Viana, V., Nagy, D., Xavier, J., Neiva, A., Ginoulhiac, M., Mateus, L. & Varela, P. (Eds.). (2022). Symmetry: Art and Science | 12th SIS-Symmetry Congress [Special Issue]. *Symmetry: Art and Science*. International Society for the Interdisciplinary Study of Symmetry. 324-331. https://doi.org/10.24840/1447-607X/2022/12-41-324

SYMMETRIES IN CLASSICAL MUSIC FROM GUILLAUME DE MACHAUT TO CESAR FRANCK

CARLOTA SIMÕES

Name: Carlota Simões (b. Guarda, 1964)
Profession: Assistant Professor, Mathematics
Fields of interest: Popularization of mathematics, history of science, science communication
Affiliation: University of Coimbra
E-mail: carlota@mat.uc.pt
Homepage: https://apps.uc.pt/mypage/faculty/uc24512/pt
Major publications:
Simões, C. (2002) Mathematical aspects in the Second Viennese School of Music, in Mathematics and Arts: Mathematical Visualization in Art and Education, pp. 105-117, Editor: Claude P. Bruter, Springer-Verlag, Berlin. https://doi:10.1007/978-3-662-04909-9
Fernandes, J., Fiolhais, C. and Simões, C. (2012) 'Mathematics and Interdisciplinarity: outreach activities at the University of Coimbra', Raising Public Awareness of Mathematics, Editors Ehrhard Behrends, Nuno Crato, José Francisco Rodrigues, pp. 241- 254, Springer. https://doi:org/10.1007/978-3-642-25710-0
Mercat, C.; Vennebush, P.; Budd, C.; Simões, C.; Struckmeier, J. (2017) "Topic Study Group No. 07: Popularization of Mathematics". In Proceedings of the 13th International Congress on Mathematical Education, 405-412. Springer International Publishing, 2017. https://doi:10.1007/978-3-319-62597-3_34

Abstract: *Reflection, translation, rotation are terms frequently used in mathematics and generally associated with the sense of sight: we can use tiles or stained glass to clarify these concepts, but it is possible to exemplify the same concepts using the sense of hearing. In the first case the x and y coordinates of the plane are, for example, the width and height of a stained-glass window, in the second case the variables are respectively time and sound height, but instead of vision, we need hearing and memory to recognize the symmetry. Musicians and composers are aware of these concepts and use them with rigor and ingenuity. We present several musical examples from different eras, analysing compositions ranging from Guillaume de Machaut (14th century) to Cesar Franck (19th century), through Mozart's joviality and Bach's genius (18th century).*

Keywords: Music, Symmetries; Bach, Mozart, Cesar Franck.

INTRODUCTION

Music is written in a way very similar to the Cartesian representation of points and functions in the plane: a vertical axis for the pitch of the sound, a horizontal axis for the different melodic lines to develop over time. See (Simões 2013) for a mathematical explanation of the following operations and their properties. Suppose that the "figure" we want to transform is a sequence of notes *S*, which we want to use both in the original form and in versions that are derived from it by symmetry. Suppose the main sequence is $S = (a_1, a_2, \ldots, a_n)$. The *retrograde of S*, *R(S),* is obtained from *S* by reading this from end to beginning, $R(S) = (a_n, \ldots, a_2, a_1)$, and corresponds to symmetry along a vertical axis. The *inversion* of *S, I(S),* is obtained from *S* by maintaining the first note, maintaining the intervals between two consecutive notes, but changing their direction, that is, making an ascending interval become a descending and vice-versa. *I(S)* correspond to symmetry along a horizontal axis. Since the notes of *I(S)* are different from those of the *S* sequence, we designate the inversion by *I(S)* = (b_1, b_2, …, b_n). The *retrograde of the inversion* of *S, RI(S),* is obtained from *S* by applying the two previous operations to it, obtaining *RI(S)* = (b_n, …, b_2, b_1) (Figure 1).

