# NUMERICAL RADIUS INEQUALITIES FOR HILBERT $C^*$-MODULES

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*Abstract.* We present a new method for studying the numerical radius of bounded operators on Hilbert $C^*$-modules. Our method enables us to obtain some new results and generalize some known theorems for bounded operators on Hilbert spaces to bounded adjointable operators on Hilbert $C^*$-module spaces.

*Keywords*: numerical radius; inner product space; $C^*$-algebra

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## 1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and denote by $B(H)$ the set of all bounded linear operators on $H$. The numerical range and numerical radius of an element $T \in B(H)$ are defined by $W(T) = \{\langle Tx, x\rangle\colon \|x\| = 1\}$ and $\omega(T) = \{|\lambda|\colon \lambda \in W(T)\}$, respectively. These concepts turn out to be useful in some situations (see, e.g., [3], [6], [7], [8]).

By a Hilbert $C^*$-module, we mean a linear space with an inner product which takes values in a $C^*$-algebra. This notion has appeared first in a work of Kaplansky (see [12]), who developed the theory for commutative unital algebras. The theory was extended to general $C^*$-algebras by Paschke (see [21]) and Rieffel (see [22]). We refer the reader to [15] for more information.

Some mathematicians studied basic properties of numerical range and numerical radius for bounded adjointable operators on Hilbert $C^*$-modules, see [10], [19], [17].

Although it is possible to prove some inequalities in Hilbert $C^*$-module spaces using standard methods, due to the different structure of Hilbert $C^*$-modules, it seems that different definitions of some concepts, which are natural extensions of some standard definitions, are needed for studying some inequalities in Hilbert $C^*$-modules.

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In this paper, we provide new definitions of the numerical range and numerical radius for bounded adjointable operators on Hilbert $C^*$-modules, which are of course the natural generalizations of these concepts to operators on Hilbert spaces. By using these definitions and applying special techniques, we prove some fundamental inequalities in the operating radius of adjointable boundary operators on Hilbert $C^*$-modules in Section 2.

In Section 3, we use the method presented in Section 2 to provide some applications of our results. We also prove some new inequalities on the numerical radius of bounded operators on Hilbert $C^*$-modules, which also extend some known inequalities in the space of bounded operators on Hilbert spaces.

## 2. Main results

We start by recalling some definitions.

**Definition 2.1.** Let $A$ be a $C^*$-algebra. A *semi-inner product $A$-module* is a linear space $E$ which is a right $A$-module with the compatible scalar multiplication

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \quad \text{for all } x \in E,\ a \in A \text{ and } \lambda \in \mathbb{C}$$

together with a map $\langle \cdot, \cdot \rangle_E \colon E \times E \to A$, which has the following properties:

(i) $\langle x, \alpha y + \beta z \rangle_E = \alpha \langle x, y \rangle_E + \beta \langle x, z \rangle_E$, $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$,

(ii) $\langle x, ya \rangle_E = \langle x, y \rangle_E a$, $x, y \in E, a \in A$,

(iii) $\langle x, y \rangle_E^* = \langle y, x \rangle_E$, $x, y \in E$.

For every $x \in E$, we put $\|x\|_E = \|\langle x, x \rangle_E\|^{1/2}$. A semi-inner product space $E$, which satisfies

$$\|x\|_E = 0 \Leftrightarrow x = 0,$$

is called an *inner product $A$-module*. A complete inner-product $A$-module is called a *Hilbert $C^*$-module*.

**Definition 2.2.** Suppose that $E$, $F$ are Hilbert $C^*$-modules. We define $\mathcal{L}(E, F)$ to be the set of all maps $t \colon E \to F$ for which there is a map $t^* \colon F \to E$ which satisfies

$$\langle tx, y \rangle_E = \langle x, t^* y \rangle_E, \quad x \in E,\ y \in F.$$

$\mathcal{L}(E, E)$ is simply denoted by $\mathcal{L}(E)$. It is known that $\mathcal{L}(E)$ is a $C^*$-algebra.

**Definition 2.3** ([20], page 89). A *state* on a $C^*$-algebra $A$ is a positive linear functional on $A$ of norm one. We denote the *state space* of $A$ by $S(A)$.

**Definition 2.4.** Suppose that $E$ is a Hilbert right $A$-module. We define the *numerical range* of $t \in \mathcal{L}(E)$ by

$$W_A(t) = \{\varrho\langle x, tx\rangle_E \colon\ x \in E,\ \varrho \in S(A),\ \varrho\langle x,\ x\rangle_E = 1\}.$$

We also define the *numerical radius* of $t \in \mathcal{L}(E)$ by

$$\omega_A(t) = \sup_{\varrho\langle x,x\rangle_E = 1} |\varrho\langle x, tx\rangle_E|.$$

Note that our definition is a natural extension of the definition of numerical range and numerical radius of bounded operators on Hilbert spaces. In fact, in this case the $C^*$-algebra $A$ is the set of complex numbers and $S(A)$ contains only the identity function on the set of complex numbers.

Hereafter, we assume that $A$ is a $C^*$-algebra and $E$ is an inner product $A$-module.

We need the following result.

**Lemma 2.5** ([15], Lemma 3.1)**.** *Suppose that $E$ is a Hilbert $C^*$-module and $t$ is a self-adjoint element of $\mathcal{L}(E)$. If $\|tx\| \geqslant k\|x\|$, $x \in E$ for some constant $k > 0$, then $t$ is invertible in $\mathcal{L}(E)$.*

In order to present the main results of this section, we need the following results.

**Lemma 2.6.** *Let $t \in \mathcal{L}(E)$ and $\varrho \in S(A)$. The following statements are equivalent:*
(a) *$\varrho\langle x, tx\rangle_E = 0$ for every $x \in E$ with $\varrho\langle x, x\rangle_E = 1$;*
(b) *$\varrho\langle x, tx\rangle_E = 0$ for every $x \in E$.*

P r o o f. The proof runs in a similar way as in the classical case of Hilbert spaces. □

**Lemma 2.7.** *Let $t \in \mathcal{L}(E)$, then $|\varrho\langle x, tx\rangle_E| \leqslant \varrho\langle x, x\rangle_E \omega_A(t)$ for every $\varrho \in S(A)$ and $x \in E$.*

P r o o f. Let $x \in E$ and $\varrho\langle x, x\rangle_E \neq 0$. Then

$$\varrho\Big\langle \frac{x}{(\varrho\langle x, x\rangle_E)^{1/2}}, \frac{x}{(\varrho\langle x, x\rangle_E)^{1/2}}\Big\rangle_E = 1,$$

so that

$$\Big|\varrho\Big\langle \frac{x}{(\varrho\langle x, x\rangle_E)^{1/2}}, t\Big(\frac{x}{(\varrho\langle x, x\rangle_E)^{1/2}}\Big)\Big\rangle_E\Big| \leqslant \omega_A(t).$$

Hence

$$|\varrho\langle x, tx\rangle_E| \leqslant \varrho\langle x, x\rangle_E \omega_A(t).$$

Let $\varrho\langle x, x\rangle_E = 0$. By the Cauchy-Schwarz inequality we have

$$|\varrho\langle x, tx\rangle_E|^2 \leqslant \varrho\langle x, x\rangle_E \, \varrho\langle tx, tx\rangle_E.$$

It follows that $|\varrho\langle x, tx\rangle_E| = 0$. Therefore

$$|\varrho\langle x, tx\rangle_E| \leqslant \varrho\langle x, x\rangle_E \omega_A(t).$$

□

It is known that $T \in B(H)$ is positive if and only if $\langle Tx, x\rangle \geqslant 0$ and $T = 0$ if and only if $\langle Tx, x\rangle = 0$ for all $x \in H$. By applying standard argument, one can show that the corresponding results hold for Hilbert $C^*$-modules.

