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Profiles in understanding the density of rational numbers among primary and secondary school students

Perfiles en la comprensión de la densidad de los números racionales en estudiantes de educación primaria y secundaria

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Abstract ∞ The present cross-sectional study investigated 953 fifth to tenth grade students' understanding of the dense structure of rational numbers. After an inductive analysis, coding the answers based on three types of items on density, a TwoStep Cluster Analysis revealed different intermediate profiles in the understanding of density along grades. The analysis highlighted qualitatively different ways of thinking: i) the idea of consecutiveness, ii) the idea of a finite number of numbers, and iii) the idea that between fractions, there are only fractions, and between decimals, there are only decimals. Furthermore, our profiles showed differences regarding rational number representation since students first recognised the dense nature of decimal numbers and then of fractions. Learners, however, were still found to have a natural number-based idea of the rational number structure by the end of secondary school, especially when they had to write a number between two pseudo-consecutive rational numbers.

Keywords ∞ Rational numbers; Density; Discreteness; Learner profiles; Fractions

Resumen ∞ En este estudio transversal sobre la densidad de los números racionales participaron 953 estudiantes desde 5º curso de educación primaria hasta 4º curso de educación secundaria. Tras un análisis inductivo, codificando las respuestas a tres tipos de ítems, se llevó a cabo un análisis clúster, que reveló diferentes perfiles intermedios en la comprensión de la densidad. Se identificaron formas de pensar diferentes: i) la idea de consecutivo, ii) la idea de número finito de números, y iii) la idea de que entre fracciones solo hay fracciones y entre decimales solo hay decimales. Además, se obtuvieron diferencias con respecto a la representación de los números racionales: los estudiantes primero reconocieron la densidad en números decimales y posteriormente, en fracciones. Se destaca que los estudiantes al final de la educación secundaria todavía tenían una idea basada en el conocimiento del número natural, especialmente cuando tenían que escribir un número entre dos números racionales pseudo-consecutivos.

Palabras clave ∞ Números racionales; Densidad; Conjunto discreto; Perfiles de estudiantes; Fracciones

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1. INTRODUCTION

Rational numbers constitute a central mathematical idea that children have to master during their pre-secondary school years. They are essential to subsequently develop an understanding of a wide range of related concepts, including proportions, ratios, and percentages, as well as more advanced concepts of algebra and calculus (Kieren, 1993). However, understanding rational numbers has been described as a student's stumbling block (Carpenter et al., 1993).

Previous research has pointed to the interference of natural numbers as a major explanation of students' difficulties in understanding rational numbers (Fischbein et al., 1985). While the concept of rational number is not a simple extension of natural number knowledge (Kieren, 1993), students sometimes treat rational numbers in the same way as natural numbers (Moss, 2005). This overreliance on natural number properties leads students to numerous errors and misconceptions when solving rational number tasks that can persist over the years (Fischbein et al., 1985; Moss, 2005; Ni & Zhou, 2005; Vamvakoussi et al., 2012).

Under this assumption, recent research has focused on examining students' tendency to inappropriately apply properties of natural numbers in rational number situations – a phenomenon denoted as natural number bias (Ni & Zhou, 2005; Van Dooren et al., 2015; Van Hoof et al., 2015). Studies addressing the natural number bias phenomenon have considered three main domains in which rational numbers differ from natural numbers: determining rational number size; conducting operations with rational numbers; and the density of rational numbers (González-Forte et al, 2020, 2021; McMullen et al., 2015).

In the present study, we focused on the domain of density, since the natural number bias is most persistent in this domain (McMullen et al., 2015) and fraction density understanding is a better predictor of algebra compared to fraction size understanding (McMullen & Van Hoof, 2020).

2. THEORETICAL AND EMPIRICAL BACKGROUND

2.1. Natural number interference in the domain of density

The natural number set is discrete since between two natural numbers there is a finite (possibly zero) number of numbers (e.g., only number 4 is between numbers 3 and 5). In contrast, the rational number set is dense since there is an infinite number of numbers between any two different rational numbers.

Previous research has shown that understanding the density of rational numbers is a complex task for primary and secondary school students (Merenluoto & Lehtinen, 2002; Vamvakoussi & Vosniadou, 2004, 2007; Van Hoof et al., 2015) and even for undergraduates (Tirosh et al., 1999).

The idea of discreteness, developed through experience with natural numbers, is considered by Vamvakoussi and Vosniadou (2004) as a "fundamental presupposition which constrains students' understanding of the structure of the set of rational numbers" (p. 457) causing numerous conceptual difficulties. For instance, some secondary school students consider that between the "pseudo-consecutive" fractions 5/7 and 6/7 there are no numbers, or that only number 1/3 exists between 1/2 and 1/4 (e.g., Merenluoto & Lehtinen, 2002). This misconception has been also observed in undergraduates (Tirosh et al., 1999). In decimal numbers, some primary school students believe that between the "pseudo-consecutive" decimals 0.59 and 0.60, it is not possible to find other numbers, or that between 1.22 and 1.24, lies only the number 1.23 (e.g., Broitman et al., 2003). This misconception has been also observed in secondary school students (Merenluoto & Lehtinen, 2002) and in undergraduates (Tirosh et al., 1999).

Difficulties in understanding the dense structure of rational numbers are also related to the fact that rational numbers can be represented as both fractions and decimals. Previous research has shown that some primary and secondary school students, and even undergraduates, treat fractions and decimal numbers as unrelated sets of numbers, rather than interchangeable representations of the same numbers (Carpenter et al., 1993; Khoury & Zazkis, 1994). For example, Markovits and Sowder (1991) showed that a large number of middle-graders ordered a series of decimal numbers and fractions separately, or they explicitly stated that ordering them in one series could not be done. Furthermore, some secondary school students tend to believe that only decimals exist between two different decimals and only fractions exist between two different fractions (Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2010).

2.2. Profiles in the understanding of density

Previous studies investigating students' understanding of the dense structure of rational numbers mainly focused on secondary school students (Vamvakoussi & Vosniadou, 2004, 2007, 2010; Vamvakoussi et al., 2011). The profiles identified in these studies were as follows:

- Students who believed that there are no numbers between two pseudo-consecutive rational numbers, and that there is a finite number of numbers between two non-pseudo-consecutive rational numbers.
- Students who believed that there is a finite number of numbers between two pseudo and non-pseudo-consecutive rational numbers.
- Students who believed that decimals are dense, whereas fractions are discrete, and *vice versa*.
- Students who believed that there is an infinite number of numbers, both between two different decimals and between two different fractions, but were reluctant to accept that there can be decimals between two different fractions, and *vice versa*.
- Students who correctly believed that there is an infinite number of numbers between any two different numbers regardless of their symbolic representation.

These range of intermediate profiles show that understanding the density of rational numbers is a gradual process, and not an all or nothing issue.

2.3. The present study

The study aims at examining individual differences in students' understanding of rational number density. Previous studies were focused on secondary school students. As far as we know, however, such studies including also primary school students – and studying them in the age range from primary to secondary – are scarce (González-Forte et al., 2018). They would be valuable in order to know how different ways of thinking about rational number density evolve from primary to secondary school. Furthermore, previous studies focused on examining individual differences in students' understanding of rational number density have included multiple-choice (Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2010), and/or open-ended question items (Vamvakoussi & Vosniadou, 2004, 2007). They asked students "how many numbers are there in between two different rational numbers given?", which measured the students' conceptual knowledge of density. None of them, nevertheless, asked students for the writing of a specific number between two given rational numbers, what serves to measure the students' procedural knowledge of density.

In the present study, we expand the research literature by performing a cross-sectional study on a large sample of primary and secondary school students, and by determining profiles based on an inductive analysis of the students' answers. We also include a wider range of item types. The inclusion of open-ended questions and writing-related items is our main analytical strategy in order to provide a more detailed and original description of the understanding of rational number density on the side of the students.

3. Method

3.1. Participants

Participants were 1,262 [Country] primary and secondary school students distributed over 5th grade (n = 205), 6th grade (n = 219), 7th grade (n = 221), 8th grade (n = 209), 9th grade (n = 198), and 10th grade (n = 210). There was approximately the same number of boys and girls in each age group. The participating schools were spread over nine cities (five primary schools and five secondary schools) and students were from mixed socio-economic backgrounds. Participants and schools were both recruited randomly and parental consent was obtained for all.

