ON FUNCTIONS WEAKLY COMPUTABLE BY PUSHDOWN PETRI NETS AND RELATED SYSTEMS

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ABSTRACT. We consider numerical functions weakly computable by grammar-controlled vector addition systems (GVASes, a variant of pushdown Petri nets). GVASes can weakly compute all fast growing functions F_α for $\alpha < \omega^\omega$, hence they are computationally more powerful than standard vector addition systems. On the other hand they cannot weakly compute the inverses F_α^{-1} or indeed any sublinear function. The proof relies on a pumping lemma for runs of GVASes that is of independent interest.

1. INTRODUCTION

Pushdown Petri nets are Petri nets extended with a pushdown stack. They have been used to model asynchronous programs [34] and, more generally, recursive programs with integer variables [2]. They sometimes appear under a different but essentially equivalent guise: stack/pushdown/context-free vector addition systems [19, 22, 26], partially blind multi-counter machines [10] with a pushdown stack, etc. It is not yet known whether reachability is decidable for pushdown Petri nets and this is one of the major open problems in computer science. However, a series of recent results improved our understanding of the computational power of these models: coverability, reachability and boundedness are TOWER-hard [19, 21], and boundedness is solvable in hyper-Ackermannian time [22].

With the present article, we contribute to this line of work. We recall *Grammar-Controlled Vector Addition Systems* [26], or GVAS, a variant model, close to Pushdown Petri nets, where the pushdown stack is replaced by a context-free restriction on the firing of rules. The runs are now naturally organized in a derivation tree, and the stack is not actually present in the configurations: this leads to a simplified mathematical treatment, where the usual monotonicity properties of VASes can be put to use.

As a step towards understanding the expressive power of these GVASes, we consider the number-theoretical functions that are *weakly computable* in this model. Restricting to weakly computing a numerical function is a natural idea when dealing with models like VASes and GVASes that lack zero-tests, or, more precisely, that cannot initiate a given action on the condition that a counter's value is zero, only on the condition that it is not zero.

This notion has been used since the early days of Petri nets and has proved very useful in hardness or impossibility proofs: For Petri nets and VASSes, the undecidability of equivalence problems, and the Ackermann-hardness of the same problems for bounded systems, have been proved using the fact that multivariate polynomials with positive integer coefficients —aka positive Diophantine polynomials— and, respectively, the fast-growing functions $(F_i)_{i \in \mathbb{N}}$ in the Grzegorczyk hierarchy, are all weakly computable [12, 28, 16]. More recently, the nonelementary complexity lower bound for VASS reachability is obtained thanks to a uniform (polynomial size) family of systems computing (exactly) $n\text{-EXP}(2)$ from n [5].

The above results rely on showing how *some* useful functions are weakly computable by Petri nets and VASSes. But not much is known about exactly which functions are weakly computable or not. It is known that all such functions are monotonic. They are all primitive-recursive. The class of weakly computable functions is closed under composition.

In this article, we show that functions weakly computable by GVASes go beyond those weakly computable by VASSes, in particular we show how to weakly compute the Fast Growing (F_α) for all $\alpha < \omega^\omega$.

A folklore conjecture states that the inverses of the fast-growing functions are not weakly computable by Petri nets. It is stated as fact in [32, p.252] but no reference is given. In this article, we settle the issue by proving that any unbounded function weakly computable by Petri nets and more generally by GVASes is in $\Omega(x)$, i.e., it eventually dominates $c \cdot x$ for some constant $c > 0$. Thus any function that is sublinear, like $x \mapsto \lfloor \sqrt{x} \rfloor$, or $x \mapsto \lfloor \log x \rfloor$, is not weakly computable by GVASes. The proof technique is interesting in its own right: it relies on a pumping lemma on runs of GVASes that could have wider applications. This pumping lemma follows from a well-quasi-ordering on the set of runs that further directs it.

Beyond Petri nets and VASSes. Petri nets and VASSes are a classic example of well-structured systems [1, 8]. In recent years, weakly computing numerical functions has proved to be a fundamental tool for understanding the expressive power and the complexity of some families of well-structured systems that are more powerful than Petri nets and VASSes [33, 14, 11]. For such systems, the hardness proofs rely on weakly computing fast-growing functions $(F_\alpha)_{\alpha \in Ord}$ that extend Grzegorczyk's hierarchy. These hardness proofs also crucially rely on weakly computing the inverses of the F_α's.

There are several extensions of Petri nets for which reachability (or coverability or boundedness) remains decidable: nets with nested zero-tests [30], recursive VASSes [3] and Branching VASSes [7], VASSes with pointers to counters [6], unordered data Petri nets [20], etc., and of course pushdown VASes and GVASes. For the latter, while coverability and reachability are still open in general, partial decidability results have been obtained by looking at sub-classes, namely GVASes with finite-index grammars [2] and GVASes of dimension one [26]. In many cases, it is not known how these extensions compare in expressive power and in complexity. We believe that weakly computable functions can be a useful tool when addressing these questions.

Related models. The GVAS model can simulate counter machines extended with nested zero-tests (from [30]), and the vector addition systems extended with a pushdown stack (from [22]). The first simulation was shown in [2] and holds even for GVASes with finite-index grammars. The second one comes from the classical transformation of a pushdown automaton into a context-free grammar that recognizes the same language. There exists still other models that extend vector addition systems with stack-related mechanisms, e.g., Mayr's Process Rewrite Systems [29] or Haddad and Poitrenaud's Recursive Petri Nets [13]. Pending some further, more formal, comparison, it seems that these models are less expressive than Pushdown VASes since they only allow limited interactions between stack and counters.

Outline of the paper. Section 2 introduces GVASes and fixes some notation. In Section 3, we introduce flow trees, a tree-shaped version of runs of GVASes for which we develop our two main tools: a well-quasi-ordering between flow trees and an Amalgamation Theorem. The following two sections explore applications of the Amalgamation Theorem in understanding the computing power of GVASes: via GVAS-definable sets in Section 4, via weakly computable function in Section 5. Finally, we show in Section 6 that GVASes can weakly compute all Fast-Growing functions F_α for $\alpha < \omega^\omega$.

2. Grammar-Controlled Vector Addition Systems

This section recalls the model of grammar-controlled vector addition systems, originally from [26]. In a nutshell, these are intersections of classical VAS with context-free grammars. Remark 2.3 relates them with the equivalent model of pushdown vector addition systems.

Vector Addition Systems. For a *dimension* $d \in \mathbb{N}$, we consider *configurations* that are vectors $\mathbf{c}, \mathbf{d}, \mathbf{x}, \mathbf{y}, \ldots$ in $\mathbb{N}^d$, and *actions* that are vectors $\mathbf{a} \in \mathbb{Z}^d$. We write $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$ for two configurations $\mathbf{x}, \mathbf{y}$ in $\mathbb{N}^d$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$. A *vector addition system* (a *VAS*) is a transition system of the form $(\mathbb{N}^d, \{\xrightarrow{\mathbf{a}}\}_{\mathbf{a} \in \mathbf{A}})$ generated by a finite set $\mathbf{A} \subseteq \mathbb{Z}^d$ of actions.

In a VAS, the one-step transition relations $\{\xrightarrow{\mathbf{a}}\}_{\mathbf{a} \in \mathbf{A}}$ are composed in a natural way: with any word $w = \mathbf{a}_1 \cdots \mathbf{a}_k \in \mathbf{A}^*$ of actions, we associate the binary relation $\xrightarrow{w}$ defined over configurations by $\mathbf{x} \xrightarrow{w} \mathbf{y}$ iff there exists a sequence $\mathbf{c}_0, \ldots, \mathbf{c}_k$ of configurations such that $\mathbf{c}_0 = \mathbf{x}$, $\mathbf{c}_k = \mathbf{y}$ and such that $\mathbf{c}_{j-1} \xrightarrow{\mathbf{a}_j} \mathbf{c}_j$ for every $1 \leq j \leq k$. Those relations are monotonic:

$$\mathbf{x} \xrightarrow{w} \mathbf{y} \text{ and } \mathbf{v} \in \mathbb{N}^d \text{ implies } \mathbf{x} + \mathbf{v} \xrightarrow{w} \mathbf{y} + \mathbf{v} . \tag{2.1}$$

Notation. When writing configurations $\mathbf{c} \in \mathbb{N}^d$, we sometimes split the vector in parts, writing e.g., $\mathbf{c} = (\mathbf{x}, \mathbf{y})$ for some $\mathbf{x} \in \mathbb{N}^{d_1}$ and $\mathbf{y} \in \mathbb{N}^{d_2}$ with $d = d_1 + d_2$. We also write $\mathbf{0}_d$ for the null vector in $\mathbb{N}^d$, often leaving the dimension implicit.

Grammar-controlled Vector Addition Systems. A d-dimensional *grammar-controlled vector addition system* (a *GVAS*) can be seen as a context-free grammar using terminals from $\mathbb{Z}^d$, or equivalently as a VAS where the valid sequences of actions are generated by a context-free grammar. Formally, a d-dimensional GVAS is some $G = (V, \mathbf{A}, R, S)$ where V is a finite set of *nonterminals*, where $\mathbf{A} \subseteq \mathbb{Z}^d$ is a finite set of terminals called *actions*, where $R \subseteq V \times (V \cup \mathbf{A})^*$ is a finite set of production *rules*, and $S \in V$ is the *start symbol*. Following the usual convention, a rule (T, u) is also written $T \to u$. We denote nonterminals from V with capital letters like $S, T, \ldots$ while symbols from the larger set $V \cup \mathbf{A}$ are denoted with $X, Y, \ldots$. Words in $(V \cup \mathbf{A})^*$ are denoted with $w, u, v, \ldots$. As usual, ε denotes the empty word.

For all words $w, w' \in (V \cup \mathbf{A})^*$, we say that $w \implies w'$ is a *derivation step* of G if there exist two words v, v' in $(V \cup \mathbf{A})^*$ and a rule (T, u) in R such that $w = vTv'$ and $w' = vuv'$. Let $\overset{*}{\implies}$ denote the reflexive and transitive closure of $\implies$. The language $L_G \subseteq \mathbf{A}^*$ generated by G seen as a grammar is defined as usual with $w \in L_G \overset{\text{def}}{\Leftrightarrow} S \overset{*}{\implies} w \in \mathbf{A}^*$. More generally, for any $u \in (V \cup \mathbf{A})^*$, the language $L_G(u)$ is $\{w \in \mathbf{A}^* \mid u \overset{*}{\implies} w\}$.

When G is a GVAS, we are interested in what sequences of actions may occur between configurations in $\mathbb{N}^d$. For this, we extend the definition of the $\overset{w}{\to}$ relation and consider $\overset{u}{\to}$ for any $u \in (V \cup \mathbf{A})^*$. Formally, we let

$$\mathbf{x} \overset{u}{\to} \mathbf{y} \overset{\text{def}}{\Leftrightarrow} \exists w \in L_G(u) : \mathbf{x} \overset{w}{\to} \mathbf{y} \,. \tag{2.2}$$

A labeled pair $\mathbf{x} \overset{u}{\to} \mathbf{y}$ is called a *run* of the GVAS, and should not be confused with the derivations $w \overset{*}{\implies} w'$ that only involve the grammar part.

Like VASes, GVASes are monotonic:

$$\mathbf{x} \overset{u}{\to} \mathbf{y} \text{ and } \mathbf{v} \in \mathbb{N}^d \text{ implies } \mathbf{x} + \mathbf{v} \overset{u}{\to} \mathbf{y} + \mathbf{v} \,. \tag{2.3}$$

The underlying grammar G is left implicit in the above notation. We sometimes write $\mathbf{x} \overset{G}{\to} \mathbf{y}$ instead of $\mathbf{x} \overset{S}{\to} \mathbf{y}$, where S is the start symbol of G, when several grammars are considered simultaneously.

Example 2.1. Let $d = 1$, $V = \{S, T\}$, and consider the 1-dimensional GVAS given by the following four rules in Backus-Naur form:

$$S \to \mathbf{1} \mid -\mathbf{1}\, S\, T \,, \qquad\qquad\qquad T \to \mathbf{0} \mid -\mathbf{1}\, T\, \mathbf{2} \,.$$

Since we shall claim in Section 5 that this GVAS weakly computes the 2^n function, let us state and prove the main properties of its runs. Formally, for every $k, k', n, n' \in \mathbb{N}$, one has:

$$k \overset{T}{\to} k' \text{ iff } k \leq k' \leq 2k \,, \qquad\qquad n \overset{S}{\to} n' \text{ iff } 1 \leq n' \leq 2^n \,. \tag{2.4}$$

Thus, even in dimension 1, the reachability relation may not be semilinear.

To prove (2.4) assume first that $k \overset{T}{\to} k'$ for some natural numbers k, k'. There exists $m \in \mathbb{N}$ such that $k \xrightarrow{-\mathbf{1}^m \mathbf{0}\, \mathbf{2}^m} k'$. In particular $m \leq k$ and $k' = k + m$. We deduce that $k \leq k' \leq 2k$. Conversely, let k, k' be two natural numbers such that $k \leq k' \leq 2k$. Observe that $T \overset{*}{\implies} -\mathbf{1}^n \mathbf{0}\, \mathbf{2}^n$ where n is defined as $k' - k$. The following relations show that $k \overset{T}{\to} k'$:

$$k \xrightarrow{-\mathbf{1}^n} k - n \overset{\mathbf{0}}{\to} k - n \xrightarrow{\mathbf{2}^n} k' \,.$$

Now, assume that $n \xrightarrow{S} n'$ for some natural numbers n, n', There exists $m \in \mathbb{N}$ such that $n \xrightarrow{-\mathbf{1}^m \mathbf{1} T^m} n'$. It follows that $m \leq n$, and from the previous paragraph, we deduce that $n' \leq (n - m + 1)2^m \leq 2^n$ by observing that $x + 1 \leq 2^x$ for every $x \in \mathbb{N}$ and by replacing x by $n - m$. Conversely, let n, n' be two natural numbers such that $1 \leq n' \leq 2^n$. Observe that $S \overset{*}{\Longrightarrow} -\mathbf{1}^n \mathbf{1} T^n$. Let us introduce a natural number m in $\{0, \ldots, n-1\}$ such that $2^m \leq n' \leq 2^{m+1}$. The following relations show that $n \xrightarrow{S} n'$:

$$n \xrightarrow{-\mathbf{1}^n} 0 \xrightarrow{\mathbf{1}} 2^0 \xrightarrow{T} 2^1 \cdots \xrightarrow{T} 2^m \xrightarrow{T} n' \xrightarrow{T^{n-1-m}} n' \, . \qquad \square$$

Example 2.2. Let G be the 2-dimensional GVAS with a single nonterminal symbol and the following three rules:

$$S \to S\,S \mid \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right) \mid \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right) \, .$$

Let $w = \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right) \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)$ and observe that $S \overset{*}{\Longrightarrow} w$. We have $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right) \xrightarrow{\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)} \left(\begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right) \xrightarrow{\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right)} \left(\begin{smallmatrix} 3 \\ 3 \end{smallmatrix} \right) \xrightarrow{\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)} \left(\begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right)$ and hence $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right) \xrightarrow{S} \left(\begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right)$. $\qquad \square$

Remark 2.3. It is well known that, from a formal language viewpoint, context-free grammars are equivalent to pushdown automata. Similarly, GVAS can be equivalently presented as VAS extended with a pushdown stack. Formally, a d-dimensional *Pushdown Vector Addition System* (a *PVAS*) is a transition system of the form $(\mathbb{N}^d \times \Gamma^*, \{\xrightarrow{\mathbf{P}}\}_{\mathbf{p} \in \mathbf{P}})$ generated by a pair $(\Gamma, \mathbf{P})$ where Γ is a finite *stack alphabet* and $\mathbf{P} \subseteq \Gamma^* \times \Gamma^* \times \mathbb{Z}^d$ is a finite set of *actions*. So configurations are now pairs $(\mathbf{x}, u)$ where $\mathbf{x} \in \mathbb{N}^d$ is as for VAS and $u \in \Gamma^*$ is a word denoting the contents of the stack. Intuitively, an action $\mathbf{p} = (\alpha, \beta, \mathbf{a})$ pops the string α from the top of the stack, then pushes the string β onto the top of the stack, and adds $\mathbf{a}$ to the vector of natural numbers. Formally, each action $(\alpha, \beta, \mathbf{a}) \in \mathbf{P}$ induces a binary relation $\xrightarrow{(\alpha,\beta,\mathbf{a})}$ on configurations defined by $(\mathbf{x}, u) \xrightarrow{(\alpha,\beta,\mathbf{a})} (\mathbf{y}, v)$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$ and there exists w such that $u = \alpha w$ and $v = \beta w$. GVASes can be translated into equivalent PVASes and vice-versa.