**Proposition 2.8.** *For every $t \in \mathcal{L}(E)$, the following statements hold.*
(a) *$t = 0$ if and only if $\langle x, tx\rangle_E = 0$ for every $x \in E$.*
(b) *$t$ is positive if and only if $\langle x, tx\rangle_E$ is positive for every $x \in E$.*
(c) *$t$ is self-adjoint if and only if $\langle x, tx\rangle_E$ is self-adjoint for every $x \in E$.*

The following lemma is an implication of part (a) of Proposition 2.8.

**Lemma 2.9.** *Let $t \in \mathcal{L}(E)$, then $t = 0$ if and only if $\varrho\langle x, tx\rangle = 0$ for every $x \in E$ and $\varrho \in S(A)$.*

Proposition 2.8 enables us to obtain the following.

**Corollary 2.10.** *Let $t \in \mathcal{L}(E)$ and $\varrho \in S(A)$, then for every $x \in E$*

$$\operatorname{Re} \varrho\langle x, tx\rangle_E = \varrho\langle x, \operatorname{Re}(t)x\rangle_E.$$

Proof. Let $b = \operatorname{Re}(t)$ and $c = \operatorname{Im}(t)$. Then $t = b + \mathrm{i}c$. By linearity of $\varrho$,

$$\varrho\langle x, tx\rangle_E = \varrho\langle x, bx\rangle_E + \mathrm{i}\varrho\langle x, cx\rangle_E, \quad x \in E.$$

By Proposition 2.8 (c), $\langle x, bx\rangle_E$, $\langle x, cx\rangle_E$ are self-adjoint. Since $\varrho$ is positive, by [20], Theorem 3.3.2, we have

$$\operatorname{Re} \varrho\langle x, tx\rangle_E = \varrho\langle x, bx\rangle_E = \varrho\langle x, \operatorname{Re}(t)x\rangle_E, \quad x \in E.$$

□

It is known that for every Hermitian element $T$ of $B(H)$, the numerical range of $T$ is a subset of the real line and if $T$ is positive it is a subset of non-negative real numbers. The next result shows that these properties hold in $\mathcal{L}(E)$.

**Theorem 2.11.** *Let $t \in \mathcal{L}(E)$, then the following statements hold.*

(a) *$t$ is self-adjoint if and only if $W_A(t)$ is a subset of the real line.*

(b) *$t$ is positive if and only if $W_A(t) \subseteq \mathbb{R}^+$.*

Proof. (a) Let $t$ be self-adjoint, $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\varrho\langle x, tx\rangle_E = \varrho\langle t^* x, x\rangle_E = \varrho\langle tx, x\rangle_E = \varrho((\langle x, tx\rangle_E)^* = \overline{\varrho\langle x, tx\rangle_E}.$$

Then $W_A(t) \subseteq \mathbb{R}$.

Conversely, suppose that $W_A(t) \subseteq \mathbb{R}$. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$, we have

$$\varrho\langle x, tx\rangle_E = \overline{\varrho\langle x, tx\rangle_E} = \varrho((\langle x, tx\rangle_E)^* = \varrho\langle tx, x\rangle_E = \varrho\langle x, t^* x\rangle_E.$$

Therefore $\varrho\langle x, (t - t^*)x\rangle_E = 0$ for every $\varrho \in S(A)$ and every $x \in E$ with $\varrho\langle x, x\rangle_E = 1$. By Lemma 2.6, $\varrho\langle x, (t - t^*)x\rangle_E = 0$ for $x \in E$ and $\varrho \in S(A)$. According to Lemma 2.9, $t = t^*$.

By using Proposition 2.8 (b) and Theorem 4.3.6 (iii) in [11], one can prove part (b) similarly. □

We know that the numerical range of an operator $T \in B(H)$ is convex. We have to admit that we could not extend this result for adjointable operators on a Hilbert $C^*$-module. Thus the following question remains open.

**Question.** *Is the numerical range of an operator $t \in \mathcal{L}(E)$ convex?*

Remark 2.12. Let $E$, $F$ be Hilbert $C^*$-modules and $t \in \mathcal{L}(E, F)$, $x \in E$. In [15], Proposition 1.2, it is shown that

$$|tx|^2 \leqslant \|t\|^2 |x|^2. \tag{2.1}$$

It follows from (2.1) that for every positive linear functional $\varrho$,

$$\varrho\langle tx, tx\rangle \leqslant \|t\|^2 \varrho\langle x, x\rangle. \tag{2.2}$$

For every $T \in B(H)$ and $x \in H$, we have (see [24])

$$\sup_{\theta \in [0, 2\pi]} \operatorname{Re}(e^{i\theta}\langle Tx, x\rangle) = |\langle Tx, x\rangle|.$$

Since for every $\varrho \in S(A)$, $\varrho\langle \cdot, \cdot\rangle_E$ is a semi-inner product, then for every $t \in \mathcal{L}(E)$ and $x \in E$,

$$\sup_{\theta \in [0, 2\pi]} \operatorname{Re}(e^{i\theta}\varrho\langle x, tx\rangle_E) = |\varrho\langle x, tx\rangle_E|. \tag{2.3}$$

The above property enables us to obtain one of the main results of this section.

**Theorem 2.13.** *For every $t \in \mathcal{L}(E)$, we have*
(a) $\frac{1}{2}\|t\| \leqslant \omega_A(t) \leqslant \|t\|$;
(b) $\omega_A(t) = \|t\|$ *if $t$ is normal.*

Proof. In order to prove (a), note that for every $\varrho \in S(A)$ and $x, y \in E$ with $\varrho\langle x,x\rangle_E \leqslant 1$ and $\varrho\langle y,y\rangle_E \leqslant 1$, we have

$$\begin{aligned} 2\operatorname{Re}\varrho(\langle x,ty\rangle_E + \langle y,tx\rangle_E) &= \operatorname{Re}\varrho(\langle x+y,t(x+y)\rangle_E - \langle x-y,t(x-y)\rangle_E) \\ &\leqslant |\varrho(\langle x+y,t(x+y)\rangle_E - \langle x-y,t(x-y)\rangle_E)| \\ &\leqslant |\varrho\langle x+y,t(x+y)\rangle_E| + |\varrho\langle x-y,t(x-y)\rangle_E| \\ &\leqslant \varrho(\langle x+y,x+y\rangle_E + \langle x-y,x-y\rangle_E)\omega_A(t) \\ &= (2\varrho\langle x,x\rangle_E + 2\varrho\langle y,y\rangle_E)\omega_A(t) \\ &\leqslant 4\omega_A(t) \end{aligned}$$

by Lemma 2.7. Then

$$\operatorname{Re}\varrho(\langle x,ty\rangle_E+\langle y,tx\rangle_E) \leqslant 2\omega_A(t), \quad \varrho \in S(A),\ \varrho(\langle x,x\rangle_E) \leqslant 1,\ \varrho(\langle y,y\rangle_E) \leqslant 1. \tag{2.4}$$