In the Spanish curriculum, the teaching of rational numbers starts in 3rd grade of primary education through the concept of fraction. In 4th grade, decimal numbers, fractions and percentages are introduced. In 5th grade, students use the number line to compare and order rational numbers that are represented as decimal numbers and fractions. Finally, in 6th grade, the meaning of equivalent fractions is taught. Then, in secondary school (from 7th to 10th grade), students are able to conduct operations with rational numbers.

3.2. Instrument

We adapted the density items of the RNST (*Rational Number Sense Test*, Van Hoof et al., 2015). Our paper-and-pencil test consisted of 17 density items. There were items which measured procedural knowledge, in which students had to write a number between two given rational numbers (called *write items*); and items which measured conceptual knowledge: items where students had to answer an open-ended question on how many numbers there were between two given fractions or two given decimal numbers (called *question items*) and, items where students had to answer a multiple-choice question about how many numbers there were between two given fractions or two decimal numbers, choosing one out of the seven answers offered (called *multiple-choice items*). Furthermore, we used both fraction items and decimal items.

There were seven write items, six question items, and four multiple-choice items. In the write items, there were four congruent items, in which students could obtain a correct answer using natural number knowledge. In these items, students had to write a number between two non-pseudo-consecutive fractions (2/7 and 6/7; 1/4 and 3/4) and between two non-pseudo-consecutive decimal numbers (2.5 and 2.7; 5.3 and 5.8). Moreover, there were three incongruent items, in which the use of natural number knowledge leads students to an incorrect answer. In these items, students had to write a number between two pseudo-consecutive decimal numbers or between two pseudo-consecutive fractions: one decimal item (3.49 and 3.50) and two fraction items: 1/3 and 2/3 (fractions with the same denominator) and 1/8 and 1/9 (fractions with the same numerator). Congruent items were only included as a "marker" to check whether students had a basic understanding at least of decimals and fractions.

In the question items, there were three fraction items: 2/5 and 3/5 (pseudo-consecutive fractions), 2/5 and 4/5 (non-pseudo-consecutive fractions with the same denominator) and 5/9 and 5/6 (non-pseudo-consecutive fractions with the same numerator) and three decimal items: 1.42 and 1.43 (pseudo-consecutive decimals), 1.9 and 1.40 (non-pseudo-consecutive decimals) and 2.3 and 2.6 (non-pseudo-consecutive decimals).

In the multiple-choice items, there were two fraction items: 1/3 and 2/3 (pseudo-consecutive fractions) and 1/6 and 4/6 (non-pseudo-consecutive factions); and two decimal items: 3.72 and 3.73 (pseudo-consecutive decimals) and 0.7 and 0.9 (non-pseudo-consecutive decimals). Figure 1 summarises the items used in the test.

Students solved the paper-and-pencil test individually. The items were presented in a randomised order in eight different versions. The multiple-choice items were always at the end of each test, since the word "infinite" appears and can help them to correctly solve the other items. There was no time limit to finish the test, as this could encourage natural number biased reasoning (Vamvakoussi et al., 2012).

Figure 1. Items used in the test

Type	Wording	Items
Write congruent	*Write a number between these two numbers. If you think that is not possible, write 'impossible'.*	2.5 and 2.7 5.3 and 5.8 2/7 and 6/7 1/4 and 3/4
Write incongruent	*Write a number between these two numbers. If you think that is not possible, write 'impossible'.*	3.49 and 3.50 1/3 and 2/3 1/8 and 1/9
Question	*How many numbers are there between…?*	1.42 and 1.43 1.9 and 1.40 2.3 and 2.6 2/5 and 3/5 2/5 and 4/5 5/9 and 5/6
Multiple-choice	*How many numbers are there between these numbers? Circle the best answer.* a) No numbers b) A finite number of decimals c) A finite number of fractions d) An infinite number of decimals e) An infinite number of fractions f) An infinite number of numbers that can be represented by several different representations, such as decimals, fractions g) None of the above, I think that…	3.72 and 3.73 0.7 and 0.9 1/3 and 2/3 1/6 and 4/6

3.3. Analysis

Firstly, four researchers individually analysed students' answers to the three different types of items, in order to identify categories according to the nature of the answer. Secondly, categories obtained were discussed until agreement was reached (triangulation process).

For the write items, we identified six categories: i) *Correct*: answers in which students correctly wrote a number between the two given numbers; ii) *Naïve*: answers where students said it was impossible, that there was no other number; iii) *Consecutiveness*: answers in which students identified that there were other numbers but still had a naïve idea of the "following" fraction (e.g., the fraction that follows 1/3 is 1/4, the next is 1/5...); iv) *Difference*: answers where students calculated and reported the difference between the two given numbers (e.g., 0.01 is between 3.49 and 3.50); v) *Rest*: answers in which students wrote a number not included between the two given numbers; vi) *Blank answers*.

For the question items, seven categories were identified: i) *Infinite*: answers where students said that there was an infinite number of numbers between the two given ones; ii) *Difference*: answers in which students calculated and reported the difference between the two given numbers (e.g., 0.3 is between 2.3 and 2.6); iii) *Naïve consecutive*: answers in which students said that there were no numbers between two pseudo-consecutive numbers (e.g., there were no numbers between 1.42 and 1.43 or between 2/5 and 3/5) and gave a finite list of consecutive numbers between two non-pseudo-consecutive numbers (e.g., only the numbers 2.4 and 2.5 are between 2.3 and 2.6 or only 3/5 is between 2/5 and 4/5) or the number of numbers of this list (e.g., there are two numbers between 2.3 and 2.6 or there is one number between 2/5 and 4/5); iv) *Finite consecutive*: answers where students gave a finite list of consecutive numbers between the numbers after adding a decimal and then counted in decimal numbers (e.g., the numbers 1.421, 1.422, 1,423..., 1.429 are between 1.42 and 1.43) or after adding a decimal in the numerator in fractions (e.g., the numbers 2.1/5, 2.2/5, 2.3/5..., 2.9/5 are between 2/5 and 3/5) or gave the corresponding number of numbers of these lists (e.g., there are nine numbers between 1.42 and 1.43 or there are nine numbers between 2/5 and 3/5); v) *Finite*: answers in which students gave other specific numbers included between the given numbers; vi) *Rest*: answers in which students gave specific numbers not included between the given numbers; vii) *Blank answers*.

For the multiple-choice items, nine categories were identified according to the chosen option: i) *Naïve*: option A; ii) *Decimal Finite*: option B; iii) *Fraction Finite*: option C; iv) *Decimal Infinite*: option D; v) *Fraction Infinite*: option E; vi) *Infinite*: option F; vii) *Finite*: option G focused on saying that there was a finite number of numbers between the two given numbers, without explicitly distinguishing between fractions or decimals; viii) *Difference*: option G focused on calculating and reporting the difference between the two given numbers; ix) *Blank answers*.

With these categories, a TwoStep Cluster Analysis (in SPSS 25) with categorical data was performed to identify groups of students (profiles) with qualitatively similar response patterns. Given the complexity of our coding scheme, many intermediate states of understanding could be expected. Therefore, we analysed the data separately for each item type, and grouped by age: the 5th and 6th grade, the 7th and 8th grade, and the 9th and 10th grade.

For the TwoStep Cluster Analysis, 309 students were eliminated since they incorrectly solved at least three of the four write congruent items. As these items can be solved even using an incorrect reasoning based on natural number properties, we believed that there was little sense in investigating answers on the actual density items (13 items) of students who failed on these items. This led to a final total of 953 participants distributed over 5th grade (n = 115), 6th grade (n = 139), 7th grade (n = 162), 8th grade (n = 173), 9th grade (n = 174), and 10th grade (n = 190).

