kappa^n_2(q;k') + \bar{r}(q,k')\right]^2} \kappa_4(p,-q,q;k')\,\mathrm{d}q\,\mathrm{d}k'\,, \tag{4.7.13}$$

which satisfies the boundary condition (4.3.1) if the integrals are finite. Note that $\kappa^n_2 \geq \kappa^1_2$ for all $n \in \mathbb{N}$, since $\kappa^1_2 > 0$, $\kappa_4 \geq 0$ and by eq. (4.3.6) the regulator contribution is positive. Hence,

$$\frac{1}{\left[\kappa^n_2(q) + \bar{r}(q)\right]^2} \leq \frac{1}{\left[m^2+k^2\right]^2} \quad \text{and} \quad \partial_k \bar{r}(q) = \frac{\|q\|^4}{k^3\left(\cosh\left[\frac{\|q\|^2}{k^2}\right] - 1\right)} \leq 2k \tag{4.7.14}$$

by eq. (4.3.8), which proves that

$$\frac{\partial_k \bar{r}(q)}{\left[\kappa^n_2(q) + \bar{r}(q)\right]^2} \leq \frac{2k}{\left[m^2+k^2\right]^2}\,. \tag{4.7.15}$$

Inserting this into the recursion relation (4.7.13) leads to

$$\begin{aligned} \kappa^n_2(p) &\leq \kappa^1_2(p) + \frac{(2\pi)^{-d}\lambda}{|m|^{d-4}} \int_k^\infty \int_{\mathbb{R}^d} \frac{k'}{\left[m^2+k'^2\right]^2} \exp\left[-\frac{2\|p\|^d + 2\|q\|^d + |m|^d}{k'\,|m|^{d-1}}\right] \mathrm{d}q\,\mathrm{d}k' \\ &= \kappa^1_2(p) + (2\pi)^{-d}\frac{s_{d-1}}{2d}\lambda\,|m|^3 \int_k^\infty \frac{k'^2}{\left[m^2+k'^2\right]^2} \exp\left[-\frac{2\|p\|^d + |m|^d}{k'\,|m|^{d-1}}\right] \mathrm{d}k'\,, \end{aligned} \tag{4.7.16}$$

where s_n denotes the surface area of the unit n-sphere. Estimating the exponential by 1 and extending the integral to $[0, \infty)$ immediately gives the result

$$\kappa_2^n(p) \leq \kappa_2^1(p) + (2\pi)^{-d} \frac{s_{d-1}}{2d} \lambda |m|^3 \int_0^\infty \frac{k'^2}{\left[m^2 + k'^2\right]^2} \mathrm{d}k' \leq \kappa_2^1(p) + (2\pi)^{-d} \pi \frac{s_{d-1}}{8d} \lambda m^2 \,, \tag{4.7.17}$$

which in a slightly more compact form reads

$$\left\| \kappa_2^n - \kappa_2^1 \right\|_{L^\infty} \leq \frac{(2\pi)^{-d} \pi s_{d-1}}{8d} \lambda m^2 := t_d \lambda m^2 \tag{4.7.18}$$

for all $n \in \mathbb{N}$. Note that the numerical factor t_d in front of λm^2 is rather small: It is $1/8$ for $d = 1$ and goes to zero rather rapidly for larger values of d.

We shall now show that the mapping $\kappa_2^n \mapsto \kappa_2^{n+1}$ given by eq. (4.7.13) actually is a contraction for values of λ not being too large. To this end, note that

$$\frac{\kappa_2^n(q) + \bar{r}(q)}{\kappa_2^1(q) + \bar{r}(q)} = \frac{\kappa_2^1(q) + \bar{r}(q)}{\kappa_2^1(q) + \bar{r}(q)} + \frac{\kappa_2^n(q) - \kappa_2^1(q)}{\kappa_2^1(q) + \bar{r}(q)} \leq 1 + t_d \lambda \frac{m^2}{\kappa_2^1(q) + \bar{r}(q)} \leq 1 + t_d \lambda \tag{4.7.19}$$

and hence

$$\begin{aligned} \left| \left[\kappa_2^{n+1}(q) + \bar{r}(q)\right]^{-2} - \left[\kappa_2^n(q) + \bar{r}(q)\right]^{-2} \right| &\leq \frac{\left[2\bar{r}(q) + \kappa_2^n(q) + \kappa_2^{n+1}(q)\right] \left|\kappa_2^n(p) - \kappa_2^{n+1}(q)\right|}{\left[\kappa_2^1(q) + \bar{r}(q)\right]^4} \\ \leq 2\left(1 + t_d\lambda\right) \frac{\left|\kappa_2^n(q) - \kappa_2^{n+1}(q)\right|}{\left[\kappa_2^1(q) + \bar{r}(q)\right]^3} &\leq 2 \frac{1 + t_d\lambda}{m^2} \frac{\left\|\kappa_2^n - \kappa_2^{n+1}\right\|_{L^\infty}}{\left[\kappa_2^1(q) + \bar{r}(q)\right]^2} \,. \end{aligned} \tag{4.7.20}$$

Using this estimate to compare two successive iterates one finally arrives at

$$\begin{aligned} &\left|\kappa_2^{n+2}(p) - \kappa_2^{n+1}(p)\right| \\ &\quad \leq 2 \frac{1 + t_d\lambda}{m^2} \frac{\left\|\kappa_2^n - \kappa_2^{n+1}\right\|_{L^\infty}}{2(2\pi)^d} \int_k^\infty \int_{\mathbb{R}^d} \frac{\partial_{k'} r(q; k')}{\left[\kappa_2^1(q) + \bar{r}(q)\right]^2} \kappa_4(p, -q, q; k') \,\mathrm{d}q \,\mathrm{d}k' \\ &\quad \leq 2 \frac{1 + t_d\lambda}{m^2} \left\|\kappa_2^n - \kappa_2^{n+1}\right\|_{L^\infty} \left\|\kappa_2^2 - \kappa_2^1\right\|_{L^\infty} \leq 2\left(1 + t_d\lambda\right) t_d\lambda \left\|\kappa_2^{n+1} - \kappa_2^n\right\|_{L^\infty} \,, \end{aligned} \tag{4.7.21}$$

or for short

$$\left\|\kappa_2^{n+2} - \kappa_2^{n+1}\right\|_{L^\infty} \leq 2\left(1 + t_d\lambda\right) t_d\lambda \left\|\kappa_2^{n+1} - \kappa_2^n\right\|_{L^\infty} \,. \tag{4.7.22}$$

The factor in front is smaller than one whenever

$$0 \leq \lambda < \frac{\sqrt{3} - 1}{2t_d} \,, \tag{4.7.23}$$

or equivalently eq. (4.3.7) is satisfied. The upper bound is a function that grows rather rapidly starting at a value of approximately 2.93 for $d = 1$. From now on, we assume λ to satisfy inequality (4.7.23). Thus, by the completeness of $L^\infty(\mathbb{R}^d)$ we have proven the convergence of the sequence $(p \mapsto \kappa_2^n(p;k))_{n\in\mathbb{N}}$ to some $p \mapsto \kappa_2(p;k)$ in $L^\infty(\mathbb{R}^d)$ for all $k \in [0,\infty)$. Also, κ_2 has to be a fixed point of the iteration map such that eq. (4.3.9) is satisfied where the right hand side is continuous with respect to k, since the integrand is non-singular for all $k' \geq 0$. Thus, κ_2 is also k-continuous on $[0,\infty)$ as well. But then the right-hand side is differentiable with respect to k on all of $[0,\infty)$, such that

$$\partial_k \kappa_2\left(p\right) = -\frac{1}{2}\left(2\pi\right)^{-d} \int_{\mathbb{R}^d} \frac{\partial_k r\left(q\right)}{\left[\kappa_2\left(q\right)+r\left(q\right)\right]^2} \kappa_4\left(p,-q,q\right) \mathrm{d}q \tag{4.7.24}$$

for all $k \in \mathbb{R}_{\geq 0}$. Hence, κ_2 satisfies the flow equation. Furthermore, the right-hand side is obviously k-differentiable such that $\partial_k^2\kappa_2$ may be expressed through κ_2 and $\partial_k\kappa_2$. Hence, $\partial_k^2\kappa_2$ is again k-differentiable. Iterating this argument then shows that κ_2 is smooth with respect to k. The p-smoothness of κ_2 is immediate from eq. (4.3.9) by the regularity of κ_4. For the $\mathcal{O}(d)$-invariance of κ_2, note that κ_4 and $\bar{r}$ as well as κ_2^1 are $\mathcal{O}(d)$-invariant. Thus, by eq. (4.7.13) each iterate κ_2^n is also $\mathcal{O}(d)$-invariant. Since the set of all $\mathcal{O}(d)$-invariant functions in $L^\infty(\mathbb{R}^d)$ is closed, the limit point κ_2 has to lie in this set as well.

4.7.4. Bounding the Higher Correlators

Let us assume that all higher correlators have been constructed by virtue of eq. (4.3.4). It then remains to find useful bounds ascertaining the correct UV limits as well as non-singular IR limits. The key to this is a proper estimate for the k-derivatives of κ_2. Before we can produce such estimates, we shall need corresponding ones for κ_4 and $\bar{r}$. Let us begin with the regulator for which we have the following relation that is easily derived from eq. (4.3.6):

$$\partial_k \bar{r}\left(q\right) = \frac{2}{k^3}\bar{r}\left(q\right)\left[\left\|q\right\|^2 + \bar{r}\left(q\right)\right] . \tag{4.7.25}$$

It hints at the following identity for all $l \in \mathbb{N}_0$ and some constants $\beta^l_{a,b} \in \mathbb{R}$:

$$\partial_k^l \bar{r}\left(q\right) = \sum_{a=1}^{l+1}\sum_{b=0}^{l+1-a} \beta^l_{a,b} k^{2-l-2a-2b} \bar{r}\left(q\right)^a \left[\left\|q\right\|^2 + \bar{r}\left(q\right)\right]^b , \tag{4.7.26}$$

which can straightforwardly be proved by induction. $\beta^l_{a,b}$ is recursively defined by

$$\beta^{l+1}_{a,b} = \left(2-l-2a-2b\right)\beta^l_{a,b} + 2a\beta^l_{a,b-1} + 2b\beta^l_{a-1,b} \quad \text{and} \quad \beta^0_{a,b} = \begin{cases} 1 & a=1, b=0\,, \\ 0 & \text{otherwise} \end{cases} \tag{4.7.27}$$

for all $l \in \mathbb{N}_0$ and $a, b \in \mathbb{Z}$. The next theorem will allow to find an estimate for such expressions.

Theorem 4.7.1. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$. Then,

$$\sup_{q\in\mathbb{R}^d} \left| \bar{r}\left(q\right)^a \left[\left\|q\right\|^2 + \bar{r}\left(q\right)\right]^b \right| \leq k^{2(a+b)} \left(1 + \frac{b}{a}\right)^b . \tag{4.7.28}$$

Proof. For $b = 0$ the statement is obvious since $\bar{r}(q) \leq k^2$. Hence, let us assume that $b \in \mathbb{N}$. Since $\bar{r}(q)^a[\|q\|^2 + \bar{r}(q)]^b$ is actually a smooth function of $\|q\|^2$, we may look for local extrema by differentiating with respect to $\|q\|^2$. Then, a necessary condition for $\|q\|^2$ at a maximum is

$$a \left[\left\|q\right\|^2 + \bar{r}\left(q\right)\right] \partial_{\|q\|^2} \bar{r}\left(q\right) + b\bar{r}\left(q\right) \left[1 + \partial_{\|q\|^2} \bar{r}\left(q\right)\right] = 0 . \tag{4.7.29}$$

Now, note that the exponential regulator also admits the following simple identity for $q \neq 0$,

$$\partial_{\|q\|^2} \bar{r}\left(q\right) = \frac{\bar{r}\left(q\right)}{\left\|q\right\|^2} \left[1 - \frac{1}{k^2} \left(\left\|q\right\|^2 + \bar{r}\left(q\right)\right)\right] \quad \text{such that} \quad 1 = \frac{\bar{r}\left(q\right)}{k^2} + \frac{a}{a+b} \frac{\left\|q\right\|^2}{k^2} \tag{4.7.30}$$

after some simple algebra. We perform a change of variables to $y = \|q\|^2/k^2$ and obtain

$$\exp y = 1 + \frac{(a+b)\,y}{a+b-ay} \tag{4.7.31}$$

as a further equivalent expression for the extremality, including the case $q = 0$. Note, that the excluded case $ay = a + b$ is irrelevant, since it does not solve eq. (4.7.30). Furthermore,

$$\frac{\partial}{\partial y} \left(1 + \frac{(a+b)\,y}{a+b-ay}\right) = \left(\frac{a+b}{a+b-ay}\right)^2 > 0 . \tag{4.7.32}$$

Hence, for $ay > a+b$ the right-hand side of eq. (4.7.31) is monotonically increasing with y and

$$\lim_{y\to\infty} \left(1 + \frac{(a+b)\,y}{a+b-ay}\right) = 1 - \frac{a+b}{a} = -\frac{b}{a} < 0 , \tag{4.7.33}$$

spoiling eq. (4.7.31). Hence, all extrema lie in the interval $[0, \frac{a+b}{a})$ and at a maximum, we have

$$\begin{aligned} \bar{r}\left(q\right)^a \left[\left\|q\right\|^2 + \bar{r}\left(q\right)\right]^b &= k^{2(a+b)} \left(\frac{y}{\exp y - 1}\right)^a \left(y + \frac{y}{\exp y - 1}\right)^b \\ &= k^{2(a+b)} \left(\frac{y}{\frac{(a+b)y}{a+b-ay}}\right)^a \left(y + \frac{y}{\frac{(a+b)y}{a+b-ay}}\right)^b \\ &= k^{2(a+b)} \left(1 - \frac{a}{a+b} y\right)^a \left(1 + \frac{b}{a+b} y\right)^b \leq k^{2(a+b)} \left(1 + \frac{b}{a}\right)^b . \end{aligned} \tag{4.7.34}$$

□

Corollary 4.7.2. Applying this estimate to the regulator derivatives, we obtain

$$\|\partial_k^n \bar{r}\|_{L^\infty} \leq k^{2-n} \sum_{a=1}^{n} \sum_{b=0}^{n+1-a} \left|\beta_{a,b}^n\right| \left(1 + \frac{b}{a}\right)^b . \tag{4.7.35}$$

Hence, there is a constant $R_n \geq 0$ such that $\|\partial_k^n \bar{r}\|_{L^\infty} \leq R_n k^{2-n}$ for all $n \in \mathbb{N}_0$.

Corollary 4.7.3. Applying the estimate to K and employing eq. (4.3.8) leads to

$$\|K\|_{L^\infty} \leq R_1 \frac{k}{\left(k^2 + m^2\right)^2} . \tag{4.7.36}$$

Having obtained the estimates for the regulator, the next step is to study κ_4.

Theorem 4.7.4. For all $l \in \mathbb{N}_0$ there exist constants $A_4^l \geq 0$ such that

$$\sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\partial_k^l \kappa_4 \left(p, q, -q\right)\right| \mathrm{d}q \leq A_4^l \frac{|m|^3 k}{k^l + |m|^l} . \tag{4.7.37}$$

Proof. As can easily be proved by induction, we have

$$\partial_k^l \kappa_4 \left(p, q, r\right) = \kappa_4 \left(p, q, r\right) \sum_{a=0}^{l} \gamma_a^l \, |m|^{a-ad} \, k^{-a-l} \left[\|p\|^d + \|q\|^d + \|r\|^d + \|p+q+r\|^d + |m|^d\right]^a \tag{4.7.38}$$

for all $l \in \mathbb{N}_0$, $p, q, r \in \mathbb{R}^d$. The constants $\gamma_a^l \in \mathbb{R}$ are determined by

$$\gamma_a^{l+1} = -\left(a + l\right) \gamma_a^l + \gamma_{a-1}^l \qquad \text{and} \qquad \gamma_a^0 = \begin{cases} 1 & a = 0 \, , \\ 0 & \text{otherwise} \end{cases} \tag{4.7.39}$$

for all $a \in \mathbb{Z}$. Expanding the above, we get

$$\begin{aligned} \left|\partial_k^l \kappa_4 \left(p, q, -q\right)\right| \leq \lambda \sum_{a=0}^{l} \sum_{b=0}^{a} \binom{a}{b} 2^b \left|\gamma_a^l\right| |m|^{4-d+a-ad} \, k^{-a-l} \left(2\|p\|^d + |m|^d\right)^{a-b} \\ \times \|q\|^{bd} \exp\left[-\frac{2\|p\|^d + 2\|q\|^d + |m|^d}{k\,|m|^{d-1}}\right] , \end{aligned} \tag{4.7.40}$$

which allows us to perform the q integral, such that

$$\begin{aligned} \int_{\mathbb{R}^d} \left|\partial_k^l \kappa_4 \left(p, q, -q\right)\right| \mathrm{d}q \leq \frac{s_{d-1}}{2d} \lambda \sum_{a=0}^{l} \sum_{b=0}^{a} \binom{a}{b} b! \left|\gamma_a^l\right| k^{1+b-a-l} \left(2\|p\|^d + |m|^d\right)^{a-b} \\ \times |m|^{3+(b-a)(d-1)} \exp\left[-\frac{2\|p\|^d + |m|^d}{k\,|m|^{d-1}}\right] . \end{aligned} \tag{4.7.41}$$

Let us again expand this, leading to

$$\int_{\mathbb{R}^d} \left|\partial_k^l \kappa_4 (p,q,-q)\right| \mathrm{d}q \leq \frac{s_{d-1}}{2d}\lambda \sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{c=0}^{a-b}\binom{a}{b}\binom{a-b}{c} b!\, 2^c \left|\gamma_a^l\right| |m|^{3+a-b-cd} \times k^{1+b-a-l} \|p\|^{cd} \exp\left[-\frac{2\|p\|^d + |m|^d}{k\,|m|^{d-1}}\right], \tag{4.7.42}$$

allowing us to produce the estimate

$$\sup_{p\in\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\partial_k^l \kappa_4 (p,q,-q)\right| \mathrm{d}q \leq \frac{s_{d-1}}{2d}\lambda \sum_{a=0}^{l}\sum_{b=0}^{a}\binom{a}{b} b! \left|\gamma_a^l\right| |m|^{3+a-b} k^{1+b-a-l}\exp\left[-\frac{|m|}{k}\right] + \frac{s_{d-1}}{2d}\lambda \sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{c=1}^{a-b}\binom{a}{b}\binom{a-b}{c} b! \left(\frac{c}{e}\right)^c \left|\gamma_a^l\right| |m|^{3+a-b-c} k^{1+b+c-a-l}\exp\left[-\frac{|m|}{k}\right]. \tag{4.7.43}$$

For $l=0$, the above reduces to the desired form,

$$\sup_{p\in\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\kappa_4 (p,q,-q)\right| \mathrm{d}q \leq \frac{s_{d-1}}{2d}\lambda \left|\gamma_0^0\right| |m|^3 k \exp\left[-\frac{|m|}{k}\right] \leq \frac{s_{d-1}}{2d}\left|\gamma_0^0\right| |m|^3 k\,. \tag{4.7.44}$$

For $l \in \mathbb{N}$ and all $a,b,c \in \mathbb{N}_0$ with $a-b-c \geq 0$ the following is valid:

$$\exp\left[-\frac{|m|}{k}\right] \leq \left[\frac{\left(\frac{|m|}{k}\right)^{a-b-c}}{(a-b-c)!} + \frac{\left(\frac{|m|}{k}\right)^{l+a-b-c}}{(l+a-b-c)!}\right]^{-1} \leq (l+a-b-c)!\frac{k^{l+a-b-c}\,|m|^{b+c-a}}{k^l + |m|^l}\,. \tag{4.7.45}$$

Inserting this into eq. (4.7.43) yields

$$\sup_{p\in\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\partial_k^l \kappa_4 (p,q,-q)\right| \mathrm{d}q \leq \frac{s_{d-1}}{2d}\lambda \sum_{a=0}^{l}\sum_{b=0}^{a}\binom{a}{b} b! \left|\gamma_a^l\right| (l+a-b)!\frac{|m|^3 k}{k^l + m^l} + \frac{s_{d-1}}{2d}\lambda \sum_{a=0}^{l}\sum_{b=0}^{a}\sum_{c=1}^{a-b}\binom{a}{b}\binom{a-b}{c} b! \left(\frac{c}{e}\right)^c \left|\gamma_a^l\right| (l+a-b-c)!\frac{|m|^3 k}{k^l + m^l}\,. \tag{4.7.46}$$

□

Finally, the relevant estimates for κ_2 can be proved:

Theorem 4.7.5. For every $n \in \mathbb{N}$ there is some $B_2^n \geq 0$ such that $\|\partial_k^n \kappa_2\|_{L^\infty} \leq B_2^n m^2/k^n$.