Putting $y = tx(\varrho\langle tx,tx\rangle_E)^{-1/2}$ in (2.4) we see that

$$\operatorname{Re}\varrho\Big(\Big\langle x, t\Big(\frac{tx}{(\varrho\langle tx,tx\rangle_E)^{1/2}}\Big)\Big\rangle_E + \Big\langle \frac{tx}{(\varrho\langle tx,tx\rangle_E)^{1/2}}, tx\Big\rangle_E\Big) \leqslant 2\omega_A(t)$$

for every $\varrho \in S(A)$ and $x \in E$ with $\varrho(\langle x,x\rangle_E) \leqslant 1$. This means that

$$\operatorname{Re}\varrho\langle x,t^2x\rangle_E + \varrho\langle tx,tx\rangle_E \leqslant 2\omega_A(t)(\varrho\langle tx,tx\rangle_E)^{1/2}, \quad \varrho \in S(A),\ \varrho(\langle x,x\rangle_E) \leqslant 1. \tag{2.5}$$

By replacing $t$ by $\mathrm{e}^{\mathrm{i}\theta/2}t$ in (2.5),

$$\operatorname{Re}\varrho\langle x,(\mathrm{e}^{\mathrm{i}\theta/2}t)^2x\rangle_E + \varrho\langle tx,tx\rangle_E \leqslant 2\omega_A(t)(\varrho\langle tx,tx\rangle_E)^{1/2}, \quad \varrho \in S(A),\ \varrho(\langle x,x\rangle_E) \leqslant 1.$$

By taking supremum over all $\theta \in [0,2\pi]$, we have

$$\sup_{\theta\in[0,2\pi]} \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}\varrho\langle x,t^2x\rangle_E) + \varrho\langle tx,tx\rangle_E \leqslant 2\omega_A(t)(\varrho\langle tx,tx\rangle_E)^{1/2},$$
$$\varrho \in S(A),\ \ \varrho(\langle x,x\rangle_E) \leqslant 1.$$

By (2.3) then $\sup_{\theta\in[0,2\pi]} \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}\varrho\langle x,t^2x\rangle_E) = |\varrho\langle x,t^2x\rangle_E|$. Therefore,

$$\begin{aligned} |\varrho\langle x,t^2x\rangle_E| + \varrho\langle tx,tx\rangle_E &\leqslant 2\omega_A(t)(\varrho\langle tx,tx\rangle_E)^{1/2} \\ &\leqslant 2\omega_A(t)\|tx\|, \quad \varrho \in S(A),\ \ \varrho(\langle x,x\rangle_E) \leqslant 1. \end{aligned}$$

Since $|\varrho\langle x,t^2x\rangle_E| \geqslant 0$ for every $\varrho \in S(A)$ and $x \in E$ with $\varrho(\langle x,x\rangle_E) \leqslant 1$, we have

$$\varrho\langle tx,tx\rangle_E \leqslant 2\omega_A(t)\|tx\|_E. \tag{2.6}$$

By taking supremum over all $\varrho \in S(A)$ we see that

$$\|tx\|_E^2 = \|\langle tx, tx\rangle_E\| \leqslant 2\omega_A(t)\|tx\|_E.$$

Therefore for every $x \in E$ with $\|x\|_E = 1$, $\|tx\|_E \leqslant 2\omega_A(t)$. Thus

$$\frac{1}{2}\|t\| \leqslant \omega_A(t).$$

For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$|\varrho\langle x, tx\rangle_E|^2 \leqslant \varrho\langle x, x\rangle_E \varrho\langle tx, tx\rangle_E \leqslant \varrho\langle x, x\rangle_E^2 \|t\|^2 = \|t\|^2$$

by (2.2). By taking supremum over all $\varrho\langle x, x\rangle_E = 1$,

$$\omega_A(t) \leqslant \|t\|.$$

(b) By (2.5), for every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$|\varrho\langle x, t^2x\rangle_E| + \varrho\langle tx, tx\rangle_E \leqslant 2\omega_A(t)(\varrho\langle tx, tx\rangle_E)^{1/2}.$$

It follows that for $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\begin{aligned}
|\varrho\langle x, t^2x\rangle_E| &\leqslant 2\omega_A(t)(\varrho\langle tx, tx\rangle_E)^{1/2} - \varrho\langle tx, tx\rangle_E \\
&= -(-2\omega_A(t)(\varrho\langle tx, tx\rangle_E)^{1/2} + \varrho\langle tx, tx\rangle_E) \\
&= -(\omega_A(t) - (\varrho\langle tx, tx\rangle_E)^{1/2})^2 + \omega_A^2(t) \\
&\leqslant \omega_A^2(t).
\end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\omega_A(t^2) \leqslant \omega_A^2(t). \tag{2.7}$$

By induction, one can show that for every $n \geqslant 1$,

$$\omega_A(t^{2^n}) \leqslant \omega_A(t)^{2^n}.$$

Since $t$ is normal,

$$\begin{aligned}
\|t\|^{2^n} &= (\|t^*t\|^{2^n})^{1/2} = \|(t^*t)^{2^n}\|^{1/2} = \|(t^*)^{2^n}t^{2^n}\|^{1/2} = \|(t^{2^n})^*t^{2^n}\|^{1/2} = \|t^{2^n}\| \\
&\leqslant 2\omega_A(t^{2^n}) \leqslant 2\omega_A(t)^{2^n}.
\end{aligned}$$

It follows that $\|t\| \leqslant 2^{1/2^n}\omega_A(t)$. By letting $n \to \infty$, we see that $\|t\| \leqslant \omega_A(t)$. This together with (a) proves (b). □

The following result follows immediately from Theorem 2.13.

**Corollary 2.14.** $\omega_A(t) = \|t\|$ *for every self-adjoint element of* $\mathcal{L}(E)$.

It is known that the numerical radius function defines an equivalent norm on $B(H)$. The next theorem shows that this result holds when the Hilbert space $H$ is replaced by an arbitrary $C^*$-algebra.

**Theorem 2.15.** $\omega_A\colon \mathcal{L}(E) \to [0,\infty)$ *defines a norm which is equivalent to the norm on* $\mathcal{L}(E)$.

P r o o f. Let $\omega_A(t) = 0$. Then for every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle_E = 1$, we have

$$|\varrho\langle x, tx\rangle_E| = 0.$$

Hence $\varrho\langle x, tx\rangle_E = 0$ for every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle_E = 1$. By Lemma 2.6, $\varrho\langle x, tx\rangle_E = 0$ for every $x \in E$ and $\varrho \in S(A)$, and by Lemma 2.9, $t = 0$. For every $\lambda \in \mathbb{C}$,

$$\omega_A(\lambda t) = \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, (\lambda t)x\rangle_E| = |\lambda| \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, tx\rangle_E| = |\lambda|\omega_A(t).$$

Let $t_1, t_2 \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle_E = 1$,

$$|\varrho\langle x, (t_1+t_2)x\rangle_E| \leqslant |\varrho\langle x, t_1x\rangle_E| + |\varrho\langle x, t_2x\rangle_E| \leqslant \omega_A(t_1) + \omega_A(t_2).$$

By taking supremum over all $\varrho\langle x,x\rangle_E = 1$,

$$\omega_A(t_1+t_2) \leqslant \omega_A(t_1) + \omega_A(t_2).$$

Thus $\omega_A(\cdot)$ defines a norm $\mathcal{L}(E)$. By Theorem 2.13 this norm is equivalent to the original norm on $\mathcal{L}(E)$. □

The following lemma is a simple consequence of the classical Jensen and Young inequalities.