For instance, the PVAS corresponding to Example 2.2 is generated by the pair $(\Gamma, \mathbf{P})$ where $\Gamma = \{S\}$ and $\mathbf{P}$ is the set of actions $\{(S, SS, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)), (S, \varepsilon, \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)), (S, \varepsilon, \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right))\}$. Corresponding to $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right) \xrightarrow{w} \left(\begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right)$ with $w = \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right) \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)$ there, we have

$$(S, \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right)) \xrightarrow{S,SS,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)} (SS, \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right)) \xrightarrow{S,\varepsilon,\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)} (S, \left(\begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right)) \xrightarrow{S,SS,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)} (SS, \left(\begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right)) \xrightarrow{S,\varepsilon,\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right)} (S, \left(\begin{smallmatrix} 3 \\ 3 \end{smallmatrix} \right)) \xrightarrow{S,\varepsilon,\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix} \right)} (\varepsilon, \left(\begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right))$$

in the PVAS.

3. Well-Quasi-Ordering Runs in GVASes

In this section we define flow trees of GVASes and show that they satisfy an amalgamation property. This property is used in the next section to provide a geometrical decomposition of GVAS sets, and in the following section to show that unbounded weakly computable functions are in $\Omega(n)$.
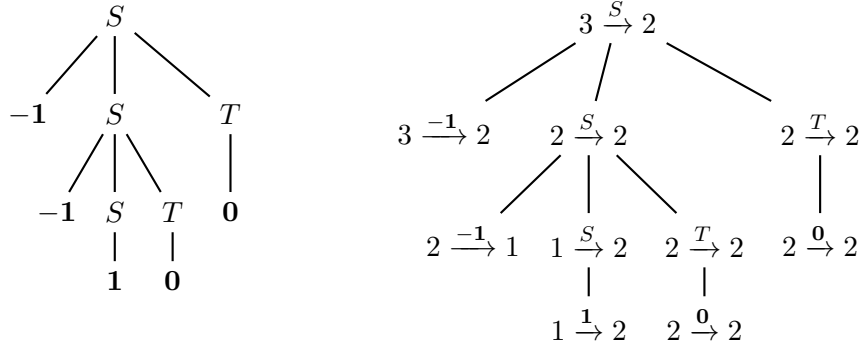
Figure 1: Continuing Example 2.1: $S \to \mathbf{1} \mid -\mathbf{1}\,S\,T$ and $T \to \mathbf{0} \mid -\mathbf{1}\,T\,\mathbf{2}$.
Left: Derivation tree witnessing $S \xRightarrow{*} w$, with $w = -\mathbf{1}-\mathbf{1}\,\mathbf{1}\,\mathbf{0}\,\mathbf{0}$.
Right: Flow tree witnessing $3 \xrightarrow{w} 2$.

Let $G = (V, \mathbf{A}, R, S)$ be a d-dimensional GVAS. *Flow trees* of G are trees that combine a transition $\mathbf{x} \xrightarrow{w} \mathbf{y}$ in the VAS part of G with a derivation tree for the corresponding $S \xRightarrow{*} w$ in the grammar part of G.

Flow trees are finite rooted ordered trees labeled with transitions of G: we write $t = \sigma[t_1, \ldots, t_\ell]$ to denote a flow tree t made of a root with ℓ subtrees $t_1, \ldots, t_\ell$. The root is labeled by a transition σ of G, of the form $\mathbf{c} \xrightarrow{X} \mathbf{d}$ with $X \in V \cup \mathbf{A}$. We write $\mathrm{root}(t) = \sigma$. Formally, $F(G)$ is the least set of trees that contains all $(\mathbf{c} \xrightarrow{\mathbf{a}} \mathbf{d})[]$ with $\mathbf{a} \in \mathbf{A}$ and $\mathbf{c} + \mathbf{a} = \mathbf{d}$, and all $(\mathbf{c} \xrightarrow{T} \mathbf{d})[t_1, \ldots, t_\ell]$ with $T \in V$ and $t_1, \ldots, t_\ell \in F(G)$ such that there is a rule $T \Longrightarrow X_1 \cdots X_\ell$ in R and configurations $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell$ with $\mathbf{c}_0 = \mathbf{c}$, $\mathbf{c}_\ell = \mathbf{d}$ and such that, for $i = 1, \ldots, \ell$, the root of t_i is labeled with $\mathbf{c}_{i-1} \xrightarrow{X_i} \mathbf{c}_i$. A *subtree* of $t = \sigma[t_1, \ldots, t_\ell]$ is either t itself or a subtree of some t_i for $i = 1, \ldots, \ell$. A (sub)tree $(\mathbf{c} \xrightarrow{X} \mathbf{d})[t_1, \ldots, t_\ell]$ is a *leaf* when $\ell = 0$: this requires that $X = \mathbf{a} \in \mathbf{A}$ is an action (and then $\mathbf{d} = \mathbf{c} + \mathbf{a}$) or that $X = T \in V$ is a non-terminal and $T \Longrightarrow \varepsilon$ is a rule in R (and then $\mathbf{d} = \mathbf{c}$).

As is standard, we use *positions* to identify occurrences of subtrees inside t. Formally, a position is a finite sequence of natural numbers, and the positions of the subtrees of t, denoted $Pos(t)$ are given inductively by

$$Pos(\sigma[t_1, \ldots, t_l]) \stackrel{\mathrm{def}}{=} \{\varepsilon\} \cup \{i.q \mid 1 \le i \le l \wedge q \in Pos(t_i)\}\,.$$

For $p \in Pos(t)$, the subtree of t at position p is denoted t/p.

Example 3.1. Recall the 1-dimensional GVAS G from Example 2.1. The grammar admits, among others, a derivation $S \xRightarrow{*} w$ for $w = -\mathbf{1}-\mathbf{1}\,\mathbf{1}\,\mathbf{0}\,\mathbf{0}$. Thus $3 \xrightarrow{w} 2$ is a transition in G.

In Fig. 1 we display a derivation tree witnessing $S \xRightarrow{*} w$ and a flow tree witnessing $\mathbf{c} \xrightarrow{w} \mathbf{d}$ for $\mathbf{c} = 3$ and $\mathbf{d} = 2$.   □

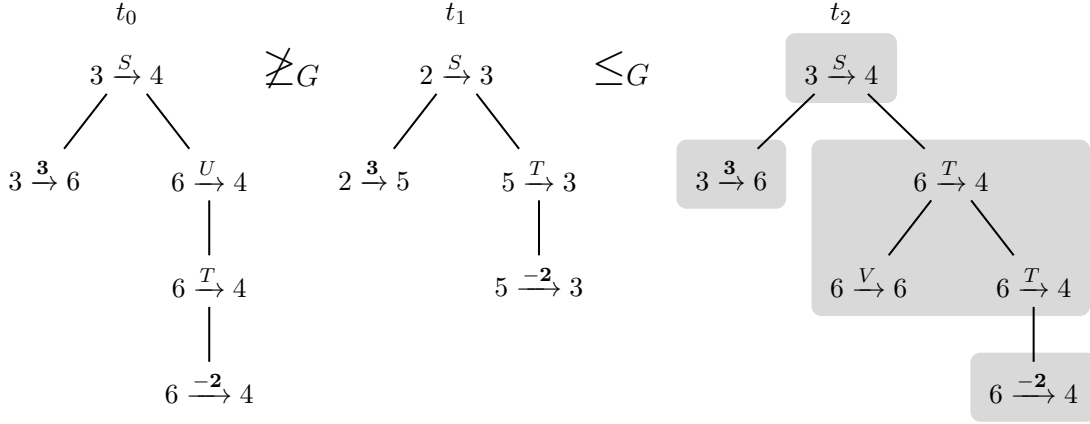We now extend to flow trees of GVASes an ordering initially introduced by Jančar for runs of VASes [15, Def. 6.4].

Figure 2: Illustration for Example 3.4: $S \to \mathbf{3}\, T \mid \mathbf{3}\, U$, $T \to -\mathbf{2} \mid V\, T$, $U \to T$ and $V \to \epsilon$.

Definition 3.2 (Ordering GVAS transitions and flow trees)**.** For two transitions $\sigma = \mathbf{c} \xrightarrow{X} \mathbf{d}$ and $\theta = \mathbf{c}' \xrightarrow{X'} \mathbf{d}'$ with $X, X' \in V \cup \mathbf{A}$, we let

$$\sigma \le \theta \stackrel{\text{def}}{\Leftrightarrow} \mathbf{c} \le \mathbf{c}' \wedge \mathbf{d} \le \mathbf{d}' \wedge X = X'\,.$$

The ordering $\le_G$ between flow trees $s, t \in F(G)$ is defined by induction on the structure of trees: $s = \sigma[s_1, \ldots, s_k] \le_G t = \theta[t_1, \ldots, t_\ell]$ if, and only if, $\sigma \le \theta$ and there exists a subtree t' of t of the form $t' = \theta'[t'_1, \ldots, t'_{\ell'}]$ with $\sigma \le \theta'$, $\ell' = k$ and $s_j \le_G t'_j$ for every $1 \le j \le k$.

This definition is well-founded and, since the subtree relation is transitive, $\le_G$ is clearly reflexive and transitive, i.e., is a quasi-ordering. In the appendix, we prove the following key property:

Lemma 3.3 (See Appendix A)**.** $(F(G), \le_G)$ *is a well-quasi-ordering.*

In other words, any infinite sequence $s_0, s_1, s_2, \ldots$ of flow trees contains an infinite increasing subsequence $s_{i_0} \le_G s_{i_1} \le_G s_{i_2} \le_G \cdots$.

Example 3.4. The flow trees shown in Fig. 2 illustrate the ordering $\le_G$, on a 1-dimensional GVAS with non-terminals S, T, U, V and the following rules: $S \to \mathbf{3}\, T \mid \mathbf{3}\, U$, $T \to -\mathbf{2} \mid V\, T$, $U \to T$ and $V \to \epsilon$. We can see that $t_1 \le_G t_2$ from the following orderings on the subtrees of t_1 and t_2: $t_1/1 \le_G t_2/1$, $t_1/2 \le_G t_2/2$ and $t_1/21 \le_G t_2/221$. It can be verified that $t_1 \not\le_G t_0$. However, $t_1 \sqsubseteq t_0$, where $\sqsubseteq$ is the standard homeomorphic embedding [1] of labeled trees. This can be seen by the following orderings: $t_1/1 \sqsubseteq t_0/1$, $t_1/2 \sqsubseteq t_0/21$ and $t_1/21 \sqsubseteq t_0/211$.

When $\sigma \le \theta$ for some $\sigma = \mathbf{c} \xrightarrow{X} \mathbf{d}$ and $\theta = \mathbf{c}' \xrightarrow{X} \mathbf{d}'$, we also write $\sigma \le^\Delta \theta$ with $\Delta = (\mathbf{c}' - \mathbf{c}, \mathbf{d}' - \mathbf{d})$. Similarly, we write $s \le_G^\Delta t$ for two flow trees s and t when $s \le_G t$ and $\mathrm{root}(s) \le^\Delta \mathrm{root}(t)$.

The pair Δ is called a *lifting*. Note that necessarily Δ belongs to $(\mathbb{N}^d)^2$ and that $\sigma \le^\Delta \theta$ and $\theta \le^{\Delta'} \rho$ entail $\sigma \le^{\Delta + \Delta'} \rho$. We write $\rho = \sigma + \Delta$ when $\sigma \le^\Delta \rho$. Two liftings $\Delta = (\mathbf{a}, \mathbf{b})$ and $\Delta' = (\mathbf{a}', \mathbf{b}')$ can be chained if $\mathbf{b} = \mathbf{a}'$. In this case we let $\Delta \cdot \Delta' \stackrel{\text{def}}{=} (\mathbf{a}, \mathbf{b}')$. Note this partial operation is associative.

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[1] defined by $s = \sigma[s_1, \ldots, s_k] \sqsubseteq t$ if, and only if, there exists a subtree $t' = \theta'[t'_1, \ldots, t'_{\ell'}]$ of t such that $\sigma \le \theta'$, $k \le \ell'$ and $s_1 \sqsubseteq t'_{j_1}, \ldots, s_k \sqsubseteq t'_{j_k}$ for some subsequence $1 \le j_1 < \cdots < j_k \le \ell'$.

When $t/p = t' \leq_G u$ we can replace t' by u inside t but this requires a bit of surgery to ensure the result is well-formed. First, for a flow tree t and a displacement $\mathbf{a} \in \mathbb{N}^d$, we let $t + \mathbf{a}$ be the tree defined via

$$\sigma[t_1, \ldots, t_\ell] + \mathbf{a} \overset{\text{def}}{=} (\sigma + (\mathbf{a}, \mathbf{a}))[t_1 + \mathbf{a}, \ldots, t_\ell + \mathbf{a}] .$$

Obviously, $t + \mathbf{a}$ is a valid flow tree, with $t \leq_G^{(\mathbf{a},\mathbf{a})} (t + \mathbf{a})$. Now, when $t/p = t' \leq_G^\Delta u$ for $\Delta = (\mathbf{a}, \mathbf{b})$, we define $t[u]_p$ by induction on p in the following way:

$$t[u]_\varepsilon \overset{\text{def}}{=} u, \qquad t[u]_{i.q} \overset{\text{def}}{=} (\sigma + \Delta)[t_1 + \mathbf{a}, \ldots, t_{i-1} + \mathbf{a}, t_i[u]_q, t_{i+1} + \mathbf{b}, \ldots, t_k + \mathbf{b}].$$

Claim 3.5. If $t/p \leq_G^\Delta u$ then $t[u]_p$ is a valid flow tree satisfying $t \leq_G^\Delta t[u]_p$.