Proof. Let us first consider the case $n=1$:

$$\|\partial_k \kappa_2\|_{L^\infty} \leq \frac{\|K\|_{L^\infty}}{2(2\pi)^d}\sup_{p\in\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\kappa_4 (p,q,-q)\right| \mathrm{d}q \leq \frac{R_1 A_4^0}{2(2\pi)^d}\frac{|m|^3 k^2}{(k^2+m^2)^2} \leq \frac{R_1 A_4^0}{4(2\pi)^d}\frac{m^2}{k}\,, \tag{4.7.47}$$

where the second inequality follows from corollary 4.7.3 and theorem 4.7.4. Let us now proceed by induction. Fix some $n \in \mathbb{N}$ and assume that the theorem holds for all $l \in \mathbb{N}_{\leq n}$. Then

$$\begin{aligned} \left\|\partial_k^{n+1}\kappa_2\right\|_{L^\infty} &\leq \frac{(2\pi)^{-d}}{2}\sum_{l=0}^{n}\binom{n}{l}\left\|\partial_k^l K\right\|_{L^\infty}\sup_{p\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\partial_k^{n-l}\kappa_4(p,q,-q)\right|\mathrm{d}q \\ &\leq \frac{(2\pi)^{-d}}{2}\sum_{l=0}^{n}\binom{n}{l}\frac{A_4^{n-l}\,|m|^3\,k}{k^{n-l}+|m|^{n-l}}\left\|\partial_k^l K\right\|_{L^\infty}. \end{aligned} \tag{4.7.48}$$

Also

$$\left\|\partial_k^l K\right\|_{L^\infty} \leq \sum_{a=0}^{l}\sum_{b=0}^{a}\binom{l}{a}\binom{a}{b}R_{1+l-a}k^{1+a-l}\left\|\partial_k^{a-b}(\kappa_2+\bar{r})^{-1}\right\|_{L^\infty}\left\|\partial_k^{b}(\kappa_2+\bar{r})^{-1}\right\|_{L^\infty}. \tag{4.7.49}$$

Equation (4.1.8) was derived in a non-commutative algebra and holds in a similar form in the commutative algebra of functions. Thus, together with eq. (4.3.8) and the induction hypothesis,

$$\begin{aligned} \left\|\partial_k^l(\kappa_2+\bar{r})^{-1}\right\|_{L^\infty} &\leq \sum_{c\in\mathcal{C}(l)}\frac{l!}{c!}\frac{1}{m^2+k^2}\prod_{a=1}^{\#c}\frac{\left\|\partial_k^{c_a}\kappa_2\right\|_{L^\infty}+\left\|\partial_k^{c_a}\bar{r}\right\|_{L^\infty}}{m^2+k^2} \\ &\leq \sum_{c\in\mathcal{C}(l)}\frac{l!}{c!}\frac{1}{m^2+k^2}\prod_{a=1}^{\#c}\left(B_2^{c_a}+R_{c_a}\right)k^{-c_a} =: B_{2,r}^l\frac{k^{-l}}{m^2+k^2} \end{aligned} \tag{4.7.50}$$

for all $l \in \mathbb{N}_{\leq n} \cup \{0\}$, where we set $B_{2,r}^0 = 1$. Inserted into the previous equation, this gives us

$$\left\|\partial_k^l K\right\|_{L^\infty} \leq \sum_{a=0}^{l}\sum_{b=0}^{a}\binom{l}{a}\binom{a}{b}R_{1+l-a}k^{1+a-l}\frac{B_{2,r}^{a-b}k^{b-a}}{m^2+k^2}\frac{B_{2,r}^{b}k^{-b}}{m^2+k^2} =: B_K^l\frac{k^{1-l}}{(m^2+k^2)^2} \tag{4.7.51}$$

for all $l \in \mathbb{N}_{\leq n} \cup \{0\}$. Finally, inserting this into eq. (4.7.48) leads to

$$\begin{aligned} \left\|\partial_k^{n+1}\kappa_2\right\|_{L^\infty} &\leq \frac{(2\pi)^{-d}}{2}\sum_{l=0}^{n}\binom{n}{l}\frac{A_4^{n-l}B_K^l\,|m|^3\,k^{2-l}}{(m^2+k^2)^2\left(k^{n-l}+|m|^{n-l}\right)} \\ &\leq \frac{m^2k^{-n-1}}{4(2\pi)^d}\sum_{l=0}^{n}\binom{n}{l}A_4^{n-l}B_K^l. \end{aligned} \tag{4.7.52}$$

□

From this proof, we also obtain the following extremely useful corollaries:

Corollary 4.7.6. For all $l \in \mathbb{N}_0$, there is a constant $B_{2,r}^l \geq 0$ such that

$$\left\|\partial_k^l(\kappa_2+\bar{r})^{-l}\right\|_{L^\infty} \leq B_{2,r}^l\frac{k^{-l}}{m^2+k^2}. \tag{4.7.53}$$

Corollary 4.7.7. For all $l \in \mathbb{N}_0$, there is a constant $B_K^l \geq 0$ such that

$$\left\|\partial_k^l K\right\|_{L^\infty} \leq B_K^l \frac{k^{1-l}}{\left(m^2+k^2\right)^2} . \tag{4.7.54}$$

Having obtained these estimates concerning κ_2, only a few estimates regarding the exponential regulator are needed before turning to ρ_{2n}. As a start, we have the following theorem.

Theorem 4.7.8. For every $l \in \mathbb{N}_0$ there is some $\bar{R}_l \geq 0$ such that $\|\partial_k^l \bar{r}\|_{L^1} \leq \bar{R}_l k^{2+d-l}$.

Proof. The use of eq. (4.7.26) yields

$$\begin{aligned} \left\|\partial_k^l \bar{r}\right\|_{L^1} &\leq \sum_{a=1}^{l} \sum_{b=0}^{l+1-a} \left|\beta_{a,b}^l\right| k^{2-l-2a-2b} \int_{\mathbb{R}^d} \bar{r}\left(q\right)^a \left[\|q\|^2 + \bar{r}\left(q\right)\right]^b \mathrm{d}q \\ &= s_{d-1} \sum_{a=1}^{l} \sum_{b=0}^{l+1-a} \left|\beta_{a,b}^l\right| k^{2+d-l} \int_0^\infty \frac{t^{2a+2b+d-1} \exp\left[t^2\right]^b}{\left(\exp\left[t^2\right]-1\right)^{a+b}} \mathrm{d}t , \end{aligned} \tag{4.7.55}$$

where the integral is finite since $a \geq 1$ and $b \geq 0$. □

Corollary 4.7.9. For all natural numbers $n \in \mathbb{N}_0$, there exist constants $C_K^n > 0$ such that

$$\|\partial_k^n K\|_{L^1} \leq C_K^n \frac{k^{d+1-n}}{\left(m^2+k^2\right)^2} . \tag{4.7.56}$$

Proof. We obviously have

$$\begin{aligned} \|\partial_k^n K\|_{L^1} &\leq \sum_{l=0}^{n} \sum_{a=0}^{l} \binom{n}{l} \binom{l}{a} \left\|\partial_k^{1+n-l} \bar{r}\right\|_{L^1} \left\|\partial_k^{l-a} \frac{1}{\kappa_2+\bar{r}}\right\|_{L^\infty} \left\|\partial_k^{a} \frac{1}{\kappa_2+\bar{r}}\right\|_{L^\infty} \\ &\leq \sum_{l=0}^{n} \sum_{a=0}^{l} B_{2,r}^{l-a} B_{2,r}^{a} \bar{R}_{1+n-l} \frac{k^{d+1-n}}{\left(m^2+k^2\right)^2} . \end{aligned} \tag{4.7.57}$$

□

Theorem 4.7.10. For all natural numbers $n \in \mathbb{N}_0$, there exist constants $\bar{C}_K^n \geq 0$ such that

$$\left|\partial_k^n \|K\|_{L^1}^{-1}\right| \leq \bar{C}_K^n \frac{\left(k^2+m^2\right)^2}{k^{d+1}} k^{-n} . \tag{4.7.58}$$

Proof. Applying eqs. (4.7.17) and (4.7.18) as well as the definition of $\bar{r}$, we have

$$\begin{aligned} \|K\|_{L^1} &\geq \frac{s_{d-1}}{k^{3-d}} \int_0^\infty \frac{t^{d+3}\left[\cosh\left(t^2\right)-1\right]^{-1}}{\left[t^2+1+\left(1+t_d\lambda\right)\lambda\frac{m^2}{k^2}\right]^2} \mathrm{d}t \\ &\geq \frac{s_{d-1}k^{d-3}}{\left[2+\left(1+t_d\lambda\right)\frac{m^2}{k^2}\right]^2} \int_0^1 \frac{t^{d+3}}{\left[\cosh\left(t^2\right)-1\right]} \mathrm{d}t \geq \frac{s_{d-1}X_d}{\max\left\{2, 1+t_d\lambda\right\}^2} \frac{k^{d+1}}{\left[k^2+m^2\right]^2} \end{aligned} \tag{4.7.59}$$

with $X_d > 0$ being the value of the integral which is finite for $d \geq 1$. By inverting both sides the theorem is true for $n = 0$. For $n \in \mathbb{N}$ note that $|\partial_k^n \|K\|_{L^1}| \leq \|\partial_k^n K\|_{L^1}$ since $K \geq 0$. Thus,

$$\left|\partial_k^n \|K\|_{L^1}^{-1}\right| \leq \sum_{c \in \mathcal{C}(n)} \frac{n!}{c!} \|K\|_{L^1}^{-1} \prod_{l=1}^{\#c} \frac{\|\partial_k^{c_l} K\|_{L^1}}{\|K\|_{L^1}} \tag{4.7.60}$$
$$\leq \sum_{c \in \mathcal{C}(n)} \frac{n!}{c!} \bar{C}_K^0 \frac{\left(k^2+m^2\right)^2}{k^{d+1}} \prod_{l=1}^{\#c} \bar{C}_k^0 C_K^{c_l} k^{-c_l} = \frac{\left(k^2+m^2\right)^2}{k^{d+1+n}} \sum_{c \in \mathcal{C}(n)} \frac{n!}{c!} \left(\bar{C}_K^0\right)^{\#c+1} \prod_{l=1}^{\#c} C_K^{c_l} . \qquad \square$$

Now, we finally turn to our estimates of the higher correlation functions.

Theorem 4.7.11. Define Δ as in eq. (4.3.11). Then, for all $n \in \mathbb{N}_{\geq 2}$, $x \in \mathbb{N}$ and $l \in \mathbb{N}_0$ there exist constants $B_{2n}^{l,x} \geq 0$ such that

$$\left\|\partial_k^l \kappa_{2n}\right\|_{L^\infty} \leq B_{2n}^{l,x} \frac{|m|^{d+(2-d)n+(n-2)(1+\Delta)}\, k^x}{(k+|m|)^{(n-2)(1+\Delta)+x+l}} . \tag{4.7.61}$$

Proof. We begin by proving the statement for κ_4 i.e. for $n = 2$. We know from eq. (4.7.38), that

$$\|\partial_k^l \kappa_4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \left|\gamma_a^l\right| \frac{|m|^{4-d+a-ad}}{k^{a+l}} \exp\left[-\frac{|m|}{k}\right] \sup_{y \in \mathbb{R}} \left(y^2 + |m|^d\right)^a \exp\left[-\frac{y^2}{k\,|m|^{d-1}}\right] . \tag{4.7.62}$$

This expands to

$$\|\partial_k^l \kappa_4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \sum_{b=0}^{a} \binom{a}{b} \left|\gamma_a^l\right| \frac{|m|^{4-d+a-bd}}{k^{a+l}} \exp\left[-\frac{|m|}{k}\right] \sup_{y \in \mathbb{R}} y^{2b} \exp\left[-\frac{y^2}{k\,|m|^{d-1}}\right] , \tag{4.7.63}$$

such that

$$\|\partial_k^l \kappa_4\|_{L^\infty} \leq \lambda \exp\left[-\frac{|m|}{k}\right] \sum_{a=0}^{l} \left(\left|\gamma_a^l\right| \frac{|m|^{4-d+a}}{k^{a+l}} + \lambda \sum_{b=1}^{a} \binom{a}{b} \left(\frac{b}{e}\right)^b \left|\gamma_a^l\right| \frac{|m|^{4-d+a-b}}{k^{a+l-b}} \right) . \tag{4.7.64}$$

But from eq. (4.7.45) we have for all $a, b \in \mathbb{N}_0$ with $a \geq b$ and all $x \in \mathbb{N}$

$$\exp\left[-\frac{|m|}{k}\right] \leq (x+l+a-b)! \frac{k^{x+l+a-b}\, |m|^{b-a}}{k^{x+l} + |m|^{x+l}} . \tag{4.7.65}$$

Inserted into the previous equation, this yields

$$\|\partial_k^l \kappa_4\|_{L^\infty} \leq \lambda \sum_{a=0}^{l} \left|\gamma_a^l\right| \left(\frac{(x+l+a)!\, k^x}{k^{x+l} + |m|^{x+l}} + \sum_{b=1}^{a} \binom{a}{b} \left(\frac{b}{e}\right)^b \frac{(x+l+a-b)!}{k^{x+l} + |m|^{x+l}} k^x \right) |m|^{4-d} . \tag{4.7.66}$$

The result then follows since $(k+|m|)^{x+l} \leq 2^{x+l-1}(k^{x+l} + |m|^{x+l})$. Let us now fix some $n \in \mathbb{N}_{\geq 2}$

and assume the theorem to be true for all $l \in \mathbb{N}_{\geq 2}$ with $l \leq n$. It needs to be shown that the theorem also holds for κ_{2n+2} as given by eq. (4.3.4). By the linearity of ρ_{2n} it suffices to show this for $\rho_{2n}\partial_k\kappa_{2n}$ and $\rho_{2n}\bar{\lambda}_c$ separately for all $c \in \bar{\mathcal{C}}(2n) \setminus \{(2n)\}$. In either case, for $l \in \mathbb{N}_0$ and a sufficiently regular Sym^*_{2n-1}-symmetric function g we have

$$\begin{aligned} \left\|\partial_k^l \rho_{2n} g\right\|_{L^\infty} \leq & \sum_{J\subseteq\{0,\dots,2n+1\}} \sum_{l=0}^{\left\lfloor \frac{2n-1-\#J}{2}\right\rfloor} \sum_{a=0}^{l}\sum_{b=0}^{a} \binom{l}{a}\binom{a}{b} \left|\alpha^{2n}_{\#J,l}\right| \\ & \times \left\|\partial_k^{l-a} g\right\|_{L^\infty} \left\|\partial_k^{a-b} K^{\otimes 2n-1-\#J-l}\right\|_{L^1} \left|\partial_k^b \left\|K\right\|_{L^1}^{-(2n-\#J-l)}\right| , \end{aligned} \tag{4.7.67}$$

where we have used that $\int_{\mathbb{R}^d} K = \left\|K\right\|_{L^1}$ since $K > 0$. Employing corollary 4.7.9, we get

$$\left\|\partial_k^a K^{\otimes b}\right\|_{L^1} \leq \sum_{\substack{\alpha\in\mathbb{N}_0^c \\ |\alpha|=a}} \frac{a!}{\alpha!} \prod_{j=1}^{b} \left\|\partial_k^{\alpha_j} K\right\|_{L^1} \leq \sum_{\substack{\alpha\in\mathbb{N}_0^b \\ |\alpha|=a}} \frac{a!}{\alpha!} \prod_{j=1}^{b} \frac{C_K^{\alpha_j} k^{d+1-\alpha_j}}{\left(m^2+k^2\right)^2} =: D_K^{a,c} \left[\frac{k^{d+1}}{\left(m^2+k^2\right)^2}\right]^b k^{-a} \tag{4.7.68}$$

for all $a, b \in \mathbb{N}_0$ (We define $K^{\otimes 0} = 1$). Furthermore, from theorem 4.7.10 one has

$$\left|\partial_k^a \left\|K\right\|_{L^1}^{-b}\right| \leq \sum_{\substack{\alpha\in\mathbb{N}_0^b \\ |\alpha|=a}} \frac{a!}{\alpha!} \prod_{j=1}^{b} \bar{C}_K^{\alpha_j} \frac{\left(k^2+m^2\right)^2}{k^{d+1}} k^{-\alpha_j} =: \bar{D}_K^{a,b} \left(\frac{\left(m^2+k^2\right)^2}{k^{d+1}}\right)^b k^{-a} \tag{4.7.69}$$