**Lemma 2.16** ([9])**.** *For* $a, b \geqslant 0$ *and* $0 \leqslant \alpha \leqslant 1$,

$$a^\alpha b^{1-\alpha} \leqslant \alpha a + (1-\alpha)b \leqslant (\alpha a^r + (1-\alpha)b^r)^{1/r} \quad \text{for } r \geqslant 1.$$

The following result shows that $\omega_A(t) = \frac{1}{2}\|t\|$ under some circumstances on $t \in \mathcal{L}(E)$.

**Theorem 2.17.** *Let $E$ be a Hilbert $C^*$-module, $t \in \mathcal{L}(E)$. If* $\operatorname{ran}(t)$ *is closed and* $t^2 = 0$, *then*

$$\omega_A(t) = \frac{1}{2}\|t\|.$$

P r o o f. Let $x$ be a unit element in $E$. Since $\operatorname{ran}(t)$ is closed, $E = \operatorname{ran}(t) \oplus \ker(t^*)$. Therefore, every $x \in E$ has a unique decomposition $x = y + z$, where $y \in \operatorname{ran}(t)$ and $z \in \ker(t^*)$. Since $t^2 = 0$, by Lemma 2.9, $\varrho\langle x, t^2 x\rangle_E = 0$ for $x \in E$ and $\varrho \in S(A)$. So $\varrho\langle t^* x, tx\rangle_E = 0$. Thus $\operatorname{ran}(t) \perp \operatorname{ran}(t^*)$. For $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\varrho\langle x, tx\rangle_E = \varrho\langle y + z, t(y + z)\rangle_E = \varrho\langle y, tz\rangle_E.$$

For every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\begin{aligned} |\varrho\langle x, tx\rangle_E|^2 &= |\varrho\langle y, tz\rangle_E|^2 \leqslant \varrho\langle y, y\rangle_E \varrho\langle tz, tz\rangle_E \leqslant \|t\|^2 \varrho\langle y, y\rangle_E \varrho\langle z, z\rangle_E \\ &\leqslant \|t\|^2 \Big(\frac{\varrho\langle y, y\rangle_E + \varrho\langle z, z\rangle_E}{2}\Big)^2 = \frac{1}{4}\|t\|^2 \varrho\langle x, x\rangle_E = \frac{1}{4}\|t\|^2 \end{aligned}$$

by (2.2) and by Lemma 2.16. By taking supremum over $\varrho\langle x, x\rangle_E = 1$,

$$\omega_A(t) \leqslant \frac{1}{2}\|t\|. \tag{2.8}$$

By Theorem 2.13,

$$\frac{1}{2}\|t\| \leqslant \omega_A(t). \tag{2.9}$$

By (2.8) and (2.9) we have $\omega_A(t) = \frac{1}{2}\|t\|$. □

## 3. Some applications

In this section we show that the results of the previous section enable us to generalize some results about the numerical radius of the operators on Hilbert spaces to the numerical radius of the operators on Hilbert $C^*$-modules.

In [14], Kittaneh proved the following statement.

**Theorem 3.1** (Kittaneh [14], Theorem 1)**.** *Let $T \in B(H)$, then*

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant \omega_A^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|.$$

The above result can be generalized as follows.

**Theorem 3.2.** *If* $t \in \mathcal{L}(E)$, *then*

$$\frac{1}{4}\|t^*t + tt^*\| \leqslant \omega_A^2(t) \leqslant \frac{1}{2}\|t^*t + tt^*\|.$$

Proof. Let $t \in \mathcal{L}(E)$. There are self-adjoint elements $b, c \in \mathcal{L}(E)$ such that $t = b + \mathrm{i}c$. For every vector $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\begin{aligned} |\varrho\langle x, tx\rangle_E|^2 &= (\varrho\langle x, bx\rangle_E)^2 + (\varrho\langle x, cx\rangle_E)^2 \geqslant \frac{1}{2}(\varrho\langle x, bx\rangle_E + \varrho\langle x, cx\rangle_E)^2 \\ &= \frac{1}{2}(\varrho\langle x, (b+c)x\rangle_E)^2. \end{aligned} \tag{3.1}$$

Since $\varrho\langle x, (b+c)x\rangle_E \in \mathbb{R}$, then $(\varrho\langle x, (b+c)x\rangle_E)^2 = |\varrho\langle x, (b+c)x\rangle_E|^2$. We have

$$\begin{aligned} \frac{1}{2}\sup_{\varrho\langle x,x\rangle_E=1}|\varrho\langle x, (b+c)x\rangle_E|^2 &= \frac{1}{2}\Big(\sup_{\varrho\langle x,x\rangle_E=1}\varrho\langle x, (b+c)x\rangle_E\Big)^2 \leqslant \sup_{\varrho\langle x,x\rangle_E=1}|\varrho\langle x, tx\rangle_E|^2 \\ &= \Big(\sup_{\varrho\langle x,x\rangle_E=1}|\varrho\langle x, tx\rangle_E|\Big)^2 \leqslant \omega_A^2(t). \end{aligned}$$

Therefore

$$\frac{1}{2}\omega_A^2(b+c) \leqslant \omega_A^2(t). \tag{3.2}$$

By Corollary 2.14, we have $\omega_A(b+c) = \|b+c\|$. Thus

$$\frac{1}{2}\|b+c\| \leqslant \omega_A^2(t).$$

Since

$$\begin{aligned} \frac{1}{2}(\varrho\langle x, (b-c)x\rangle_E)^2 &= \frac{1}{2}(\varrho\langle x, bx\rangle_E - \varrho\langle x, cx\rangle_E)^2 \leqslant \frac{1}{2}(|\varrho\langle x, bx\rangle_E| + |\varrho\langle x, cx\rangle_E|)^2 \\ &\leqslant \varrho\langle x, bx\rangle_E^2 + \varrho\langle x, cx\rangle_E^2 = |\varrho\langle x, tx\rangle_E|^2 \leqslant \omega_A^2(t), \end{aligned}$$

we have

$$\frac{1}{2}\omega_A^2(b-c) \leqslant \omega_A^2(t).$$

By Corollary 2.14, we have $\omega_A(b-c) = \|b-c\|$. Hence

$$\frac{1}{2}\|b-c\|^2 \leqslant \omega_A^2(t).$$

Moreover,

$$\begin{aligned} 2\omega_A^2(t) &\geqslant \frac{1}{2}\|b+c\|^2 + \frac{1}{2}\|b-c\|^2 = \frac{1}{2}\|(b+c)^2\| + \frac{1}{2}\|(b-c)^2\| \\ &\geqslant \frac{1}{2}\|(b+c)^2 + (b-c)^2\| = \|b^2 + c^2\|. \end{aligned}$$