Proof. By induction on p. If $p = \varepsilon$ the claim holds trivially. Assume $p = i.q$ with $1 \leq i \leq k$ and let $u' = t_i[u]q$. By induction hypothesis, $t_i \leq_G^\Delta u'$. This implies that $t[u]_p$ is a well-defined flow tree. Since furthermore $t_j \leq_G t_j + \mathbf{a}$ when $1 \leq j < i$, and symmetrically, $t_j \leq_G t_j + \mathbf{b}$ when $i < j \leq k$, we see that $t \leq_G t[u]_p$. Finally, we observe that $\text{root}(t) \leq^\Delta \text{root}(t[u]_p)$. $\square$

Lemma 3.6. *Let $t = \sigma[t_1, \ldots, t_k]$ and assume $t_i \leq_G^{\Delta_i} u_i$ for $i = 1, \ldots, k$. If $\Delta_1 \cdot \Delta_2 \cdots \Delta_k = \Delta$ is defined, then $u = (\sigma + \Delta)[u_1, \ldots, u_k]$ is a valid flow tree satisfying $t \leq_G^\Delta u$.*

Proof. Since $\Delta = \Delta_1 \cdots \Delta_k$ is defined, we can write $\Delta_i = (\mathbf{a}_{i-1}, \mathbf{a}_i)$ and $\Delta = (\mathbf{a}_0, \mathbf{a}_k)$. Assume $\sigma = \mathbf{c}_0 \xrightarrow{X} \mathbf{c}_k$, with $\text{root}(t_i) = \mathbf{c}_{i-1} \xrightarrow{Y_i} \mathbf{c}_i$ for $i = 1, \ldots, k$. Then $\text{root}(u_i) = (\mathbf{c}_{i-1} + \mathbf{a}_{i-1}) \xrightarrow{Y_i} (\mathbf{c}_i + \mathbf{a}_i)$ and u is a valid transition. That $t \leq_G u$ is immediate. $\square$

Theorem 3.7 (Amalgamation). *If $s \leq_G^{\Delta_1} t_1$ and $s \leq_G^{\Delta_2} t_2$ then there exists s' s.t. $t_1 \leq_G^{\Delta_2} s'$ and $t_2 \leq_G^{\Delta_1} s'$ (further entailing $s \leq_G^{\Delta_1+\Delta_2} s'$).*

Proof. By induction on s. Assume $s = \sigma[s_1, \ldots, s_k]$. Since $s \leq_G^{\Delta_1} t_1$, there is a subtree $t_1/p = t^1 = \rho_1[t_1^1, \ldots, t_k^1]$ of t_1 such that $\sigma \leq \rho_1$ and $s_j \leq_G t_j^1$ for all $j = 1, \ldots, k$. Assume that $\sigma \leq^{\Delta_1'} \rho_1$ and that $s_j \leq_G^{\Gamma_j} t_j^1$ for $j = 1, \ldots, k$. Since s and t^1 are valid flow trees, we deduce that $\Delta_1' = \Gamma_1 \cdots \Gamma_k$. Symmetrically, from $s \leq_G^{\Delta_2} t_2$, we know that there is a subtree $t^2 = t_2/q$ of t_2, of the form $t^2 = \rho_2[t_1^2, \ldots, t_k^2]$ with $\sigma \leq^{\Delta_2'} \rho_2$, $s_j \leq_G^{\Gamma_j'} t_j^2$ for $j = 1, \ldots, k$, and $\Delta_2' = \Gamma_1' \cdots \Gamma_k'$.

By the induction hypothesis, there exists flow trees $s_1', \ldots, s_k'$ such that $t_j^1 \leq_G^{\Gamma_j'} s_j'$ and $t_j^2 \leq_G^{\Gamma_j} s_j'$ for all $j = 1, \ldots, k$. We now define

$$u \overset{\text{def}}{=} (\rho_2 + \Delta_1')[s_1', \ldots, s_k'], \qquad u' \overset{\text{def}}{=} t_2[u]_q, \qquad s' \overset{\text{def}}{=} t_1[u']_p,$$

and claim that these are valid flow trees, s' being the flow tree witnessing the Lemma.

To begin with, and since $\Delta_1' = \Gamma_1 \cdots \Gamma_k$, u is well-formed by Lemma 3.6 and satisfies $t^2 \leq_G^{\Delta_1'} u$. Since $\rho_2 + \Delta_1' = \rho_1 + \Delta_2'$, and since $\Delta_2' = \Gamma_1' \cdots \Gamma_k'$, one also has $t^1 \leq_G^{\Delta_2'} u$.

Then, and since $t^2 \leq_G^{\Delta_1'} u$, we have $t_2 = t_2[t^2]_q \leq_G^{\Delta_1'} t_2[u]_q = u'$ as in Claim 3.5. Thus the root of u' is $\sigma + \Delta_2 + \Delta_1' = \rho_1 + \Delta_2$. We deduce $t^1 \leq_G^{\Delta_2} u'$, relying on $t^1 \leq_G u$. As in Claim 3.5, we obtain $t_1 \leq_G^{\Delta_2} t_1[u']_p = s'$, proving the first half of the Lemma.

On the other hand, from $t_2 \leq_G u'$ we get $t_2 \leq_G^{\Delta_1} s'$ by just checking that the root of t_2, i.e., $\sigma + \Delta_2$, is smaller than the root of s', i.e., $\sigma + \Delta_1 + \Delta_2$. This provides the other half and completes the proof. $\square$

4. GVAS-Definable Predicates

We explore in this section a natural notion of computable sets and relations for the GVAS model, defined as projections of reachability sets. The context-free grammar ingredient of GVASes is essential in the proof that the class of computable sets is closed under intersection, while the Amalgamation Theorem proves that computable sets are finite union of shifted periodic sets.

Definition 4.1. A n-dimensional *GVAS-definable predicate* is a subset $\mathbf{X}$ of $\mathbb{N}^n$ such that there exists a d-dimensional GVAS G with $d = n + \ell$ for some $\ell \in \mathbb{N}$ such that:

$$\mathbf{X} = \left\{ \mathbf{x} \in \mathbb{N}^n \mid \exists \mathbf{e} \in \mathbb{N}^\ell : \mathbf{0}_d \xrightarrow{G} (\mathbf{x}, \mathbf{e}) \right\} . \tag{†}$$

When (†) holds, we say that G defines $\mathbf{X}$ using ℓ auxiliary counters.

The class of GVAS-definable predicates is clearly closed under union, cartesian product [2], and by projecting away some components. We will provide additional closure properties in the remainder of this section. GVAS-definable predicates form a rich class that strictly contains all Presburger sets, i.e., subsets of $\mathbb{N}^n$ that are definable in $\mathsf{FO}(\mathbb{N}; +)$, the first-order theory of natural numbers with addition.

Remark 4.2. Presburger sets are GVAS-definable predicates. The proof is obtained by introducing the class of semilinear sets as follows. A *linear set* of $\mathbb{N}^n$ is a set of the form $\{\mathbf{b} + \lambda_1 \mathbf{p}_1 + \cdots + \lambda_k \mathbf{p}_k \mid \lambda_j \in \mathbb{N}\}$ where $\mathbf{b}$ and $\mathbf{p}_1, \ldots, \mathbf{p}_k$ are vectors in $\mathbb{N}^n$. A *semilinear set* of $\mathbb{N}^n$ is a finite union of linear sets of $\mathbb{N}^n$. Let us recall that a subset of $\mathbb{N}^n$ is Presburger if, and only if, it is semilinear [9]. Since the class of GVAS-definable predicates is closed under union, it is sufficient to show that every linear set is GVAS-definable. We associate with a linear set $\mathbf{X} = \{\mathbf{b} + \lambda_1 \mathbf{p}_1 + \cdots + \lambda_k \mathbf{p}_k \mid \lambda_j \in \mathbb{N}\}$ the n-dimensional GVAS G that generates the regular language $\mathbf{b}\mathbf{p}_1^* \ldots \mathbf{p}_k^*$. Notice that G defines the linear set $\mathbf{X}$ (using no auxiliary counter).

Example 4.3. Let G be the 2-dimensional GVAS given by the following four rules:

$$S \to \binom{0}{1} \mid \binom{1}{0} S T , \qquad\qquad T \to \binom{0}{0} \mid \binom{0}{-1} T \binom{0}{2} .$$

This GVAS is a variant of the 1-dimensional GVAS given in Example 2.1. Analogously to that example, it can be shown that G defines the set $\mathbf{X} = \{(x, y) \in \mathbb{N}^2 \mid 1 \leq y \leq 2^x\}$, which is not semilinear.

A geometrical decomposition of the GVAS-definable predicates can be shown thanks to the *periodic sets*. A subset $\mathbf{P}$ of $\mathbb{N}^d$ is said to be *periodic* [27] if it contains the zero vector, and if $\mathbf{x} + \mathbf{y} \in \mathbf{P}$ for every $\mathbf{x}, \mathbf{y} \in \mathbf{P}$. Note that a periodic $\mathbf{P}$ is not necessarily finitely generated, hence not necessarily semilinear (see Example 4.5). The following Proposition extends the known decomposition of Presburger sets into linear sets.

Proposition 4.4. *Every GVAS-definable predicate* $\mathbf{X} \subseteq \mathbb{N}^n$ *can be decomposed into a finite union of sets of the form* $\mathbf{b} + \mathbf{P}$ *where* $\mathbf{b} \in \mathbb{N}^n$ *and* $\mathbf{P}$ *is a periodic subset of* $\mathbb{N}^n$.

Proof. There exists a d-dimensional GVAS G that defines the set $\mathbf{X}$ with ℓ auxiliary counters. Let us consider the set T of flow trees t such that $\mathrm{root}(t) = (\mathbf{0}_d, S, (\mathbf{x}, \mathbf{e}))$ for some $\mathbf{x} \in \mathbb{N}^n$, $\mathbf{e} \in \mathbb{N}^\ell$ and where S is the start symbol of G. For such a flow tree t in T, we denote by $\mu(t)$

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[2]defined via $\mathbf{X} \times \mathbf{Y} \stackrel{\mathrm{def}}{=} \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbf{X}, \ \mathbf{y} \in \mathbf{Y}\}$.

the vector $\mathbf{x}$. With each $s \in T$ we associate the set $\uparrow s = \{t \in T \mid s \leq_G t\}$. Since $(T, \leq_G)$ is a wqo, there exists a finite subset T_0 of T such that

$$T = \bigcup_{s \in T_0} \uparrow s \,.$$

Given $s \in T$, we introduce the set $\mathbf{P}_s = \{\mu(t) - \mu(s) \mid t \in \uparrow s\}$. Theorem 3.7 shows that $\mathbf{P}_s$ is a periodic set. Now, just observe that the following equality holds:

$$\mathbf{X} = \bigcup_{s \in T_0} \mu(s) + \mathbf{P}_s \,.$$

The proposition is proved. $\qquad\qquad\square$

Example 4.5. Continuing Example 4.3, the set $\mathbf{X} = \{(x, y) \in \mathbb{N}^2 \mid 1 \leq y \leq 2^x\}$ may be decomposed into $\mathbf{X} = (0, 1) + \mathbf{P}$ where $\mathbf{P}$ is the periodic set $\mathbf{P} = \{(x, y) \in \mathbb{N}^2 \mid 0 \leq y < 2^x\}$.

The rest of this section discusses various closure properties of GVAS-definable predicates. We start with boolean operations. As mentioned previously, GVAS-definable predicates are closed under union. In order to prove closure under intersection, we first provide a technical lemma that shows how auxiliary counters of a GVAS can be assumed to be zero at the end of the computation.

Lemma 4.6. *For every d-dimensional GVAS G and for every subset I of $\{1, \ldots, d\}$, there is a $(d + 1)$-dimensional GVAS G_I such that for every $\mathbf{x} \in \mathbb{N}^d$ and for every $c \in \mathbb{N}$, we have:*

$$\mathbf{0}_{d+1} \xrightarrow{G_I} (\mathbf{x}, c) \text{ iff } \mathbf{0}_d \xrightarrow{G} \mathbf{x} \wedge c = 0 \wedge \bigwedge_{i \in I} \mathbf{x}[i] = 0 \,. \tag{4.1}$$

Proof. The idea of the proof is to put the counters in I "on a budget" (see, e.g., [33] for details on the budgeting construction) and to harness the expressive power given by context-free grammars to non-deterministically initialize the total budget, simulate G with the given budget, and finally check that the budget is fully restored at the end of the computation, which guarantees that the counters in I are zero.

Formally, let us introduce the function Δ_I that maps vectors $\mathbf{x}$ of $\mathbb{Z}^d$ to the number $\Delta_I(\mathbf{x}) = \sum_{i \in I} \mathbf{x}[i]$. We also introduce the mapping $\mu_I : \mathbb{Z}^d \to \mathbb{Z}^{d+1}$ defined by $\mu_I(\mathbf{x}) = (\mathbf{x}, -\Delta_I(\mathbf{x}))$. This mapping is extended over words of actions as a word morphism, and over languages by $\mu_I(L) = \{\mu_I(w) \mid w \in L\}$. Let us introduce the actions $\mathbf{a}_+ = (\mathbf{0}_d, 1)$, and $\mathbf{a}_- = (\mathbf{0}_d, -1)$. In linear time, from G we can define a $(d + 1)$-dimensional GVAS G_I that generates the following language:

$$L_{G_I} = \bigcup_{k \in \mathbb{N}} \mathbf{a}_+^k \mu_I(L_G) \mathbf{a}_-^k \,.$$

Let us prove that this GVAS satisfies the lemma. We consider $\mathbf{x} \in \mathbb{N}^d$ and $c \in \mathbb{N}$.