The insertion of these two inequalities into eq. (4.7.67) reveals the important intermediate result

$$\begin{aligned} \left\|\partial_k^l \rho_{2n} g\right\|_{L^\infty} \leq & \sum_{J\subseteq\{1,\dots,2n+2\}} \sum_{j=0}^{\left\lfloor \frac{2n-1-\#J}{2}\right\rfloor} \sum_{a=0}^{l}\sum_{b=0}^{a} \binom{l}{a}\binom{a}{b} \left|\alpha^{2n}_{\#J,j}\right| D_K^{a-b,2n-1-\#J-j} \\ & \times \bar{D}_K^{b,2n-\#J-j} \frac{\left(m^2+k^2\right)^2}{k^{d+1+a}} \left\|\partial_k^{l-a} g\right\|_{L^\infty} := \sum_{a=0}^{l} E_{2n}^{l,a} \frac{\left(m^2+k^2\right)^2}{k^{d+1+a}} \left\|\partial_k^{l-a} g\right\|_{L^\infty} . \end{aligned} \tag{4.7.70}$$

The divergent behaviour for $k \to 0$ elucidates the need for the extremely strong IR regularity of κ_4 as imposed in eq. (4.3.2). Now, consider the case $g = \partial_k \kappa_{2n}$ and $x \in \mathbb{N}$:

$$\begin{aligned}
\left\|\partial_k^l \rho_{2n} \partial_k \kappa_{2n}\right\|_{L^\infty} &\leq \sum_{a=0}^{l} E_{2n}^{l,a} \frac{\left(m^2+k^2\right)^2}{k^{d+1+a}} \left\|\partial_k^{l+1-a} \kappa_{2n}\right\|_{L^\infty} \\
&\leq \sum_{a=0}^{l} E_{2n}^{l,a} B_{2n}^{l+1-a,x+d+1+a} \frac{\left(m^2+k^2\right)^2}{k^{d+1+a}} \frac{|m|^{d+(2-d)n+(n-2)(1+\Delta)}\, k^{x+d+1+a}}{(k+|m|)^{(n-2)(1+\Delta)+l+2+d+x}} \\
&= \sum_{a=0}^{l} E_{2n}^{l,a} B_{2n}^{l+1-a,x+d+1+a} \frac{\left(m^2+k^2\right)^2 |m|^{d-3-\Delta}}{(k+|m|)^{d+1-\Delta}} \frac{|m|^{d+(2-d)(n+1)+(n-1)(1+\Delta)}\, k^{x}}{(k+|m|)^{(n-1)(1+\Delta)+l+x}} \\
&\leq \sum_{a=0}^{l} E_{2n}^{l,a} B_{2n}^{l+1-a,x+d+1+a} \frac{|m|^{d+(2-d)(n+1)+(n-1)(1+\Delta)}\, k^{x}}{(k+|m|)^{(n-1)(1+\Delta)+l+x}} \,.
\end{aligned} \tag{4.7.71}$$

This is the expected result and also shows that the use of these methods requires $d-3-\Delta \geq 0$. Otherwise, the last inequality would not generally hold. It remains to estimate the $g = \bar{\lambda}_c$ terms. Letting $c \in \bar{\mathcal{C}}(2n) \setminus \{(2n)\}$, clearly, $\|\partial_k^l \bar{\lambda}_c\|_{L^\infty} \leq \|\partial_k^l \lambda_c\|_{L^\infty}$, so that we get

$$\left\|\partial_k^l \rho_{2n} \bar{\lambda}_c\right\|_{L^\infty} \leq \sum_{a=0}^{l} E_{2n}^{l,a} \frac{\left(m^2+k^2\right)^2}{k^{d+1+a}} \left\|\partial_k^{l-a} \lambda_c\right\|_{L^\infty} \,. \tag{4.7.72}$$

for $l \in \mathbb{N}_0$. Estimating $\left\|\partial_k^{l-a} \lambda_c\right\|_{L^\infty}$ is rather cumbersome with

$$\left\|\partial_k^l \lambda_c\right\|_{L^\infty} \leq \sum_{a=0}^{l} \sum_{b=0}^{a} \binom{l}{a} \binom{a}{b} \left\|\partial_k^{l-a} K\right\|_{L^1} \left\|\partial_k^{a-b} \left(\left[\kappa_2 + r\right]^{-1}\right)^{\otimes \#c-1}\right\|_{L^\infty} \left\|\partial_k^{b} \bigotimes_{j=1}^{\#c} \kappa_{2+c_j}\right\|_{L^\infty} \tag{4.7.73}$$

for all $l \in \mathbb{N}_0$. However, using corollary 4.7.9 and corollary 4.7.6 one obtains

$$\begin{aligned}
\left\|\partial_k^l \lambda_c\right\|_{L^\infty} \leq &\sum_{a=0}^{l} \sum_{b=0}^{a} \binom{l}{a} \binom{a}{b} C_K^{l-a} \frac{k^{d+1-l+a}}{\left(m^2+k^2\right)^2} \sum_{\substack{\alpha \in \mathbb{N}_0^{\#c-1} \\ |\alpha| = a-b}} \frac{(a-b)!}{\alpha!} \left(\prod_{j=1}^{\#c-1} B_{2,r}^{\alpha_j} \frac{k^{-\alpha_j}}{m^2+k^2}\right) \\
&\times \sum_{\substack{\beta \in \mathbb{N}_0^{\#c} \\ |\beta| = b}} \frac{b!}{\beta!} \prod_{i=1}^{\#c} \left\|\partial_k^{\beta_i} \kappa_{2+c_i}\right\|_{L^\infty} := \sum_{\substack{\beta \in \mathbb{N}_0^{\#c} \\ |\beta| \leq l}} \frac{F_c^{l,\beta} k^{d+1-l+|\beta|}}{\left(m^2+k^2\right)^{\#c+1}} \prod_{j=1}^{\#c} \left\|\partial_k^{\beta_j} \kappa_{2+c_j}\right\|_{L^\infty} \,.
\end{aligned} \tag{4.7.74}$$

Inserting this result into eq. (4.7.72) yields

$$\begin{aligned}\left\|\partial_k^l \rho_{2n}\bar{\lambda}_c\right\|_{L^\infty} &\leq \sum_{a=0}^{l} \sum_{\substack{\beta\in\mathbb{N}_0^{\#c}\\|\beta|\leq l-a}} E_{2n}^{l,a} F_c^{l-a,\beta} \frac{k^{|\beta|-l}}{\left(m^2+k^2\right)^{\#c-1}} \prod_{j=1}^{\#c} \left\|\partial_k^{\beta_j}\kappa_{2+c_j}\right\|_{L^\infty} \\ &:= \sum_{\substack{\beta\in\mathbb{N}_0^{\#c}\\|\beta|\leq l}} \frac{G_c^{l,\beta} k^{|\beta|-l}}{\left(m^2+k^2\right)^{\#c-1}} \prod_{j=1}^{\#c} \left\|\partial_k^{\beta_j}\kappa_{2+c_j}\right\|_{L^\infty} ,\end{aligned} \tag{4.7.75}$$

such that it just remains to estimate $\|\partial_k^{\beta_n}\kappa_{2+c_n}\|_{L^\infty}$. To that end, let $x\in\mathbb{N}$ and fix some multi-index $X\in\mathbb{N}_0^{\#c}$ with $|X| = x+l$. Invoking the induction hypothesis, we conclude that

$$\left\|\partial_k^{\beta_j}\kappa_{2+c_j}\right\| \leq B_{2+c_j}^{\beta_j,X_j} \frac{|m|^{d+(2-d)\left(\frac{c_j}{2}+1\right)+\left(\frac{c_j}{2}-1\right)(1+\Delta)}\, k^{X_j}}{(k+|m|)^{\left(\frac{c_j}{2}-1\right)(1+\Delta)+X_j+\beta_j}} \tag{4.7.76}$$

for all even $c_j\in\mathbb{N}_{\leq 2n-2}$ and all $\beta_j\in\mathbb{N}_0$. In particular, this translates to

$$\begin{aligned}\prod_{j=1}^{\#c}\left\|\partial_k^{\beta_j}\kappa_{2+c_j}\right\|_{L^\infty} &\leq \frac{|m|^{(1-\Delta)\#c+(3+\Delta-d)n}\, k^{x+l}}{(k+|m|)^{(1+\Delta)(n-\#c)+x+b+l}} \prod_{j=1}^{\#c} B_{2+c_j}^{\beta_j,X_j} \\ &= \frac{|m|^{(1-\Delta)(\#c-1)}}{(k+|m|)^{(1-\Delta)(\#c-1)}} \frac{|m|^{(3+\Delta-d)n+1-\Delta}\, k^{x+l}}{(k+|m|)^{n+\Delta n-2\#c+x+b+l+1-\Delta}} \prod_{j=1}^{\#c} B_{2+c_j}^{\beta_j,X_j} \\ &\leq \frac{|m|^{(3+\Delta-d)n+1-\Delta}\, k^{x+l}}{(k+|m|)^{n+\Delta n-2\#c+x+b+l+1-\Delta}} \prod_{j=1}^{\#c} B_{2+c_j}^{\beta_j,X_j} \\ &\leq \frac{|m|^{(3+\Delta-d)n+1-\Delta}\, k^{x+l}\left(k^2+m^2\right)^{\#c-1}}{(k+|m|)^{n+\Delta n+x+l-1-\Delta}\, k^b} 2^{\#c-1} \prod_{j=1}^{\#c} B_{2+c_j}^{\beta_j,X_j} ,\end{aligned} \tag{4.7.77}$$

with $|c| = 2n$ and $|\beta| = b$. Here, it may be seen that it was important to choose $\Delta\leq 1$. Otherwise, the second inequality would in general not hold. Thus, the largest Δ that is possible using these methods is $\max\{d-3,1\}$ which precisely corresponds to the choice made in eq. (4.3.11). We may now insert this result into eq. (4.7.75) obtaining

$$\left\|\partial_k^l \rho_{2n}\bar{\lambda}_c\right\|_{L^\infty} \leq 2^{\#c-1} \frac{|m|^{(3+\Delta-d)n+1-\Delta}\, k^x}{(k+|m|)^{(n-1)(\Delta+1)+x+l}} \sum_{b=0}^{l} \sum_{\substack{\beta\in\mathbb{N}_0^{\#c}\\|\beta|=b}} G_c^{l,\beta} \prod_{j=1}^{\#c} B_{2+c_j}^{\beta_j,X_j} . \tag{4.7.78}$$

The right-hand side precisely corresponds to the one of eq. (4.7.61) with n replaced by $n+1$. □

5. Asymptotically Safe QED

This chapter is derived from [GZ20].[1]

While Quantum Electrodynamics is the most precisely tested part of the QFT of the Standard Model of particle physics (see, e.g., [HFG08]), it has been known early on that its perturbative structure is plagued by a singularity in the running coupling at finite scales, the so-called Landau pole [LAK54; Lan55]. Attempts to search for a cure for this consistency problem in the nonperturbative strong-coupling domain also date back to the early days of QFT [GL54; BJ69; JB73]. In absence of a convincing solution, QED is considered to be a 'trivial' theory, in the sense that the theory is assumed to be a consistent QFT only at the prize of having no interactions (see e.g. [Lüs90]).

In fact, evidence for triviality has been provided by lattice simulations [Goc+98; KKL01; KKL02] as well as nonperturbative functional methods [GJ04], though the resulting picture is more involved (and different from the triviality arising, e.g., in ϕ^4 theory [LW88; LW89; Has+87]): If QED was in a strong-coupling regime at a high-energy scale Λ, interactions would trigger chiral symmetry breaking [Mir85; Aok+97] much in the same way as in QCD. As a consequence, such a strong-coupling realisation of QED would go along with electron masses of the order of the high scale $m \sim \Lambda$ in contradistinction to the observed small mass of the electron and its approximate chiral symmetry in comparison with generic standard-model scales. Therefore, the Landau pole representing a strong-coupling regime is not connected by a line of constant physics with QED as observed in Nature [Goc+98]; nevertheless, the existence of a chiral-symmetry breaking phase imposes a scale Λ_{max} up to which QED as an effective field theory can be maximally extended [GJ04]. In pure QED, this scale has been estimated as $\Lambda_{\text{max,QED}} \simeq 10^{278}$GeV. As far as an ultraviolet completion of QED is concerned, the conclusion is similar to that of naive perturbation theory: A simple high-energy completion of QED does not seem to exist.

From the modern perspective of the Standard Model, QED is merely the low-energy remnant of the electroweak sector of the Standard Model as a consequence of the Brout-Englert-Higgs mechanism. However, the hypercharge $U(1)$ factor of the gauge group of the SM exhibits a high

[1]The most recent arXiv version (v3) agrees with the presentation in this thesis. Compared to arXiv version 2 as well as the published version, we have corrected eq. (5.2.5) by a term that was missing, yielding a subdominant correction to the values in table 5.3 on the few percent level. Also the figures have been updated correspondingly though the corrections are hardly visible. A typo in the code has been fixed that changed the quantitative scale estimate in eq. (5.4.6) and eq. (5.4.7) substantially compared to the previous and published versions.

energy renormalisation group (RG) behaviour qualitatively similar to QED; the high-energy location of the corresponding Landau pole of perturbation theory suggests the existence of a scale of maximum UV extent of the Standard Model of $\Lambda_{\text{max,SM}} \simeq 10^{40}$GeV. It is fair to say that the physical relevance of such a scale remains unclear, since it is much larger than the Planck scale where the renormalisation behaviour of the particle physics sector is expected to be modified by quantum gravitational effects. Still, this problem appears to be generic for models with $U(1)$ factors; in fact the Landau pole typically moves to smaller scales for new physics models with a larger sector of U(1)-charged scalar or fermionic particles and can thus easily drop below the Planck scale.

Within QED-like (asymptotically non-free) theories, analytic properties of the 't Hooft expansion at large N_{f} [PP84; Gra96] have been used in combination with high-order perturbation theory to actively search for UV fixed points [Shr14], and are currently studied with renewed interest using novel resummation techniques [AS18; Ant+18; Don+20; Don+19], aiming at addressing the fate of these theories in the deep UV. Proposed solutions of the problem of high-energy incompleteness caused by a $U(1)$ factor typically go much beyond the particle content of pure QED-like systems. One example is given by asymptotically safe particle physics models [LS14] which require a large number of additional vector-like fermions [Man+17] but go along with a nonperturbative scalar sector [Pel+18]; the mechanisms that help controlling UV fixed points in non-abelian gauge-Yukawa models have recently been shown to be, in principle, also available in corresponding abelian systems but definite answers require a non-perturbative analysis [Hel20]. A natural solution might be given by an embedding of the $U(1)$ factor into a unified non-abelian group, provided that a suitable physical spectrum arises [MST19; Son19]. A rather interesting possibility has been discussed within the combined system of QED and gravitational fluctuations based on the asymptotic safety scenario of quantum gravity [HR11; CE17; EV18; EHW18; EHW20], since the combined system can develop a UV fixed point, for which the low-energy QED coupling becomes a predictable quantity.

Returning to a pure QED perspective, it has recently been observed within an effective-field theory analysis that a finite Pauli term (the spin-field coupling) can be sufficient to screen the perturbative Landau pole [DGM17] and render the minimal gauge coupling finite. Within the effective field theory paradigm, this suggests that QED triviality could be an artefact of truncating the effective field theory at leading order. If so, high-energy completion would still require an embedding into a "new-physics" framework which remains unknown at this point.

In the present paper, we explore the possibility whether QED could be asymptotically safe in a theory space larger than what has so far been considered in lattice simulations or functional methods. Inspired by [DGM17], we include the Pauli coupling κ parameterising the unique dimension-5 operator to lowest-derivative order and thus a next-to-leading order term in an operator expansion of the effective action. A reason to disregard this term in earlier studies might have been given by the fact that the Pauli term breaks chiral symmetry explicitly (apart from perturbative non-renormalisability). By contrast, high-energy studies typically assume asymptotic symmetry [LW74], as the electron mass being the source of chiral symmetry breaking (in pure QED) is implicitly assumed to be irrelevant in comparison to all other momentum scales at high energies. Counterexamples to this scenario have been constructed only recently in the context of non-abelian Higgs-(Yukawa) models [Gie+13; GZ15; GZ17; Gie+19a; Gie+19b], exhibiting mass scales that grow proportionally to an (RG) scale; see [GS10; GRS10] for earlier toy-model examples.

In fact, using modern functional renormalisation group techniques, we find evidence for the existence of interacting RG fixed points in the theory space spanned also by the Pauli coupling. RG trajectories that emanate from such fixed points correspond to high-energy complete realisations of QED with a fixed set of physical parameters and a full predictive power for the long-range behaviour of the theory. The dynamics induced by the Pauli coupling exhibits several interesting features: For increasing Pauli coupling, its RG flow turns from irrelevant to relevant, i.e., the power-counting scaling is compensated by quantum fluctuations. Also the running of the gauge coupling e is driven towards asymptotic freedom (whereas κ is asymptotically safe). We observe several fixed points that qualitatively differ by the presence or absence of a finite value for the electron mass (measured in units of the RG scale), by the number of relevant directions corresponding to the number of physical parameters, and by the properties of the long-range physics. We identify RG trajectories that interconnect the physical values for the low-energy parameters of real QED with one of the UV fixed points, thereby constructing a high-energy complete version of QED with only photon and electron degrees of freedom.

Our paper is structured as follows: In section 5.1, we introduce the subspace of the QED theory space to be screened for the existence of fixed points. Section 5.2 presents our results for the RG flow equation in that subspace. In section 5.3, we present the results of our RG fixed points search and classify the resulting universality classes. Section 5.4 is devoted to a construction of UV complete trajectories and an analysis of the resulting long-range properties. In section 5.5, we conclude and discuss possible implications of our results for pure QED in the context of an embedding into a standard-model like theory. Further technical details are summarised in section 5.6.

5.1. QED with a Pauli Term

Let us consider pure QED, consisting of an electromagnetic $U(1)$ gauge field A_μ interacting with a massive electron that is described by a Dirac spinor ψ. In addition to the standard kinetic terms, the mass term, and the minimal coupling, we also consider a Pauli term, parameterising the (anomalous) coupling of the electron to the electromagnetic field. Using Euclidean spacetime and Dirac-space conventions, the bare action reads

$$S = \int_x \bar{\psi} i \not{D}[A] \psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\bar{m}\bar{\psi}\psi + i\bar{\kappa}\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi, \tag{5.1.1}$$

with $D_\mu[A] = \partial_\mu - i\bar{e}A_\mu$ denoting the covariant derivative, and $\bar{m}$, $\bar{e}$, $\bar{\kappa}$ representing the bare mass and couplings. Note that the factors of i in front of the mass term and the Pauli spin coupling arise from the Euclidean description; the action satisfies Osterwalder-Schrader reflection positivity. The action features a local $U(1)$ (vector) gauge invariance. In addition to the electron mass term $\sim \bar{m}$, also the Pauli coupling $\sim \bar{\kappa}$ breaks the global chiral symmetry explicitly.