Therefore

$$\frac{1}{4}\|t^*t + tt^*\| \leqslant \omega_A^2(t).$$

For $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\begin{aligned}|\varrho\langle x, tx\rangle_E|^2 &= \varrho\langle x, bx\rangle_E^2 + \varrho\langle x, cx\rangle_E^2 \leqslant \varrho\langle x, x\rangle_E \varrho\langle bx, bx\rangle_E + \varrho\langle x, x\rangle_E \varrho\langle cx, cx\rangle_E \\ &= \varrho\langle x, b^2x\rangle_E + \varrho\langle x, c^2x\rangle_E = \varrho\langle x, (b^2+c^2)x\rangle_E \leqslant \omega_A(b^2+c^2) \\ &= \|b^2+c^2\| = \frac{1}{2}\|t^*t + tt^*\|\end{aligned}$$

by Corollary 2.14. It follows that $\omega_A^2(t) \leqslant \frac{1}{2}\|t^*t + tt^*\|$. $\square$

In [1], the authors obtained the following statement.

**Theorem 3.3** ([1], Theorem 2.1)**.** *Let $T \in B(H)$, then*

$$\omega^4(T) \leqslant \frac{1}{4}\omega^2(T^2) + \frac{1}{8}\omega(T^2P + PT^2) + \frac{1}{16}\|P\|^2,$$

*where $P = T^*T + TT^*$.*

In order to find a generalization of the above theorem, we need the following.

**Lemma 3.4.** *Let $t \in \mathcal{L}(E)$, then*

$$\omega_A(t) = \sup_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\| = \frac{1}{2}\sup_{\theta\in[0,2\pi]} \|t + \mathrm{e}^{\mathrm{i}\theta}t^*\|.$$

Proof. For any $t \in \mathcal{L}(E)$,

$$\begin{aligned}\omega_A(t) &= \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, tx\rangle_E| = \sup_{\varrho\langle x,x\rangle_E=1}\sup_{\theta\in[0,2\pi]} \varrho\langle x, \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E \\ &\leqslant \sup_{\varrho\langle x,x\rangle_E=1}\sup_{\theta\in[0,2\pi]} |\varrho\langle x, \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E| = \sup_{\theta\in[0,2\pi]}\sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E| \\ &= \sup_{\theta\in[0,2\pi]} \omega_A(\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)) = \sup_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\|\end{aligned}$$

by (2.3) and Corollary 2.14. For every $\theta \in [0, 2\pi]$,

$$\begin{aligned}\|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\| = \omega_A(\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)) &= \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, \operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E| \\ &= \sup_{\varrho\langle x,x\rangle_E=1} |\varrho \operatorname{Re}\langle x, (\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E| \leqslant \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, (\mathrm{e}^{\mathrm{i}\theta}t)x\rangle_E| \\ &= \sup_{\varrho\langle x,x\rangle_E=1} |\varrho\langle x, tx\rangle_E| = \omega_A(t)\end{aligned}$$

by Corollary 2.14. So $\sup\limits_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\| \leqslant \omega_A(t)$. Hence

$$\begin{aligned}\omega_A(t) &= \sup_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\| = \frac{1}{2}\sup_{\theta\in[0,2\pi]} \|\mathrm{e}^{\mathrm{i}\theta}t + \mathrm{e}^{-\mathrm{i}\theta}t^*\| \\ &= \frac{1}{2}\sup_{\theta\in[0,2\pi]} \|t + \mathrm{e}^{-2\mathrm{i}\theta}t^*\| = \frac{1}{2}\sup_{\theta'\in[0,2\pi]} \|t + \mathrm{e}^{\mathrm{i}\theta'}t^*\|,\end{aligned}$$

where we put $-2\theta = \theta'$. $\square$

We can now generalize Theorem 3.3 for Hilbert $C^*$-modules.

**Theorem 3.5.** *Let $t \in \mathcal{L}(E)$, then*

$$\omega_A^4(t) \leqslant \frac{1}{4}\omega_A^2(t^2) + \frac{1}{8}\omega_A(t^2p + pt^2) + \frac{1}{16}\|p\|^2,$$

*where $p = t^*t + tt^*$.*

P r o o f. By Lemma 3.4, we have

$$\omega_A(t) = \sup_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}t)\|.$$

The rest of the proof is similar to the proof of Theorem 2.1 in [1]. □

Sattari et al. in [23] obtained the following statement.

**Theorem 3.6** ([23], Theorem 2.4)**.** *Let $T \in B(H)$, then*

$$\omega^{2r}(T) \leqslant \frac{1}{2}(\omega^r(T^2) + \|T\|^{2r}).$$

The following result shows that the above theorem is true for bounded operators on Hilbert $C^*$-modules.

**Theorem 3.7.** *Let $t \in \mathcal{L}(E)$, then*

$$\omega_A^{2r}(t) \leqslant \frac{1}{2}(\omega_A^r(t^2) + \|t\|^{2r}).$$

P r o o f. In [2], Theorem 2, it is shown that for every semi-inner product $(H, \langle\cdot,\cdot\rangle)$,

$$|\langle a,b\rangle| \leqslant |\langle a,e\rangle\langle e,b\rangle| + |\langle a,b\rangle - \langle a,e\rangle\langle e,b\rangle| \leqslant \langle a,a\rangle^{1/2}\langle b,b\rangle^{1/2} \tag{3.3}$$

for every $a, b, e \in H$ with $\langle e,e\rangle = 1$. Observing that

$$|\langle a,e\rangle\langle e,b\rangle| - |\langle a,b\rangle| \leqslant |\langle a,b\rangle - \langle a,e\rangle\langle e,b\rangle|, \tag{3.4}$$

we have by (3.3) and (3.4) that

$$\begin{aligned}|\langle a,e\rangle\langle e,b\rangle| &\leqslant \langle a,a\rangle^{1/2}\langle b,b\rangle^{1/2} - |\langle a,b\rangle - \langle a,e\rangle\langle e,b\rangle| \\ &\leqslant \langle a,a\rangle^{1/2}\langle b,b\rangle^{1/2} + |\langle a,b\rangle| - |\langle a,e\rangle\langle e,b\rangle|\end{aligned}$$

for every $a, b, e \in H$ with $\langle e,e\rangle = 1$. Therefore

$$|\langle a,e\rangle\langle e,b\rangle| \leqslant \frac{1}{2}(\langle a,a\rangle^{1/2}\langle b,b\rangle^{1/2} + |\langle a,b\rangle|), \quad a,b,e \in H,\ \langle e,e\rangle = 1. \tag{3.5}$$