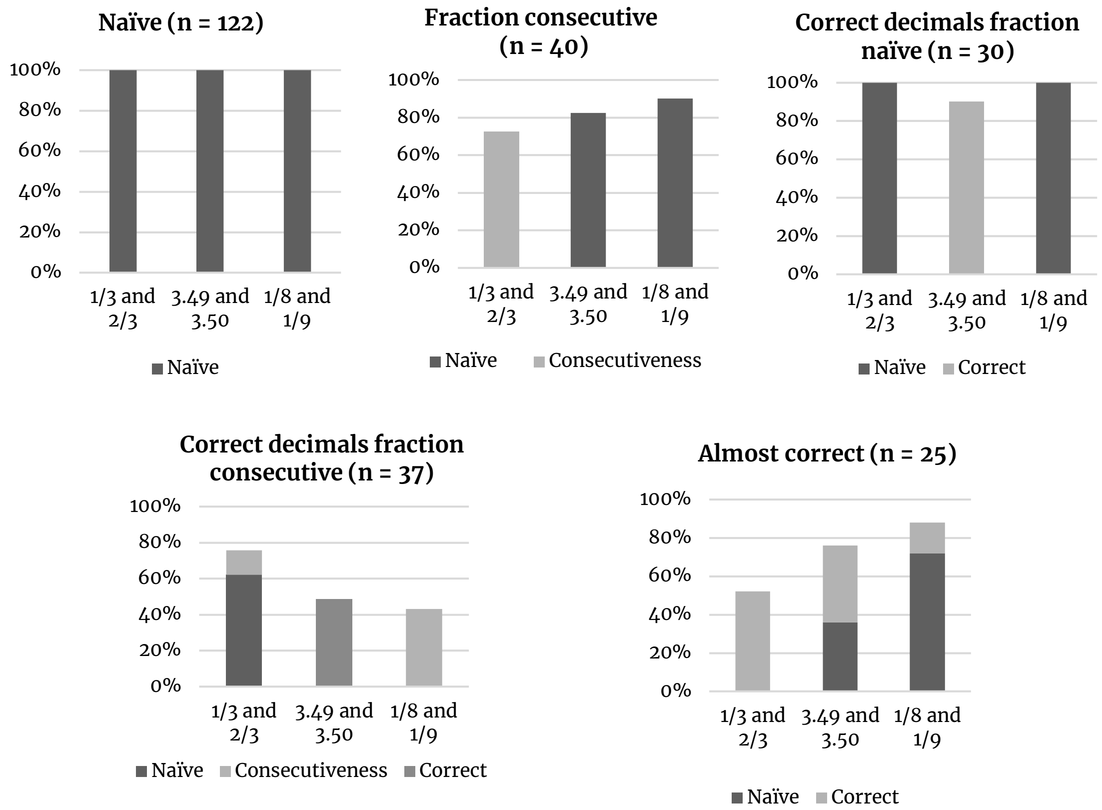
4. Results

In this section, we first determine the number of profiles in each type of item and characterise them. The number of profiles was always based on a lower value of the *Bayesian Information Criterion* (BIC) as well as an interpretative viewpoint. Second, we show the evolution of these profiles from 5th to 10th grade.

4.1. Profiles in each type of item

With regard to the write items, in 5th and 6th grade, the five profiles-solution provided the best description of the answers (see Figure 2). The X-axis consists of the three test items, and the Y-axis consists of the frequency percentages of the most widely used categories identified in the inductive analysis.

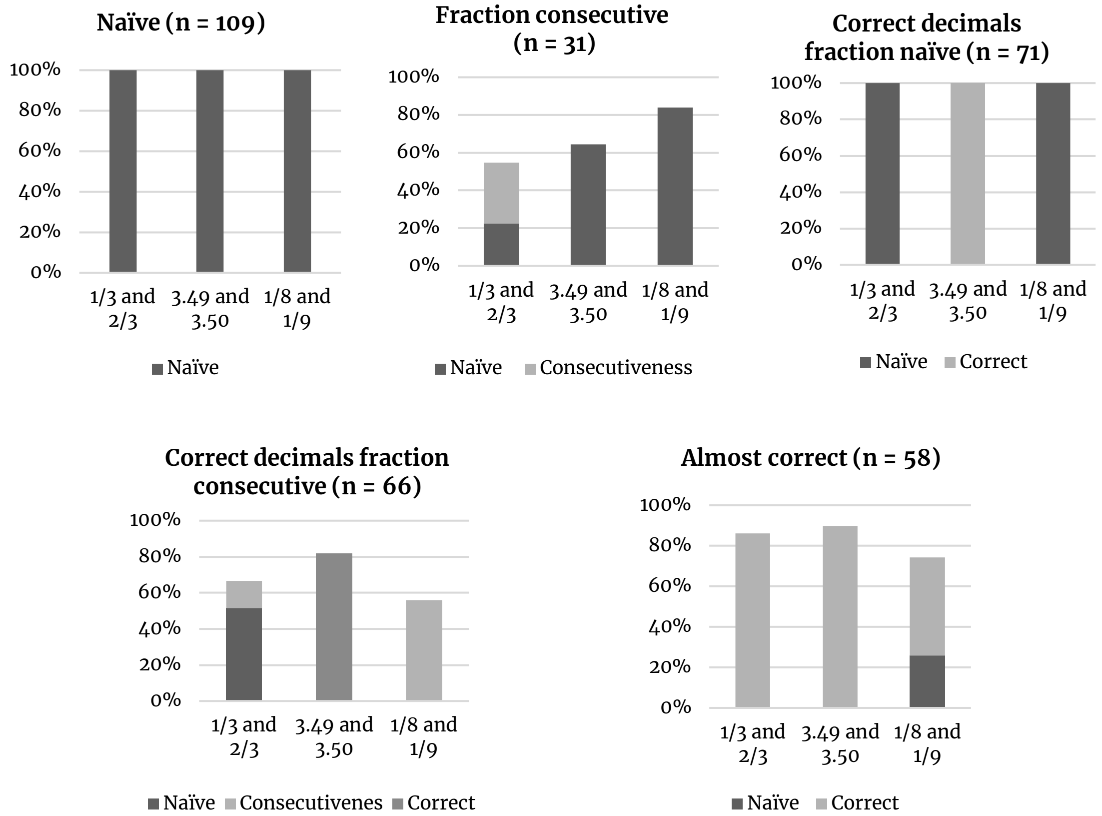
Figure 2. Profiles in the write items in 5th and 6th grade



- *Naïve*: students who considered that it was impossible to write a number between two pseudo-consecutive numbers.
- *Fraction consecutive*: students who considered that it was impossible to write a number between two pseudo-consecutive numbers, but who, in fractions with the same denominator (i.e. 1/3 and 2/3), started to think that it could be other numbers, and incorrectly answered in a naïve consecutive way (i.e. 1/4, 1/5, 1/6...).
- *Correct decimals fraction naïve*: students who correctly wrote a number between two pseudo-consecutive decimals, but considered that it was impossible to write a number between two pseudo-consecutive fractions.
- *Correct decimals fraction consecutive*: students who correctly wrote a number between two pseudo-consecutive decimals. In fractions, they considered that it was impossible to write a number, with the exception of some students who answered in a naïve consecutive way. They answered 1/4, 1/5, 1/6... between 1/3 and 2/3 and 2/8, 3/8, 4/8...between 1/8 and 1/9.
- *Almost correct*: students who correctly wrote a number between two pseudo-consecutive decimals. In fractions, they correctly wrote a number between two pseudo-consecutive fractions with the same denominator (1/3 and 2/3), but in fractions with the same numerator (1/8 and 1/9), they considered that it was impossible to write a number.

The same five profiles-solution also seemed most appropriate in 7th and 8th grade (see Figure 3).

Figure 3. Profiles in the write item in 7th and 8th grade



Finally, a solution with five profiles also seemed most appropriate in 9th and 10th grade (see Figure 4).

In these grades, the *Fraction consecutive* profile was not detected, but we identified a new profile:

- *Correct*: students who correctly wrote a number between two pseudo-consecutive numbers.

Regarding question items, in 5th and 6th grade, we chose a five profiles-solution (see Figure 5).

- *Naïve*: students who considered that there was no other number between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers.
- *Decimal finiters*: students who began to consider that there was a finite number of numbers between two pseudo and non-pseudo-consecutive decimals (a subgroup of students still considered that there was no other number between two pseudo-consecutive decimals). However, they considered that there was no other number between two pseudo-consecutive fractions.