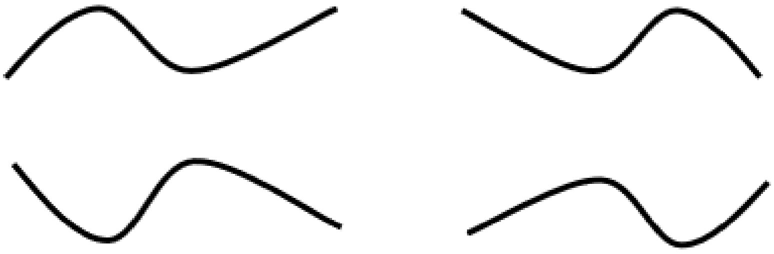

Figure 1 Representation of a melody *S* and its transformations by symmetry *R(S), I(S), RI(S)*.

There is yet another symmetry operation in music: the *transposition* of *S* by k halftones, S_k, is obtained from *S* by "adding" k halftones to all the notes of *S*. This transformation corresponds to the translation in the plane (Figure 2).

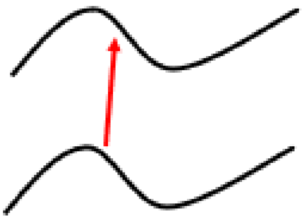

Figure 2 Representation of a melody *S* and its transposition S_k.

We can as well try to go directly from the original sequence *S* to the retrograde of the inversion *RI(S)* applying a 180° rotation to the sheet music: from a given melody we obtain a new one, in which ascending intervals become descending and vice versa, and where the length of the intervals, roughly speaking, remains the same. It is, however, necessary to add a clef to the staff, defining the position of one single note and thus making possible to know the name of all the other notes represented on the staff (Figure 3).

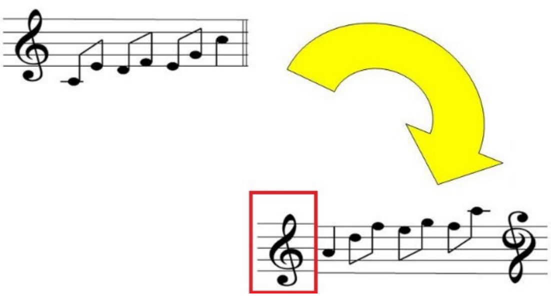

Figure 3 An initial melody and the one obtained by rotation of the musical staff.

MUSICAL EXAMPLES

Transposition and canon

We all know the nursery rhyme *Brother John*. We call it a *canon*: a song with several voices, in which a melodic line is sung by a first voice, which will be repeated by other voices that enter one after the other, each one repeating what the other has just sung. A magnificent example of canon is the fourth movement of the *Sonata in A Major* (1886) by César Franck (1822-1890) (Figure 4). Another example is the *Canon in D* by Johann Pachelbel (1653-1706) (Figure 5). In these examples, *Brother John*, *Sonata in A* by Franck and *Canon in D* by Pachelbel, the melodic line undergoes just a translation in time: the second voice reproduces the same melody, starting it on the same note.

Figure 4 Violin Sonata in A major, IV. Allegretto poco mosso (1886), by César Franck (1822-1890).

Figure 5 Canon in D by Johann Pachelbel (1653-1706), original manuscript.
Staatsbibliothek zu Berlin, image retrieved from Wikimedia.

But there are canons that shift the second voice both in time and in pitch. The *Goldberg Variations* by Johann Sebastian Bach (1685-1750) (Bach 1741), consisting of a total of 30 variations on the same theme, includes nine canons. Furthermore, all variations corresponding to a multiple of three, except the last one (Variation 30), are a canon. Moreover: the variation number 3*n* is a canon at the interval *n*, see Table 1.

Table 1. The *Goldberg Variations* by Johann Sebastian Bach (Bach 1741).

Aria Variatio 1. Variatio 2. *Variatio 3. Canone all'Unisono* Variatio 4. Variatio 5. *Variatio 6. Canone alla Seconda* Variatio 7. al tempo di Giga Variatio 8. *Variatio 9. Canone alla Terza* Variatio 10. Fughetta Variatio 11. *Variatio 12 Canone alla Quarta* Variatio 13. Variatio 14. *Variatio 15. Canone alla Quinta*	Variatio 16. Ouverture Variatio 17. *Variatio 18. Canone alla Sesta* Variatio 19. Variatio 20. *Variatio 21. Canone alla Settima* Variatio 22. alla breve Variatio 23. *Variatio 24. Canone all'Ottava* Variatio 25. adagio Variatio 26. *Variatio 27. Canone alla Nona* Variatio 28. Variatio 29. Variatio 30. Quodlibet Aria da Capo

The palindrome symmetry

A melody with the symmetry of a palindrome can be played from beginning to end or end to beginning, and the musical result is the same. This is the symmetry we find in the *Minuetto al Roverso* of the *Symphony No. 47 in G Major* of Joseph Haydn (1732-1809), represented in Figure 6 by one single voice that must be read by two voices simultaneously, one from beginning to end, another from end to beginning, or in the *Canon 1* of the *Musical Offering* (1747) by Johann Sebastian Bach (Figure 7).