Since $(E, \varrho\langle\cdot,\cdot\rangle)$ is a semi-inner product space for every $\varrho \in S(A)$, if we put $a = t^*x$, $b = tx$ and $e = x$ with $\varrho(\langle x, x\rangle) = 1$, we get by (3.5) that

$$\begin{aligned}|\varrho\langle t^*x, x\rangle_E \varrho\langle x, tx\rangle_E|^r &\leqslant \Big(\frac{(\varrho\langle t^*x, t^*x\rangle_E)^{1/2}(\varrho\langle tx, tx\rangle_E)^{1/2} + |\varrho\langle t^*x, tx\rangle_E|}{2}\Big)^r \\ &= \Big(\frac{(\varrho\langle x, x\rangle_E)^{1/2}\|t^*\|(\varrho\langle x, x\rangle_E)^{1/2}\|t\| + |\varrho\langle x, t^2x\rangle_E|}{2}\Big)^r \\ &= \Big(\frac{\|t\|^2 + |\varrho\langle x, t^2x\rangle_E|}{2}\Big)^r \\ &\leqslant \frac{1}{2}(\|t\|^{2r} + |\varrho\langle x, t^2x\rangle_E|^r) \leqslant \frac{1}{2}(\|t\|^{2r} + \omega_A^r(t^2))\end{aligned}$$

by (2.2) and Lemma 2.16. Since $|\varrho\langle t^*x, x\rangle_E \varrho\langle x, tx\rangle_E| = |\varrho\langle x, tx\rangle_E|^2$, we have

$$|\varrho\langle x, tx\rangle_E|^{2r} \leqslant \frac{1}{2}(\|t\|^{2r} + \omega_A^r(t^2)).$$

By taking supremum over all $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$,

$$\omega_A^{2r}(t) \leqslant \frac{1}{2}(\|t\|^{2r} + \omega_A^r(t^2)).$$

□

Next, we need the following result.

**Lemma 3.8** ([16], McCarty inequality). *Let $T \in B(H)$, $T \geqslant 0$ and $x \in H$, then*
(i) $\langle Tx, x\rangle^r \leqslant \|x\|^{2(1-r)}\langle T^r x, x\rangle$ *for* $r \geqslant 1$,
(ii) $\langle Tx, x\rangle^r \geqslant \|x\|^{2(1-r)}\langle T^r x, x\rangle$ *for* $0 < r \leqslant 1$.

The following result follows immediately from Lemma 3.8.

**Corollary 3.9.** *Let $t \in \mathcal{L}(E)$, $t \geqslant 0$ and $x \in E$, then for every $\varrho \in S(A)$*
(i) $(\varrho\langle x, tx\rangle_E)^r \leqslant \|x\|^{2(1-r)}\varrho\langle x, t^r x\rangle_E$ *for* $r \geqslant 1$ *and*
(ii) $(\varrho\langle x, tx\rangle_E)^r \geqslant \|x\|^{2(1-r)}\varrho\langle x, t^r x\rangle_E$ *for* $0 < r \leqslant 1$.

**Lemma 3.10** ([13], Cauchy-Schwarz inequality). *Let $T \in B(H)$ and $0 \leqslant \alpha \leqslant 1$, then*

$$|\langle x, Ty\rangle|^2 \leqslant \langle x, |T|^{2\alpha}x\rangle\langle y, |T^*|^{2(1-\alpha)}y\rangle, \quad x, y \in H.$$

The following result is a consequence of Lemma 3.10.

**Corollary 3.11.** *For $\varrho \in S(A)$, $\varrho\langle\cdot,\cdot\rangle_E$ is a semi-inner product. Suppose that $t \in \mathcal{L}(E)$ and $0 \leqslant \alpha \leqslant 1$, then*

$$|\varrho\langle x, ty\rangle_E|^2 \leqslant \varrho\langle x, |t|^{2\alpha}x\rangle_E \varrho\langle y, |t^*|^{2(1-\alpha)}y\rangle_E, \quad x, y \in E.$$

*If $\alpha = \frac{1}{2}$, then*

$$|\varrho\langle x, ty\rangle_E|^2 \leqslant \varrho\langle x, |t|x\rangle_E \varrho\langle y, |t^*|y\rangle_E, \quad x, y \in E.$$

The above results enable us to state the following.

**Theorem 3.12.** *Let $t \in \mathcal{L}(E)$, $r \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, then*

$$\omega_A^{2r}(t) \leqslant \|\alpha(t^*t)^r + (1-\alpha)(tt^*)^r\|.$$

P r o o f. Let $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$. Then

$$\begin{aligned}
|\varrho\langle x, tx\rangle_E|^2 &\leqslant \varrho\langle x, |t|^{2\alpha}x\rangle_E \varrho\langle x, |t^*|^{2(1-\alpha)}x\rangle_E \leqslant (\varrho\langle x, |t|^2x\rangle_E)^\alpha (\varrho\langle x, |t^*|^2x\rangle_E)^{1-\alpha} \\
&\leqslant (\alpha\varrho\langle x, |t|^2x\rangle_E^r + (1-\alpha)\varrho\langle x, |t^*|^2x\rangle_E^r)^{1/r} \\
&\leqslant (\alpha\varrho\langle x, |t|^{2r}x\rangle_E + (1-\alpha)\varrho\langle x, |t^*|^{2r}x\rangle_E)^{1/r} \\
&= (\varrho\langle x, (\alpha|t|^{2r} + (1-\alpha)|t^*|^{2r})x\rangle_E)^{1/r} \\
&\leqslant \|\alpha|t|^{2r} + (1-\alpha)|t^*|^{2r}\|^{1/r} = \|\alpha(t^*t)^r + (1-\alpha)(tt^*)^r\|^{1/r}
\end{aligned}$$

by Lemma 2.16 and Corollaries 3.9 and 3.11. □

In 2009, Dragomir obtained the following statement.

**Theorem 3.13** ([5], Theorem 2)**.** *For any $T, S \in B(H)$, any $0 < \alpha < 1$ and $r \geqslant 1$, we have the inequality*

$$\omega^{2r}(S^*T) \leqslant \|\alpha(T^*T)^{r/\alpha} + (1-\alpha)(S^*S)^{r/(1-\alpha)}\|.$$

In the following we show that $B(H)$ can be replaced by $\mathcal{L}(E)$ in Dragomir's theorem.

**Theorem 3.14.** *For any $t, s \in \mathcal{L}(E)$, any $0 < \alpha < 1$ and $r \geqslant 1$, we have the inequality*

$$\omega_A^{2r}(s^*t) \leqslant \|\alpha(t^*t)^{r/\alpha} + (1-\alpha)(s^*s)^{r/(1-\alpha)}\|.$$