Figure 4. Profiles in the write items in 9th and 10th grade

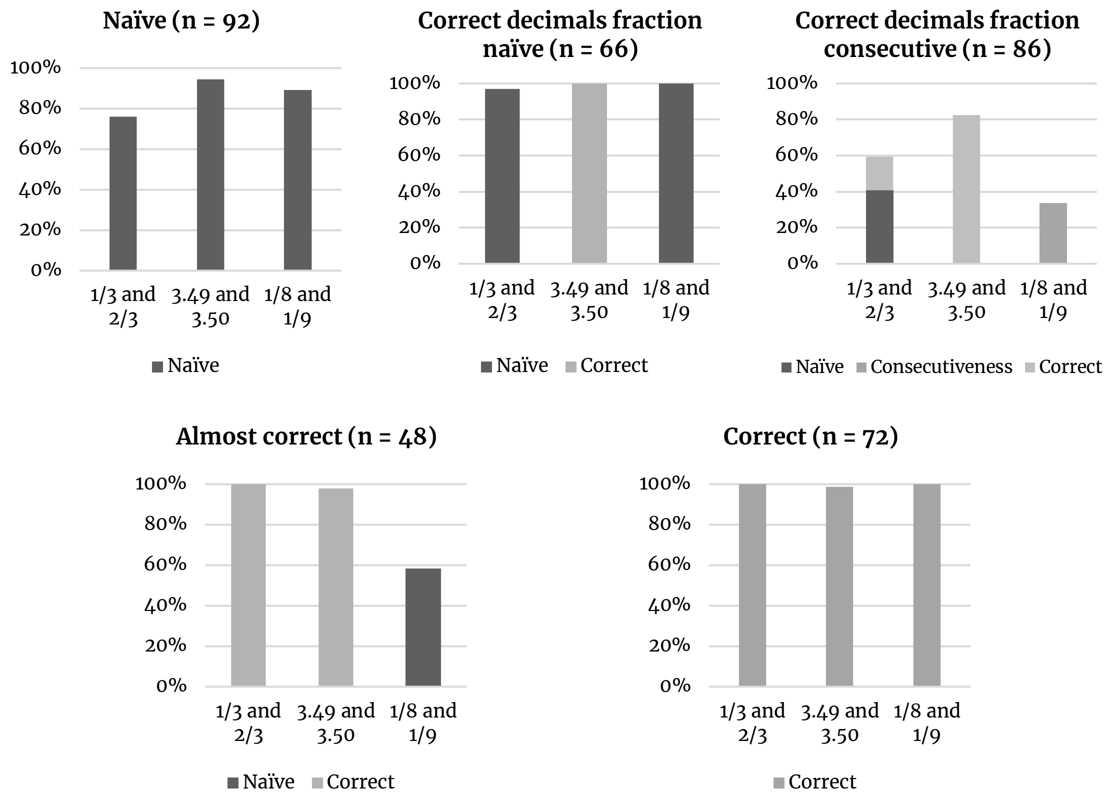
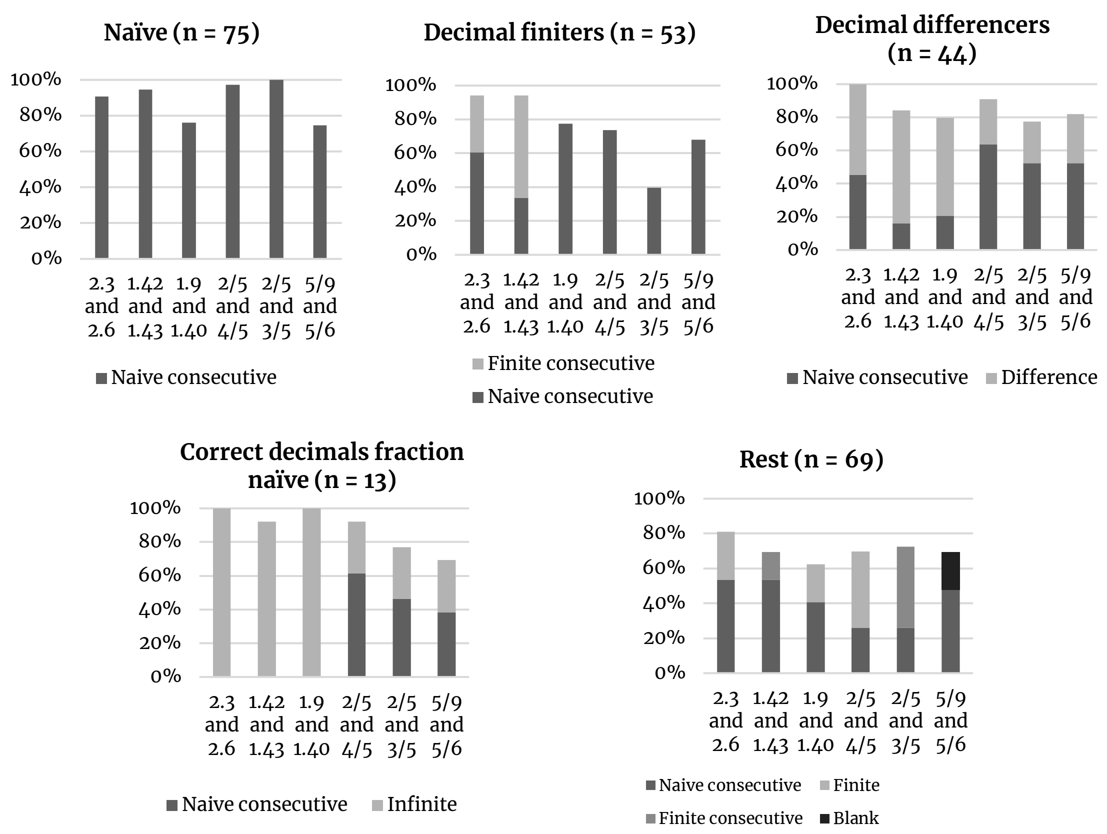


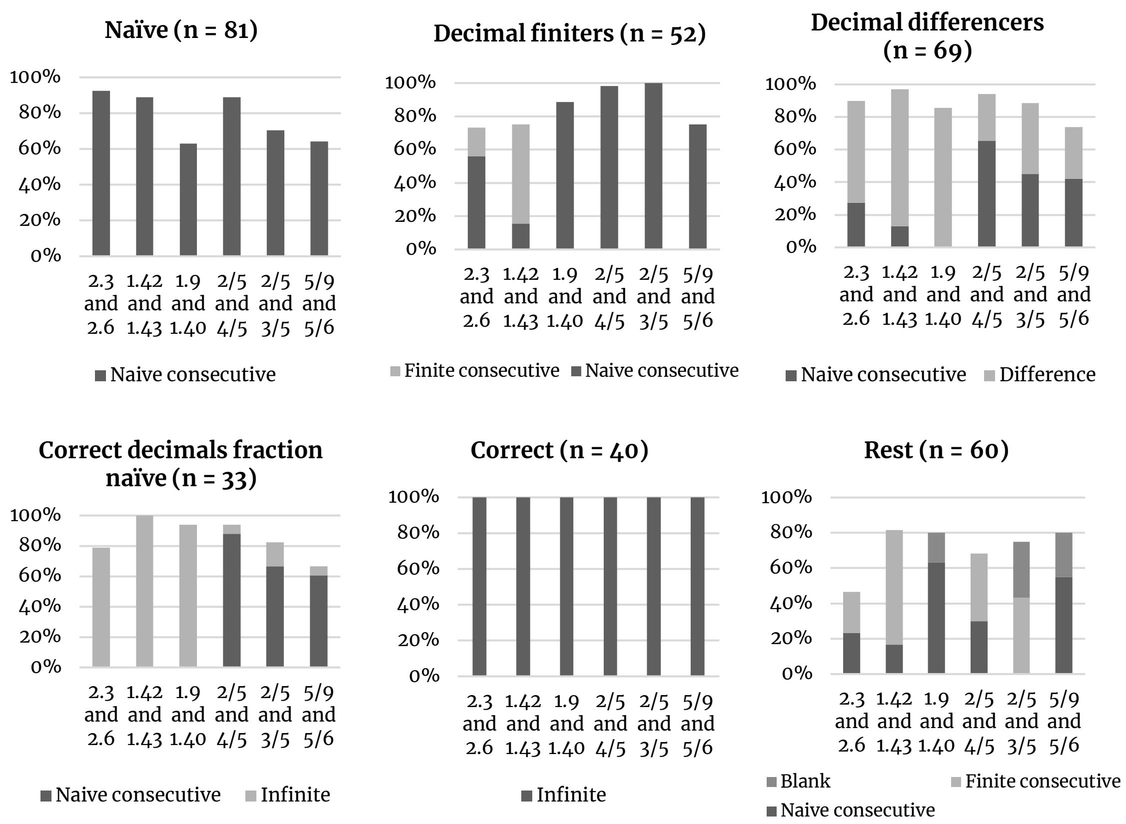
Figure 5. Profiles in the question items in 5th and 6th grade



- *Decimal differencers*: students who calculated the difference between two decimals but considered that there was no other number between two pseudo-consecutive fractions, and that there was a finite number of numbers between two non-pseudo-consecutive fractions. A subgroup of students, however, also calculated the difference in fractions.
- *Correct decimals fraction naïve*: students who considered that there was an infinite number of numbers between two different decimals, but that there was no other number between two pseudo-consecutive fractions, and a finite number of numbers between two non-pseudo-consecutive fractions. A subgroup of students, however, started recognising that there was an infinite number of numbers between fractions.
- *Rest*: students with a generally low performance who solved the items following no recognisable pattern.