Let us briefly sketch the reasoning behind a conventional perturbative RG analysis: Based on the assumption that the theory is close to the Gaußian fixed point at an initial high-energy scale Λ with all couplings $\bar{e}, \bar{\kappa}, \dots \lesssim \mathcal{O}(1)$ (suitably measured in units of Λ), the Pauli term as a dimension-5 operator as well as all possible higher order couplings $\bar{\kappa}, \dots$ are expected to be governed by their power-counting dimension (possibly amended by logarithmic corrections). As a result, the Pauli coupling is expected to scale as $\bar{\kappa} \sim k/\Lambda$ towards lower scales $k \ll \Lambda$, exhibiting RG irrelevance for the long-range physics. Higher-order operators are expected to be correspondingly power-suppressed. By contrast, the dimensionless RG-marginal gauge coupling runs logarithmically, as is captured by the β function for the suitably renormalised coupling e (see below),

$$\beta_e = k\frac{de}{dk} = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5). \tag{5.1.2}$$

The running of the coupling obtained from the integrated β_e function exhibits a logarithmic decrease of the coupling towards lower scales k and a Landau-pole singularity towards the UV; the latter signals the break-down of the perturbative reasoning towards higher energies.

In this discussion, we have already implicitly assumed the mass to be smaller than any of the scales k, Λ (or loop momenta). This assumption characterises the *deep Euclidean region* where a possible finite mass can be ignored. The finiteness of the renormalised mass m then only becomes relevant at low scales $k \sim m$, where threshold effects lead to a decoupling of massive particles from the flow.

Upon embedding the pure QED sector into the Standard Model, the corresponding mass term arises from the Higgs mechanism and is seeded by the Yukawa coupling to the Higgs field. The latter is an RG-marginal coupling as well and preserves chiral symmetry. This, together with the assumption of asymptotic symmetry [LW74] justifies the procedure to ignore particle masses in the high-energy analysis of standard-model like theories.

This work is devoted to an analysis of the nonperturbative RG flow in pure QED theory space including the Pauli term. The anticipated existence of an interacting RG fixed point can invalidate simple power-counting arguments for the Pauli coupling. If so, high-energy quantum fluctuations could render the Pauli term RG relevant and exert a strong influence on the high-energy behaviour of the gauge coupling. In addition to strong-coupling effects, explicit chiral symmetry breaking triggered by the Pauli term makes it necessary to consider the flow of the mass term on the same footing as the couplings. Since a finite mass term can generically induce decoupling, it remains a nontrivial question as to whether RG trajectories exist along which the high-energy behaviour can be separated from the physical low-energy electron mass scale.

5.2. RG Flow Equations

Whereas the action in eq. (5.1.1) could be straightforwardly treated with standard effective field theory methods in the deep Euclidean region, the fact that the Pauli term breaks chiral symmetry suggests to use a formalism where all sources of symmetry breaking including the mass term are treated on the same footing. In order to study the RG flow beyond the bias of the deep Euclidean region, we use the functional RG formulated in terms of the Wetterich equation[2]

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left[(\partial_t R_k) \left(\Gamma_k^{(2)} + R_k \right)^{-1} \right], \tag{5.2.1}$$

where $t = \ln(k/\Lambda)$ for some UV scale Λ denotes "RG time" defined in terms of a scale k, separating the modes with momenta $\lesssim k$ to be integrated out from those with momenta $\gtrsim k$ already integrated out. In the present work, we study the flow of the system in a truncated theory space spanned by the action

$$\Gamma_k = \int_x \left[\bar{\psi} \left(i Z_\psi \partial\!\!\!/ + \bar{e} A\!\!\!/ - i\bar{m} + i\bar{\kappa}\sigma_{\mu\nu}F^{\mu\nu} \right) \psi + \frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + \frac{Z_A}{2\xi} \left(\partial_\mu A^\mu \right)^2 \right]. \tag{5.2.2}$$

Here, the wave function normalisations Z_ψ and Z_A as well as all couplings and mass parameters are considered as k dependent. While the gauge parameter ξ could also be studied as a k-dependent parameter, we choose the Landau gauge $\xi = 0$ in practice, as it is a fixed point of the RG flow [EHW96; LP98]. This truncation is complete to lowest order in a derivative expansion (1st order for fermions, 2nd order for photons) and to dimension-5 in a power-counting operator expansion. A next order in derivatives would include the operators $\bar{\psi} D\!\!\!/ D\!\!\!/ \psi$ (dimension-5), $F_{\mu\nu} \Box F^{\mu\nu}$ (dimension-6); a next order in the operator expansion includes four-fermion terms as studied in [Aok+97; GJW04; GJ04].

It is convenient to express the RG flow in terms of dimensionless and renormalised parameters:

$$e = \frac{k^{\frac{d}{2}-2}\bar{e}}{Z_\psi \sqrt{Z_A}}, \quad \kappa = \frac{k^{\frac{d}{2}-1}\bar{\kappa}}{Z_\psi \sqrt{Z_A}}, \quad m = \frac{\bar{m}}{Z_\psi k}. \tag{5.2.3}$$

[2] In this chapter we use a slightly different notation compared to theorem 3.1.18. The absence of a field-independent term is immaterial as we shall expand the Wetterich equation in powers of the fields.

and calculate their β functions in the Landau gauge $\xi \to 0$ using eq. (5.2.1). For this, we use standard methods for the operator expansion of the Wetterich equation [BTW02; GW02; Gie12; Bra12] in order to project onto the operators of eq. (5.2.2), and employ FeynCalc [MBD91; SMO16] for some of the tensor manipulations. The results for these β functions are rather involved as a result of the absence of chiral symmetry and the possible finiteness of the mass term. For generality, we list the results for a generic spacetime dimension d:

$$\begin{aligned}\partial_t e = {}& e\Big(\tfrac{d}{2}-2+\eta_\psi+\tfrac{\eta_A}{2}\Big)-4v_d\tfrac{(d-4)(d-1)}{d}e^3\, l_d^{(1,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2)-16v_d\tfrac{(d-2)(d-1)}{d}e\kappa^2\, l_d^{(2,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2)\\ &-32v_d\tfrac{d-1}{d}e^2\kappa m\, l_d^{(1,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2)-4v_d\tfrac{(d-2)(d-1)}{d}e^3m^2\, l_d^{(\mathrm{B},\mathrm{F}^2)}(0,m^2)\\ &-16v_d\tfrac{(d-4)(d-1)}{d}e\kappa^2m^2\, l_d^{(2,\mathrm{B},\mathrm{F}^2)}(0,m^2)\,,\end{aligned} \tag{5.2.4}$$

$$\begin{aligned}\partial_t \kappa = {}& \kappa\Big(\tfrac{d}{2}-1+\eta_\psi+\tfrac{\eta_A}{2}\Big)+16v_d\tfrac{(d-4)(d-1)}{d}\kappa^3\, l_d^{(2,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2)-4v_d\Big(3\tfrac{(d-6)(d-2)}{d}+1\Big)e^2\kappa\, l_d^{(1,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2)\\ &+4v_d\, e^3m\,\Big[\tfrac{d-3}{d}[l_d^{(1,\mathrm{B},\tilde{\mathrm{F}}_1,\mathrm{F})}(0,m^2,m^2)-l_d^{(1,\mathrm{B},\mathrm{F}_1,\tilde{\mathrm{F}})}(0,m^2,m^2)]-\tfrac{(d-4)(d-1)}{2d}l_d^{(\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2)\Big]\\ &+16v_d\, e\kappa^2m\,\Bigg[\tfrac{5(d-4)(d-3)}{2d}\, l_d^{(1,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2)+\tfrac{d-3}{d}\, l_d^{(2,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}}_1)}(0,m^2,m^2)-\tfrac{d-3}{d}\, l_d^{(2,\mathrm{B},\mathrm{F}_1,\tilde{\mathrm{F}})}(0,m^2,m^2)\\ &\qquad-\tfrac{d+2}{d}\, l_d^{(1,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2)\Bigg]\\ &+16v_d\left(1-\tfrac{(d-4)^2}{d}\right)\kappa^3m^2\, l_d^{(1,\mathrm{B},\mathrm{F}^2)}(0,m^2)+4v_d\tfrac{(d-4)(d-1)}{d}e^2\kappa m^2\, l_d^{(\mathrm{B},\mathrm{F}^2)}(0,m^2)\,,\end{aligned} \tag{5.2.5}$$

$$\begin{aligned}\partial_t m = {}& -m\big(1-\eta_\psi\big)-16v_d(d-1)e\kappa\, l_d^{(1,\mathrm{B},\tilde{\mathrm{F}})}(0,m^2)\\ &+16v_d(d-1)m\kappa^2\, l_d^{(1,\mathrm{B},\mathrm{F})}(0,m^2)-4v_d(d-1)e^2m\, l_d^{(\mathrm{B},\mathrm{F})}(0,m^2)\,.\end{aligned} \tag{5.2.6}$$

Here, $v_d = [2^{d+1}\pi^{d/2}\Gamma(d/2)]^{-1}$, and the functions $l_d^{\cdot\cdot}(\dots)$ parameterise threshold effects arising from the massive fermion propagator; in addition to the explicitly highlighted mass dependence, they can also depend on the anomalous dimensions

$$\eta_A = -\partial_t \ln Z_A, \quad \eta_\psi = -\partial_t \ln Z_\psi \tag{5.2.7}$$

as a consequence of "RG improvement" implementing a resummation of large classes of diagrams. The threshold functions approach finite non-negative constants for $m, \eta_\psi, \eta_A \to 0$, and vanish for $m \to \infty$ manifesting the decoupling of massive fermion modes. As we encounter threshold functions that go beyond those tabulated in the literature as a consequence of the absence of chiral symmetry as well as the presence of a momentum dependent vertex, we have introduced a new systematic notation here, which we explain in detail in section 5.6.

These flow equations are autonomous coupled ordinary differential equations which depend on the anomalous dimensions of the fields. The latter are determined by the flow of the kinetic terms, yielding algebraic equations of the form

$$\begin{aligned}\eta_\psi = {} & 4v_d \tfrac{(d-2)(d-1)}{d} e^2\, l_d^{(\mathrm{B},\tilde{\mathrm{F}})}(0,m^2) - 8v_d \tfrac{d-1}{d} e^2\, l_d^{(1,\mathrm{B},\tilde{\mathrm{F}}_1)}(0,m^2) + 16v_d \tfrac{(d-4)(d-1)}{d} \kappa^2\, l_d^{(1,\mathrm{B},\tilde{\mathrm{F}})}(0,m^2) \\ & - 32v_d \tfrac{d-1}{d} \kappa^2\, l_d^{(2,\mathrm{B},\tilde{\mathrm{F}}_1)}(0,m^2) + 32v_d \tfrac{d-1}{d} e\kappa m\, l_d^{(1,\mathrm{B},\mathrm{F}_1)}(0,m^2)\,,\end{aligned} \tag{5.2.8}$$

$$\begin{aligned}\eta_A = {} & 8v_d \tfrac{d_\gamma N_\mathrm{f}}{d+2} e^2\, l_d^{(2,\tilde{\mathrm{F}}_1^2)}(m^2) + 16v_d d_\gamma N_\mathrm{f} \kappa^2 m^2 l_d^{(\mathrm{F}^2)}(m^2) - 16v_d \tfrac{d-4}{d} d_\gamma N_\mathrm{f} \kappa^2\, l_d^{(1,\tilde{\mathrm{F}}^2)}(m^2) \\ & - 64v_d \tfrac{d_\gamma N_\mathrm{f}}{d} e\kappa m\, l_d^{(1,\tilde{\mathrm{F}},\mathrm{F}_1)}(m^2,m^2) + 8v_d \tfrac{d_\gamma N_\mathrm{f}}{d} e^2 m^2\, l_d^{(1,\mathrm{F}_1^2)}(m^2)\,,\end{aligned} \tag{5.2.9}$$

where d_γ denotes the dimensionality of the representation of the Dirac algebra ($d_\gamma = 4$ in physical QED), and N_f is the number of Dirac fermion flavors. For an understanding of the coupling dependence of the flows, the following two discrete $\mathbb{Z}_2$ symmetries are relevant: We observe that the action (5.2.2) is invariant under a simultaneous discrete axial transformation $\psi \to e^{i\frac{\pi}{2}\gamma_5}\psi$, $\bar{\psi} \to \bar{\psi}e^{i\frac{\pi}{2}\gamma_5}$ and a sign flip of $\bar{\kappa} \to -\bar{\kappa}$ and $\bar{m} \to -\bar{m}$. This $\mathbb{Z}_2$ symmetry is also visible in all flow equations and anomalous dimensions, as they remain invariant under a simultaneous sign flip of κ and m. Furthermore, charge conjugation on the level of couplings is represented by simultaneous sign flip of e and κ which is also an invariance of all β functions and anomalous dimensions.

While each term in the flow equations reflects the one-loop structure of the Wetterich equation – visible in terms of the explicitly highlighted polynomial coupling dependence – the flows still exhibits various nonperturbative features: the flow of κ coupling to the e and m equations effectively corresponds to feeding back higher-order diagrams, the anomalous dimensions in the threshold functions also yield higher-order resummations, and the dependence of the threshold functions on the running mass is also a nonperturbative effect. As a simple check, it is straightforward to rediscover the perturbative limit. For this, we drop all κ terms and take the deep Euclidean limit $m \to 0$. In the flow equation (5.2.4) for e only the anomalous dimensions in the first scaling term remain in this limit. Further, we observe that $\eta_\psi \to 0$ in this limit as the seemingly remaining terms $\sim e^2$ cancel by virtue of properties of the threshold functions. This is in agreement with the standard perturbative result in the Landau gauge. The only non-trivial term in the flow is carried by the anomalous dimension η_A of the photon, finally leading to eq. (5.1.2) to lowest order in the coupling.

5.3. Fixed Points and Universality Classes

The scenario of asymptotic safety relies on the existence of an interacting non-Gaußian fixed point of the renormalisation group. Summarising all dimensionless couplings including the mass parameter into a vector $\mathbf{g}$, with $\mathbf{g} = (e, \kappa, m)$ in the present case, a fixed point satisfies $\partial_t \mathbf{g}|_{\mathbf{g}=\mathbf{g}^*} = 0$, realising the concept of (quantum) scale invariance. In the vicinity of a fixed point, the RG flow to linear order is governed by the properties of the stability matrix $\mathbf{B}_{ij}$, the eigenvalues of which are related to the RG critical exponents θ_I,

$$\mathbf{B}_{ij} = \frac{\partial}{\partial g_j}\partial_t g_i \bigg|_{\mathbf{g}=\mathbf{g}^*}, \quad \theta_I = -\mathrm{eig}\,\mathbf{B}. \tag{5.3.1}$$

The number of positive critical exponents corresponds to the number of RG relevant directions. (A zero eigenvalue $\theta_I = 0$ corresponds to an RG marginal direction with higher-orders beyond the linearised regime deciding about marginal relevance or irrelevance.) The number of relevant and marginally relevant directions counts the number of physical parameters that need to be fixed in order to predict the long-range behaviour of the theory. In the presence of several fixed points, each fixed point defines a universality class: for all RG flow trajectories passing through the vicinity of the fixed point, the long-range behaviour is universally governed by these (marginally) relevant directions. Those RG trajectories that emanate from a fixed point are UV complete: a theory can be extended to arbitrarily high scales with the long-range physics remaining fixed; such a trajectory defines a "line of constant physics".

The previously mentioned $\mathbb{Z}_2$ symmetries of the flows translate to relations among possible fixed points of the RG flow: given any fixed point (e^*, κ^*, m^*), we can construct the following set of points which are also fixed points of the RG, describing one and the same universality class

$$(-e^*, -\kappa^*, m^*), (e^*, -\kappa^*, -m^*), (-e^*, \kappa^*, -m^*). \tag{5.3.2}$$

For the concrete evaluation of the flows and the search for fixed points, we concentrate on the relevant case of four spacetime dimensions $d = 4$, the irreducible representation of the Dirac algebra $d_\gamma = 4$ and a single fermion flavor $N_\mathrm{f} = 1$. For simplicity, we use the the linear regulator [Lit00; Lit01] for which all threshold functions can be evaluated analytically, yielding rational functions of the mass arguments given in section 5.6.

As an internal consistency check, we define the leading-order (LO) evaluation of our flows in terms of ignoring the dependence of the threshold functions on the anomalous dimensions; i.e., we drop the higher-loop RG improvement provided by these resummations, but keep the anomalous dimensions in the scaling terms as they contribute to leading-loop level. If our truncation is reliable, we expect the LO results to agree qualitatively and semi-quantitatively with those of the full truncation for the following reasons: First, the size of the anomalous dimensions can be viewed as a measure for the validity of the derivative expansion as the anomalous dimensions quantify the running of the kinetic (derivative) terms. Second, anomalous dimensions also quantify the deviations from canonical scaling; hence, if the anomalous dimensions are sufficiently small, higher-order operators can be expected to remain RG irrelevant. As a self-consistency criterion, we thus require the anomalous dimensions to be sufficiently small, $|\eta_{A,\psi}| \lesssim \mathcal{O}(1)$. In this LO approximation, we find the fixed points displayed in table 5.1. The first line in table 5.1 characterises the trivial Gaußian fixed point $\mathcal{A}$ with the mass correspond-

	e^*	κ^*	m^*	multiplicity	n_{phys}	θ_{max}	η_ψ	η_{A}
$\mathcal{A}$:	0	0	0	–	1	1.00	0.00	0.00
$\mathcal{B}$:	0	4.98	0.283	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	2.6478	−1.24	0.319
$\mathcal{C}$:	0	4.06	0	$\mathbb{Z}_2$	3	2.00	−1.00	0.00

Table 5.1.: Fixed points of the RG flow evaluated to leading order (LO) as described in the text.

	e	κ	m	multiplicity	n_{phys}	θ_{max}	η_ψ	η_{A}
$\mathcal{A}$:	0	0	0	–	1	1.00	0.00	0.00
$\mathcal{B}$:	0	5.09	0.328	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	3.10	−1.38	0.53
$\mathcal{C}$:	0	3.82	0	$\mathbb{Z}_2$	3	2.25	−1.00	0.00

Table 5.2.: Fixed points of the given RG flow truncation.

ing to the only relevant RG direction with power-counting critical exponent $\theta_m = \theta_{\text{max}} = 1$. The Pauli coupling is RG irrelevant at this fixed point $\theta_\kappa = -1$ and the gauge coupling is marginal $\theta_e = 0$ with the next order given in terms of eq. (5.1.2) classifying the gauge coupling as marginally irrelevant; this reflects perturbative triviality: there is no UV-complete trajectory in QED emanating from the Gaußian fixed point that corresponds to an interacting theory at low energies.