P r o o f. Let $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$. Then

$$\begin{aligned}|\varrho\langle x, s^*ty\rangle_E|^2 &= |\varrho\langle sx, ty\rangle_E|^2 \leqslant \varrho\langle sx, sx\rangle_E \varrho\langle tx, tx\rangle_E = \varrho\langle x, s^*sx\rangle_E \varrho\langle x, t^*tx\rangle_E \\ &= \varrho\langle x, ((s^*s)^{1/(1-\alpha)})^{1-\alpha}x\rangle_E \varrho\langle x, ((t^*t)^{1/\alpha})^{\alpha}x\rangle_E \\ &\leqslant (\varrho\langle x, (s^*s)^{1/(1-\alpha)}x\rangle_E)^{1-\alpha}(\varrho\langle x, (t^*t)^{1/\alpha}x\rangle_E)^{\alpha} \\ &\leqslant (1-\alpha)\varrho\langle x, (s^*s)^{1/(1-\alpha)}x\rangle_E + \alpha\varrho\langle x, (t^*t)^{1/\alpha}x\rangle_E \\ &\leqslant ((1-\alpha)(\varrho\langle x, (s^*s)^{1/(1-\alpha)}x\rangle_E)^r + \alpha(\varrho\langle x, (t^*t)^{1/(1-\alpha)}x\rangle_E)^r)^{1/r} \\ &\leqslant ((1-\alpha)\varrho\langle x, (s^*s)^{r/(1-\alpha)}x\rangle_E + \alpha\varrho\langle x, (t^*t)^{r/(1-\alpha)}x\rangle_E)^{1/r} \\ &= (\varrho\langle x, ((1-\alpha)(s^*s)^{r/(1-\alpha)} + \alpha(t^*t)^{r/(1-\alpha)})x\rangle_E)^{1/r} \\ &\leqslant \|(1-\alpha)(s^*s)^{r/(1-\alpha)} + \alpha(t^*t)^{r/(1-\alpha)}\|^{1/r}\end{aligned}$$

by Lemma 2.16, Corollary 3.9 (i) and Corollary 3.11. By taking supremum over $\varrho\langle x, x\rangle_E = 1$,

$$\omega_A^{2r}(s^*t) \leqslant \|\alpha(t^*t)^{r/\alpha} + (1-\alpha)(s^*s)^{r/(1-\alpha)}\|.$$

□

In 2005, Dragomir proved the following statement.

**Theorem 3.15** ([4]). *Let $T$, $S$ be two bounded linear operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If $r \geqslant 0$ and $\|T - S\| \leqslant r$, then*

$$\Big\|\frac{T^*T + S^*S}{2}\Big\| \leqslant \omega(T^*S) + \frac{1}{2}r^2.$$

The following theorem generalizes the above result for operators on Hilbert $C^*$-modules.

**Theorem 3.16.** *Suppose that $t, s \in \mathcal{L}(E)$, then*

$$\Big\|\frac{t^*t + s^*s}{2}\Big\| \leqslant \omega_A(s^*t) + \frac{1}{2}\|t - s\|^2.$$

P r o o f. For any $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$, we get

$$\begin{aligned}\|t-s\|^2 &= \|(t-s)^*(t-s)\| = \omega_A((t-s)^*(t-s)) \\ &\geqslant \varrho\langle x, (t-s)^*(t-s)x\rangle_E = \varrho\langle (t-s)(x), (t-s)(x)\rangle_E \\ &= \varrho\langle x, t^*tx\rangle_E - 2\,\mathrm{Re}\,\varrho\langle sx, tx\rangle_E + \varrho\langle x, s^*sx\rangle_E \\ &= \varrho\langle x, (t^*t + s^*s)x\rangle_E - 2\,\mathrm{Re}\,\varrho\langle x, s^*tx\rangle_E\end{aligned} \tag{3.6}$$

by Lemma 2.14.

By (3.6),

$$\varrho\langle x,(t^*t+s^*s)x\rangle_E \leqslant \|t-s\|^2 + 2\operatorname{Re}\varrho\langle x,s^*tx\rangle_E \leqslant \|t-s\|^2 + 2|\varrho\langle x,s^*tx\rangle_E| \leqslant \|t-s\|^2 + 2\omega_A(s^*t).$$

By taking supremum over $\varrho\langle x,x\rangle_E = 1$,

$$\omega_A(t^*t+s^*s) \leqslant \|t-s\|^2 + 2\omega_A(s^*t).$$

Since $t^*t+s^*s$ is self-adjoint, by Corollary 2.14 then

$$\|t^*t+s^*s\| = \omega_A(t^*t+s^*s) \leqslant \|t-s\|^2 + 2\omega_A(s^*t).$$

Therefore

$$\Big\|\frac{t^*t+s^*s}{2}\Big\| \leqslant \omega_A(s^*t) + \frac{1}{2}\|t-s\|^2.$$

□

The following theorem is established by Dragomir in [4].

**Theorem 3.17.** *Let $T$, $S$ be two bounded linear operators on a Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then*

$$\Big\|\frac{T+S}{2}\Big\|^2 \leqslant \frac{1}{2}\Big(\Big\|\frac{T^*T+S^*S}{2}\Big\| + \omega(S^*T)\Big).$$

The next result shows that Theorem 3.17 is true for operators on Hilbert $C^*$-modules.

**Theorem 3.18.** *Let $t,s \in \mathcal{L}(E)$. Then*

$$\Big\|\frac{t+s}{2}\Big\|^2 \leqslant \frac{1}{2}\Big(\Big\|\frac{t^*t+s^*s}{2}\Big\| + \omega_A(s^*t)\Big).$$

P r o o f. Let $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x,x\rangle_E = 1$. Then

$$\begin{aligned}2\omega_A(s^*t) + \|t^*t+s^*s\| &= 2\omega_A(s^*t) + \omega_A(t^*t+s^*s)\\ &= 2\sup_{\varrho\langle x,x\rangle_E=1}|\varrho\langle x,(s^*t)x\rangle_E| + \sup_{\varrho\langle x,x\rangle_E=1}\varrho\langle x,(t^*t+s^*s)x\rangle_E\\ &\geqslant \sup_{\varrho\langle x,x\rangle_E=1}(2|\varrho\langle x,(s^*t)x\rangle_E| + \varrho\langle x,(t^*t+s^*s)x\rangle_E)\\ &\geqslant 2|\varrho\langle x,(s^*t)x\rangle_E| + \varrho\langle x,(t^*t+s^*s)x\rangle_E\\ &\geqslant 2\operatorname{Re}\varrho\langle x,(s^*t)x\rangle_E + \varrho\langle x,(t^*t+s^*s)x\rangle_E\\ &= \varrho\langle (t+s)x,(t+s)x\rangle_E = \varrho\langle x,(t+s)^*(t+s)x\rangle_E\end{aligned}$$

by Corollary 2.14.

By taking supremum over $\varrho\langle x, x\rangle_E = 1$ we have

$$\omega_A((t+s)^*(t+s)) \leqslant 2\omega_A(s^*t) + \|t^*t + s^*s\|. \tag{3.7}$$

Since $(t+s)^*(t+s)$ is self-adjoint, by Corollary 2.14 then

$$\omega_A((t+s)^*(t+s)) = \|(t+s)^*(t+s)\| = \|t+s\|^2. \tag{3.8}$$

By (3.7) and (3.8) we have

$$\|t+s\|^2 \leqslant 2\omega_A(s^*t) + \|t^*t + s^*s\|.$$

Thus $2\|\frac{1}{2}(t+s)\|^2 \leqslant \|\frac{1}{2}(t^*t + s^*s)\| + \omega_A(s^*t)$. □

Mirmostafaee et al. in [18] proved for the following statement.

**Theorem 3.19** ([18], Theorem 3.1). *Let $T_j \in B(H)$ have the Cartesian decomposition $T_j = B_j + \mathrm{i}C_j$ for $j = 1, \dots, n$ and $r \geqslant 1$, then*

$$\omega^r\Bigl(\sum_{j=1}^{n} T_j\Bigr) \leqslant \bigl(\sqrt{2}n\bigr)^{r-1} \sum_{j=1}^{n} \||B_j|^{2r} + |C_j|^{2r}\|^{1/2}.$$

Now, we are ready to state the following extension of Theorem 3.19.