In 7th and 8th grade, we chose a solution with six profiles (see Figure 6).

Figure 6. Profiles in the question items in 7th and 8th grade

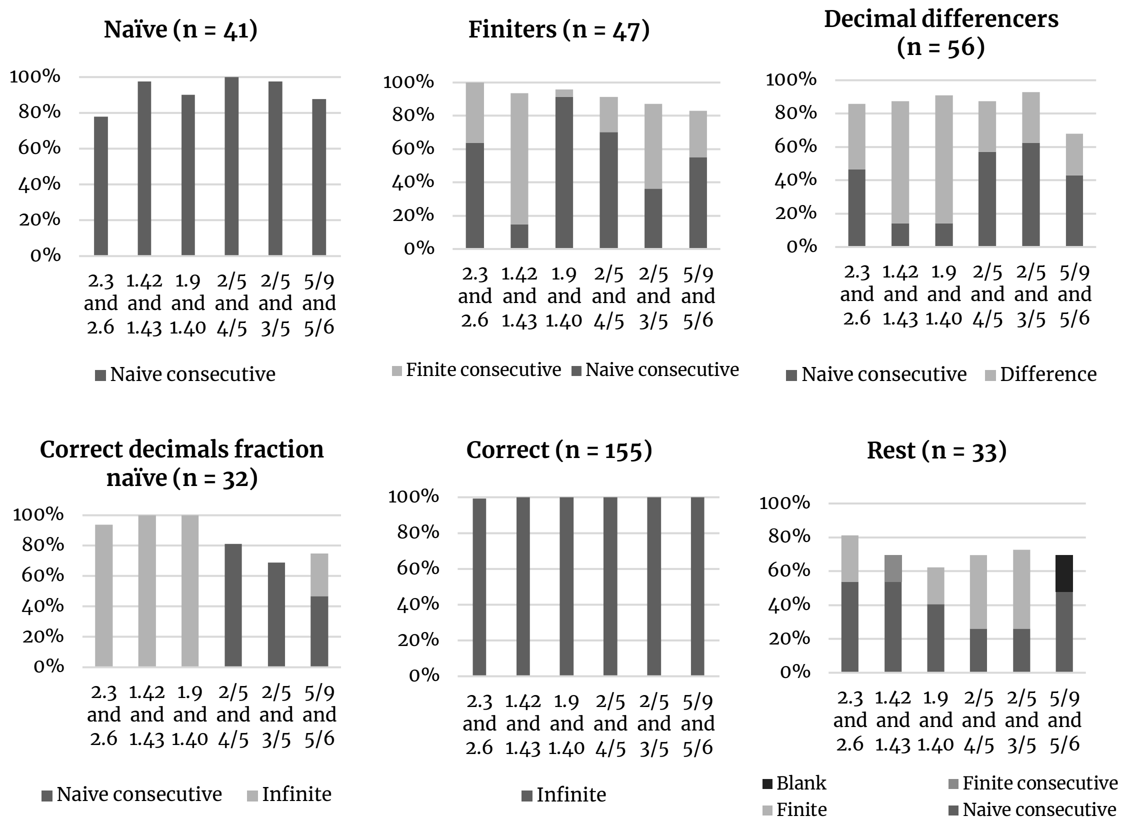


The same profiles as in 5th and 6th grade were identified. We also detected a new one:

- *Correct*: students who considered that there was an infinite number of numbers between two different fractions and two different decimals.

In 9th and 10th grade, we also chose a six-profile solution (see Figure 7).

Figure 7. Profiles in the question items in 9th and 10th grade



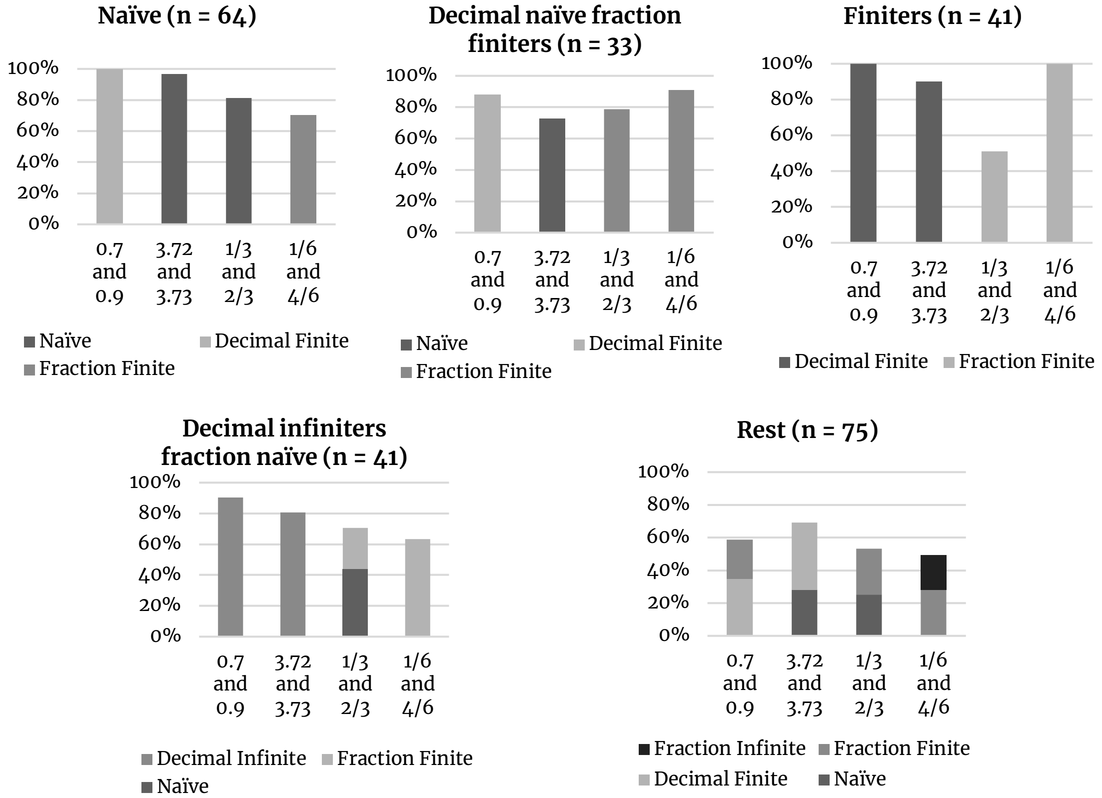
In these grades, the *Decimal finiters* profile was not detected, but we identified a new profile:

- *Finiters*: students who started to consider that there was a finite number of numbers between two pseudo and non-pseudo-consecutive decimals and fractions.

With regard to the multiple-choice items, in 5^{th} and 6^{th} grade, we chose the five-profiles solution (see Figure 8).

- *Naïve*: students who considered that there were no numbers between two pseudo-consecutive numbers, and that there was a finite number of numbers between two non-pseudo-consecutive numbers.
- *Decimal naïve fraction finiters*: students who considered that there were no numbers between two pseudo-consecutive decimals, and a finite number of decimals between two non-pseudo-consecutive decimals. In fractions, they considered that there was a finite number of fractions between two different fractions.
- *Finiters*: students who considered that there was a finite number of numbers between two different fractions and two different decimal numbers.