In the LO approximation, we find two further non-Gaußian fixed points labelled by $\mathcal{B}$ and $\mathcal{C}$ at finite values of the Pauli coupling $\kappa^* > 0$, with $\mathcal{B}$ also featuring a finite (dimensionless) mass parameter $m^* > 0$. Taking the aforementioned discrete symmetries into account, these fixed points occur in multiplicities according to their nontrivial $\mathbb{Z}_2$ reflections as listed in eq. (5.3.2).

In addition, these fixed points differ by their number of relevant directions, n_{phys} counting the number of physical parameters. The largest critical exponent is listed in table 5.1 as θ_{max}. The table also lists the anomalous dimensions at the fixed point which both satisfy the self-consistency criterion $|\eta_{A,\psi}| \lesssim \mathcal{O}(1)$. It is instructive to take a closer look at the fixed point $\mathcal{C}$ which exhibits an anomalous fermion dimension $\eta_\psi = -1$ (and $\eta_A = 0$). This value of η_ψ corresponds precisely to the amount required to convert the power-counting irrelevant Pauli coupling to a marginal coupling, cf. the dimensional scaling terms in eq. (5.2.5); the scaling dimension of Dirac fermions near fixed point $\mathcal{C}$ thus is similar to that of a scalar boson near the Gaußian fixed point. In absence of further fluctuation terms, the Pauli coupling would run logarithmically; however, the fluctuation terms turn it into a relevant power-law running.

Finally, we observe no non-Gaußian fixed point at finite values of the gauge coupling within the LO approximation. This is in line with the conclusion of many literature studies that have not found a UV completion in the theory space spanned by the standard QED bare action.

Let us now turn to the fixed-point analysis of the full truncation without any further approximation. In fact, we again find the same set of fixed points, see table 5.2. The Gaußian fixed point $\mathcal{A}$, of course, remains unaffected by the improved approximation. We observe the identical qualitative features such as multiplicities and number of physical parameters n_{phys}, with quantitative changes of our estimates for the (non-universal) fixed-point values for $\mathcal{B}$ and $\mathcal{C}$ as well as for the (universal) critical exponents. The quantitative improvements arising from the full truncation in contrast to the LO approximation are on the $\mathcal{O}(10\%)$ level. This is self-consistent with the modification of the threshold functions upon inclusion of anomalous dimensions as a consequence of higher-order resummations.

The location of the fixed points and the corresponding phase diagram in the (κ, m) plane for $e = 0$ is displayed in fig. 5.1 with arrows pointing towards the IR. This figure also illustrates that fixed point $\mathcal{C}$ is fully IR repulsive in this plane, whereas $\mathcal{A}$ and $\mathcal{B}$ exhibit one attractive direction visible in this projection. Fixed points $\mathcal{B}$ and $\mathcal{C}$ are both IR repulsive also in the direction of the coupling e (not shown in the figure), whereas $\mathcal{A}$ is marginally attractive as dictated by eq. (5.1.2). Apart from the separatrices, all trajectories emanating from the non-Gaußian fixed points $\mathcal{B}$ and $\mathcal{C}$ towards the region of smaller Pauli coupling eventually approach the basin of attraction at $|m| \to \infty$ (corresponding to the formal fixed point of the free photon gas and non-propagating electrons). This scaling of the dimensionless mass parameter corresponds to a physical electron mass approaching a constant value. Subsequently, the flow of all physical observables measured in units of a physical scale freezes out, and the observables acquire their long-range values. Incidentally, we note that the full truncation also exhibits six other non-Gaußian fixed points and their $\mathbb{Z}_2$ reflections. However, these fixed points have large (≥ 6) anomalous dimensions and therefore clearly lie outside the validity regime of our approximation. The fact, that these fixed points do not appear in the LO approximation demonstrates that they do not pass the self-consistency test of our approximation scheme. Therefore, we identify them as artefacts of the approximation and dismiss them in our further analysis.

In summary, we have discovered two new non-Gaußian fixed points $\mathcal{B}$ and $\mathcal{C}$ (and their corresponding $\mathbb{Z}_2$ reflections) of the RG flow of QED in a truncated theory space including the Pauli coupling. They can be associated with two new QED universality classes parameterised by $n_{\text{phys},\mathcal{B}} = 2$ and $n_{\text{phys},\mathcal{C}} = 3$ physical parameters. The existence of these fixed points with a finite number of physical parameters is a prerequisite for constructing a UV-complete asymptotically safe version of QED.

5.4. Long-range Properties and Physical Trajectories

Let us now construct RG trajectories related to the different fixed points. A crucial question is as to whether asymptotically safe trajectories can be constructed that are in agreement with the observed QED long-range physics.

As a warm up, we first analyse the RG flow in the vicinity of the Gaußian fixed point (perturbative QED) without paying attention to a possibly existing UV completion. For this, we

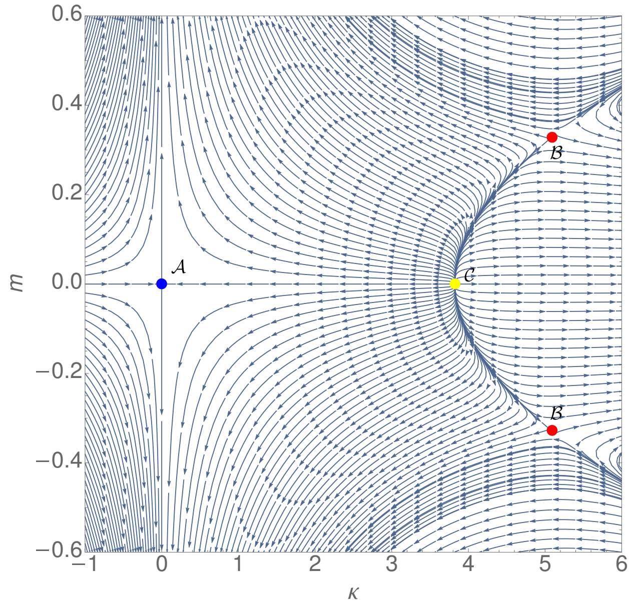

Figure 5.1.: Phase diagram in the plane of dimensionless parameters (κ, m) at $e = 0$ showing the Gaußian fixed point $\mathcal{A}$, and the non-Gaußian fixed points $\mathcal{B}$ (including a $\mathbb{Z}_2$ reflection) and $\mathcal{C}$. Arrows denote the RG flow towards the IR. The strongly repulsive direction at the Gaußian fixed point $\mathcal{A}$ towards large values of $|m|$ corresponds to the dimensional scaling of a mass parameter describing the decoupling of the massive modes. In units of a physical mass scale, the flow of all physical observables freezes out along this direction towards their long-range values. This massive phase can be reached from all fixed points.

assume e to be perturbatively small over a considered range of scales, say $k \in [0, \Lambda]$. The perturbative initial condition for κ is more subtle: as κ is irrelevant, it is tempting to assume that the initial condition $\kappa_{k=\Lambda}$ at a the high scale does not matter too much, as long as it is in the perturbative domain. However, the long-range value of κ is related to the celebrated result of the anomalous magnetic moment, i.e., the g factor of the electron; To one-loop accuracy, we have [Sch48],

$$a_e := -\frac{4}{e}\kappa m\Big|_{k=0} = \frac{g-2}{2} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2). \tag{5.4.1}$$

From the standard QED computation, it is obvious that this result is independent of the electron mass, which drops out of the corresponding projection of the electron-photon vertex. Now, the initial conditions at $k = \Lambda$ have to be chosen such that they correspond to the effective action $\Gamma_{k=\Lambda}$ which we would obtain from, say, the path integral in the presence of the IR cutoff $k = \Lambda$. In particular, the bare value for $\bar{\kappa}\bar{m}|_{k=\Lambda}$ is expected to be finite, since all fluctuations with momenta above $k = \Lambda$ already have to be included. As k acts like a mass parameter for all modes, we anticipate that $\bar{\kappa}\bar{m}|_{k=\Lambda}$ may already be close to its physical value (5.4.1), since mass scales drop out of this classic result. The details may depend on the chosen regulator.

This necessity of choosing "loop-improved" initial conditions [EHW98] also becomes visible in the flow equations. In fact, the flow of the dimensionless combination

$$\partial_t(\kappa m) = \kappa\partial_t m + m\partial_t\kappa, \tag{5.4.2}$$

shows features of a marginal coupling as the dimensional scaling terms drop out. In the perturbative domain, this flow agrees with that of $\partial_t(\bar{\kappa}\bar{m})$ up to subleading anomalous-dimension terms. While the perturbative flow is characterised by the standard log-like running of e and the power-counting dimensional running of $m = \bar{m}/k$ (with small perturbative corrections), the flow of κ can be characterised by eq. (5.4.2). Anticipating $\kappa \sim e^3$ in line with eq. (5.4.1) and dropping higher-order terms, we find to leading perturbative order:

$$\partial_t(\kappa m) = v_4 e^3 m^2 [l_4^{(1,\mathrm{B},\tilde{\mathrm{F}}_1,\mathrm{F})}(0,m^2,m^2) - l_4^{(1,\mathrm{B},\mathrm{F}_1,\tilde{\mathrm{F}})}(0,m^2,m^2)] + \mathcal{O}(\kappa^3, e\kappa^2, e^2\kappa)\,. \tag{5.4.3}$$

Integrating this flow equation to leading order for a fixed e and $\bar{m}$ from $k = 0$ to Λ, we find in the limit $\Lambda \gg \bar{m}$ using the linear regulator:

$$\begin{aligned} \bar{\kappa}\bar{m}|_{k=0} \simeq \kappa m|_{k=0} &= \kappa m|_{k=\Lambda} - \int_0^\Lambda \frac{dk}{k}\ \text{RHS of eq. (5.4.3)} \\ &= \kappa m|_{k=\Lambda} - \frac{e^3}{32\pi^2}\frac{1}{6}\frac{\bar{m}^2}{\Lambda^2}. \end{aligned} \tag{5.4.4}$$

We observe that the flow in our current massive regularisation scheme does not induce a significant running of κm for $\Lambda \gg \bar{m}$. This confirms our expectation that the proper description of the anomalous magnetic moment a_e of the electron is essentially encoded in the boundary condition for Γ_k at $k = \Lambda$. Note that this boundary condition is not an independent parameter of the theory but can be worked out from a standard perturbative loop computation upon inclusion of the regulator term. In practice, we fix the physical flow such that $\kappa m|_{k=0}$ corresponds to the observed experimental value for a_e, see below.

Let us now turn to a discussion of the long-range properties of the system in the universality classes defined by the nontrivial fixed points $\mathcal{B}$ and $\mathcal{C}$. In contrast to the Gaußian fixed point $\mathcal{A}$, the fixed-points $\mathcal{B}$ and $\mathcal{C}$ allow for the construction of UV-complete RG trajectories. However, UV completeness does not guarantee that these universality classes exhibit a proper QED-type long-range behaviour. For this, it is of central interest whether one can find an RG trajectory connecting the fixed point regimes in the UV with physical long-range behaviour defined by the IR values for all couplings. The number of relevant directions n_{phys} defines the dimensionality of the set of UV-complete RG trajectories emanating from the fixed point.

Let us start with fixed point $\mathcal{B}$ with $n_{\text{phys},\mathcal{B}} = 2$ relevant directions and critical exponents ($\theta_1 = \theta_{\text{max}} = 3.10$, $\theta_2 = 2.13$, $\theta_3 = -0.81$). This implies that if we fix two parameters out of our set of couplings (e, κ, m) the third one is a definite prediction of the universality class.

UV fixed point	$a_e = -4\frac{\kappa m}{e}$ in the IR
$(0, 5.09, 0.328)$	≈ -18.55
$(0, -5.09, 0.328)$	≈ 14.01

Table 5.3.: The long-range prediction for the anomalous magnetic moment for UV-complete trajectories emanating from two $\mathbb{Z}_2$ copies of fixed point $\mathcal{B}$ connected to a long-range coupling of $e \approx 0.3$, i.e, $\alpha \simeq 1/137$.

In practice, we fix one parameter such that $e \approx 0.3$ in the IR corresponding to the physical value of the coupling $\alpha \simeq 1/137$. The second parameter is implicitly chosen by initiating the flow at some scale Λ in the vicinity of the fixed point. This scale Λ can then be expressed in terms of the resulting dimensionful electron mass $\bar{m}_e = mk|_{k\to 0}$ which defines our physical mass units. The long-range Pauli coupling κ is then a prediction of the universality class. As an interesting subtlety, there is not just one RG trajectory, but there are actually two corresponding to a relative sign choice between our IR condition $e \simeq +0.3$ and the discrete $\mathbb{Z}_2$ symmetries. These two trajectories are physically distinct as they go along with a different sign for the correction to the anomalous magnetic moment.[3] These two trajectories correspond to distinct tangent vectors to the flow at the corresponding $\mathbb{Z}_2$ reflections of the fixed point $\mathcal{B}$ as listed in table 5.3. Our corresponding long-range prediction for the anomalous magnetic moment a_e of the electron by following the flow from the fixed points towards the deep infrared is also listed in this table. As is obvious, these predictions do clearly not match with the physical value

$$a_e = -4\frac{\kappa m}{e} \approx 0.00116. \tag{5.4.5}$$

(For instance, the trajectory ending up with $a_e \approx -18.55$ corresponds to the separatrix emanating from fixed point $\mathcal{B}$ in the upper half-plane of fig. 5.1 and then running towards $m \to \infty$ at finite e.) We conclude that physical QED is not in the universality class of fixed point $\mathcal{B}$. Though this universality class would not be plagued by a Landau-pole problem and potentially represent a consistent QFT at all scales, its long-range properties would be rather unusual: since the quantum corrections to the magnetic moment even overwhelm the Dirac value of $g = 2$, strong-magnetic fields are likely to induce tachyonic modes in the spectrum of the quantum-corrected Dirac operator rendering strong and spatially extended magnetic fields unstable (similar to the Nielsen-Olesen unstable mode in nonabelian gauge theories [NO78]). Still, this version of a new QED universality class is interesting as it presents an example that QED could seem asymptotically free[4] in its gauge coupling, since $e^* = 0$, at the expense of an asymp-

[3] Another way of phrasing this subtlety is that if we consider flows emanating from fixed point $\mathcal{B}$ as listed in table 5.2, we have the choice of flowing towards $e \approx 0.3$ or $e \approx -0.3$ which are both compatible with $\alpha \simeq 1/137$.

[4] This estimate of the gauge coupling appearing asymptotically free may be modified in a larger truncation. Since the Pauli coupling is non-Gaußian, it is well possible that it feeds back into the gauge coupling through

totically safe Pauli term. This is another example for a close connection between *paramagnetic dominance* and the UV behaviour of a system [NR13].

We finally study the universality class corresponding to fixed point $\mathcal{C}$ where only the Pauli coupling acquires a nonzero value $|\kappa| \approx 3.82$. This fixed point features three relevant directions, $n_{\text{phys},\mathcal{C}} = 3$, and hence we expect $\mathcal{C}$ to have open neighbourhoods in the (e, κ, m) space as parts of its basin of attraction. Whether or not there exist RG trajectories emanating from $\mathcal{C}$ that are compatible with the physical QED long-range properties still remains a quantitative question to be studied. Even though $n_{\text{phys},\mathcal{C}} = 3$ agrees with the number of physical parameters that we wish to match, there is a priori no reason why the physical domain belongs to the "IR window" of such a fixed point.

A straightforward construction starting at a UV scale in the vicinity of the fixed point – though possible in principle – is numerically challenging, since the large critical exponents ($\theta_1 = \theta_{\text{max}} = 2.25$, $\theta_2 = 1.79$, $\theta_3 = 0.413$) indicate that a substantial amount of fine-tuning of the initial conditions would be necessary to yield specific IR values. This numerical issue can be circumvented by constructing the flow from the IR towards the UV, since the fixed point is fully UV attractive in the present truncation.

In practice, we initiate the flow close to our physical IR boundary conditions: e.g., at the mass threshold scale defined by $k = \Lambda_m$ where $m = 1$ and the couplings being close to their IR values $e \approx 0.3$ and $\kappa \approx \kappa_{\text{phys}}$ satisfying eq. (5.4.5). For k towards smaller scales, the flow quickly freezes out as a consequence of the decoupling of massive electron modes.

Running the RG flow numerically from Λ_m towards the UV, we arrive at the $\mathbb{Z}_2$ reflection of fixed point $\mathcal{C}$ with $\kappa_{\text{UV}} \approx -3.82$ without any further fine-tuning. We can vary the e and κ values at Λ_m by at least 10% and still hit the same UV fixed point as is demonstrated in fig. 5.2 where different colours correspond to different initial conditions in the IR; vice versa, the existence of this set of trajectories illustrates that, e.g., the IR value of the fine-structure constant $\alpha \simeq 1/137$ is not particularly distinguished, but merely one out of a larger interval of possible IR values. This is also visible in the phase diagram in the (e, κ) plane at $m = 0$ displayed in fig. 5.3. A wide range of trajectories emanating from fixed point $\mathcal{C}$ towards smaller values of κ approach the small κ region at some finite value of the gauge coupling. At the same time, generic initial conditions lead to finite values of the physical mass and thus to a decoupling or freeze-out behaviour towards $m \to \infty$ for the dimensionless mass parameter. This dominant IR flow orthogonal to the (e, κ) plane appears as a seeming singularity at $\kappa = 0$ in fig. 5.3. In summary, the physical IR values of the fine-structure constant and the anomalous magnetic moment of the electron can easily be accommodated in the set of trajectories emanating from

higher-order operators rendering also the fixed-point value of the gauge coupling nonzero [EG11; Eic12]. This is a rather general mechanism which has lead to the notion of a "shifted Gaußian fixed point" representing a partial near-Gaußian fixed point in a sub-set of couplings. Despite the overall non-Gaußianity of the system, the shifted Gaußian sub-system behaves as if it were Gaußian.