**Theorem 3.20.** *Let $t_j \in \mathcal{L}(E)$ have the Cartesian decomposition $t_j = b_j + \mathrm{i}c_j$ for $j = 1, \dots, n$ and $r \geqslant 1$, then*

$$\omega_A^r\Bigl(\sum_{j=1}^{n} t_j\Bigr) \leqslant \bigl(\sqrt{2}n\bigr)^{r-1} \sum_{j=1}^{n} \||b_j|^{2r} + |c_j|^{2r}\|^{1/2}. \tag{3.9}$$

P r o o f. According to Boher's inequality (see [9]) for every finite positive numbers $a_1, \dots, a_n$ and $r \geqslant 1$,

$$\Bigl(\sum_{i=1}^{n} a_i\Bigr)^r \leqslant n^{r-1} \sum_{i=1}^{n} a_i^r. \tag{3.10}$$

For every $1 \leqslant j \leqslant n$, $\varrho \in S(A)$ and $x \in E$ with $\varrho\langle x, x\rangle_E = 1$, we have

$$\begin{aligned}
|\varrho\langle x, t_j x\rangle_E|^2 &\leqslant \varrho\langle x, |t_j|x\rangle_E \varrho\langle x, |t_j^*|x\rangle_E \leqslant \Bigl(\frac{\varrho\langle x, |t_j|x\rangle_E + \varrho\langle x, |t_j^*|x\rangle_E}{2}\Bigr)^2 \\
&\leqslant \frac{\varrho\langle x, |t_j|x\rangle_E^2 + \varrho\langle x, |t_j^*|x\rangle_E^2}{2} \leqslant \frac{\varrho\langle x, |t_j|^2x\rangle_E + \varrho\langle x, |t_j^*|^2x\rangle_E}{2} \\
&= \frac{\varrho\langle x, (|t_j|^2 + |t_j^*|^2)x\rangle_E}{2}
\end{aligned}$$

by Corollaries 3.9 and 3.11.

It follows that for every $x \in E$ and $\varrho \in S(A)$ with $\varrho\langle x, x\rangle_E = 1$, we have

$$\begin{aligned}
\Big|\varrho\Big\langle x, \sum_{j=1}^{n} t_j x\Big\rangle_E\Big|^r &\leqslant \Big(\sum_{j=1}^{n}\Big(\frac{1}{2}\varrho\langle x, (t_j^* t_j + t_j t_j^*)x\rangle_E\Big)^{1/2}\Big)^r \\
&= \Big(\sum_{j=1}^{n} \varrho\langle x, (|b_j|^2 + |c_j|^2)x\rangle_E^{1/2}\Big)^r \\
&\leqslant \sum_{j=1}^{n} n^{r-1}(\varrho\langle x, |b_j|^2 x\rangle_E + \varrho\langle x, |c_j|^2 x\rangle)^{r/2} \\
&\leqslant \sum_{j=1}^{n} n^{r-1}(2^{r-1}(\varrho\langle x, |b_j|^2 x\rangle_E^r + \varrho\langle x, |c_j|^2 x\rangle^r))^{1/2} \\
&\leqslant \sum_{j=1}^{n} \big(\sqrt{2}n\big)^{r-1}(\varrho\langle x, |b_j|^{2r} x\rangle_E + \varrho\langle x, |c_j|^{2r} x\rangle_E)^{1/2} \\
&\leqslant \sum_{j=1}^{n} \big(\sqrt{2}n\big)^{r-1}\Big(\sup_{\varrho\in S(A)} \varrho\langle x, |b_j|^{2r} + |c_j|^{2r} x\rangle_E\Big)^{1/2} \\
&= \sum_{j=1}^{n} \big(\sqrt{2}n\big)^{r-1}\|\langle x, |b_j|^{2r} + |c_j|^{2r} x\rangle_E\|^{1/2} \\
&\leqslant \sum_{j=1}^{n} \big(\sqrt{2}n\big)^{r-1}\||b_j|^{2r} + |c_j|^{2r}\|^{1/2}
\end{aligned}$$

by (3.10) and Corollary 3.11. By taking supremum over all $\varrho \in S(A)$ and $x \in E$ with $\varrho(\langle x, x\rangle) = 1$ we get (3.9). □

In order to obtain an application of the above result, we need the following.

**Lemma 3.21** ([9], Hardy's inequalities). *If $r > 1$ and $(a_n)$ are positive real numbers such that $0 < \sum_{n=1}^{\infty} a_n^r < \infty$, then*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{j=1}^{n} a_j\Big)^r \leqslant \Big(\frac{r}{r-1}\Big)^r \sum_{n=1}^{\infty} a_n^r.$$

**Theorem 3.22.** *Let $\{t_n\}$ be a sequence in $\mathcal{L}(E)$ and $r > 1$. Suppose that for every $j \geqslant 1$, $t_j = a_j + \mathrm{i}b_j$ is the Cartesian decomposition of $t_j$ and $0 < \sum_{n=1}^{\infty} \||b_n|^{2r} + |c_n|^{2r}\|^{r/2} < \infty$, then*

$$\sum_{n=1}^{\infty} \omega_A^r\Big(\frac{t_1 + \ldots + t_n}{n}\Big) \leqslant 2^{(r-1)/2}\Big(\frac{r}{r-1}\Big)^r \sum_{n=1}^{\infty} \||b_n|^{2r} + |c_n|^{2r}\|^{r/2}.$$

P r o o f. Let $t_j = b_j + \mathrm{i}c_j$ where $b_j, c_j \in \mathcal{L}(E)_{sa}$. By Theorem 2.5,

$$\omega_A^r\Bigl(\sum_{j=1}^n t_j\Bigr) \leqslant (\sqrt{2}n)^{r-1}\sum_{j=1}^n \||b_j|^{2r} + |c_j|^{2r}\|^{1/2} \tag{3.11}$$

and thus

$$\omega_A^r\Bigl(\frac{t_1+\ldots+t_n}{n}\Bigr) \leqslant (\sqrt{2})^{r-1}\frac{1}{n}\sum_{j=1}^n \||b_j|^{2r} + |c_j|^{2r}\|^{1/2}. \tag{3.12}$$

Hence

$$\begin{aligned}\sum_{n=1}^\infty \omega_A^r\Bigl(\frac{t_1+\ldots+t_n}{n}\Bigr) &\leqslant 2^{(r-1)/2}\sum_{n=1}^\infty \frac{1}{n}\sum_{j=1}^n \||b_j|^{2r} + |c_j|^{2r}\|^{1/2}\\ &\leqslant 2^{(r-1)/2}\Bigl(\frac{r}{r-1}\Bigr)^r\sum_{n=1}^\infty \||b_n|^{2r} + |c_n|^{2r}\|^{r/2}\end{aligned}$$

by Lemma 3.21. □

## 4. Concluding remarks

We presented a new method for studying the numerical range of bounded operators on Hilbert $C^*$-modules. We proved that some results concerning numerical radius of bounded operators on Hilbert spaces remain true when the operators are defined on a Hilbert $C^*$-module. It seems that our method is also applicable for proving other inequalities for operators on Hilbert $C^*$-modules.

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