Assume first that $\mathbf{0}_{d+1} \xrightarrow{G_I} (\mathbf{x}, c)$. In that case, there exists $k \in \mathbb{N}$ and $w \in L_G$ such that $\mathbf{0}_{d+1} \xrightarrow{\mathbf{a}_+^k \mu_I(w) \mathbf{a}_-^k} (\mathbf{x}, c)$. Observe that we have

$$\mathbf{0}_{d+1} \xrightarrow{\mathbf{a}_+^k} (\mathbf{0}_d, k) \xrightarrow{\mu_I(w)} (\mathbf{x}, c + k) \xrightarrow{\mathbf{a}_-^k} (\mathbf{x}, c) \,.$$

Since $\mu_I(w)$ preserves the sum of the counters in I and of the last counter, it follows that $\Delta_I(\mathbf{0}_d) + k = \Delta_I(\mathbf{x}) + c + k$. Thus $c + \Delta_I(\mathbf{x}) = 0$. It follows that $c = 0$ and $\mathbf{x}[i] = 0$ for every $i \in I$. Moreover, from $\mathbf{0}_d \xrightarrow{w} \mathbf{x}$ we derive $\mathbf{0}_d \xrightarrow{G} \mathbf{x}$.

Conversely, let us assume that $\mathbf{0}_d \xrightarrow{G} \mathbf{x}$, $c = 0$ and $\mathbf{x}[i] = 0$ for every $i \in I$. There exists $w \in L_G$ such that $\mathbf{0}_d \xrightarrow{w} \mathbf{x}$. There exists $k \in \mathbb{N}$ large enough such that $(\mathbf{0}_d, k) \xrightarrow{\mu_I(w)} (\mathbf{x}, k - \Delta_I(\mathbf{x})) = (\mathbf{x}, k)$. It follows that $\mathbf{0}_{d+1} \xrightarrow{\mathbf{a}_+^k \mu_I(w) \mathbf{a}_-^k} (\mathbf{x}, 0) = (\mathbf{x}, c)$. Thus $\mathbf{0}_{d+1} \xrightarrow{G_I} (\mathbf{x}, c)$. $\qquad \square$

We are now ready to prove that GVAS-definable predicates are closed under intersection.[3]

Lemma 4.7. *The class of GVAS-definable predicates is closed under intersection.*

Proof. Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^n$ be GVAS-definable. Since the class of GVAS-definable predicates is closed under cartesian product, it follows that $\mathbf{X} \times \mathbf{Y}$ is also GVAS-definable. Hence, there exists a d-dimensional GVAS H with ℓ auxiliary counters that defines that set. Let us consider the mapping $\mu : \mathbb{Z}^d \to \mathbb{Z}^n \times \mathbb{Z}^d$ defined by $\mu(\mathbf{x}, \mathbf{y}, \mathbf{e}) = (\mathbf{0}_n, \mathbf{x}, \mathbf{y}, \mathbf{e})$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$ and $\mathbf{e} \in \mathbb{Z}^\ell$. The mapping μ is extended as a word morphism. We introduce the action $\mathbf{a}_i$ in $\mathbb{Z}^{n+d}$ defined as follows:

$$\mathbf{a}_i = (\mathbf{i}_{i,n}, -\mathbf{i}_{i,n}, -\mathbf{i}_{i,n}, \mathbf{0}_\ell) \,.$$

Obviously, we can build a $(d+n)$-dimensional GVAS G such that $L_G = \mu(L_H)\mathbf{a}_1^* \cdots \mathbf{a}_d^*$. Now, let $I = \{n+1, \ldots, 3n\}$ and let us apply Lemma 4.6 on G and I. We obtain a $(d+n+1)$-GVAS that defines $\mathbf{X} \cap \mathbf{Y}$. $\qquad \square$

Remark 4.8. The class of GVAS-definable predicates is not closed under taking complements, see Proposition 5.6.

We now investigate closure under sum [4] and under the associated Kleene star, which we call periodic hull. Formally, the *periodic hull* of a subset $\mathbf{X} \subseteq \mathbb{N}^n$ is the set of finite sums of vectors in $\mathbf{X}$. It turns out that the class of GVAS-definable predicates is closed under sum and periodic hull. Closure under sum can be proved along the same lines as closure under intersection (see Lemma 4.7). The detailed proof is left as an exercise. Closure under periodic hull is more involved and requires well-behaved GVASes.

In the definition of a GVAS-definable predicate $\mathbf{X}$ given in (†), the vector $\mathbf{e}$ can be seen as auxiliary counters that can have arbitrary values at the end of the computation. We say that a d-dimensional GVAS G defining a predicate $\mathbf{X} \subseteq \mathbb{N}^n$ using ℓ auxiliary counters is *auxiliary-resetting* if for every $(\mathbf{x}, \mathbf{e}) \in \mathbb{N}^n \times \mathbb{N}^\ell$ such that $\mathbf{0}_d \xrightarrow{G} (\mathbf{x}, \mathbf{e})$ we have $\mathbf{e} = \mathbf{0}_\ell$. We also say that G is *output-increasing* if every action of G is an element of $\mathbb{N}^n \times \mathbb{Z}^\ell$ meaning that the output counters $\mathbf{x}$ cannot be decremented during a computation.

Let us first prove that GVAS-definable predicates can be defined by auxiliary-resetting output-increasing GVAS. To do so, we introduce, for every $k \in \mathbb{N}$, the unit vector $\mathbf{i}_{i,k}$ of $\mathbb{N}^k$ defined by $\mathbf{i}_{i,k}[j] = 0$ if $j \neq i$ and $\mathbf{i}_{i,k}[i] = 1$.

Lemma 4.9. *For every d-dimensional GVAS G defining a set $\mathbf{X} \subseteq \mathbb{N}^n$, there is a $(d+n+1)$-dimensional auxiliary-resetting output-increasing GVAS defining $\mathbf{X}$.*

Proof. We introduce the mapping $\mu : \mathbb{Z}^d \to \mathbb{Z}^{n+d}$ defined by $\mu(\mathbf{a}) = (\mathbf{0}_n, \mathbf{a})$. This mapping is extended as a word morphism. Observe that there exists a $(d+n)$-dimensional GVAS G' satisfying the following equality:

$$L_{G'} = \mu(L_G)(\mathbf{i}_{1,n}, -\mathbf{i}_{1,n}, \mathbf{0}_\ell)^* \cdots (\mathbf{i}_{n,n}, -\mathbf{i}_{n,n}, \mathbf{0}_\ell)^*(\mathbf{0}_n, \mathbf{0}_n, -\mathbf{i}_{1,\ell})^* \cdots (\mathbf{0}_n, \mathbf{0}_n, -\mathbf{i}_{\ell,\ell})^* \,.$$

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[3]By contrast, we believe that "VAS-definable" predicates are not closed under intersection (unless one requires auxiliary counters to be zero at the end of the computation). This conjecture remains to be proved.

[4]defined via $\mathbf{X} + \mathbf{Y} \stackrel{\text{def}}{=} \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathbf{X}, \, \mathbf{y} \in \mathbf{Y}\}$.

This GVAS G' satisfies $\mathbf{0}_{d+n} \xrightarrow{G'} (\mathbf{x}, \mathbf{0}_n, \mathbf{0}_\ell)$ with $\mathbf{x} \in \mathbb{N}^n$ if, and only if, there exists $\mathbf{e} \in \mathbb{N}^\ell$ such that $\mathbf{0}_d \xrightarrow{G} (\mathbf{x}, \mathbf{e})$. Moreover, actions of that GVAS are in $\mathbb{N}^n \times \mathbb{Z}^d$. By applying the construction given in the proof of Lemma 4.6 on G' with $I = \{n, \ldots, d+n\}$ observe that we get a $(d + n + 1)$-dimensional auxiliary-resetting output-increasing GVAS defining $\mathbf{X}$. $\square$

Corollary 4.10. *The class of GVAS-definable predicates is closed under periodic hull.*

Proof. Assume that a predicate $\mathbf{X} \subseteq \mathbb{N}^n$ is GVAS-definable. Lemma 4.9 shows that $\mathbf{X}$ is defined by a d-dimensional auxiliary-resetting output-increasing GVAS G using ℓ auxiliary counters. As context-free languages are closed under Kleene star, there exists a GVAS G' satisfying $L_{G'} = L_G^*$. We show that G' defines the periodic hull of $\mathbf{X}$. Let $\mathbf{y} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ with $k \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbf{X}$ and let us prove that $\mathbf{0}_d \xrightarrow{G'} (\mathbf{y}, \mathbf{0}_\ell)$. We introduce $\mathbf{y}_j = \mathbf{x}_1 + \cdots + \mathbf{x}_j$ for every $j \in \{0, \ldots, k\}$. Let $i \in \{1, \ldots, k\}$. Since $\mathbf{x}_i \in \mathbf{X}$ and G is auxiliary-resetting, we get $\mathbf{0}_d \xrightarrow{G} (\mathbf{x}_i, \mathbf{0}_\ell)$. By monotony, we can add on both sides the vector $(\mathbf{y}_{i-1}, \mathbf{0}_\ell)$ and derive $(\mathbf{y}_{i-1}, \mathbf{0}_\ell) \xrightarrow{G} (\mathbf{y}_i, \mathbf{0}_\ell)$. We get $(\mathbf{y}_0, \mathbf{0}_\ell) \xrightarrow{G} (\mathbf{y}_1, \mathbf{0}_\ell) \cdots \xrightarrow{G} (\mathbf{y}_k, \mathbf{0}_\ell)$. Since $\mathbf{y}_0 = \mathbf{0}_n$ and $\mathbf{y}_k = \mathbf{y}$, we have proved that $\mathbf{0}_d \xrightarrow{G'} (\mathbf{y}, \mathbf{0}_\ell)$. Conversely, let $\mathbf{y} \in \mathbb{N}^n$ and $\mathbf{e} \in \mathbb{N}^\ell$ such that $\mathbf{0}_d \xrightarrow{G'} (\mathbf{y}, \mathbf{e})$ and let us prove that $\mathbf{y}$ is in the periodic hull of $\mathbf{X}$. Since $L_{G'} = L_G^*$, we have $(\mathbf{y}_0, \mathbf{e}_0) \xrightarrow{G} (\mathbf{y}_1, \mathbf{e}_1) \cdots \xrightarrow{G} (\mathbf{y}_k, \mathbf{e}_k)$ for some sequence $(\mathbf{y}_0, \mathbf{e}_0), \ldots, (\mathbf{y}_k, \mathbf{e}_k)$ such that $(\mathbf{y}_0, \mathbf{e}_0) = \mathbf{0}_d$ and $(\mathbf{y}_k, \mathbf{e}_k) = (\mathbf{y}, \mathbf{e})$. Since G is output-increasing we deduce that $\mathbf{x}_i$, defined as $\mathbf{x}_i = \mathbf{y}_i - \mathbf{y}_{i-1}$, is in $\mathbb{N}^n$ and satisfies $(\mathbf{0}_n, \mathbf{e}_{i-1}) \xrightarrow{G} (\mathbf{x}_i, \mathbf{e}_i)$. As G is auxiliary-resetting, by induction we deduce that $\mathbf{e}_i = \mathbf{0}$ for every i. It follows that $\mathbf{x}_i \in \mathbf{X}$. As $\mathbf{y} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$, we conclude that $\mathbf{y}$ is in the periodic hull of $\mathbf{X}$. $\square$

To conclude this section, we discuss closure under relational composition and under the associated Kleene star (namely, the reflexive-transitive closure). For the purpose of GVAS-definability, we view binary relations on $\mathbb{N}^n$ as subsets of $\mathbb{N}^{2n}$. The class of GVAS-definable binary relations on $\mathbb{N}^n$ is closed under relational composition. This claim follows from closure of GVAS-definable predicates under cartesian product, intersection and projection. However, GVAS-definable binary relations are not closed under reflexive-transitive closure in general, as the following example shows.

Example 4.11. Consider the binary relation $\mathbf{R}$ on $\mathbb{N}$ defined by $\mathbf{R} = \{(x, 2x) \mid x \in \mathbb{N}\}$. The binary relation $\mathbf{R}$ is clearly GVAS-definable. However, its reflexive-transitive closure $\mathbf{R}^* = \{(x, 2^k x) \mid x, k \in \mathbb{N}\}$ is not GVAS-definable. Indeed, if $\mathbf{R}^*$ were GVAS-definable then, by Proposition 4.4, there would exist some $\mathbf{b} \in \mathbb{N}^2$, some periodic $\mathbf{P} \subseteq \mathbb{N}^2$ and two distinct powers $2 \leq 2^k < 2^\ell$ such that

$$(1, 2^k), (1, 2^\ell) \in \mathbf{b} + \mathbf{P} \subseteq \mathbf{R}^* .$$

We now use the assumption that $\mathbf{P}$ is periodic and derive a contradiction. Let us write $\mathbf{b} = (b_1, b_2)$. Note that $b_1 \leq 1$ and $b_2 \leq 2^k$ since $(1, 2^k) \in \mathbf{b} + \mathbf{P}$. There are three cases.
- If $\mathbf{b} = \mathbf{0}$ then $(1, 2^k)$ and $(1, 2^\ell)$ are both in $\mathbf{P}$, hence, $(2, 2^k + 2^\ell)$ is also in $\mathbf{P}$ by periodicity, and so $(2, 2^k + 2^\ell) \in \mathbf{R}^*$. This is impossible since $2^k + 2^\ell$ is not a power of two (as $k \neq \ell$).
- If $b_1 > 0$ then $b_1 = 1$, hence, $(1, 2^\ell) = (1, b_2) + (0, p_2)$ for some $(0, p_2) \in \mathbf{P}$. Note that $p_2 > 0$ since $b_2 \leq 2^k < 2^\ell$. Since $\mathbf{b} + \mathbf{P} \subseteq \mathbf{R}^*$, we get by periodicity that $(1, b_2 + np_2) \in \mathbf{R}^*$ for every $n \in \mathbb{N}$. This means that $b_2 + np_2$ is a power of two for every $n \in \mathbb{N}$, which is impossible as $p_2 > 0$.

- If $b_1 = 0$ and $b_2 > 0$ then $(1, 2^\ell) = (0, b_2) + (1, p_2)$ for some $(1, p_2) \in \mathbf{P}$. Since $\mathbf{b} + \mathbf{P} \subseteq \mathbf{R}^*$, we get by periodicity that $(n, b_2 + np_2) \in \mathbf{R}^*$ for every $n \in \mathbb{N}$. Taking $n = b_2 + 1$, we derive that $(b_2 + 1, b_2 + (b_2 + 1)p_2) \in \mathbf{R}^*$. This is impossible since $b_2 > 0$ and y is a multiple of x for every $(x, y) \in \mathbf{R}^*$.

5. Weakly Computable Functions

There is a classical notion of number-theoretical functions weakly computable by Petri nets [12]. In this section, we extend the idea to GVASes.

As we argued in the introduction, the notion of weakly computable functions has recently gained new relevance with the development of well-structured systems that go beyond Petri nets and VASSes in expressive power, while sharing some of their characteristics.

The expected way for a GVAS to compute a numerical function $f : \mathbb{N} \to \mathbb{N}$ is to start with some input number n stored in a designated input counter and, from that configuration, eventually reach a configurations with $f(n)$ in a designated output counter. In order for that GVAS to be correct (as a computer for f), it should be impossible that it reaches a value differing from $f(n)$ in the output counter. In that case, we say that the GVAS *strongly computes* f. This notion of correctness is fine with other models like Minsky machines but it is too strong for GVASes and does not lead to an interesting family of computable functions. In fact, GVAS are essentially nondeterministic devices, and the above notion of strongly computing some function does not accommodate nondeterminism nicely.