Figure 8. Profiles in the multiple-choice items in 5th and 6th grade



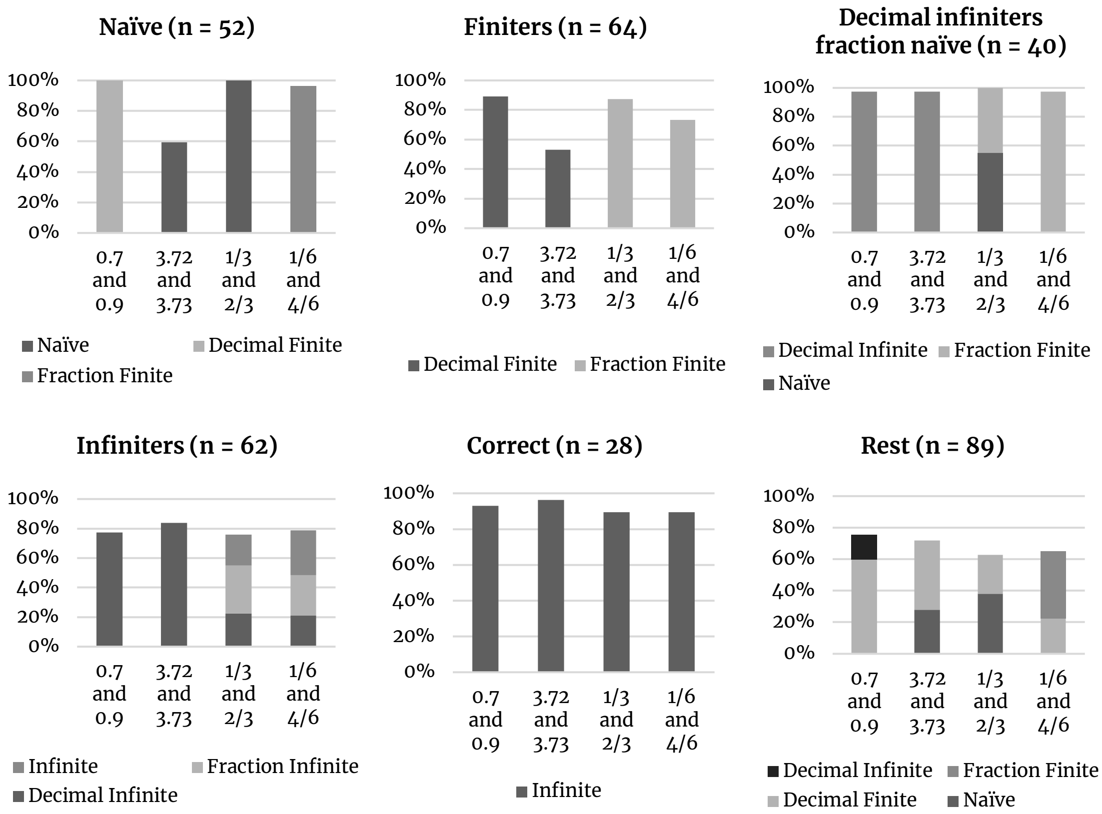
- *Decimal infiniters fraction naïve*: students who considered that there was an infinite number of decimals between two different decimal numbers, but that in factions, there were no numbers between two pseudo-consecutive fractions, and a finite number of fractions between two non-pseudo-consecutive fractions. A subgroup of students started to recognise that there was a finite number of fractions between two pseudo-consecutive fractions.
- *Rest*: students with a generally low performance who solved the items following no recognisable pattern.

In 7th and 8th grade, we chose a solution with six profiles (see Figure 9).

In these grades, the *Decimal naïve fraction finiters* profile was not detected, but we identified two new profiles:

- *Infiniters*: students who considered that there was an infinite number of decimals between two different decimals. Between two different fractions, there were: students who considered that there was an infinite number of decimals; students who considered that there was an infinite number of fractions; and students who considered that there was an infinite number of numbers that can be represented by several different representations, such as decimals and fractions.

Figure 9. Profiles in the multiple-choice items in 7th and 8th grade



- *Correct*: students who considered that between two different fractions and two different decimals there was an infinite number of numbers that can be represented by several different representations, such as decimals and fractions.

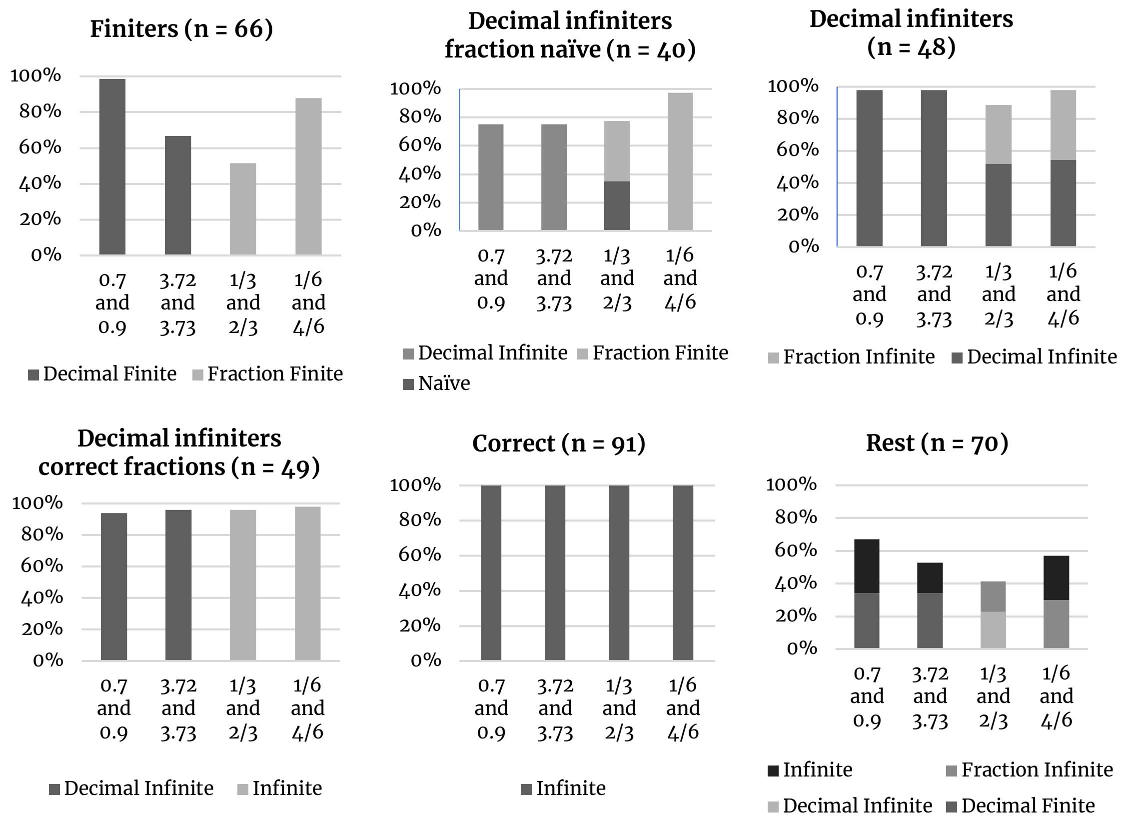
In 9th and 10th grade, we also chose a solution with six profiles (see Figure 10).

In these grades, the *Naïve* and *Infiniters* were not detected, but we identified two new profiles:

- *Decimal infiniters*: students who considered that there was an infinite number of decimals between two different fractions and decimals. A subgroup of students considered that there was an infinite number of fractions between two different fractions.

- *Decimal infiniters correct fractions*: students who considered that there was an infinite number of decimals between two different decimals. Between two different fractions, they considered that there was an infinite number of numbers that could be represented by several different representations, such as decimals and fractions.

Figure 10. Profiles in the multiple-choice items in 9th and 10th grade



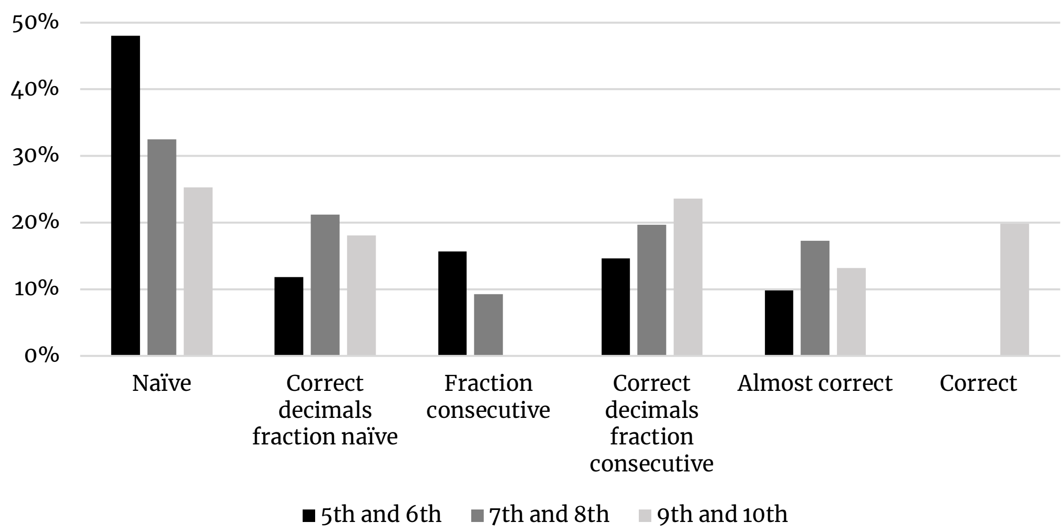
The profiles obtained for each of the three types of items show some similarities. First, there was always one group of students whose reasoning was biased by natural number knowledge (*Naïve*), even when they were presented with the correct choice. These students considered that there were no numbers between two pseudo-consecutive rational numbers. Second, there was always a group of students who correctly solved the decimal items but considered it impossible to find a number between two pseudo-consecutive fractions (*Correct decimals fraction naïve* in write and question items/*Decimal infiniters fraction naïve* in multiple-choice items). Therefore, they understood the dense nature of decimals better than fractions.