Figure 6: Minuetto al Roverso, *Symphony n. 47* in G Major, Joseph Haydn (1732-1809), retrieved from Wikimedia.

Figure 7 Canon 2, Musical Offering, by Johann Sebastian Bach (1685 -1750), retrieved from Wikimedia.

And some centuries before Haydn or Bach, we find the rondeau *Ma fin est mon commencement* by Guillaume de Machaut (1300-1377) (Figure 8), in which even the lyrics are close to a palindrome:

Ma fin est mon commencement
Et mon commencement ma fin
Est teneure vraiement
Ma fin est mon commencement.
Mes tiers chans trois fois seulement
Se retrograde et einsi fin.
Ma fin est mon commencement
Et mon commencement ma fin.

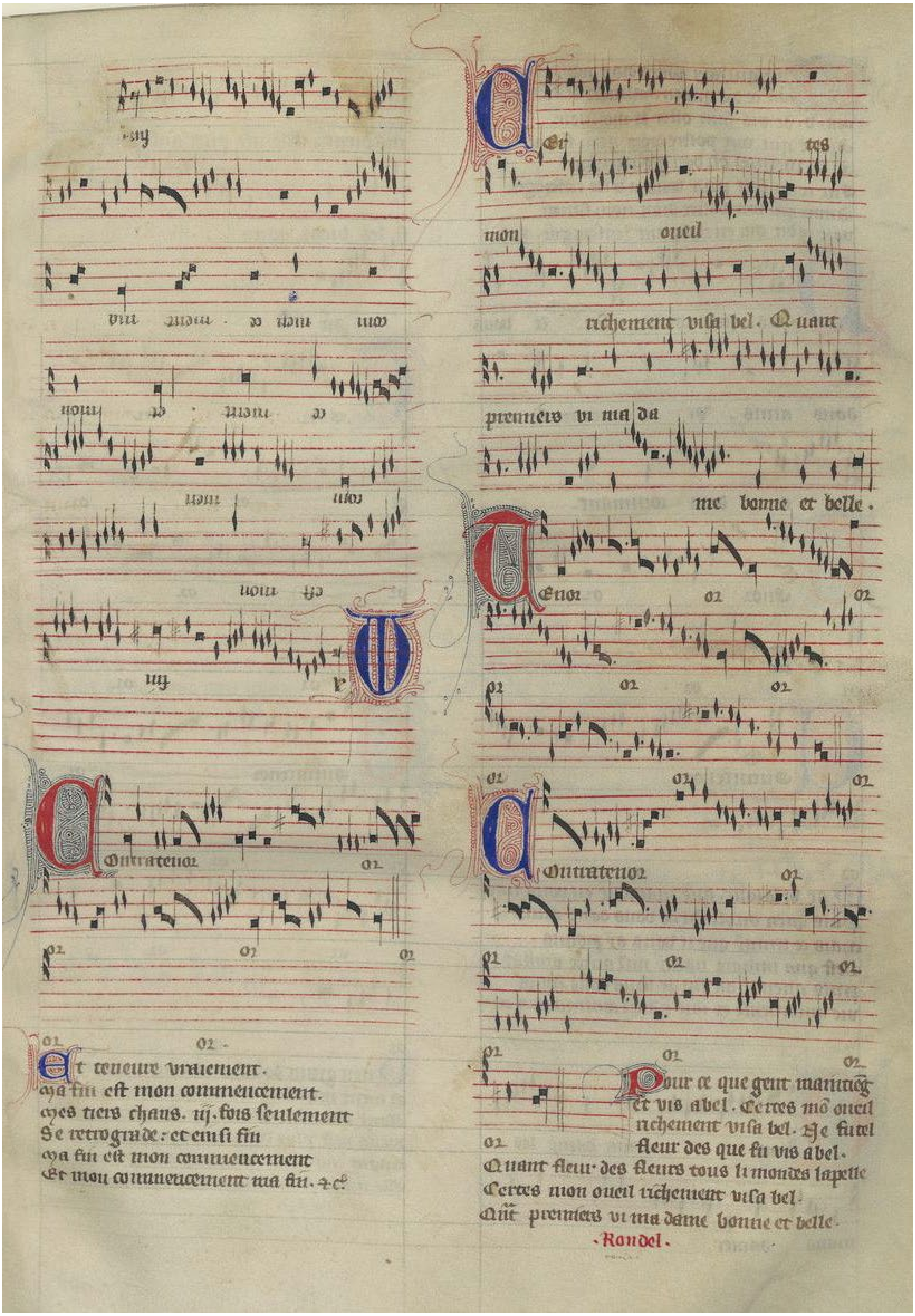

Figure 8 Ma fin est mon commencement, Guillaume de Machaut (1300-1377), retrieved from Wikimedia.

The symmetry of the horizontal mirror

The most famous composition by Paganini (1782 – 1840) is almost certainly *Capriccio n. 24* for violin. A century later, Rachmaninoff (1873-1943) honoured Paganini with his *Rhapsody on a Theme by Paganini* (1934), a set of 24 variations on Paganini's *Capriccio 24*. In Variation 18 of Rachmaninoff's *Rhapsody*, it is almost impossible to recognize Paganini's original theme without looking carefully at both music sheets. Rachmaninoff uses the inversion of Paganini's theme as the theme of his composition, that is, the melody that is obtained after applying the symmetry of reflection with a horizontal axis to Paganini's theme (Figure 9).

Figure 9 Themes from Paganini’s Capriccio n. 24 (above), and Rachmaninoff’s Variation 18 (bellow).

The rotation symmetry

The way we represent music on a musical staff allows one to apply a 180º rotation to the staff of a given melody obtaining a new melody, in which ascending intervals become descending and vice versa, and where intervals, roughly speaking, remain the same (see Figure 3). This is the idea behind a piece for two violins, attributed to Wolfgang Amadeus Mozart (1756 - 1791) called *Der Spiegel Duet* (Figure 10), in which just one sheet music is used for both players, in from of each other, one reading the sheet in one direction, the other reading it up-side-down. See (Simões 2006) for a more detailed analysis of this piece.