With this in mind, and in the setting of VASes, Rabin defined a notion of "weakly computing f" that combines the following two principles:

Completeness: For any $n \in \mathbb{N}$, there is a computation with input n and output $f(n)$;
Safety: Any computation from input n to some output r satisfies $r \le f(n)$.

This leads to our definition of weak GVAS computers, where the input and output counters are the first two components.

Definition 5.1 (Weak GVAS computers)**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a total function. A *weak GVAS computer* (with ℓ auxiliary counters) for f is a d-dimensional GVAS G with $d = 2 + \ell$ that satisfies the following two properties:

$$\forall n : \exists n', \mathbf{e} : (n, 0, \mathbf{0}_\ell) \xrightarrow{G} (n', f(n), \mathbf{e}) \,, \tag{CO}$$

$$\forall n, n', r, \mathbf{e} : (n, 0, \mathbf{0}_\ell) \xrightarrow{G} (n', r, \mathbf{e}) \text{ implies } r \le f(n) \,. \tag{SA}$$

We say that f is *weakly computable*, or WC, if there is a weak GVAS computer for it.

For convenience, Definition 5.1 assumes that the input is given in the first counter of G, and that the result is found in the second counter. Note that G may use its ℓ last counters for auxiliary calculations. We focus on total functions over the natural numbers rather than total functions over the vectors of natural numbers to simplify the presentation. However, results given in this section can be easily extended to this more general setting.

Example 5.2 (A weak computer for exponentiation)**.** Example 2.1 shows that the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2^n$ is WC. □

Only monotonic functions can be weakly computed in the above sense. This is an immediate consequence of the monotonicity of GVASes (see (2.3)). Recall that a total function $f : \mathbb{N} \to \mathbb{N}$ is *non-decreasing* if $n \le m$ implies $f(n) \le f(m)$.

Proposition 5.3 (Monotonicity of WC functions). *If f is WC then f is non-decreasing.*

Proof. Assume that $n \le m$ and pick any weak GVAS computer G for f. By (CO), we have $(n, 0, \mathbf{0}_\ell) \xrightarrow{G} (n', f(n), \mathbf{e})$ for some $n' \in \mathbb{N}$ and $\mathbf{e} \in \mathbb{N}^\ell$. By monotonicity, it follows that $(n + (m - n), 0, \mathbf{0}_\ell) \xrightarrow{G} (n' + (m - n), f(n), \mathbf{e})$. We get $f(n) \le f(m)$ by (SA). $\square$

We may now relate WC computability with GVAS-definability.

Lemma 5.4. *A total function $f : \mathbb{N} \to \mathbb{N}$ is WC if, and only if, f is non-decreasing and the following set is GVAS-definable.*

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \le f(x)\} .$$

Proof. Assume first that f is WC. There exists a weak GVAS computer (with ℓ auxiliary counters) for f given as a d-dimensional GVAS G with $d = 2 + \ell$ that satisfies (CO) and (SA). Let us consider the mapping $\mu : \mathbb{Z}^d \to \mathbb{Z}^{d+1}$ defined by $\mu(a, b, \mathbf{e}) = (0, b, a, \mathbf{e})$ for every $a, b \in \mathbb{Z}$, and $\mathbf{e} \in \mathbb{Z}^\ell$. The mapping μ is extended as a word morphism. Let us show that a GVAS G' such that $L_{G'} = (1, 0, 1, \mathbf{0}_\ell)^* \mu(L_G)(0, -1, 0, \mathbf{0}_\ell)^*$ is defining the set $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \le f(x)\}$. Let (x, y) in that set. From (CO), there exists a word $\sigma \in L_G$, $x' \in \mathbb{N}$ and $\mathbf{e} \in \mathbb{N}^\ell$ such that $(x, 0, \mathbf{0}_\ell) \xrightarrow{\sigma} (x', f(x), \mathbf{e})$. The word $\sigma' = (1, 0, 1, \mathbf{0}_\ell)^x \mu(\sigma)(0, -1, 0, \mathbf{0})^{f(x)-y}$ shows that $(0, 0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma'} (x, y, x', \mathbf{e})$. As $\sigma' \in L_{G'}$, we get $(0, 0, 0, \mathbf{0}_\ell) \xrightarrow{G'} (x, y, x', \mathbf{e})$. Conversely, assume that $(0, 0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma'} (x, y, x', \mathbf{e})$ for some $x, y, x' \in \mathbb{N}$, $\mathbf{e} \in \mathbb{N}^\ell$ and $\sigma' \in L_{G'}$, and let us prove that $y \le f(x)$. By definition of G', there exists $n, m \in \mathbb{N}$ and a word $\sigma \in L_G$ such that $\sigma' = (1, 0, 1, \mathbf{0}_\ell)^n \mu(\sigma)(0, -1, 0, \mathbf{0}_\ell)^m$. It follows that $(n, 0, n, \mathbf{0}_\ell) \xrightarrow{\mu(\sigma)} (x, y + m, x', \mathbf{e})$. Since actions occurring in $\mu(\sigma)$ cannot modify the first counter, we get $n = x$. Moreover, $(x, 0, \mathbf{0}_\ell) \xrightarrow{\sigma} (x', y + m, \mathbf{e})$. From (SA), we derive $y + m \le f(x)$. Hence $y \le f(x)$. We have proved that $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \le f(x)\}$ is GVAS-definable.

Conversely, let us assume that f is non-decreasing and that $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \le f(x)\}$ is GVAS-definable. There exists a d-dimensional GVAS G with $d = 2 + \ell$ such that:

$$\{(x, y) \mid y \le f(x)\} = \{(x, y) \mid \exists \mathbf{e} \in \mathbb{N}^\ell : (0, 0, \mathbf{0}_\ell) \xrightarrow{G} (x, y, \mathbf{e})\} .$$

Let us consider the mapping $\mu : \mathbb{Z}^d \to \mathbb{Z}^{d+1}$ defined by $\mu(a, b, \mathbf{e}) = (-a, b, a, \mathbf{e})$ for every $a, b \in \mathbb{Z}$, and $\mathbf{e} \in \mathbb{Z}^\ell$. The mapping μ is extended as a word morphism. Let us show that a GVAS G' such that $L_{G'} = \bigcup_{k \in \mathbb{N}} (1, 0, 0, \mathbf{0}_\ell)^k \mu(L_G)(-1, 0, 0, \mathbf{0}_\ell)^k$ is a weak GVAS computer for f. Let us first consider $x \in \mathbb{N}$. By definition of G, there exists a word $\sigma \in L_G$ and $\mathbf{e} \in \mathbb{N}^\ell$ such that $(0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma} (x, f(x), \mathbf{e})$. Notice that for k large enough, we have $(k + x, 0, 0, \mathbf{0}_\ell) \xrightarrow{\mu(\sigma)} (k, f(x), x, \mathbf{e})$. The word $\sigma' = (1, 0, 0, \mathbf{0}_\ell)^k \mu(\sigma)(-1, 0, 0, \mathbf{0}_\ell)^k$ is such that $(x, 0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma'} (0, f(x), x, \mathbf{e})$. Hence (CO) is satisfied by G'. Finally, let us assume that $(x, 0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma'} (z, y, x', \mathbf{e})$ for a word $\sigma' \in L_{G'}$ and $x', y, z \in \mathbb{N}$ and $\mathbf{e} \in \mathbb{N}^\ell$. There exists $k \in \mathbb{N}$ and $\sigma \in L_G$ such that $\sigma' = (1, 0, 0, \mathbf{0}_\ell)^k \mu(\sigma)(-1, 0, 0, \mathbf{0}_\ell)^k$. It follows that $(x + k, 0, 0, \mathbf{0}_\ell) \xrightarrow{\mu(\sigma)} (z + k, y, x', \mathbf{e})$. By definition of μ, since the effect of the sum of the first and third counters is zero, we get $x + k + 0 = z + k + x'$. Hence $x' \le x$ and in particular $f(x') \le f(x)$. Moreover, we have $(0, 0, \mathbf{0}_\ell) \xrightarrow{\sigma} (x', y, \mathbf{e})$. By definition of G, we get $y \le f(x')$.

We have proved that $y \leq f(x)$. Hence (SA) is satisfied by G'. We have proved that G' is a weak GVAS computer for f $\qquad\square$

By combining Lemma 5.4 and the decomposition of GVAS-definable sets given by Proposition 4.4, we obtain two interesting, albeit negative, results on WC functions and GVAS-definable sets.

Proposition 5.5. *Let f be an unbounded WC function. Then there exists a rational number $c > 0$ and some $z \in \mathbb{Z}$ such that $f(n) \geq cn + z$ for every $n \in \mathbb{N}$.*

Proof. Lemma 5.4 shows that the set $\mathbf{X}$ defined as $\{(n, m) \mid m \leq f(n)\}$ is GVAS-definable. Proposition 4.4 shows that $\mathbf{X}$ can be decomposed into a finite union of sets of the form $(a, b) + \mathbf{P}$ where $(a, b) \in \mathbb{N}^2$ and $\mathbf{P}$ is a periodic subset of $\mathbb{N}^2$. Since f is unbounded, there exists $(p, q) \in \mathbf{P}$ such that $q > 0$. It follows $(a, b) + k(p, q) \in \mathbf{X}$ for every $k \in \mathbb{N}$. In particular $f(a + kp) \geq b + kq$ for every $k \in \mathbb{N}$. As $f(a) \in \mathbb{N}$ and $q > 0$, we deduce that $p > 0$. Let us consider $n \in \mathbb{N}$ such that $n \geq a$ and observe that there exists $k \in \mathbb{N}$ such that:

$$k \leq \frac{n - a}{p} < k + 1 \, .$$

It follows that $a + kp \leq n$ and in particular $f(a + kp) \leq f(n)$. Hence $f(n) \geq b + kq \geq b + (\frac{n-a}{p} - 1)q$. Introducing $c = \frac{q}{p}$, we deduce that $f(n) - cn \geq b - c(a + p)$ for every $n \geq a$. We have proved the lemma with any $z \in \mathbb{Z}$ satisfying $z \leq f(n) - cn$ for every $0 \leq n < a$ and $z \leq b - c(a + p)$. $\qquad\square$

Proposition 5.6. *The complement of a GVAS-definable set is not always GVAS-definable.*

Proof. Recall from Example 5.2 that the function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = 2^n$ is WC. We derive from Lemma 5.4 that $\mathbf{X} \stackrel{\text{def}}{=} \{(n, m) \mid m \leq 2^n\}$ is GVAS-definable. Assume, by way of contradiction, that the complement $\mathbf{Y} \stackrel{\text{def}}{=} \{(n, m) \mid 2^n < m\}$ is GVAS-definable. From a GVAS defining $\mathbf{Y}$, we easily derive a GVAS defining $\mathbf{Z} \stackrel{\text{def}}{=} \{(n, m) \mid 2^m \leq n + 1\}$, by swapping the first two counters and then decrementing the first counter by two at the end. It follows from Lemma 5.4 that the mapping $g : \mathbb{N} \rightarrow \mathbb{N}$ defined by $g(n) = \lfloor \log_2(n + 1) \rfloor$ is WC, contradicting Proposition 5.5 since g is unbounded and sublinear. Hence $\mathbf{Y}$, i.e., $\mathbb{N}^2 \setminus \mathbf{X}$, cannot be GVAS-definable. $\qquad\square$

6. Hyper-Ackermannian GVAS

In this section we construct GVASes that weakly compute functions from the Fast Growing Hierarchy. Our main result is the following.

Theorem 6.1. *The Fast Growing functions $(F_\alpha)_{\alpha < \omega^\omega}$ are weakly computable (by GVASes).*

Note that these are exactly the multiply-recursive functions F_α. They include functions that are not primitive-recursive (the F_α for $\omega \leq \alpha < \omega^\omega$) and that are thus not weakly computable by VASSes (see [17, section 2]). We do not know whether F_{ω^ω} is weakly computable by a GVAS, or whether there exist WC functions that are not multiply-recursive.

The rest of this section proves Theorem 6.1. The detailed proof illustrates how the GVAS model makes it manageable to define complex constructions precisely, and to formally prove their correctness. By contrast, observe how in less abstract models e.g., the Timed-Arc

Petri Nets of [14], only schematic constructions are given for weakly computing functions, and only an outline for a correctness proof can be provided.

We follow notation and definitions from [31] and consider functions $F_\alpha : \mathbb{N} \to \mathbb{N}$ indexed by an ordinal $\alpha < \epsilon_0$ (though we shall only build GVASes for functions with $\alpha < \omega^\omega$). Any such ordinal can be written in Cantor normal form (CNF) $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ with $\alpha > \alpha_1 \geq \cdots \geq \alpha_m$. When $m = 0$, α is 0. When $\alpha_m = 0$, α is a successor of the form $\beta + \omega^0$, i.e., $\beta + 1$, and when $\alpha_m > 0$, α is a limit ordinal. When $\alpha \neq 0$, we often decompose α under the form $\alpha = \gamma + \omega^{\alpha_m}$ so that the smallest summand in α's CNF is exposed. CNFs are often written more concisely using coefficients, as in $\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_m} \cdot c_m$, with now $\alpha > \alpha_1 > \cdots > \alpha_m$ and $\omega > c_1, \ldots, c_m > 0$.

With each limit ordinal $\lambda < \epsilon_0$, one associates a fundamental sequence $(\lambda(n))_{n < \omega}$ such that $\lambda = \sup_n \lambda(n)$. These are defined inductively as follows.

$$(\gamma + \omega^{\beta+1})(n) = \gamma + \omega^\beta \cdot (n+1) \,, \tag{L1}$$

$$(\gamma + \omega^\lambda)(n) = \gamma + \omega^{\lambda(n)} \,. \tag{LL}$$

For instance, Eq. (L1) gives $\omega(n)$, i.e., $\omega^1(n) = \omega^0 \cdot (n+1) = n+1$ and $(\omega^3 \cdot 6 + \omega^2 \cdot 3)(n) = \omega^3 \cdot 6 + \omega^2 \cdot 2 + \omega \cdot (n+1)$. Similarly, Eq. (LL) gives $\omega^\omega(n) = \omega^{\omega(n)} = \omega^{n+1}$. Note that the fundamental sequences satisfy $\lambda(0) < \cdots < \lambda(n) < \lambda(n+1) < \cdots < \lambda$ for any limit ordinal λ and index n.