In addition, some profiles showed a transition from a naïve to a correct idea of density. In the write items, this was revealed through the idea of "Consecutiveness": students who considered that numbers existed between fractions, but they applied a naïve idea of the next number (*Fraction consecutive* and *Correct decimals fraction consecutive*). And in the question and multiple-choice items, this transition was revealed through the idea of a finite number of numbers (*Finiters* and *Decimal finiters*), or the difference (*Decimal differencers*). Furthermore, in the multiple-choice items, one group of students did not consider that there were decimal numbers between fractions and vice versa (*Infiniters*). The correct idea of density was observed when students correctly solved the write items, and when they considered that there was an infinite number of numbers in question and multiple-choice items (*Correct*).

4.2. Evolution of the profiles

Figure 11 shows the evolution of each profile from 5th to 10th grade in the write items. The *Naïve* profile decreased as the grades advanced. However, the most naïve natural number bias was still present at the end of secondary school, both in fractions and decimals. The *Fraction consecutive* profile also decreased as the grades advanced, disappearing in 9th and 10th grade. This profile shows that some students were only reluctant to consider that there were no numbers between 1/3 and 2/3 using a naïve idea of the following number.

Figure 11. Evolution of the profiles in the write items

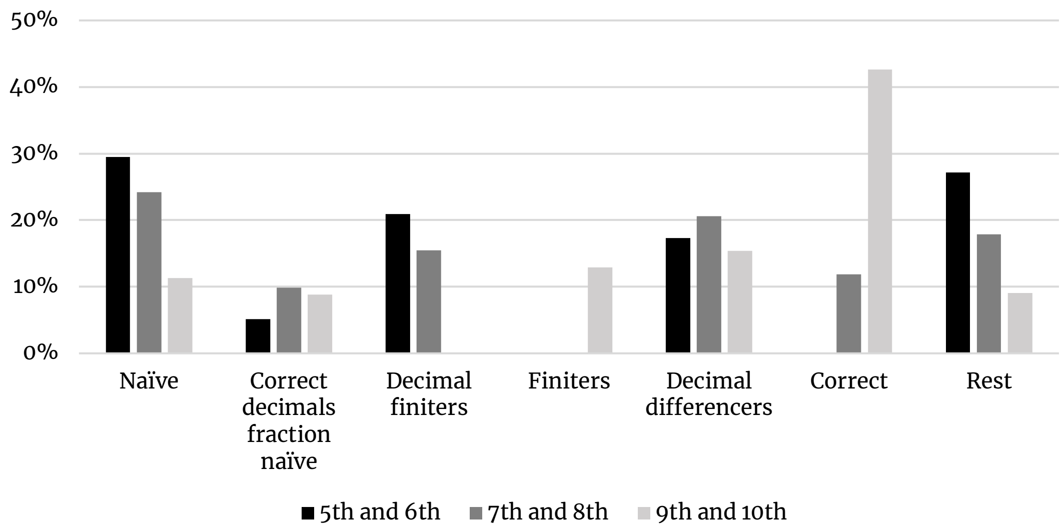


The *Correct decimals fraction naïve* profile increased from 5th and 6th grade to 7th and 8th grade, and then decreased in 9th and 10th, while the *Correct decimals fraction consecutive* profile increased as the grades advanced. Both profiles provide evidence of the difference between rational number representations: students first understood the density of decimal numbers and later the density of fractions. In addition, results seem to show other profiles (different from naïve) exist before the correct idea of density is reached: some students apply an incorrect reasoning based on "Consecutiveness".

Finally, the *Almost correct* profile increased from 5th and 6th grade to 7th and 8th grade, and then decreased in 9th and 10th grade where the *Correct* profile appeared. First of all, this result reveals the differences between the two fraction items. Writing a number between two fractions with the same denominator was easier than writing a number between two fractions with the same numerator. Second, it reflects the fact that few students understood the dense structure of rational numbers, even by the end of secondary school.

Figure 12 shows the evolution of each profile from 5th to 10th grade in the question items. The *Naïve* profile decreased as the grades advanced. As in the other items, the most naïve natural number bias seems not to have disappeared in the last grades of the secondary school, neither regarding fractions nor decimal numbers. The *Decimal finiters* profile also decreased along grades, disappearing in 9th and 10th grade, where it was replaced by a *Finiters* profile. This result seems to show that students started to believe that there was a finite number of numbers between two different decimals and then between two different fractions.

Figure 12. Evolution of the profiles in the question items

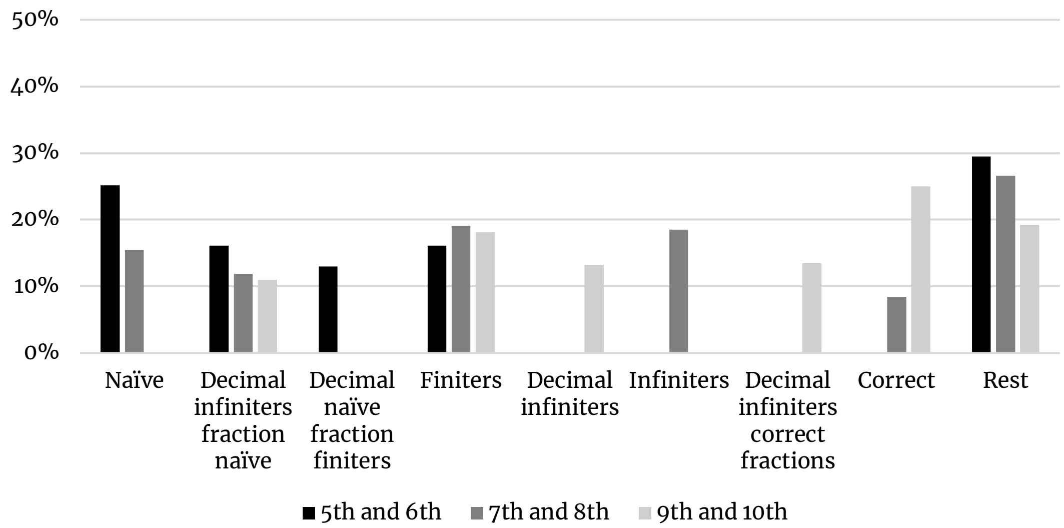


The decrease of the *Naïve* and *Decimal finiters* profiles corresponded to an increase of the *Correct* profile and of the *Correct decimals fraction naïve* profile, which again shows that there were differences in the understanding of density between fractions and decimals. In fact, decimal infinity was reached even in some primary school students. Finally, the *Decimal differencers* profile remained stable along grades.

Figure 13 shows the evolution of each profile from 5th to 10th grade in the multiple-choice items. The *Naïve* profile decreased as the grades advanced. In this case, the most naïve natural numbers bias seems to disappear in the last grades of the secondary school in these types of items. Furthermore, the *Decimal naïve fraction finiters* only appeared in 5th and 6th grade, so it seems that the most naïve natural number bias is first overcome in decimal numbers. The *Finiters* profile remained stable along grades. This profile seems to show the transition from the most naïve idea (there are no numbers) to an advanced idea of discreteness (there is a finite number of numbers). However, they still failed to recognise that rational numbers are dense.

The *Decimal infiniters fraction naïve* profile (students who recognised infinity only in decimals) decreased as the grades advanced. Furthermore, as this profile includes a subgroup of students who started to recognise that there was a finite number of fractions between two pseudo-consecutive fractions, it seems that the profiles that precede the understanding of fraction density correspond to an advanced idea of discreteness (there is a finite number of numbers).

Figure 13. Evolution of the profiles in the multiple-choice items



The *Infiniters* profile appeared only in 7th and 8th grade and seems to show the transition from discreteness to infinity in fractions. However, this profile did not appear in 9th and 10th grade, where the *Decimal infiniters* profile and the *Decimal infiniters correct fractions* profile appeared. This shows differences regarding the representation of rational numbers. While students were better able to consider that there was an infinite number of decimals between decimals, some of them considered that there was an infinite number of fractions between fractions, others considered that there was an infinite number of decimals, and others that there was an infinite number of rational numbers that could be represented as fractions and decimals.