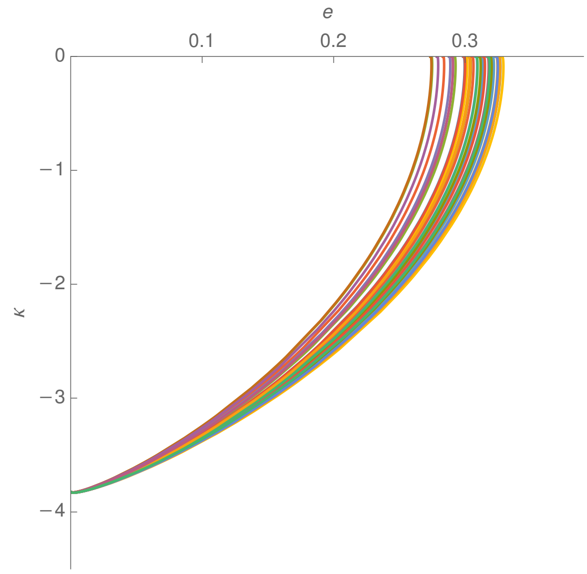

Figure 5.2.: RG trajectories towards the UV of 30 points close to $(0.3, \kappa_{\text{phys}}, 1)$ projected to the (e, κ) plane. All flows converge to the UV fixed point $(0, -3.82, 0)$.

the non-Gaußian fixed point $\mathcal{C}$. Figure 5.4 shows such a flow for intermediate values of e and κ approximately corresponding to physical IR values of the fine-structure constant and the electron anomalous magnetic moment. The dimensionless mass parameter (green line) exhibits a massive decoupling behaviour in the IR near the initial scale $k = \Lambda_m$. It is interesting to observe that the flow of the gauge coupling e first shows the characteristic increase towards higher energies in accordance with the perturbative running of eq. (5.1.2) (hardly visible in the plot), but finally features asymptotic freedom with e approaching its fixed point value $e^* \to 0$ in the deep UV. The Pauli coupling κ (orange) first remains perturbatively small in the IR but then undergoes a transition to its non-Gaußian fixed-point regime. In fig. 5.4, we have introduced the scale Λ_c as the scale where the flow of e has its steepest slope towards asymptotic freedom. At about the same scale, κ quickly flows towards κ^*. If these flows were indeed physical (in the sense of pure QED being a fundamental theory), Λ_c would denote the scale where perturbative calculations break down because of the Pauli coupling κ becoming an RG relevant operator. For RG flows approximately satisfying physical boundary conditions, we find

$$\frac{\Lambda_\text{c}}{\bar{m}_e} \approx 46329 \tag{5.4.6}$$

in the IR limit. In more conventional units this is equivalent to

$$\Lambda_\text{c} \approx 46329 \cdot \bar{m}_e \approx 23.67\,\text{GeV}. \tag{5.4.7}$$

By varying the IR boundary conditions for the Pauli coupling, i.e., varying the electron anomalous magnetic moment of eq. (5.4.5) on the $\mathcal{O}(10\%)$ level by hand, we observe that Λ_c varies

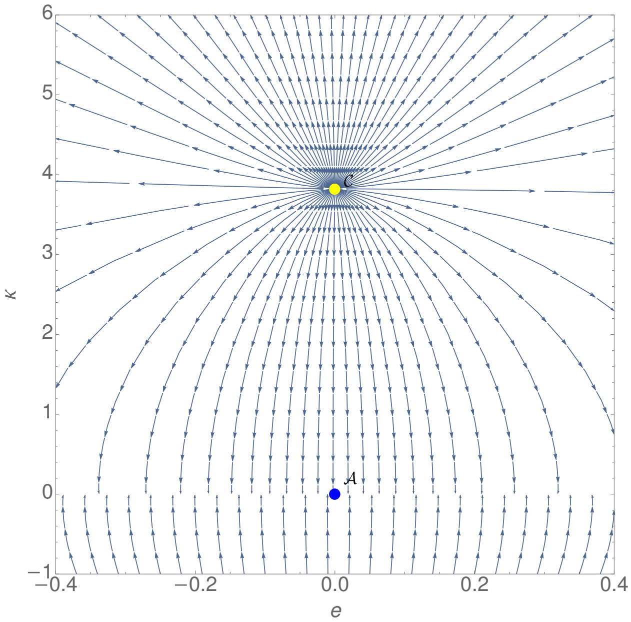

Figure 5.3.: Phase diagram in the plane of dimensionless parameters (e, κ) at $m = 0$ showing the non-Gaußian fixed point $\mathcal{C}$ and the Gaußian fixed point $\mathcal{A}$. Flows emanating from $\mathcal{C}$ towards smaller values of κ span a wide range of finite gauge couplings e in the IR, also accommodating the physical value $|e| \simeq 0.3$. (The phase diagram near the $\kappa = 0$ axis exhibits a seeming singularity which is lifted by a strong flow of m towards decoupling, implying that all trajectories freeze-out and end at $\kappa = 0$, generically at finite values of $-4\frac{\kappa m}{e}$.)

approximately linearly with a_e.

It is interesting to see that this transition scale is much larger than the intrinsic mass scale $\bar{m}_e$ of QED and somewhat below the electroweak scale.

5.5. Conclusions

We have studied the renormalisation flow of QED in a subspace of theory space that includes the Pauli spin-field coupling. In contrast to the reduced subspace defined by perturbatively renormalisable operators, the enlarged subspace features two non-Gaußian fixed points of the RG in addition to the Gaußian free-field fixed point. The existence of such interacting fixed points allows for the construction of RG trajectories approaching the fixed points towards high energies thus representing UV-complete realisations of QED within the scenario of asymptotic safety. Each fixed point defines a different universality class of QED labelled by a set of critical exponents and a corresponding number of physical parameters.

One of the newly discovered fixed points (fixed point $\mathcal{C}$) allows for the construction of UV-complete RG trajectories that can be interconnected with the long-range physics of QED as observed in Nature. In this scenario, the UV-failure of perturbation theory as indicated by the

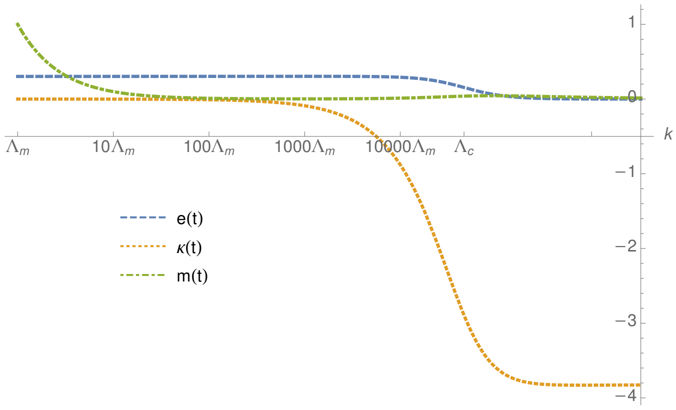

Figure 5.4.: RG flow towards (a $\mathbb{Z}_2$ reflection of) the UV fixed point $\mathcal{C}$ $(0, -3.82, 0)$. The dimensionless electron mass parameter m (green) exhibits the massive decoupling in the IR near $k = \Lambda_m$, the gauge coupling (blue) is asymptotically free towards high energies, and the Pauli coupling κ (orange) features a transition to the fixed point regime near the scale $k = \Lambda_\mathrm{c}$. Note that the k axis is logarithmic.

Landau-pole singularity is resolved by a controlled approach of the renormalisation flow towards the fixed point with a finite value of the Pauli coupling and a vanishing value of the gauge coupling in our approximation. In pure QED, we estimate this transition to occur at a crossover scale Λ_c somewhat below the electroweak scale. The RG flow below this transition scale towards long-range physics remains essentially perturbative. A particularity of this universality class is that it features $n_\mathrm{phys} = 3$ physical parameters to be fixed. In our considerations, we use the anomalous magnetic moment of the electron in addition to the gauge coupling and the electron mass as additional input. In this sense, this UV-complete version of QED has less predictive power than perturbative QED. However, the latter has to be considered as an effective field theory requiring the implicit assumption that all possible higher-order operators are sufficiently small at some high scale. By contrast, QED in universality class $\mathcal{C}$ controls all further higher-order operators by virtue of the fixed point.

The other newly discovered fixed point $\mathcal{B}$ also allows for UV-complete versions of QED fixed by only two physical parameters and thus has the same predictive power as perturbative QED. However, our estimates of the corresponding long-range physics feature rather large values for the anomalous magnetic moment of the electron which are incompatible with observation. If pure QED was a correct description of Nature, low-energy observations would already rule out a UV completion of QED in universality class $\mathcal{B}$.

Our estimates of the RG flow in the enlarged QED theory space are based on the functional RG which can address both perturbative as well as nonperturbative regimes. Our truncation of theory space is complete to lowest nontrivial order in a combined operator and derivative expansion. While higher-order computations will eventually be required to check the convergence of this expansion scheme, we have performed an intrinsic consistency check by quantify-ing the contributions of derivative operators in terms of anomalous dimensions. A comparison of leading-order to next-to-leading order results shows variations on the $\mathcal{O}(10\%)$ level for nonperturbative quantities, while qualitative results remain unchanged. At the non-Gaußian fixed points, the anomalous dimensions become large enough to turn the perturbatively irrelevant Pauli term into a relevant operator, but remain sufficiently small preserve the ordering of operators according to their power-counting dimension apart from $\mathcal{O}(1)$ shifts. In summary, we consider our results as first evidence for an asymptotically safe realisation of QED.

Our study had also been motivated by a recent analysis of the Pauli coupling and its influence on the UV-running of the QED gauge coupling within effective field theory [DGM17]. We confirm the conclusion of [DGM17] that the Landau pole can be screened by the Pauli coupling. In addition, we find that the running of the Pauli coupling itself can be UV stabilised by fluctuations leading to the existence of the fixed points. We also observe that it is important to treat the mass parameter on the same level as the couplings, since one of the fixed points occurs at a finite dimensionless mass parameter, invalidating the standard assumption of asymptotic symmetry.

The resulting scenario of asymptotically safe QED also fits into the picture developed in [NR13], observing that strong ultra-local paramagnetic interactions can dominate the RG behaviour of coupling flows. We hope that our findings serve as an inspiration for searches for non-Gaußian fixed points in QED using other nonperturbative methods: Within functional methods, vertex expansions offer a powerful expansion scheme; in fact, vertex structures overlapping with the Pauli term are found to play an important role in the strong coupling region of QCD [MPS15]. New lattice searches would need to go beyond the standard bare QED lattice action and also require an explicit parameterisation of the Pauli term and a corresponding independent coupling; for an example of asymptotic safety discovered on the lattice in a scalar model, see [WKW14]. Studying the existence of these fixed points would also be an interesting target for the conformal bootstrap along the lines of [LP20].

Whether or not the mechanisms and universality classes observed in the present work on pure QED can analogously be at work in the Standard Model remains to be investigated. Because of the chiral symmetry of the Standard Model, the analogue of the Pauli term corresponds to a dimension-6 operator also involving the Higgs field. Nevertheless, if asymptotic symmetry is not present in the UV as in the models of [Gie+13; GZ15; GZ17; Gie+19a; Gie+19b], analogous mechanisms as revealed here in pure QED can be at work and thus pave novel ways towards an asymptotically safe completion of the Standard Model.

5.6. A Systematic Notation for Threshold Functions

A widely used nomenclature for the threshold functions that parameterise the decoupling of massive modes, has already been introduced in early applications of the Wetterich equation, see, e.g., [JW96]. However, the present model requires a large number of threshold functions which have not been considered so far, because of the explicit breaking of chiral symmetry and because of the momentum dependence of the Pauli coupling. We therefore suggest a more comprehensive nomenclature of threshold functions that covers all cases typically studied in the literature, as well as the new cases required by this project, and leaves room for further generalisations. We define the threshold functions used in section 5.2 as follows:

$$\begin{aligned} l_d^{([n],X^{[x_p]}_{[x_d]},Y^{[y_p]}_{[y_d]},\dots)}&(\omega_X,\omega_Y,\dots;\eta_X,\eta_Y,\dots) = \\ &(-1)^{1+x_d\,x_p+y_d\,y_p+\dots}\,\frac{k^{-2n-d+2x_p(1+x_d)+2y_p(1+y_d)+\dots}}{4v_d} \\ &\times\int\frac{\mathrm{d}^dp}{(2\pi)^d}\,(p^2)^n\,\tilde{\partial}_t\left[\left(\frac{\partial}{\partial p^2}\right)^{x_d} G_X(\omega_X)\right]^{x_p}\left[\left(\frac{\partial}{\partial p^2}\right)^{y_d} G_Y(\omega_Y)\right]^{y_p}\dots \end{aligned} \tag{5.6.1}$$

Here, parameters in brackets are optional and are understood to have standard defaults ($n = 0, x_d = 0, y_d = 0, \dots, x_p = 1, y_p = 1, \dots$). The sign conventions are such that all threshold functions are positive for finite mass parameters $\omega_{X,Y,\dots}$ and vanishing anomalous dimensions $\eta_{X,Y,\dots}$). As conventional in the literature, the modified scale derivative is understood to act on the regulator terms only, see, e.g., [BTW02; Gie12; Bra12].

Moreover, $G_X(\omega)$ denotes the inverse regularised propagator of type X, i.e

$$G_{\mathrm{B}}(\omega) = \frac{1}{P_{\mathrm{B}}+\omega k^2}\,, \qquad G_{\mathrm{F}}(\omega) = \frac{1}{P_{\mathrm{F}}+\omega k^2}\,, \qquad G_{\tilde{\mathrm{F}}}(\omega) = \frac{1+r_{\mathrm{F}}}{P_{\mathrm{F}}+\omega k^2}\,, \tag{5.6.2}$$

where

$$P_{\mathrm{B}} = p^2\left[1+r_{\mathrm{B}}\left(\frac{p^2}{k^2}\right)\right]\,, \qquad P_{\mathrm{F}} = p^2\left[1+r_{\mathrm{F}}\left(\frac{p^2}{k^2}\right)\right]^2 \tag{5.6.3}$$

and $r_{\mathrm{B}}, r_{\mathrm{F}}$ are the boson and fermion regulator shape functions respectively.

Our convention covers many widely used threshold functions as well as some that have been defined for specific studies. For instance, in comparison to the notation used in [Gie+13], we have the following correspondence

$$l_d^{(\mathrm{B}^{n_1},\mathrm{F}^{n_2})} = l_{n_1,n_2}^{(\mathrm{FB})d}\,, \qquad l_d^{(1,\mathrm{F}_1^2)} = m_2^{(\mathrm{F})d}\,, \qquad l_d^{(2,\tilde{\mathrm{F}}_1^2)} = m_4^{(\mathrm{F})d}\,, \qquad l_d^{(\mathrm{B},\tilde{\mathrm{F}})} = a_3^d\,. \tag{5.6.4}$$

Let us finally list the explicit forms of the threshold functions as they are needed for the present work in $d = 4$, employing the linear regulator [Lit00; Lit01] for r_{B} and r_{F}:

$$
\begin{aligned}
&l_4^{(1,\mathrm{B},\mathrm{F}_1})(0,m^2) = \frac{5-\eta_\psi}{5\left(1+m^2\right)^2} \qquad l_4^{(\mathrm{F}^2)}(m^2) = \frac{5-\eta_\psi}{5\left(1+m^2\right)^3} \qquad l_4^{(1,\tilde{\mathrm{F}}^2)}(m^2) = \frac{\left(5-\eta_\psi\right)\left(1-m^2\right)}{10\left(1+m^2\right)^3} \\
&l_4^{(1,\tilde{\mathrm{F}},\mathrm{F})}(0,m^2,m^2) = \frac{\left(6-\eta_\psi\right)\left(3-m^2\right)}{30\left(1+m^2\right)^3} \quad l_4^{(1,\mathrm{F}_1^2)}(0,m^2) = \frac{1}{(1+m^2)^4} \quad l_4^{(1,\mathrm{B},\mathrm{F}_1,\tilde{\mathrm{F}})}(0,m^2,m^2) = \frac{4-\eta_\psi}{4\left(1+m^2\right)^3}
\end{aligned}
\tag{5.6.5}
$$

$$
\begin{aligned}
l_4^{(\mathrm{B},\tilde{\mathrm{F}})}(0,m^2) &= \frac{1}{60\left(1+m^2\right)^2}\left[60-5\eta_\psi+5\left(4+\eta_\psi\right)m^2-8\eta_A\left(1+m^2\right)\right] \\
l_4^{(1,\mathrm{B},\tilde{\mathrm{F}}_1)}(0,m^2) &= \frac{1}{30\left(1+m^2\right)^2}\left[-2\eta_A\left(1+m^2\right)+10\left(3-m^2\right)-5\eta_\psi\left(1-m^2\right)\right] \\
l_4^{(1,\mathrm{B},\tilde{\mathrm{F}})}(0,m^2) &= \frac{1}{210\left(1+m^2\right)^2}\left[7\eta_\psi\left(-1+m^2\right)-12\eta_A\left(1+m^2\right)+42\left(3+m^2\right)\right] \\
l_4^{(2,\mathrm{B},\tilde{\mathrm{F}}_1)}(0,m^2) &= \frac{1}{70\left(1+m^2\right)^2}\left[-2\eta_A\left(1+m^2\right)+28\left(2-m^2\right)-7\eta_\psi\left(1-m^2\right)\right] \\
l_4^{(2,\tilde{\mathrm{F}}_1^2)}(0,m^2) &= \frac{1-m^2}{4\left(1+m^2\right)^4}\left[4-\eta_\psi+2m^2-\eta_\psi m^2\right] \\
l_4^{(\mathrm{B},\mathrm{F}^2)}(0,m^2) &= \frac{1}{60\left(1+m^2\right)^3}\left[-12\eta_\psi-5\eta_A\left(1+m^2\right)+30\left(3+m^2\right)\right] \\
l_4^{(\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2) &= \frac{1}{12\left(1+m^2\right)^3}\left[24-4\eta_\psi\left(1-m^2\right)-3\eta_A\left(1+m^2\right)\right] \\
l_4^{(1,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2) &= \frac{1}{210\left(1+m^2\right)^3}\left[-7\eta_\psi\left(3-m^2\right)-12\eta_A\left(m^2+1\right)+42\left(m^2+5\right)\right] \\
l_4^{(2,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2,m^2) &= \frac{1}{168\left(1+m^2\right)^3}\left[112-8\eta_\psi\left(1-m^2\right)-7\eta_A\left(1+m^2\right)\right] \\
l_4^{(2,\mathrm{B},\mathrm{F}^2)}(0,m^2,m^2) &= \frac{1}{360\left(1+m^2\right)^3}\left[-20\eta_\psi-9\eta_A\left(1+m^2\right)+90\left(3+m^2\right)\right] \\
l_4^{(1,\mathrm{B},\tilde{\mathrm{F}}_1,\mathrm{F})}(0,m^2,m^2) &= \frac{1}{60\left(1+m^2\right)^3}\left[-5\eta_\psi\left(3-2m^2\right)+20\left(4-m^2\right)-4\eta_A\left(1+m^2\right)\right] \\
l_4^{(\mathrm{B},\mathrm{F},\tilde{\mathrm{F}})}(0,m^2,m^2) &= \frac{1}{60\left(1+m^2\right)^3}\left[-5\eta_\psi\left(3-m^2\right)-8\eta_A\left(1+m^2\right)+20\left(5+m^2\right)\right] \\
l_4^{(\mathrm{B},\mathrm{F}^2)}(0,m^2) &= \frac{1}{60\left(1+m^2\right)^3}\left[-12\eta_\psi-5\eta_A\left(1+m^2\right)+30\left(m^2+3\right)\right] \\
l_4^{(1,\mathrm{B},\tilde{\mathrm{F}}^2)}(0,m^2) &= \frac{1}{60\left(1+m^2\right)^3}\left[60-6\eta_\psi\left(1-m^2\right)-5\eta_A\left(1+m^2\right)\right] \\
l_4^{(2,\mathrm{B},\mathrm{F},\tilde{\mathrm{F}}_1)}(0,m^2) &= \frac{1}{210\left(1+m^2\right)^3}\left[210-84m^2-6\eta_A\left(1+m^2\right)+7\eta_\psi\left(3m^2-4\right)\right] \\
l_4^{(1,\mathrm{B},\mathrm{F}^2)}(0,m^2) &= \frac{1}{168\left(1+m^2\right)^3}\left[-16\eta_\psi-7\eta_A\left(1+m^2\right)+56\left(3+m^2\right)\right] \\
l_4^{(1,\mathrm{B},\mathrm{F})}(0,m^2) &= \frac{1}{168\left(1+m^2\right)^2}\left[-8\eta_\psi-7\eta_A\left(1+m^2\right)+56\left(m^2+2\right)\right] \\
l_4^{(\mathrm{B},F)}(0,m^2) &= \frac{1}{60\left(1+m^2\right)^2}\left[-6\eta_\psi-5\eta_A\left(1+m^2\right)+30\left(2+m^2\right)\right]
\end{aligned}
\tag{5.6.6}
$$