We may now define our fast growing functions $F_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha < \epsilon_0$ by induction on the α index.

$$F_0(x) = x + 1 \,, \qquad \overbrace{}^{x+1 \text{ times}} \tag{F0}$$

$$F_{\alpha+1}(x) = F_\alpha^{\omega(x)}(x) = \overbrace{F_\alpha(\cdots(F_\alpha}^{x+1 \text{ times}}(x))\cdots) \,, \tag{F1}$$

$$F_\lambda(x) = F_{\lambda(x)}(x) \,. \tag{FL}$$

As shown —e.g., in [31]— these functions are *strictly expansive* and *monotonic*, i.e., for all ordinals $\alpha < \epsilon_0$ and all $n, n' \in \mathbb{N}$:

$$n < F_\alpha(n) \,, \tag{FX}$$

$$n \leq n' \implies F_\alpha(n) \leq F_\alpha(n') \,. \tag{FM}$$

Given two ordinals in Cantor normal form $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ and $\alpha' = \omega^{\beta'_1} + \cdots + \omega^{\beta'_n}$, we denote by $\alpha \oplus \alpha'$ their *natural sum* $\sum_{k=1}^{m+n} \omega^{\gamma_k}$, where $\gamma_1 \geq \ldots \geq \gamma_{m+n}$ is a reordering of $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$. The F_α functions are not monotonic in the ordinal index, i.e., $\alpha \leq \alpha'$ does not always entail $F_\alpha(n) \leq F_{\alpha'}(n)$, see [31, section A.2]. However, our construction relies on similar monotonicity properties, albeit for special cases of α and α', that we now state.

Lemma 6.2. *For any ordinals $\alpha, \alpha' < \epsilon_0$ and any $n \in \mathbb{N}$, $F_\alpha(n) \leq F_{\alpha \oplus \alpha'}(n)$.*

Lemma 6.3. *For any ordinal $\alpha < \epsilon_0$ and limit ordinal $\lambda < \omega^\omega$, for any $m, n \in \mathbb{N}$, if $m \leq n$ then $F_{\alpha \oplus \lambda(m)}(n) \leq F_{\alpha \oplus \lambda}(n)$.*

For these two results, detailed proofs are given in the appendix. We note that Lemma 6.2 is a rewording of Lemma 2.2a from [4], however that paper uses a different definition for the fundamental sequences $(\lambda(n))_{n \in \mathbb{N}}$, resulting in slightly different F_α functions, hence the need of an independent proof. Similarly, Lemma 6.3 is a generalization of Lemma VI.5 from [22], using different notation and allowing a simpler proof.

We now define weak GVAS computers for the F_α functions such that $\alpha < \omega^\omega$. Our construction is in two steps: we first pick an arbitrary exponent $d \in \mathbb{N}$ and define G_d, a GVAS with a structure suitable for correctness proofs. We then obtain a weak GVAS computer for F_α by slightly modifying G_d, provided $\alpha < \omega^d$. The whole construction is an adaptation into the GVAS framework of the pushdown VAS from [22].

The dimension of G_d is $d + 2$ and we use $d + 2$ counters named $r, \overline{r}, \kappa_0, \ldots, \kappa_{d-1}$, in this order. The set of actions $\mathbf{A} \subseteq \mathbb{Z}^{d+2}$ consists of all vectors $\mathbf{d}_x$ and $\mathbf{i}_x$ where x is one of the $d + 2$ counters: formally $\mathbf{d}_x$ is the vector that decrements x, while $\mathbf{i}_x \stackrel{\text{def}}{=} -\mathbf{d}_x$ increments it. For instance, $\mathbf{d}_{\kappa_0} = (0, 0, -1, \mathbf{0}_{d-1})$ and $\mathbf{i}_r = (1, 0, \mathbf{0}_d)$. The set of non-terminals of G_d is $V = \{\mathtt{F}, \mathtt{Rec}, \mathtt{Pop}, \mathtt{Lim}_1, \ldots, \mathtt{Lim}_{d-1}\}$. The start symbol is $\mathtt{F}$. The other non-terminals are used for intermediate steps (see the rules below).

The first two counters, r and $\overline{r}$, are used to manipulate the arguments of the functions being computed. The other d counters are used as a data structure representing an ordinal $\alpha < \omega^d$. Formally, with any d-tuple $\langle c_0, \ldots, c_{d-1} \rangle$ of natural numbers, we associate the ordinal $\alpha = \omega^{d-1} \cdot c_{d-1} + \cdots + \omega^0 \cdot c_0$. We will follow the convention of writing the contents of the counters of our GVAS in the form $\langle n, m, \alpha \rangle$, where n and m are the value of r and $\overline{r}$, respectively, and where α is the ordinal associated with the values in $\kappa_0, \ldots, \kappa_{d-1}$.

The rules of G_d are given below. The rules involving the $\mathtt{Lim}_i$ non-terminals are present for every $i \in \{1, \ldots, d-1\}$.

$$\mathtt{F} \to \mathbf{i}_r \,, \tag{R1}$$

$$\mathtt{F} \to \mathbf{d}_{\kappa_0} \ \mathtt{Rec} \ \mathtt{F} \ \mathbf{i}_{\kappa_0} \,, \tag{R2}$$

$$\mathtt{F} \to \mathbf{d}_{\kappa_i} \mathbf{i}_{\kappa_{i-1}} \ \mathtt{Lim}_i \ \mathbf{d}_{\kappa_{i-1}} \mathbf{i}_{\kappa_i} \,, \tag{R3i}$$

$$\mathtt{Rec} \to \mathtt{Pop} \,, \tag{R4}$$

$$\mathtt{Rec} \to \mathbf{d}_r \mathbf{i}_{\overline{r}} \ \mathtt{Rec} \ \mathtt{F} \,, \tag{R5}$$

$$\mathtt{Pop} \to \varepsilon \,, \tag{R6}$$

$$\mathtt{Pop} \to \mathbf{i}_r \mathbf{d}_{\overline{r}} \ \mathtt{Pop} \,, \tag{R7}$$

$$\mathtt{Lim}_i \to \mathtt{Pop} \ \mathtt{F} \,, \tag{R8i}$$

$$\mathtt{Lim}_i \to \mathbf{d}_r \mathbf{i}_{\overline{r}} \mathbf{i}_{\kappa_{i-1}} \ \mathtt{Lim}_i \ \mathbf{d}_{\kappa_{i-1}} \,. \tag{R9i}$$

Our first goal is to prove that G_d has computations of the form $\langle n, 0, \alpha \rangle \xrightarrow{\mathtt{F}} \langle F_\alpha(n), 0, \alpha \rangle$, for any $n \in \mathbb{N}$ and $\alpha < \omega^d$. We start with a lemma exposing some specific sentential forms that can be derived from $\mathtt{F}$ and $\mathtt{Pop}$. As will be clear from the proof of Lemma 6.5, these derivations (namely (D0), (D1) and (DL)) correspond to our inductive definition of the fast growing functions F_α (namely (F0), (F1) and (FL)).

Lemma 6.4. *For every $n \in \mathbb{N}$ and $0 < i < d$, G_d admits the following derivations:*

$$\mathtt{F} \stackrel{*}{\Longrightarrow} \mathbf{i}_r \,, \tag{D0}$$

$$\mathtt{F} \stackrel{*}{\Longrightarrow} \mathbf{d}_{\kappa_0} \ (\mathbf{d}_r \mathbf{i}_{\overline{r}})^n \ \mathtt{Pop} \ (\mathtt{F})^{n+1} \ \mathbf{i}_{\kappa_0} \,, \tag{D1}$$

$$\mathtt{F} \stackrel{*}{\Longrightarrow} \mathbf{d}_{\kappa_i} \mathbf{i}_{\kappa_{i-1}} \ (\mathbf{d}_r \mathbf{i}_{\overline{r}} \mathbf{i}_{\kappa_{i-1}})^n \ \mathtt{Pop} \ \mathtt{F} \ (\mathbf{d}_{\kappa_{i-1}})^{n+1} \ \mathbf{i}_{\kappa_i} \,, \tag{DL}$$

$$\mathtt{Pop} \stackrel{*}{\Longrightarrow} (\mathbf{i}_r \mathbf{d}_{\overline{r}})^n \,. \tag{DP}$$

Proof. Derivation (D0) is an immediate consequence of rule (R1) and derivation (DP) similarly follows from rules (R6) and (R7). To prove derivation (D1), we use

$$\mathtt{F} \xrightarrow{\text{(R2)}} \mathbf{d}_{\kappa_0}\ \mathtt{Rec\ F\ i}_{\kappa_0} \xrightarrow{\text{(R5)}} \cdots \xrightarrow{\text{(R5)}} \mathbf{d}_{\kappa_0}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}})^n\ \mathtt{Rec\ (F)}^n\ \mathtt{F\ i}_{\kappa_0}$$
$$\xrightarrow{\text{(R4)}} \mathbf{d}_{\kappa_0}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}})^n\ \mathtt{Pop\ (F)}^{n+1}\ \mathbf{i}_{\kappa_0}\ .$$

Finally, derivation (DL) is obtained with

$$\mathtt{F} \xrightarrow{\text{(R3}i)} \mathbf{d}_{\kappa_i}\mathbf{i}_{\kappa_{i-1}}\ \mathtt{Lim}_i\ \mathbf{d}_{\kappa_{i-1}}\mathbf{i}_{\kappa_i}$$
$$\xrightarrow{\text{(R9}i)} \cdots \xrightarrow{\text{(R9}i)} \mathbf{d}_{\kappa_i}\mathbf{i}_{\kappa_{i-1}}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}}\mathbf{i}_{\kappa_{i-1}})^n\ \mathtt{Lim}_i\ (\mathbf{d}_{\kappa_{i-1}})^n\ \mathbf{d}_{\kappa_{i-1}}\mathbf{i}_{\kappa_i}$$
$$\xrightarrow{\text{(R8}i)} \mathbf{d}_{\kappa_i}\mathbf{i}_{\kappa_{i-1}}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}}\mathbf{i}_{\kappa_{i-1}})^n\ \mathtt{Pop\ F}\ (\mathbf{d}_{\kappa_{i-1}})^{n+1}\ \mathbf{i}_{\kappa_i}\ . \qquad \square$$

Lemma 6.5. *For every ordinal $\alpha < \omega^d$ and every $n \in \mathbb{N}$, G_d has a computation $\langle n, 0, \alpha \rangle \xrightarrow{\mathtt{F}} \langle F_\alpha(n), 0, \alpha \rangle$.*

Proof. We first observe that G_d has a computation $\langle 0, n, \alpha \rangle \xrightarrow{\mathtt{Pop}} \langle n, 0, \alpha \rangle$ for every ordinal $\alpha < \omega^d$ and every $n \in \mathbb{N}$. This computation exists because G_d admits derivation (DP), i.e., $\mathtt{Pop} \xrightarrow{*} (\mathbf{i}_r \mathbf{d}_{\overline{r}})^n$. We now prove the lemma by induction on α.

For the base case $\alpha = 0$, we use derivation (D0), i.e., $\mathtt{F} \xrightarrow{*} \mathbf{i}_r$, yielding the following computation:

$$\begin{pmatrix} n \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathtt{F}} \begin{pmatrix} n+1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_0(n) \\ 0 \\ 0 \end{pmatrix}.$$

In the case of a successor ordinal $\alpha = \beta + 1$, we use derivation (D1), i.e., $\mathtt{F} \xrightarrow{*} \mathbf{d}_{\kappa_0}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}})^n\ \mathtt{Pop\ (F)}^{n+1}\ \mathbf{i}_{\kappa_0}$, leading to the following computation (recall that $F_\alpha(n) = F_\beta^{n+1}(n)$ by Eq. (F1)):

$$\begin{pmatrix} n \\ 0 \\ \alpha \end{pmatrix} \xrightarrow{\mathbf{d}_{\kappa_0}} \begin{pmatrix} n \\ 0 \\ \beta \end{pmatrix} \xrightarrow{(\mathbf{d}_r \mathbf{i}_{\overline{r}})^n} \begin{pmatrix} 0 \\ n \\ \beta \end{pmatrix} \xrightarrow{\mathtt{Pop}} \begin{pmatrix} n \\ 0 \\ \beta \end{pmatrix} \xrightarrow[\text{ind. hyp.}]{\mathtt{F}^{n+1}} \begin{pmatrix} F_\beta^{n+1}(n) \\ 0 \\ \beta \end{pmatrix} \xrightarrow{\mathbf{i}_{\kappa_0}} \begin{pmatrix} F_\beta^{n+1}(n) \\ 0 \\ \beta+1 \end{pmatrix} = \begin{pmatrix} F_\alpha(n) \\ 0 \\ \alpha \end{pmatrix}.$$

Finally, in the case of a limit ordinal $\alpha = \lambda$, say of the form $\lambda = \gamma + \omega^i$ where $0 < i < d$, we use derivation (DL), i.e., $\mathtt{F} \xrightarrow{*} \mathbf{d}_{\kappa_i}\mathbf{i}_{\kappa_{i-1}}\ (\mathbf{d}_r \mathbf{i}_{\overline{r}}\mathbf{i}_{\kappa_{i-1}})^n\ \mathtt{Pop\ F}\ (\mathbf{d}_{\kappa_{i-1}})^{n+1}\ \mathbf{i}_{\kappa_i}$. Before inspecting the computation below, note that if $\lambda = \gamma + \omega^i$ is represented by the values in $\kappa_0, \ldots, \kappa_{d-1}$, then one obtains a representation for γ by decrementing κ_i. Then, by incrementing $(n+1)$ times κ_{i-1}, one obtains $\gamma + \omega^{i-1} \cdot (n+1)$ which is $\lambda(n)$ by (L1). This leads to the following computation (recall that $F_\lambda(n) = F_{\lambda(n)}(n)$ by Eq. (FL)):

$$\begin{pmatrix} n \\ 0 \\ \lambda \end{pmatrix} \xrightarrow{\mathbf{d}_{\kappa_i}} \begin{pmatrix} n \\ 0 \\ \gamma \end{pmatrix} \xrightarrow{\mathbf{i}_{\kappa_{i-1}}(\mathbf{d}_r \mathbf{i}_{\overline{r}}\mathbf{i}_{\kappa_{i-1}})^n} \begin{pmatrix} 0 \\ n \\ \lambda(n) \end{pmatrix} \xrightarrow{\mathtt{Pop}} \begin{pmatrix} n \\ 0 \\ \lambda(n) \end{pmatrix} \xrightarrow[\text{ind. hyp.}]{\mathtt{F}} \begin{pmatrix} F_{\lambda(n)}(n) \\ 0 \\ \lambda(n) \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} F_{\lambda(n)}(n) \\ 0 \\ \lambda(n) \end{pmatrix} = \begin{pmatrix} F_\lambda(n) \\ 0 \\ \lambda(n) \end{pmatrix} \xrightarrow{(\mathbf{d}_{\kappa_{i-1}})^{n+1}} \begin{pmatrix} F_\lambda(n) \\ 0 \\ \gamma \end{pmatrix} \xrightarrow{\mathbf{i}_{\kappa_i}} \begin{pmatrix} F_\lambda(n) \\ 0 \\ \lambda \end{pmatrix}.$$

In all three cases, G_d has a computation $\langle n, 0, \alpha \rangle \xrightarrow{\mathtt{F}} \langle F_\alpha(n), 0, \alpha \rangle$. $\qquad \square$

We showed in Lemma 6.5 that there are computations of G_d that end in $F_\alpha(n)$. This corresponds to the completeness of weak computers. We will now show the safety part, i.e., that no successful computation of G_d may reach a value greater than $F_\alpha(n)$.