Finally, the correct idea of density was only reached by the *Correct* profile, which did not appear in 5th and 6th grade, and increased from 7th and 8th grade to 9th and 10th grade. This result shows that primary school students did not understand the dense structure of rational numbers, even when they were presented with the correct choice.

In summary, the *Naïve* profile decreased as the grades advanced in all three types of items. However, the profile only disappeared in 9th and 10th grade in the multiple-choice items (in which they could choose the correct answer based on recognition). Therefore, students were still constrained by the discreteness of the natural numbers by the end of secondary school. Furthermore, the natural number bias was greater in write items, where students had to write a number between two given rational numbers, than in question and multiple-choice items, where students could answer that there was a finite number of numbers. Second, the decrease of the *Naïve* profile coincided with the appearance of other profiles different than the *Correct* profile, thus revealing other ways of thinking before understanding the idea of density. In other words, they show intermediate profiles in the understanding of density. These different profiles were described in the previous section.

5. Discussion and conclusions

The present study revealed a wide range of intermediate profiles in students' density understanding, providing evidence of the transition between the most naïve idea of the dense structure of rational numbers and the most sophisticated one. The clearest natural number bias, denoted as *Naïve* profile, was frequent in 5^{th} and 6^{th} grade and decreased along grades, but it had not disappeared by the end of secondary school. This result confirms the prior findings of Vamvakoussi and Vosniadou (2007, 2010). The *Correct* profile on the other hand, did not appear in 5^{th} and 6^{th} graders, but was present in 7^{th} and 8^{th} grade and more frequent in 9^{th} and 10^{th} grade. By the end of secondary school, however, half the students had this profile.

An intermediate profile that represents the transition from discreteness to infinity was revealed by the presence of the *Correct decimals fraction naïve* profile (*Decimal infiniters fraction naïve* profile in multiple-choice items). This profile represented students who already had a good understanding of the density of decimals, but not of fractions yet. This result shows differences between rational number representations: students first understood the dense nature of decimal numbers and later the density of fractions, confirming previous research (e.g., McMullen & Van Hoof, 2020).

Furthermore, some other intermediate profiles were revealed. The idea of consecutiveness clearly appeared in the write items. These students considered that there were numbers between fractions, but they applied a naïve idea regarding the next number (profiles: *Fraction consecutive* and *Correct decimals fraction consecutive*). The *Fraction consecutive* profile decreased as the grades advanced, disappearing in 9^{th} and 10^{th} grade but increased the *Correct decimals fraction consecutive* profile. Therefore, by the end of secondary school, the idea of consecutiveness was still present. Moreover, the *Almost correct* profile reflected students who recognised that it was possible to find a number between two different rational numbers, except when both fractions had the same numerator, where it was considered impossible. This result shows differences between fraction items. Some students were able to write a number between two fractions with the same denominator (1/3 and 2/3), but they considered that it was impossible between two fractions with the same numerator (1/8 and 1/9). These difficulties may be due to the distance effect (DeWolf & Vosniadou, 2015). In the item 1/8 and 1/9 the distance effect is stronger, since the numerical value of both fractions is very close, which makes it difficult to find a number in between. In this item, it is also difficult, from a procedural perspective, to construct a fraction that is situated between these two given fractions.

In the question items, some groups of students answered a finite number of numbers (*Finiters* and *Decimal finiters*), or they referred to the difference (*Decimal differencers*). The *Decimal finiters* profile (identified from 5^{th} to 8^{th} grade) represented students who had overcome the naïve idea of discreteness in decimal numbers. However, this profile was not identified in 9^{th} and 10^{th} grade, where the *Finiters* profile appeared. These last students had overcome the naïve discreteness both in fractions and decimal numbers. Therefore, this result shows there are differences between the rational number representations. Students, first, recognise that it is possible to find a finite number of numbers between two pseudo-consecutive decimals and then between fractions. Moreover, the *Decimal differencers* profile, which was present to approximately the same extent in each grade, was evidenced by a group of students who determined the number of numbers between the two given fractions or decimal numbers by subtracting both numbers.

In the multiple-choice items, one group of students considered it was possible to find a finite number of numbers between fractions in 5th and 6th grades (*Decimal naïve fraction finiters*). However, this profile did not appear in the rest of the grades. Another group was the *Finiters*, who considered that there was a finite number of numbers, both in fractions and decimals. This profile was present to approximately the same extent in each grade. Finally, some profiles were closer to the most sophisticated idea of density. In 7th and 8th grade, the *Infiniters* represented a group of students who recognised infinity, but they believed that only fractions existed between fractions and only decimals between decimals (Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2010). However, this profile was not found in 9th and 10th grade, where a group of students appeared who considered that there was an infinite number of decimal numbers between two different fractions and two different decimals (*Decimal infiniters*). There was also a group of students who did not recognise that fractions could exist between two different decimal numbers, but did recognise that decimals could exist between fractions (*Decimal infiniters correct fractions*). These three last profiles show the transition towards the understanding of density, showing difficulties with rational number representations: students treat fractions and decimal numbers as more or less unrelated sets of numbers, rather than as completely interchangeable representations of the same numbers (Khoury & Zazkis, 1994).

Our results have theoretical and methodological implications. First, the profiles were identified after an inductive analysis of a large sample of students' answers, leading to the characteristics of each profile and the frequency of the profiles in each grade. Second, our research addressed a broader age range compared to previous research, allowing us to examine the evolution of profiles from primary to secondary school. Finally, contrary to previous research, we combined different types of items representing both conceptual and procedural knowledge of density. This allowed us to detect a more extensive range of intermediate profiles.

This study also has educational implications. The results demonstrated that the natural number bias was still present towards the end of secondary school. Therefore, instructional efforts are necessary to supress this natural number bias even into secondary school (Van Hoof et al., 2015). Moreover, the intermediate profiles obtained in this study revealed a wide range of qualitatively different incorrect ways of thinking about the dense structure of rational numbers. This finding could benefit primary and secondary school teachers, making them aware of students' incorrect ways of thinking as they teach rational numbers. Finally, teachers must emphasise that decimal numbers and fractions are representations of the same rational number. In this sense, students would more easily understand that both fractions and decimals are dense, and therefore it is always possible to find a fraction between two decimal numbers and *vice versa*.

Further research could adopt longitudinal study designs to examine how learners' individual understanding of rational number density progresses over time. Such studies would allow clarifying the possible transitions between profiles. Moreover, we are aware that qualitative studies focusing on students' verbalisations would also be valuable to gain a deeper understanding of our results. In this sense, it would be interesting in future studies to conduct interviews with students representing different profiles. Such studies would not only contribute qualitative evidence to our results but also deepen our understanding of students' exhibited reasoning.

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Profiles in understanding the density of rational numbers among primary and secondary school students

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Previous research has pointed out the students' tendency to inappropriately apply properties of natural numbers in rational number situations (natural number bias) as a major explanation of students' difficulties in understanding rational numbers. In this study, we focus on one of the three domains in which rational numbers differ from natural numbers: the density of rational numbers. Previous research has shown that understanding the density of rational numbers is a complex task for primary and secondary school students. For instance, some secondary school students consider that between the "pseudo-consecutive" fractions 5/7 and 6/7 there are no numbers, or that only the number 1/3 exists between 1/2 and 1/4. The present cross-sectional study investigated 953 fifth to tenth grade students' understanding of the dense structure of rational numbers. First, we carried out an inductive analysis, coding the answers based on three types of items about density: i) writing a number between two different rational numbers given, ii) answering open-ended questions about how many numbers there are between two different rational numbers given, and, iii) answering multiple-choice items in which some of the answers reveal specific misunderstandings. Then, we performed a TwoStep Cluster Analysis, which revealed a wide range of intermediate profiles in learners' density understanding, providing evidence of the transition between the most naïve idea of the dense structure of rational numbers and the most sophisticated one. The analysis highlighted qualitatively different ways of thinking: i) the idea of consecutiveness, ii) the idea of a finite number of numbers, and iii) the idea that between fractions, there are only fractions, and between decimals, there are only decimals. Furthermore, our profiles showed differences regarding rational numbers representation since students first recognised the dense nature of decimal numbers and then of fractions. Learners, however, were still found to have a natural number-based idea of the rational number structure by the end of secondary school, especially when they had to write a number between two pseudo-consecutive rational numbers.