6. Gaußian Integrability of the Regularised ϕ^4 Theory

As hinted at in section 2.1, the free, massive QFT of a scalar field in d spacetime dimensions can be modelled by a Gaußian measure on the space $\mathcal{S}(\mathbb{R}^d)^*_\beta$ of tempered distributions. With regard to possible interactions it is necessary to regularise the theory and a specific method was presented in section 2.2. Following the corresponding steps, it is clear that a necessary condition for that regularisation scheme to work, is the finiteness of the integral

$$\int_{\mathcal{S}(\mathbb{R}^d)} \exp\left[-S_n^{\text{int}}(\phi)\right] \mathrm{d}\nu_n(\phi) \ , \tag{6.0.1}$$

where ν_n denotes the regularised free model and S_n^{int} the interacting part of the classical action – both objects considered at a finite regularisation index $n \in \mathbb{N}$. It is known (at least perturbatively) that in four dimensions there will be a mass counterterm with negative sign that diverges as $n \to \infty$ i.e. in the limit of vanishing regularisation [PS95]. In particular, this means that for large n we cannot absorb such a counterterm into ν_n because Gaußian measures à la eq. (2.1.5) with negative values for m^2 do not exist. Consequently, the corresponding counterterm has to be considered as part of S_n^{int}. This leads to integrals of the form

$$\int_{\mathcal{S}(\mathbb{R}^d)} \exp\left[-\lambda_n \int_{\mathbb{R}^d} \phi(x)^4 \,\mathrm{d}^d x + \Delta m_n^2 \int_{\mathbb{R}^d} \phi(x)^2 \,\mathrm{d}^d x\right] \mathrm{d}\nu_n(\phi) \ , \tag{6.0.2}$$

which have to be finite in order for the regularisation to work. If these integrals were finite for all possible values for $\lambda_n > 0$ and $\Delta m^2 \in \mathbb{R}$, we would also immediately have the slightly stronger $L^q(\nu_n)$ integrability which was demanded in axiom 3.1.1. This is non-trivial because the $L^2(\mathbb{R}^d)$ and $L^4(\mathbb{R}^d)$ norms are inequivalent on $\mathcal{S}(\mathbb{R}^d)$. One can also see this problem as arising from the insistence on not using a finite volume regularisation because in such a setting the L^2 norm would indeed be bounded by a multiple of the L^4 norm.

A mathematically abstract and generalised version of this question was treated in [HJZ22] and - for our purposes - boils down to the following theorem. The given proof is an adaptation of the one presented in [HJZ22].

Theorem 6.0.1. Let μ be a Radon Gaußian probability measure on a nuclear space X and p and

q two continuous seminorms on X with the property that the natural map $\iota^p_{p+q} : X_{p+q} \to X_p$ is injective. Then

$$\int_X \exp\left[-p(x)^4 + q(x)^2\right] \mathrm{d}\mu(x) < \infty\,. \tag{6.0.3}$$

Before proving this theorem, we need one short lemma.

Lemma 6.0.2. Let μ be a centred Radon Gaußian probability measure on a locally space X and (e_n) an orthonormal basis of the corresponding Cameron-Martin space $H(\mu)$. Then, the linear operators $P_n : X \to X$ given by

$$P_n : x \mapsto \sum_{m=1}^{n} \left(R_\mu^{-1} e_m\right)(x)\, e_m \tag{6.0.4}$$

for all $n \in \{1, \ldots, \dim H(\mu)\}$ are $(\mathcal{B}(X)_\mu, \mathcal{B}(X))$-measurable. The definition of R_μ is given in definition A.4.11. If $\dim H(\mu) < \infty$, then $P_{\dim H(\mu)} = \mathrm{id}_X$ μ-almost everywhere and otherwise the pushforward measures $\mu \circ P_n^{-1}$ converge weakly to μ.

Proof. The infinite-dimensional case is treated in [Bog98, Proposition 3.8.12] and the finite-dimensional case is clear from [Bog98, Theorem 3.6.1]. □

Proof of theorem 6.0.1. By the nuclearity of X there is a continuous Hilbert seminorm r on X with $p + q \leq r$ such that the natural map $\iota^{p+q}_r : X_r \to X_{p+q}$ is nuclear. At the same time there is another Hilbert seminorm $s \geq r$ such that the natural map $\iota^r_s : X_s \to X_r$ is nuclear. Letting $H(\mu)$ denote the Cameron-Martin space of μ with its Hilbert space topology, it is also known that the inclusion map $\iota^X_\mu : H(\mu) \to X$ and the natural map $\iota^s_X : X \to X_s$ are continuous. Consequently, the natural map $\pi^{p+q}_\mu : H(\mu) \to H(\mu)_{p+q}$ factors through these maps with

$$\pi^{p+q}_\mu = \iota^{p+q}_r \circ \iota^r_s \circ \iota^s_X \circ \iota^X_\mu = \iota^{p+q}_r \circ \left(\iota^r_s \circ \iota^s_X \circ \iota^X_\mu\right)\,, \tag{6.0.5}$$

where $k = \iota^{p+q}_r$ is a compact linear map and $j = \iota^r_s \circ \iota^s_X \circ \iota^X_\mu$ is a nuclear map between Hilbert spaces. It follows that $j^* \circ j$ is a (non-negative) Hilbert-Schmidt operator such that the eigenvalues (λ_n^2) of $j^* \circ j$ are summable.

Letting $J \subseteq X_r$ denote the closed range of $j : H(\mu) \to X_r$, J is a Hilbert subspace of the Hilbert space X_r. Thus, there is an orthonormal basis (e_n) of $H(\mu)$ and an orthonormal set $(f_n) \subset J$ such that

$$j = \sum_{n=1}^{\dim H(\mu)} \lambda_n \langle e_n, \cdot \rangle_{H(\mu)} f_n\,. \tag{6.0.6}$$

By rescaling j and k on finite-dimensional subspaces, we can without loss of generality assume that $\lambda_n^2 < 1/2$.

The key observation is now that

$$C = \sup_{x \in J} F(x) := \sup_{x \in J} \left[-p(kx)^4 + q(kx)^2 - \frac{1}{2} \|x\|_J^2 \right] < \infty\,. \tag{6.0.7}$$

To prove this, assume the contrary, i.e. suppose that $\lim_{n\to\infty} F(x_n) = \infty$ for some sequence (x_n) in J. We may also clearly assume that $q(kx_n) > 0$ for all $n \in \mathbb{N}$. It is then obvious that the sequence $\|x_n\|_J / q(kx_n)$ is bounded such that there is a subsequence (y_n) of (x_n) and some $\bar{y} \in J$ such that $y_n / q(ky_n)$ converges weakly to $\bar{y}$. At the same time

$$0 = \lim_{n\to\infty} \frac{p(ky_n)}{q(ky_n)} = \lim_{n\to\infty} p\left(k \frac{y_n}{q(ky_n)}\right) = p(k\bar{y})\ , \tag{6.0.8}$$

since k maps weakly convergent sequences to convergent sequences. By assumption, ι^p_{p+q} is injective such that $p(k\bar{y}) = 0$ implies $q(k\bar{y}) = 0$. But then

$$1 = \lim_{n\to\infty} \frac{q(ky_n)}{q(ky_n)} = \lim_{n\to\infty} q\left(k \frac{y_n}{q(ky_n)}\right) \neq q(k\bar{y}) = 0 \tag{6.0.9}$$

is a contradiction.

Define P_n for the basis (e_n) as in lemma 6.0.2 as well as the linear operators $\Lambda_n : \mathbb{R}^n \to H(\mu)$ with $\Lambda_n : x \mapsto \sum_{m=1}^n x_m e_m$ for all $n \in \{1, \ldots, \dim H(\mu)\}$. It then follows from lemma A.4.15 (see [HJZ22] for details) that

$$\begin{aligned}
&\int_X \exp\left[-p(x)^4 + q(x)^2\right] \mathrm{d}\mu(x) \\
&\quad\leq \liminf_{n\to\dim H(\mu)} \int_X \exp\left[-p(x)^4 + q(x)^2\right] \mathrm{d}\left(\mu \circ P_n^{-1}\right)(x) \\
&\quad= \liminf_{n\to\dim H(\mu)} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left[-p(kj\Lambda_n x)^4 + q(kj\Lambda_n x)^2 - \frac{1}{2}\sum_{m=1}^n x_m^2\right] \mathrm{d}^n x \\
&\quad\leq e^C \liminf_{n\to\dim H(\mu)} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left[-\frac{1}{2}\sum_{m=1}^n \left(1 - \lambda_m^2\right) x_m^2\right] \mathrm{d}^n x \\
&\quad= e^C \prod_{m=1}^{\dim H(\mu)} \frac{1}{\sqrt{1-\lambda_m^2}}\,.
\end{aligned} \tag{6.0.10}$$

For $\dim H(\mu) < \infty$ this is clearly finite. Otherwise, since $\lambda_m^2 < 1/2$ and $-\ln\sqrt{1-t} < t$ for all $t \in [0, 1/2]$,

$$\prod_{m=1}^{\dim H(\mu)} \frac{1}{\sqrt{1-\lambda_m^2}} = \exp\left[-\sum_{m=1}^{\dim H(\mu)} \ln\sqrt{1-\lambda_m^2}\right] \leq \exp\left[\sum_{m=1}^{\dim H(\mu)} \lambda_m^2\right] < \infty\,. \tag{6.0.11}$$

□

With regard to the application to ϕ^4 theory, it is appropriate to discuss the assumptions of theorem 6.0.1: $\mathcal{S}(\mathbb{R}^d)$ is indeed a nuclear space and, in particular, a Radon space (see remark 3.4.1), such that every considered Borel Gaußian measure on it is a Radon measure. p may be identified with (a multiple of) the L^4 norm on $\mathcal{S}(\mathbb{R}^d)$ and q correspondingly with (a multiple of) the L^2 norm which are both continuous on $\mathcal{S}(\mathbb{R}^d)$.

Lemma 6.0.3. Let $p = \|\cdot\|_{L^4(\mathbb{R}^d)}$ and $q = \|\cdot\|_{L^2(\mathbb{R}^d)}$ on $\mathcal{S}(\mathbb{R}^d)$. Then the natural map $\iota^p_{p+q} : \mathcal{S}(\mathbb{R}^d)_{p+q} \to \mathcal{S}(\mathbb{R}^d)_p$ is injective.

Proof. Let $f \in \ker \iota^p_{p+q}$. Then every representing sequence in $\mathcal{S}(\mathbb{R}^d)$ is Cauchy in q and a null sequence in p. Hence, every corresponding subsequence has a subsequence (f_n) that converges pointwise almost everywhere to the zero function on $\mathbb{R}^d$. Since (f_n) is Cauchy in q, the limit in $L^2(\mathbb{R}^d)$ must then also be the zero function such that (f_n) is indeed a null sequence in q. Hence, $f = 0$. □

7. Summary and Outlook

Regularisation and renormalisation are major constituents in the understanding of Quantum Field Theories and their phenomenology. They appear naturally from the very beginning in the definition of the path integral. We have introduced a regularisation scheme that is particularly suitable to the application of the Wetterich equation for the quantum effective action. The regularisation scheme ensures that the effective average action is well-defined on the Cameron-Martin space of the underlying free field theory. Together with the boundary conditions given by the classical limit along with a regularised propagator, theorem 3.1.18 provides a mathematically rigorous understanding of the differential equation governing the flow from the classical action to the quantum effective action for scalar Quantum Field Theories.

In situations where a sufficient amount of control of the solutions of this equation is exhibited, theorem 3.4.13 provides sufficient conditions that are also necessary in the sense of the discussion presented in the beginning of section 3.4 for the existence of a non-regularised limit of a full Quantum Field Theory. In particular, proving sufficiently strong bounds on the quantum effective action that are uniform in the regularisation parameter would solve the renormalisation problem.

In view of this tremendously difficult problem we have considered the asymptotic safety hypothesis that postulates the validity of the Wetterich equation in the limit of vanishing regularisation by ascribing particular asymptotics to dimensionful coupling constants. This hypothesis was tested on a slightly extended model of Quantum Electrodynamics that had already been suggested in [Wei95]. In particular, we truncated the operator expansion of the right-hand side of the Wetterich equation and found non-trivial fixed points of the dimensionless coupling constants that define the theory. One of these fixed points passed our consistency checks and does in fact admit a RG trajectory that is compatible with basic predictions from Quantum Electrodynamics and lies in the perturbative regime up to the electroweak scale.

As final step, we verified that ϕ^4 theory can indeed be regularised using the presented scheme which effectively removes the necessity of introducing a finite volume. Moreover, because the scheme works without an underlying discretisation, the full rotational invariance can be maintained and is not reduced to a discrete symmetry group.

The results presented in this work suggest a diverse set of directions for future research.

- *Further studies on the possible asymptotic safety of QED:*

The results of chapter 5 present the astonishing possibility of a UV completion of QED. Analysing this feature in a larger section of theory space including (at least) four-fermion operators would provide further valuable conclusions on this matter. From an effective field theory perspective, the results for $d \leq 3$ dimensions and larger N_f could, theoretically, also show new features in e.g. the physics of thin sheets or wires. Research in these directions is currently being undertaken by Holger Gies and Kevin Tam.

- *Deriving the Wetterich Equation for Quantum Mechanics:*
  In Quantum Mechanics a free particle propagating through time is commonly modelled by the Wiener measure on the space $C([t_i, t_f], \mathbb{R}^{d-1})$ of trajectories in space between an initial time t_i and a final time t_f. To the author it seems very likely that similar techniques can be applied to such models proving the existence of a corresponding Wetterich equation. A combination with field theory is also conceivable such as e.g. the Nelson model [Spo04].

- *Deriving the Wetterich Equation for fermionic fields:*
  Fermionic path integrals are notoriously difficult to define and for an introduction, we refer to [Fel+02]. In particular, the infinite dimensionality of the underlying Grassmann algebra poses some trouble such that often a lattice regularisation is used that in every step only needs to handle a finite-dimensional Grassmann algebra (see e.g. [Sal07]). One goal of the approach shown in this work was to preserve the smooth features of spacetime making the infinite-dimensional nature unavoidable. Hence, one would need a Euclidean field theory of fermions for which as was shown in [FO74] there does not appear to exist a natural construction. It might be hoped however, that a strong regularisation scheme as the one presented for scalar fields can solve this problem. Some obstacles have already been cleared like the issue of fermion doubling in a continuum setting (see e.g. [Meh90; vW96; Wet11]).

- *Deriving a Non-regularised Version of the Wetterich Equation:* As we discussed in section 3.5, one might hope that the limit of vanishing regularisation can be taken in a way that retains the meaning of the Wetterich equation. Because theorem 3.4.13 already provides a notion of convergence of quantum effective actions, the question under what circumstances the Wetterich equation survives the corresponding limit, appears to be well-posed. The major difficulty would lie in the existence of a useful domain for the limit object $\bar{\Gamma}_k^\infty$ as well as establishing sufficiently uniform behaviour of the corresponding derivatives. It would be instructive to see whether the existence of a non-regularised Wetterich equation could be interpreted as a physical property of a Quantum Field Theory.

- *Numerical analysis of the Wetterich equation along with quantifiable effects of the regular-*

isation:
Theorem 3.1.18 gives the differential equation in terms of a trace-class operator which is particularly suited for numerical computations. Hence, a simple strategy could be to choose a finite set of orthonormal vectors in the Cameron-Martin space and model the effective average action as a function on their finite-dimensional span. Some care must be taken because it is a priori not guaranteed that the trace-class operators at different scales diagonalise over the same orthonormal set.

- *Mathematical analysis of ϕ^4 theory:*
  It was recently shown that ϕ^4 theory in four spacetime dimensions is a trivial theory when viewed from a lattice regularisation perspective [AD21]. Being able to prove (or disprove) the same from rigorous bounds extracted from the Wetterich equation would pose a significant step forward in the understanding of Quantum Field Theories.

- *Mathematical abstraction of the Wetterich equation:*
  From a mathematical perspective the presented setup could immediately be extended to any Fréchet nuclear space other than $\mathcal{S}(\mathbb{R}^d)$. Furthermore, while some proofs relied on the metrisability of $\mathcal{S}(\mathbb{R}^d)$ (e.g. lemma 3.4.5) and others on the nuclearity (e.g. lemma 3.4.10) it seems likely that many proofs can be generalised to less restrictive settings.

Appendices

A. Mathematical Notations, Conventions and Theorems

Most of the mathematical theorems and notations used in this work are well-known and standardised. For completeness, the author lists the (in his view) most important notions in this appendix.

A.1. Fourier Transform

Definition A.1.1. We define the Fourier transform $\hat{f}$ of a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\hat{f}\,(p) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left[-ipx\right] f\,(x)\,\mathrm{d}x\,, \tag{A.1.1}$$

whenever the integral converges. The corresponding unitary operator on $L^2(\mathbb{R}^d)_{\mathbb{C}}$ is denoted by $\mathcal{F}$ and the function $x \to \exp[-ipx]/(2\pi)^{d/2}$ interpreted as a tempered distribution is denoted by $\mathcal{F}_p$.

A.2. Locally Convex Spaces

Given a real vector space V we let $V_{\mathbb{C}}$ denote its complexification.

Definition A.2.1. A subset $A \subseteq V$ of a real vector space is **convex** if for all $x, y \in A$ and all $t \in (0,1)$, $tx + (1-t)y \in A$. It is **balanced** if $x \in A$ implies $-x \in A$. A balanced and convex set is called **absolutely convex**.