Lemma 6.6. *For all $n, n', m, m' \in \mathbb{N}$, $\alpha, \alpha' < \omega^d$, and $0 < i < d$, the following hold:*

$$\langle n, m, \alpha \rangle \xrightarrow{\mathtt{F}} \langle n', m', \alpha' \rangle \implies \alpha' = \alpha \wedge n' + m' \leq F_\alpha(n + m)\,, \tag{6.1}$$

$$\langle n, m, \alpha \rangle \xrightarrow{\mathtt{Rec}} \langle n', m', \alpha' \rangle \implies \alpha' = \alpha \wedge n' + m' \leq F_\alpha^n(n + m)\,, \tag{6.2}$$

$$\langle n, m, \alpha \rangle \xrightarrow{\mathtt{Pop}} \langle n', m', \alpha' \rangle \implies \alpha' = \alpha \wedge n' + m' = n + m\,, \tag{6.3}$$

$$\langle n, m, \alpha \rangle \xrightarrow{\mathtt{Lim}_i} \langle n', m', \alpha' \rangle \implies \alpha' = \alpha \wedge n' + m' \leq F_{\alpha \oplus (\omega^{i-1} \cdot n)}(n + m)\,. \tag{6.4}$$

Proof. By structural induction on the flow trees witnessing the transitions.

Top rule is (R1) $\mathtt{F} \to \mathbf{i}_r$**:** Then the flow tree has the following shape (in this and following illustrations, *we only display the top node of each immediate subtree* of the flow tree under consideration):

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathtt{F}} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

$$\Big|$$

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathbf{i}_r} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

Using action $\mathbf{i}_r$ in the subtree implies $n' = n + 1$, and also $\alpha' = \alpha$ and $m' = m$. With (FX), we deduce $n' + m' = n + m + 1 \leq F_\alpha(n + m)$ as required by (6.1).

Top rule is (R2) $\mathtt{F} \to \mathbf{d}_{\kappa_0} \mathtt{Rec}\ \mathtt{F}\ \mathbf{i}_{\kappa_0}$**:** We note that the first action, $\mathbf{d}_{\kappa_0}$, can only be fired if α is a successor ordinal $\beta + 1$. Then decrementing κ_0 transforms α into β, and the flow tree has the following form.

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathtt{F}} \begin{pmatrix} n' \\ m' \\ \beta'+1 \end{pmatrix}$$

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathbf{d}_{\kappa_0}} \begin{pmatrix} n \\ m \\ \beta \end{pmatrix} \quad \begin{pmatrix} n \\ m \\ \beta \end{pmatrix} \xrightarrow{\mathtt{Rec}} \begin{pmatrix} n_1 \\ m_1 \\ \beta_1 \end{pmatrix} \quad \begin{pmatrix} n_1 \\ m_1 \\ \beta_1 \end{pmatrix} \xrightarrow{\mathtt{F}} \begin{pmatrix} n' \\ m' \\ \beta' \end{pmatrix} \quad \begin{pmatrix} n' \\ m' \\ \beta' \end{pmatrix} \xrightarrow{\mathbf{i}_{\kappa_0}} \begin{pmatrix} n' \\ m' \\ \beta'+1 \end{pmatrix}$$

Invoking the induction hypothesis on the second and third subtrees yields

$$\beta_1 = \beta\,, \qquad\qquad\qquad n_1 + m_1 \leq F_\beta^n(n + m)\,,$$
$$\beta' = \beta_1\,, \qquad\qquad\qquad n' + m' \leq F_{\beta_1}(n_1 + m_1)\,.$$

Combining these results, we obtain $\beta' + 1 = \alpha$ as needed, and

$$n' + m' \leq F_{\beta_1}(n_1 + m_1) \leq F_{\beta_1}(F_\beta^n(n + m)) \qquad\qquad \text{by (FM)}$$
$$= F_\beta^{n+1}(n + m)$$
$$\leq F_\beta^{n+m+1}(n + m) \qquad\qquad \text{by (FX)}$$
$$= F_{\beta+1}(n + m) = F_\alpha(n + m)\,. \qquad\qquad \text{by (F1)}$$

Finally, $n' + m' \leq F_\alpha(n + m)$ as required by (6.1).

Top rule is (R3i) $\mathtt{F} \to \mathbf{d}_{\kappa_i} \mathbf{i}_{\kappa_{i-1}}\ \mathtt{Lim}_i\ \mathbf{d}_{\kappa_{i-1}} \mathbf{i}_{\kappa_i}$**:** The flow tree has the following form.

$$\binom{n}{m}{\alpha} \xrightarrow{\text{F}} \binom{n'}{m'}{\alpha'}$$

$$\binom{n}{m}{\alpha} \xrightarrow{\mathbf{d}_{\kappa_i}\mathbf{i}_{\kappa_{i-1}}} \binom{n_1}{m_1}{\alpha_1} \qquad \binom{n_1}{m_1}{\alpha_1} \xrightarrow{\text{Lim}_i} \binom{n_2}{m_2}{\alpha_2} \qquad \binom{n_2}{m_2}{\alpha_2} \xrightarrow{\mathbf{d}_{\kappa_{i-1}}\mathbf{i}_{\kappa_i}} \binom{n'}{m'}{\alpha'}$$

Firing the first two actions requires decrementing κ_i, hence α is some $\alpha_0 \oplus \omega^i$. After these actions, one has $n_1 = n$, $m_1 = m$ and $\alpha_1 = \alpha_0 \oplus \omega^{i-1}$. Similarly, the last two actions require decrementing κ_{i-1}, hence α_2 is some $\alpha_3 \oplus \omega^{i-1}$ and one has $n' = n_2$, $m' = m_2$ and $\alpha' = \alpha_3 \oplus \omega^i$. One obtains $\alpha_1 = \alpha_2$, hence $\alpha' = \alpha$, with the induction hypothesis, as well as

$$
\begin{aligned}
n' + m' = n_2 + m_2 &\leq F_{\alpha_1 \oplus (\omega^{i-1} \cdot n_1)}(n_1 + m_1) && \text{by ind. hyp.} \\
&= F_{(\alpha_0 \oplus \omega^{i-1}) \oplus (\omega^{i-1} \cdot n)}(n + m) \\
&= F_{\alpha_0 \oplus (\omega^i(n))}(n + m) \\
&\leq F_{\alpha_0 \oplus \omega^i}(n + m) && \text{by Lemma 6.3} \\
&= F_\alpha(n + m) \,,
\end{aligned}
$$

as required by (6.1).

Top rule is (R4) $\text{Rec} \to \text{Pop}$**:** Then the flow tree has the following form.

$$\binom{n}{m}{\alpha} \xrightarrow{\text{Rec}} \binom{n'}{m'}{\alpha'}$$

$$\binom{n}{m}{\alpha} \xrightarrow{\text{Pop}} \binom{n'}{m'}{\alpha'}$$

The induction hypothesis gives $\alpha' = \alpha$ and $n' + m' = n + m$. We deduce $n' + m' \leq F_\alpha^n(n + m)$, as required by (6.2), by invoking (FX).

Top rule is (R5) $\text{Rec} \to \mathbf{d}_r \mathbf{i}_{\bar{r}} \, \text{Rec} \, \text{F}$**:** Then $n > 0$ and the flow tree has the following form.

$$\binom{n}{m}{\alpha} \xrightarrow{\text{Rec}} \binom{n'}{m'}{\beta}$$

$$\binom{n}{m}{\alpha} \xrightarrow{\mathbf{d}_r \mathbf{i}_{\bar{r}}} \binom{n-1}{m+1}{\alpha} \qquad \binom{n-1}{m+1}{\alpha} \xrightarrow{\text{Rec}} \binom{n_1}{m_1}{\beta_1} \qquad \binom{n_1}{m_1}{\beta_1} \xrightarrow{\text{F}} \binom{n'}{m'}{\beta}$$

Here we can use the induction hypothesis on the second subtree, yielding

$$\beta_1 = \alpha \,, \qquad n_1 + m_1 \leq F_\alpha^{n-1}(n - 1 + m + 1) = F_\alpha^{n-1}(n + m) \,,$$

and on the third subtree, yielding

$$\beta = \beta_1 \,, \qquad n' + m' \leq F_{\beta_1}(n_1 + m_1) = F_\alpha(n_1 + m_1) \,.$$

Combining these and invoking (FM), yields $\beta = \alpha$ and $n' + m' \leq F_\alpha(n_1 + m_1) \leq F_\alpha(F_\alpha^{n-1}(n + m)) = F_\alpha^n(n + m)$ as required by (6.2).

Top rule is (R6) $\texttt{Pop} \to \varepsilon$**:** Then the flow tree $(n, m, \alpha) \xrightarrow{\texttt{Pop}} (n', m', \alpha')$ is a leaf, entailing $\alpha' = \alpha$ and $n' + m' = n + m$ as required by (6.3).

Top rule is (R7) $\texttt{Pop} \to \mathbf{i}_r \mathbf{d}_{\overline{r}} \texttt{Pop}$**:** Then the flow tree has the form.

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\texttt{Pop}} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathbf{i}_r \mathbf{d}_{\overline{r}}} \begin{pmatrix} n+1 \\ m-1 \\ \alpha \end{pmatrix} \qquad \begin{pmatrix} n+1 \\ m-1 \\ \alpha \end{pmatrix} \xrightarrow{\texttt{Pop}} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

The induction hypothesis on the second subtree gives $\alpha' = \alpha$ and $n' + m' = n + 1 + m - 1 = n + m$ as required by (6.3).

Top rule is (R8i) $\texttt{Lim}_i \to \texttt{Pop F}$**:** The flow tree has the following form.

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\texttt{Lim}_i} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\texttt{Pop}} \begin{pmatrix} n_1 \\ m_1 \\ \alpha_1 \end{pmatrix} \qquad \begin{pmatrix} n_1 \\ m_1 \\ \alpha_1 \end{pmatrix} \xrightarrow{\texttt{F}} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

On these subtrees, the induction hypothesis yields $\alpha' = \alpha_1 = \alpha$ and $n_1 + m_1 = n + m$. Furthermore we have

$$n' + m' \le F_{\alpha_1}(n_1 + m_1) \qquad \qquad \text{by ind. hyp.}$$
$$\le F_{\alpha \oplus (\omega^{i-1} \cdot n)}(n + m) , \qquad \qquad \text{by Lemma 6.2}$$

as required by (6.4).

Top rule is (R9i) $\texttt{Lim}_i \to \mathbf{d}_r \mathbf{i}_{\overline{r}} \mathbf{i}_{\kappa_{i-1}} \texttt{Lim}_i\ \mathbf{d}_{\kappa_{i-1}}$**:** The flow tree has the following form.

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\texttt{Lim}_i} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

$$\begin{pmatrix} n \\ m \\ \alpha \end{pmatrix} \xrightarrow{\mathbf{d}_r \mathbf{i}_{\overline{r}} \mathbf{i}_{\kappa_{i-1}}} \begin{pmatrix} n_1 \\ m_1 \\ \alpha_1 \end{pmatrix} \qquad \begin{pmatrix} n_1 \\ m_1 \\ \alpha_1 \end{pmatrix} \xrightarrow{\texttt{Lim}_i} \begin{pmatrix} n_2 \\ m_2 \\ \alpha_2 \end{pmatrix} \qquad \begin{pmatrix} n_2 \\ m_2 \\ \alpha_2 \end{pmatrix} \xrightarrow{\mathbf{d}_{\kappa_{i-1}}} \begin{pmatrix} n' \\ m' \\ \alpha' \end{pmatrix}$$

With its three actions, the first subtree implies

$$n_1 = n - 1 , \qquad\qquad m_1 = m + 1 , \qquad\qquad \alpha_1 = \alpha \oplus \omega^{i-1} .$$

Similarly, the last subtree yields

$$n' = n_2 , \qquad\qquad m' = m_2 , \qquad\qquad \alpha_2 = \alpha' \oplus \omega^{i-1} .$$

With the second subtree, the induction hypothesis yields $\alpha_2 = \alpha_1$ and $n_2 + m_2 \le F_{\alpha_1 \oplus (\omega^{i-1} \cdot n_1)}(n_1 + m_1)$. Combining these gives $\alpha' = \alpha$ and $n' + m' = n_2 + m_2 \le F_{\alpha_1 \oplus (\omega^{i-1} \cdot n_1)}(n_1 + m_1) = F_{(\alpha \oplus \omega^{i-1}) \oplus (\omega^{i-1} \cdot (n-1))}(n + m) = F_{\alpha \oplus (\omega^{i-1} \cdot n)}(n + m)$ as required by (6.4). $\qquad\square$

Now, for any ordinal $\alpha < \omega^d$, we may extend G_d and obtain a GVAS G_{F_α} that weakly computes F_α. This new GVAS inherits the counters, actions, non-terminals and rules of G_d. It furthermore includes two additional non-terminals, $\mathtt{S}$ and $\mathtt{Pop}'$, and associated rules. The start symbol will be $\mathtt{S}$ and, if α's CNF is $\sum_{i=d-1}^{0} \omega^i \cdot c_i$, the extra rules are:

$$\mathtt{S} \to \mathbf{i}_{\kappa_0}^{c_0} \mathbf{i}_{\kappa_1}^{c_1} \cdots \mathbf{i}_{\kappa_{d-1}}^{c_{d-1}} \; \mathtt{F} \; \mathtt{Pop}' \,, \tag{R0}$$

$$\mathtt{Pop}' \to \varepsilon \,, \tag{R10}$$

$$\mathtt{Pop}' \to \mathbf{d}_r \mathbf{i}_{\overline{r}} \; \mathtt{Pop}' \,. \tag{R11}$$

It is clear that, since there are no new rules for the non-terminals inherited from G_d, the properties stated in Lemmas 6.4 to 6.6 hold for G_{F_α} as they hold for G_d. Note also that $\mathtt{Pop}'$ behaves as $\mathtt{Pop}$ but exchanging the roles of r and $\overline{r}$.

Lemma 6.7. G_{F_α} *weakly computes* F_α.