Figure 10 Der Spiegel Duet, attributed to Mozart (public domain - https://musescore.com/user/18878/scores/2617441).

CONCLUSION

Composers of all times use mathematical concepts in their compositions, often without realizing it. Symmetries is just one of many mathematical aspects we find in music. We presented examples of symmetries mainly from 17th century (Bach, Mozart, Haydn), but we could as well have presented the same concepts taken to the extreme by the Second Viennese School of Music of the 20th century (Simões 2002) or have developed Plato theories relating different scales with the character of their listeners, proposed in his *Republic* during the 4th century BC.

REFERENCES

Bach, J. S (1741). Clavier Übung, (Goldberg Variations, BWV 988) Nurberg.

Simões, C. (2002). 'Mathematical aspects in the Second Viennese School of Music', Mathematics and Arts: Mathematical Visualization in Art and Education, pp. 105–117, Editor. Claude P. Bruter, Springer-Verlag, Berlin. https://doi:10.1007/978-3-662-04909-9

Simões, C. (2006). 'Padrões matemáticos na obra de Mozart', Encontro MÚSICA e MATEMÁTICA - Actas, Helena Mena Matos e João Nuno Tavares (coord. editorial), pp. 64-76, Universidade do Porto e Casa da Música, Porto.

Simões. C. (2013) "Do canto gregoriano a Jacques Brel - simetrias e estruturas matemáticas em música" - Educação e Matemática, Revista da Associação de Professores de Matemática, Nº 125, pp. 23-33.

Carlota SIMÕES

Carlota Simões has a PhD in Mathematics from the University of Twente (The Netherlands). She is professor at the Mathematics Department of the Faculty of Science and Technology of the University of Coimbra (UC) and member of the Centre of Physics of the UC. From 2007 to 2019, she was part of the board of the Science Museum of the University of Coimbra (Vice-Director from 2007 till 2015 and Director from 2015 till 2019). Presently, she is Member of the Arts, Letters and Sciences Class of the Academia de Marinha. Her main interests are history and communication of science.