Definition A.2.2. A **locally convex space** X is defined as a real vector space with the topology induced by a family $(p_\alpha)_{\alpha\in I}$ of seminorms where we shall make the additional assumption that the topology is Hausdorff. In short, a net $(x_\beta)_{\beta\in J}$ in X converges to some $y \in X$ if and only if

$$\lim_{\beta\in J} p_\alpha(x_\beta - y) = 0 \tag{A.2.1}$$

for all $\alpha \in I$ which in particular implies the continuity of all p_α themselves. The Hausdorff criterion translates to the implication that $z = 0$ whenever $p_\alpha(z) = 0$ for all $\alpha \in I$.

Remark A.2.3. Note that the family of seminorms generating the topology of a locally convex space is never unique. One can e.g. replace any seminorms p and q by their sum or pointwise maximum or positive multiples thereof.

Definition A.2.4. A net $(x_\beta)_{\beta\in J}$ in a locally convex space X with generating seminorms $(p_\alpha)_{\alpha\in I}$ is a **Cauchy net** if for every $\alpha \in I$ and $\epsilon > 0$ there is a $\beta \in J$ such that for all $\gamma, \delta \in J_{\geq\beta}$

$$p_\alpha(x_\gamma - x_\delta) < \epsilon\,. \tag{A.2.2}$$

If for such a net $J = \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence**.

Definition A.2.5. A locally convex space X is **complete** if every Cauchy net in X also converges in X. If every Cauchy sequence in X converges in X, it is called **sequentially complete**.

Definition A.2.6. A locally convex space X is a **Fréchet space** if it is complete and its topology can be generated by a countable set of seminorms. A Fréchet space is also metrisable, that is, its topology can be generated by a translation invariant metric.

Definition A.2.7. Let (X, p) be a seminormed space and consider the set $\mathcal{C}(X)$ of all Cauchy sequences in X and let $(x_n) \sim (y_n)$ whenever $\lim_{n\to\infty} p(x_n - y_n) = 0$. Then $\mathcal{C}(X)/\sim$ is a vector space with the obvious operations, $\bar{p}([(x_n)]_\sim) = \lim_{n\to\infty} p(x_n)$ is a well-defined norm on $\mathcal{C}(X)/\sim$ and we shall denote the resulting normed space by X_p. X_p is complete and is called the **completion** of (X, p). There is also a **natural map** $\pi^p : X \to X_p, x \mapsto [(x, x, \dots)_\sim]$ which is linear, continuous and has dense range.

Remark A.2.8. Let (X, p) be a seminormed space and Y a complete locally convex space. Then, by the Hahn-Banach theorem, every continuous linear operator $L : X \to Y$ extends uniquely to the completion X_p in the sense that there is a unique continuous linear operator $\bar{L} : X_p \to Y$ such that $L = \bar{L} \circ \pi^p$. Consequently, for every seminorm $q \geq p$ on X there is a unique continuous, linear **natural map** $\pi^p_q : X_q \to X_p$ with $\pi^p = \pi^p_q \circ \pi^q$.

Definition A.2.9. A linear operator $L : A \to B$ between Banach spaces A and B is continuous if and only if it is **bounded**, i.e. if the range of the closed unit ball U in A under L is bounded in B. It is called **compact** if $L(U)$ is precompact in B which also implies its continuity.

Corollary A.2.10. Let $L : A \to B$ be a compact linear operator between Banach spaces A and B. Then the sequence $(La_n)_{n\in\mathbb{N}}$ is convergent in B for every weakly convergent sequence $(a_n)_{n\in\mathbb{N}}$ in A.

Definition A.2.11. A subset $B \subseteq X$ of a locally convex space X with a generating family $(p_\alpha)_{\alpha\in I}$ of seminorms is **bounded** if $\sup_{x\in B} p_\alpha(x) < \infty$ for all $\alpha \in I$.

Definition A.2.12. The **topological dual space** X^* of a locally convex space X is defined as the real vector space of all **continuous linear functionals** $X \to \mathbb{R}$ on X. Moreover, X^* **separates points** of X, i.e. for any two distinct points $x, y \in X$ there exists a $\phi \in X^*$ with $\phi(x) \neq \phi(y)$.

Definition A.2.13. The topological dual space X^* of a locally convex space X can also be equipped with a locally convex topology. Consider the seminorms

$$p_B(\phi) = \sup_{x \in B} |\phi(x)| \tag{A.2.3}$$

for all bounded subsets B of X. Then X together with these seminorms p_B define the **strong topological dual space** X^*_β of X.

Remark A.2.14. The strong topological dual space B^* of a Banach space B is again a Banach space in the sense that B^* is complete and p_U as defined in eq. (A.2.3), where $U \subset B$ is the open unit ball in B, generates the topology of B^*. Hence, we shall always assume that B^* is equipped with the norm p_U.

Likewise, the strong topological dual space H^* of a Hilbert space H is again a Hilbert space with the corresponding norm. Furthermore, the linear map

$$\iota : H \to H^*, v \mapsto \langle v, \cdot \rangle \tag{A.2.4}$$

is an **isometry** in the sense that

$$p_U(\iota v) = \|\iota v\|_{H^*} = \|v\|_H \tag{A.2.5}$$

for all $v \in H$. Consequently, ι is continuous and injective and by Riesz's theorem also surjective with isometric and continuous inverse. Hence, ι gives a natural identification of H^* and H such that we shall implicitly assume that $H \simeq H^*$ from here on.

Definition A.2.15. For every locally convex space X there is a canonical linear **evaluation map** into its strong bidual given by $j : X \to (X^*_\beta)^*_\beta, x \mapsto j_x$ with $j_x : X^*_\beta \to \mathbb{R}, \phi \mapsto \phi(x)$. If j is a homeomorphism of topological vector spaces it offers a natural identification of X with $(X^*_\beta)^*_\beta$ [SW99]. In such cases X is called **reflexive** and we shall implicitly identify it with its strong bidual.

Theorem A.2.16 (Bourbaki-Alaoglu theorem [NB10, Theorem 15.2.4])**.** Let X be a reflexive space. Then every bounded set in X is weakly precompact.

Theorem A.2.17 (Banach-Alaoglu theorem [Ban32, p. 123, Théorème 3])**.** Let X be a separable Banach space. Then every bounded sequence in X^* has a weakly convergent subsequence.

Theorem A.2.18 ([Meg12, Theorem 1.12.11])**.** Let X be a Banach space. If X^* is separable, so is X.

Definition A.2.19. For a continuous linear map $L : X \to Y$ where X and Y are locally convex spaces, the **adjoint (or transpose)** $L^* : Y^*_\beta \to X^*_\beta, \psi \mapsto \psi \circ L$ is also continuous and linear [NB10, Theorem 8.11.3]. If either X or Y is a Hilbert space we shall implicitly identify the dual spaces in accordance with remark A.2.14.

Definition A.2.20. A bounded linear operator $L : H \to H$ on a separable Hilbert space H is called **Hilbert-Schmidt** if there is some orthonormal basis (e_n) of H such that $\sum_n \|Le_n\|^2 < \infty$. Moreover, it is called **nuclear** if $\sum_n |\langle e_n, Le_n\rangle| < \infty$ for all orthonormal bases (e_n) of H. In either case it is well-known that the given sums are then independent of the choice of orthonormal basis and that L is compact. Generalising the above, we call a bounded linear operator $L : H \to J$ between two separable Hilbert spaces **Hilbert-Schmidt** if $L^* \circ L$ is nuclear.

Definition A.2.21. A locally convex space X is **nuclear** if its topology can be induced by a family of Hilbert seminorms and for every continuous Hilbert seminorm p on X there is a continuous Hilbert seminorm $q \geq p$ on X such that the natural map π^p_q is Hilbert-Schmidt.

Theorem A.2.22 ([SW99, p. 101, Corollary])**.** A nuclear space X satisfies the **Heine-Borel property**: The compact sets are precisely those that are closed and bounded.

Definition A.2.23. For $d \in \mathbb{N}$ and open subsets $U \subseteq \mathbb{R}^d$ we let $\boldsymbol{C^\infty(U)}$ denote the real vector space of infinitely often continuously differentiable real-valued functions on U. Furthermore, for $\alpha \in \mathbb{N}_0^d$ and $f \in C^\infty(U)$ we define

$$\partial_\alpha f(x) = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f\right)(x) \qquad \text{and} \qquad x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} \tag{A.2.6}$$

for all $x \in U$.

Definition A.2.24. For $d \in \mathbb{N}$ we define the real vector space of **Schwartz functions** as

$$\mathcal{S}\left(\mathbb{R}^d\right) = \left\{ f \in C^\infty\left(\mathbb{R}^d\right) \mid \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_\beta f(x)| < \infty \right\}. \tag{A.2.7}$$

The seminorms $p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_\beta f(x)|$ define a locally convex topology on $\mathcal{S}(\mathbb{R}^d)$ turning it into a reflexive Fréchet nuclear space [SW99, p. 107, Example 5]. Its strong dual space $\mathcal{S}(\mathbb{R}^d)^*_\beta$ of **tempered distributions** is also a reflexive nuclear space [SW99, p. 173, Example 1].

A.3. Convex Functions

We let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with the usual topology, i.e. neighbourhood bases of $-\infty$ and ∞ are given by $[-\infty, a)$ and $(a, \infty]$ for all $a \in \mathbb{R}$ respectively. A convex function is **proper**, if it does not attain the value $-\infty$ and is not equal to the constant function ∞.

Definition A.3.1. Let X be a Hausdorff, locally convex topological vector space and $f : X \to \bar{\mathbb{R}}$ a proper convex and lower semicontinuous function. Then, the **convex conjugate (Legendre-Fenchel transform)** $f^c : X^* \to \bar{\mathbb{R}}$ of f is defined as

$$\phi \mapsto \sup_{T \in X} \left[\phi(T) - f(T)\right] \tag{A.3.1}$$

for all $\phi \in X^*$. If we equip X^* with a topology τ at least as fine as the weak-$*$ topology. it is also proper convex and lower semicontinuous. We may then also define $(f^c)^c : (X^*, \tau)^* \to \bar{\mathbb{R}}$ and by the well-known **Fenchel-Moreau theorem** $(f^c)^c|_X = f$ [Zal02].

Theorem A.3.2 ([Zal02, Theorem 2.2.9])**.** Let $f : X \to \mathbb{R}$ be a convex function on a Hausdorff, locally convex space X. If f is bounded from above on some open subset of X, then f is continuous.

Theorem A.3.3 ([Zal02, Theorem 2.2.20])**.** Let $f : X \to \mathbb{R}$ be a convex and lower semicontinuous function on X where X is either a Banach space or a reflexive space. Then f is continuous.

Lemma A.3.4. Let X be a Fréchet space. Furthermore, let $f_n : X \to \bar{\mathbb{R}}$ be a sequence of convex and lower semicontinuous functions converging pointwise to a function $f : X \to \mathbb{R}$. Then f is continuous.

Proof. By the pointwise convergence, f is clearly convex. Hence, by theorem A.3.2 it suffices to show that Z is bounded from above on some open subset of X. Let

$$A_{K,N} = \bigcap_{n \in \mathbb{N}_{\geq N}} f_n^{-1}\left((-\infty, K]\right) \tag{A.3.2}$$

for $K, N \in \mathbb{N}$. Then all $A_{K,N}$ are closed by the lower semicontinuity of f_n. Furthermore, $\cup_{K,N \in \mathbb{N}} A_{K,N} = X$ because $\lim_{n\to\infty} f_n(x) = f(x) < \infty$ for all $x \in X$. By the Baire category theorem, some $A_{K,N}$ contains an open set, i.e. there exists $N, K \in \mathbb{N}$, $x \in X$ and an open neighbourhood $U \subseteq X$ of zero such that

$$\sup_{y \in U} f_n(x+y) \leq K \tag{A.3.3}$$

for all $n \in \mathbb{N}_{\geq N}$. Thus f is bounded from above on $x + U$. □

A.4. Measure Theory

Most definitions in this appendix are taken or adapted from [Bog98].

Definition A.4.1. A **measure** μ on a σ-algebra $\mathcal{A}$ on a set X is defined as a non-negative function $\mu : \mathcal{A} \to [0, \infty]$ with

- $\mu(\{\}) = 0$,
- $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for every sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{A}$.

If $\mu(X) < \infty$ we say that μ is **finite**. If there is a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n = X$, μ is called **σ-finite**. The triple $(X, \mathcal{A}, \mu)$ is referred to as a **measure space**.

Definition A.4.2. Given a topological space X, we let $\mathcal{B}(X)$ denote its **Borel** σ-algebra, i.e. the smallest σ-algebra containing all open subsets of X. A member of $\mathcal{B}(X)$ is called a **Borel set**. A measure μ on $\mathcal{B}(X)$ is called a **Borel measure on** X.

Given a locally convex space X, we define the **cylindrical σ-algebra** $\mathcal{E}(X) \subseteq \mathcal{B}(X)$ to be the smallest σ-algebra with respect to which every function in X^* is measurable.

Definition A.4.3. Let μ be a Borel measure on a topological space X. Then μ is said to have **full topological support** if every open set $U \subseteq X$ has nonzero measure.

Definition A.4.4. For any measure μ on $\mathcal{E}(X)$ where X is a locally convex space, we define its **characteristic function** as

$$\hat{\mu}(\phi) = \int_X \exp\left[i\phi(x)\right] \mathrm{d}\mu(x) \tag{A.4.1}$$

for all $\phi \in X^*$.

Definition A.4.5. For any measure μ on $\mathcal{E}(X)$ where X is a locally convex space, we define its **moment-generating function** $Z : X^* \to \bar{\mathbb{R}}$ as

$$Z(\phi) = \int_X \exp\left[\phi(x)\right] \mathrm{d}\mu(x) \tag{A.4.2}$$

for all $\phi \in X^*$.

The following lemma is immediate from Hölder's inequality and Fatou's lemma.

Lemma A.4.6. The moment-generating function of a finite measure is logarithmically convex, proper convex and lower semicontinuous whenever X^* is equipped with a topology at least as fine as the weak-$*$ topology.

Definition A.4.7. Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X and $f : X \to Y$ a function into another set Y equipped with a σ-algebra $\mathcal{A}'$. If $f^{-1}(A') \in \mathcal{A}$ whenever $A' \in \mathcal{A}'$, then $f_*\mu := \mu \circ f^{-1}$ is a measure on $\mathcal{A}'$ called the **pushforward measure of μ under f**.

Definition A.4.8. A measure μ on $\mathcal{E}(X)$ where X is a locally convex space is a **centred Gaußian measure** if the pushforward measures $\mu \circ \phi^{-1}$ are centred Gaußian Borel measures on $\mathbb{R}$ for every $\phi \in X^*$. A Borel measure μ on X is a centred Gaußian measure if its restriction to $\mathcal{E}(X)$ is.

Definition A.4.9. Let X be a topological space. A finite Borel measure μ is a **Radon measure** if, for every Borel set $B \subseteq X$ and every $\epsilon > 0$, there exists a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \epsilon$.

Lemma A.4.10 ([Bog98, Appendix 3])**.** A Radon measure on a locally convex space is uniquely determined by its characteristic function.

Definition A.4.11. Let μ be a Radon Gaußian measure on a locally convex space X. Note that $X^* \subset L^2(\mu)$ by definition and denote by X^*_μ the closure of X^* in $L^2(\mu)$. Given $f \in X^*_\mu$, there exists a unique $R_\mu f \in X$ such that [Bog98, Theorem 3.2.3]

$$\phi\left(R_\mu f\right) = \int_X \phi f \mathrm{d}\mu \qquad \text{for all } \phi \in X^* . \tag{A.4.3}$$

We now define the **Cameron-Martin space** of μ as the range $H(\mu) = R_\mu(X^*_\mu)$, which is turned into a separable Hilbert space by the inner product induced by $L^2(\mu)$ [Bog98, Theorem 3.2.7]. Then, $R_\mu : X^*_\mu \to H(\mu)$ is a Hilbert space isomorphism.

Theorem A.4.12 (Cameron-Martin [Bog98, Corollary 2.4.3, Remark 3.1.8])**.** Let μ be a Radon Gaußian measure on a locally convex space X and $h \in H(\mu)$ an element of its Cameron-Martin space. Then the pushforward measure $\mu_h = \mu \circ \tau_h^{-1}$ with $\tau_h : X \to X, x \mapsto x - h$ is equivalent to μ with the corresponding Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mu_h}{\mathrm{d}\mu}(x) = \exp\left[\left(R_\mu^{-1} h\right)(x) - \|h\|^2_{H(\mu)}\right] \tag{A.4.4}$$

for all $x \in X$.

Definition A.4.13. A sequence $(\omega_n)_{n\in\mathbb{N}}$ of Radon measures on a topological space X **converges weakly** to another Radon measure ω if

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}\omega_n = \int_X f \,\mathrm{d}\omega \tag{A.4.5}$$

for all bounded continuous functions $f : X \to \mathbb{R}$.

Theorem A.4.14 (Portmanteau theorem [Bog98, Theorem 3.8.2])**.** A sequence $(\mu_n)_{n\in\mathbb{N}}$ of Radon probability measures on a locally convex space X converges weakly to a Radon probability measure μ on X precisely when either (and then both) of the following conditions is satisfied:

- $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for every open set $U \subseteq X$,
- $\liminf_{n\to\infty} \mu_n(C) \leq \mu(C)$ for every closed set $C \subseteq X$.

Lemma A.4.15. Let X be a locally convex space, $(\mu_n)_{n\in\mathbb{N}}$ a sequence of Borel probability measures on X weakly converging to a Radon measure μ on X and $f : X \to \mathbb{R}$ a lower semicontinuous function that is bounded from below. Then

$$\int_X f \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f \mathrm{d}\mu_n \,. \tag{A.4.6}$$

Proof. From [Bog07, Corollary 8.2.5] this is true if f is bounded. For unbounded f, set $f_m = \max\{f, m\}$ which is also lower semicontinuous. Then,

$$\int_X f \mathrm{d}\mu = \sup_{m\in\mathbb{N}} \int_X f_m \mathrm{d}\mu \leq \sup_{m\in\mathbb{N}} \liminf_{n\to\infty} \int_X f_m \mathrm{d}\mu_n \leq \liminf_{n\to\infty} \int_X f \mathrm{d}\mu_n \,. \tag{A.4.7}$$

□

Definition A.4.16. A sequence $(\mu_n)_{n\in\mathbb{N}}$ of finite Borel measures on a topological space X is **uniformly tight** if for any $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that $\mu_n(X \setminus K) < \epsilon$ for all $n \in \mathbb{N}$.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

Bei der Auswahl und Auswertung des Materials sowie beim Verfassen der Publikationen [GZ20; HJZ22] haben mir die nachstehend aufgeführten Personen unentgeltlich geholfen:

1. Prof. Dr. Holger Gies
2. Dr. Benjamin Hinrichs
3. Daan Willem Janssen

Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

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Ort, Datum Jobst Ziebell