Proof. We start with the completeness part of Definition 5.1. For this it is needed to show that, for any $n \in \mathbb{N}$, G_{F_α} has a computation of the form $(n, 0, \mathbf{0}_d) \xrightarrow{\mathtt{S}} (n', F_\alpha(n), \mathbf{e})$. For this we use the rule (R0) and exhibit the following computation:

$$(n, 0, \mathbf{0}_d) \xrightarrow{\mathbf{i}_{\kappa_0}^{c_0} \mathbf{i}_{\kappa_1}^{c_1} \cdots \mathbf{i}_{\kappa_{d-1}}^{c_{d-1}}} (n, 0, \alpha) \xrightarrow{\mathtt{F}} (F_\alpha(n), 0, \alpha) \xrightarrow{\mathtt{Pop}'} (0, F_\alpha(n), \alpha) \,.$$

The first part of that computation just relies on our convention for reading $\kappa_{d-1}, \dots, \kappa_0$ as the encoding of an ordinal, the second (crucial) part is given by Lemma 6.5, and the last part is by an analog of derivation (DP) for $\mathtt{Pop}'$.

For the safety part, we consider an arbitrary computation of the form $(n, 0, \mathbf{0}_d) \xrightarrow{\mathtt{S}} (n', r, \mathbf{e})$. The only rule for $\mathtt{S}$ is (R0), so there must exist some steps of the form

$$(n, 0, \mathbf{0}_d) \xrightarrow{\mathbf{i}_{\kappa_0}^{c_0} \mathbf{i}_{\kappa_1}^{c_1} \cdots \mathbf{i}_{\kappa_{d-1}}^{c_{d-1}}} (n, 0, \alpha) \xrightarrow{\mathtt{F}} (n', m', \alpha') \xrightarrow{\mathtt{Pop}'} (n'', r, \mathbf{e}) \,.$$

Necessarily, this satisfies $n' + m' \le F_\alpha(n)$ by (6.1). And since $\mathtt{Pop}'$ behaves like $\mathtt{Pop}$, we have $n'' + r = n' + m'$, as in (6.3). All this entails $r \le F_\alpha(n)$ as required by (SA). $\qquad\square$

7. Concluding Remarks

We proved that Grammar-controlled VASes or Pushdown VASes cannot weakly compute numerical functions that are sublinear. This was recently shown for plain VASes [24]. We also proved that GVASes can weakly compute the fast-growing functions F_α for all $\alpha < \omega^\omega$ while VASes can only weakly compute F_α for $\alpha < \omega$.

This research is motivated by verification questions for well-structured systems, in particular VASes and their extensions. In this area, weakly computable functions have traditionally been used to prove hardness results. Recent hardness proofs for well-structured systems crucially rely on the ability to weakly compute both fast-growing and slow-growing functions.

This work raises some new questions that are left for future work, including whether GVASes can weakly compute F_{ω^ω} and whether slow-growing functions can be weakly computed in other VAS extensions like the VASes with nested zero-tests of [30].

Another open question is the decidability of the boundedness problem for GVASes. Boundedness is decidable for PVASes [22] but the two problems do not coincide: on the one hand, in GVASes we only consider sequences of actions that are the yields of complete

derivation trees of a grammar, corresponding to configurations in PVAS that have empty stack content; on the other hand unboundedness in PVASes can come from unbounded counters or unbounded stack, while in GVASes only counters are measured. Indeed, the counter-boundedness problem for PVASes reduces to the boundedness problem for GVASes and is still open, while the stack-boundedness problem was shown decidable in [25].

The reachability problem for GVASes is also a source of open problems. Recently, the complexity of the reachability problem for plain VASes was proved to be between $\mathbf{F}_3$ and $\mathbf{F}_\omega$ [5, 23] in the complexity hierarchy set up by Schmitz [31]. Improving the $\mathbf{F}_3$ lower bound in the case of GVASes is an open question, as is the decidability status of the reachability problem.

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Appendix A. Well-quasi-ordering flow trees

We now prove Lemma 3.3, stating that $\leq_G$ is a well-quasi-ordering of $F(G)$, this for any GVASS G. A simple way to prove this is to reformulate $\leq_G$ as an homeomorphic embedding on flow trees labeled with enriched information equipped with further labels.

With a flow tree $t = \sigma[t_1, \ldots, t_k]$ we associate a tree $\mathrm{adorn}(t)$ made of a root labeled with $\langle \mathrm{root}(t), \mathrm{root}(t_1), \ldots, \mathrm{root}(t_k) \rangle$ and having $\mathrm{adorn}(t_1), \ldots, \mathrm{adorn}(t_k)$ as immediate subtrees. The nodes of $\mathrm{adorn}(t)$ are labeled by tuples of the form $\langle (\mathbf{c}_0, X, \mathbf{c}_k), (\mathbf{c}_0, X_1, \mathbf{c}_1), \ldots, (\mathbf{c}_{k-1}, X_k, \mathbf{c}_k) \rangle$ where $\mathbf{c}_0, \ldots, \mathbf{c}_k \in \mathbb{N}^d$ are configurations and where $X \Longrightarrow X_1 \cdots X_k$ is a rule in R, or $X \in \mathbf{A}$ is a terminal action and $k = 0$. Such a tuple is called an *instance* of the rule $X \Longrightarrow X_1 \cdots X_k$ (or of the action $X \in \mathbf{A}$). Given two instances $\lambda = \langle (\mathbf{c}_0, X, \mathbf{c}_k), (\mathbf{c}_0, X_1, \mathbf{c}_1), \ldots, (\mathbf{c}_{k-1}, X_k, \mathbf{c}_k) \rangle$ and $\lambda' = \langle (\mathbf{d}_0, Y, \mathbf{d}_\ell), (\mathbf{d}_0, Y_1, \mathbf{d}_1), \ldots, (\mathbf{d}_{\ell-1}, Y_\ell, \mathbf{d}_\ell) \rangle$, we write $\lambda \leq \lambda'$ when $(X \Longrightarrow X_1 \cdots X_k)$ and $(Y \Longrightarrow Y_1 \cdots Y_\ell)$ are the same rule or action —entailing $k = \ell$— and when $\mathbf{c}_j \leq \mathbf{d}_j$ for all $1 \leq j \leq k$. Suppose s is a flow tree with immediate subtrees $s_1, \ldots, s_k$. We write $\mathrm{adorn}(s) \sqsubseteq \mathrm{adorn}(t)$ if there is a subtree t' of t with immediate subtrees $t_1, \ldots, t_l$ such that $\mathrm{root}(\mathrm{adorn}(s)) \leq \mathrm{root}(\mathrm{adorn}(t'))$ (entailing $k = l$) and inductively $\mathrm{adorn}(s_j) \sqsubseteq \mathrm{adorn}(t_j)$ for all $1 \leq j \leq k$. It is easy to see that on derived trees of flow trees, $\sqsubseteq$ coincides with the standard homeomorphic embedding of labeled trees.

Lemma A.1. *$s \leq_G t$ if, and only if, $\mathrm{root}(s) \leq \mathrm{root}(t) \wedge \mathrm{adorn}(s) \sqsubseteq \mathrm{adorn}(t)$.*

Proof. We assume $s = \sigma[s_1, \ldots, s_k]$, $t = \theta[t_1, \ldots, t_\ell]$ and prove the claim by structural induction.

$\Longrightarrow$: Assume $s \leq_G t$. Thus $\mathrm{root}(s) \leq \mathrm{root}(t)$ and, by definition of $\leq_G$, t contains a subtree $t' = \theta'[t'_1, \ldots, t'_k]$ with

$$\sigma \leq \theta' \wedge s_1 \leq_G t'_1 \wedge \cdots \wedge s_k \leq_G t'_k \,.$$

Now this entails $\mathrm{root}(s) \leq \mathrm{root}(t')$ and $\mathrm{root}(s_i) \leq \mathrm{root}(t'_i)$ for all i, as well as (by ind. hyp.) $\mathrm{adorn}(s_i) \sqsubseteq \mathrm{adorn}(t'_i)$ for all i. Hence $\mathrm{adorn}(s) \sqsubseteq \mathrm{adorn}(t')$, entailing $\mathrm{adorn}(s) \sqsubseteq \mathrm{adorn}(t)$.
$\Longleftarrow$: We assume $\mathrm{root}(s) \leq \mathrm{root}(t)$ and $\mathrm{adorn}(s) \sqsubseteq \mathrm{adorn}(t)$. Hence there is a subtree t' of t with immediate subtrees $t'_1, \ldots, t'_k$ such that $\mathrm{root}(\mathrm{adorn}(s)) \leq \mathrm{root}(\mathrm{adorn}(t'))$ and $\mathrm{adorn}(s_i) \sqsubseteq \mathrm{adorn}(t'_i)$ for all $i = 1, \ldots, k$. From $\mathrm{root}(\mathrm{adorn}(s)) \leq \mathrm{root}(\mathrm{adorn}(t'))$, we infer that $\mathrm{root}(s_i) \leq \mathrm{root}(t'_i)$ for all $i = 1, \ldots, k$. Now one witnesses $s \leq_G t$ by observing that $s_i \leq_G t_i$ by ind. hyp. $\qquad\square$

Since the instances of rules are well-quasi-ordered by $\leq$ (there are only finitely many rules), $\sqsubseteq$ is a well-quasi-ordering by Kruskal's Tree Theorem [18]. With Lemma A.1 we immediately infer that $\leq_G$ is a well-quasi-ordering.

Appendix B. Monotonicity for Fast-Growing functions

We give detailed proofs for the two monotonicity lemmas stated after the definitions of the F_α functions in section 6.

Lemma 6.2. *For any ordinals $\alpha, \alpha' < \epsilon_0$ and any $n \in \mathbb{N}$, $F_\alpha(n) \leq F_{\alpha \oplus \alpha'}(n)$.*

Proof. By induction on α', then on α. We first observe that, if the claim holds for some given α and α', then it entails $F_\alpha^m(n) \leq F_{\alpha \oplus \alpha'}^m(n)$ for any $m > 0$ as a consequence of monotonicity, i.e., (FM).
    We now consider several cases for α and α':
- If $\alpha' = 0$, then $\alpha \oplus \alpha' = \alpha$ and the claim holds trivially.
- If $\alpha = 0$ then the claim becomes $F_0(n) \leq F_{\alpha'}(n)$, which holds since $F_0(n) = n + 1$ by (F0) and $n + 1 \leq F_{\alpha'}(n)$ by (FX).
- If $\alpha' = \beta' + 1$ is a successor then $\alpha \oplus \alpha'$ is $(\alpha \oplus \beta') + 1$ and we have $F_\alpha(n) \leq F_{\alpha \oplus \beta'}(n)$ by ind. hyp., $\leq F_{\alpha \oplus \beta'}^{n+1}(n)$ by (FX), $= F_{\alpha \oplus \beta' + 1}(n) = F_{\alpha \oplus \alpha'}(n)$, and we are done.
- If $\alpha = \beta + 1$ is a successor then $\alpha \oplus \alpha'$ is $(\beta \oplus \alpha') + 1$ and we have $F_\alpha(n) = F_\beta^{n+1}(n) \leq F_{\beta \oplus \alpha'}^{n+1}(n)$ by ind. hyp., $= F_{\alpha \oplus \alpha'}(n)$.
- The only remaining possibility is that both α and α' are limit ordinals. Then $(\alpha \oplus \alpha')(n)$ is $\alpha \oplus \alpha'(n)$ or $\alpha(n) \oplus \alpha'$, depending on which limit has the CNF with smallest last summand. In the first case we have $F_\alpha(n) \leq F_{\alpha \oplus \alpha'(n)}(n)$ by ind. hyp. since $\alpha'(n) < \alpha'$, $= F_{(\alpha \oplus \alpha')(n)}(n) = F_{\alpha \oplus \alpha'}(n)$. In the second case we have $F_\alpha(n) = F_{\alpha(n)}(n) \leq F_{\alpha(n) \oplus \alpha'}(n)$ by ind. hyp. since $\alpha(n) < \alpha$, $= F_{(\alpha \oplus \alpha')(n)}(n) = F_{\alpha \oplus \alpha'}(n)$. $\qquad\square$

Lemma 6.3. *For any ordinal $\alpha < \epsilon_0$ and limit ordinal $\lambda < \omega^\omega$, for any $m, n \in \mathbb{N}$, if $m \leq n$ then $F_{\alpha \oplus \lambda(m)}(n) \leq F_{\alpha \oplus \lambda}(n)$.*

Proof. Let us decompose λ under the form $\lambda = \delta + \omega^{k+1}$ so that $\lambda(m) = \delta + \omega^k \cdot (m+1)$. We prove the lemma by induction on α. We first observe that, if the claim holds for some given α, then $F_{\alpha \oplus \lambda(m)}^p(n) \leq F_{\alpha \oplus \lambda}^p(n)$ for every $p > 0$ and every $m, n \in \mathbb{N}$ such that $m \leq n$. This observation, which is easily proved by induction on p, is a consequence of strict expansivity and monotonicity, i.e., (FX) and (FM), respectively.

- If $\alpha = 0$ we have $F_{\alpha \oplus \lambda(m)}(n) = F_{\lambda(m)}(n) \leq F_{\lambda(n)}(n)$ by Lemma 6.2 since $\lambda(n) = \lambda(m) \oplus \omega^k \cdot (n - m)$, $= F_\lambda(n) = F_{\alpha \oplus \lambda}(n)$ by (FL).
- If $\alpha = \beta + 1$ is a successor, we have $F_{\alpha \oplus \lambda(m)}(n) = F_{(\beta \oplus \lambda(m))+1}(n) = F_{\beta \oplus \lambda(m)}^{n+1}(n)$ by (F1), $\leq F_{\beta \oplus \lambda}^{n+1}(n)$ by ind. hyp., $= F_{(\beta \oplus \lambda)+1}(n) = F_{\alpha \oplus \lambda}(n)$ again by (F1).
- If $\alpha = \gamma + \omega^\beta$ is a limit, then $\alpha \oplus \lambda$ is a limit too and we can compare ω^β and ω^{k+1}, the last summands of α and λ. There are two subcases:
  - If $0 < \beta \leq k$ then $(\alpha \oplus \lambda)(n) = \alpha(n) \oplus \lambda$. Moreover, $\alpha \oplus \lambda(m)$ is a limit since $0 < k$, and $(\alpha \oplus \lambda(m))(n) = \alpha(n) \oplus \lambda(m)$. We deduce $F_{\alpha \oplus \lambda(m)}(n) = F_{\alpha(n) \oplus \lambda(m)}(n)$ by (FL), $\leq F_{\alpha(n) \oplus \lambda}(n)$ by ind. hyp., $= F_{(\alpha \oplus \lambda)(n)}(n) = F_{\alpha \oplus \lambda}(n)$ again by (FL).
  - If $k + 1 \leq \beta$ then $(\alpha \oplus \lambda)(n) = \alpha \oplus \lambda(n)$. We deduce $F_{\alpha \oplus \lambda(m)}(n) \leq F_{\alpha \oplus \lambda(n)}(n)$ by Lemma 6.2 since $m \leq n$, $= F_{(\alpha \oplus \lambda)(n)}(n) = F_{\alpha \oplus \lambda}(n)$ by (FL) and we are done.　□