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Monte Carlo Calculations on the Gauge Potts Model*

H. Fanchiotti, S.J. Sciutto, and H. Vucetich
Laboratorio de Física Teórica, Departamento de Física, Universidad Nacional de La Plata, La Plata, República Argentina

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The q-state gauge Potts model P_q in d-dimensions has been studied using Monte Carlo techniques. For $d=2$ no phase transitions were detected. The P_2 model for $d=3$ shows a second order phase transition. On the other hand, all the $d=3$ ($q+2$) and $d=4$ cases studied show first order phase transitions. In these cases, it was possible to estimate transition coupling parameters as well as latent heat. For selected cases, a study of the behavior of the Wilson loop factor was done.

1. Introduction

In studying simple gauge models, one can acquire a great amount of experience that could be useful when attacking more realistic (and generally, more difficult) models. The gauge Potts model [1, 2] is certainly a very good example of such a case. There exist several attempts, and different approaches, for solving this model, such as series expansion [1, 2, 3, 4] mean field techniques [5, 6], and variational methods [7]. However, it is worthwhile mentioning that different approaches give in same cases, different results.

The main purpose of this paper is to report the results of Monte Carlo [8] calculations on the q-state gauge Potts model, P_q, in d-dimensions, for q up to 12 and $d=2$, 3 and 4.

The previous Monte Carlo studies of the Potts model, refer to the case of a global symmetry [9]. There are also studies of more general models which reduce to a Potts model in particular cases [10]. We present a more exhaustive study of the model, which includes the analysis of the q-dependence of transition coupling parameters and latent heats, and the influence of the dimensionality of the lattice on the phase structure. Moreover, we have done a study of the behavior of the corresponding Wilson loop [11] in several cases. In Sect. 2 there is a brief description of the model, together with the definitions of the magnitudes used throughout the paper. Section 3 contains the results of the calculations and our conclusions are included in Sect. 4.

2. The Model

The euclidean action of the gauge Potts model is

$$S=-\beta \sum_{\text{plaq.}} \delta_{U_p,1} \tag{1}$$

where plaq. denotes a primitive square (plaquette) on the d-dimensional lattice, β, is the coupling parameter, $U_p=U_1 U_2 U_3 U_4$ is the product of link variables U (taking the values $\exp(i\cdot 2\pi\cdot k/q)$, $k=0, 1, 2, \ldots, q-1$) around the plaquette and δ is the standard Kronecker symbol.

The action (1) allows the definition of the partition function:

$$Z(\beta)=\sum_{\{\text{conf.}\}} \exp(-S(\text{conf.})) \tag{2}$$

where the sum runs over all possible configurations of the system, that is, all possible values of the link variables U_i.

The magnitudes of interest of Monte Carlo calculations are:

The average action per plaquette (the "internal energy") that reads:

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$$E=\frac{\langle S\rangle}{N_p} \tag{3}$$

with N_p being the number of plaquettes on the lattice.
The "specific heat"

$$C=-\beta^2\frac{d}{d\beta}\left(\frac{E}{\beta}\right)=(\langle S^2\rangle-\langle S\rangle^2)/N_p \tag{4}$$

And the Wilson loop W, defined by

$$W=\langle \mathrm{Re}(U_1 U_2 \ldots U_n)\rangle \tag{5}$$

where the links $1, 2, \ldots, n$ form a closed loop in the lattice.
We recall that at a first order phase transition, the internal energy E, is discontinuous in β. On the other hand, at a second order one E is continuous and the specific heat C, diverges if the lattice is infinite. For finite lattices, C presents a peaked curve which sharpens as lattice size increases. It is also known that in an infinite lattice the Wilson loop W approaches zero when the loop is increased, following an area (perimeter) law in a disordered (ordered) phase [11, 12]. This behavior can be observed in a finite lattice if the size of the loops considered is small when compared with the total lattice size.

3. Results

We present our results for the Potts model for the already mentioned values of q, and for two, three or four-dimensional lattices with periodic boundary conditions. All the Monte Carlo computations were done using the Metropolis algorithm [13]. First we have determined the phase structure of the different systems. The "thermal cycle method" [12] (calculating E for different values of β as usual) was used to detect first-order phase transitions, and the "specific heat divergence" method [14] for second order ones. Neither first nor second order phase transitions were detected in the two dimensional systems, in agreement with previous theoretical results for infinite lattices [15]. We should mention that $q=2$, 3, 4, 5, 6, 8, 10, 12; $N^2=36^2$ and $\beta\in[0,5]$ were the values of q, the lattice size and the interval of β considered, respectively.
The P_2 model, being equivalent to the Z_2 model has been widely studied [12, 14, 16] and our results are in agreement with these previous analysis. For example, in $d=3$, P_2 shows a second order phase transition for $\beta_c=1.400\pm0.025$.
All the other cases studied in three and four dimensions show first-order phase transitions. The results

Table 1. Geometric parameters, β-interval and transition parameters for the three-dimensional systems studied

q	N	β-interval	β_c-interval	ΔE_l
3	11	[0.01; 3.00]	[1.43; 1.90]	0.216 ± 0.037
4	11	[0.01; 3.00]	[1.43; 1.97]	0.318 ± 0.047
6	11	[0.01; 4.00]	[1.59; 2.15]	$0.49\ \pm0.10$
8	11	[0.01; 5.00]	[1.63; 2.38]	$0.65\ \pm0.13$
10	11	[0.01; 5.00]	[1.70; 2.38]	$0.76\ \pm0.10$
12	11	[0.01; 5.00]	[1.73; 2.51]	$0.95\ \pm0.14$

Table 2. The same, but for four-dimensional systems

q	N	β-interval	β_c-interval	ΔE_l
3	8	[0.01; 3.00]	[0.83; 1.33]	0.315 ± 0.047
4	6	[0.01; 3.00]	[0.87; 1.59]	0.429 ± 0.067
5	8	[0.01; 4.00]	[0.92; 1.70]	0.517 ± 0.079
6	6	[0.01; 4.00]	[0.98; 1.74]	0.610 ± 0.081
8	6	[0.01; 5.00]	[1.08; 1.89]	0.740 ± 0.084
10	6	[0.01; 5.00]	[1.07; 2.13]	$0.88\ \pm0.10$
12	6	[0.01; 5.00]	[1.17; 2.20]	$0.90\ \pm0.10$

are summarized in Tables 1 and 2. The transition values for the coupling parameter and the corresponding latent heat, ΔE_l were obtained from direct measurement on the thermal cycles, and for that reason they should be regarded as rough estimations.
In $d=4$ the gauge Potts model is self-dual allowing the parametrization of the coupling parameter in the self-dual point by [3]:

$$\beta_c=\ln(1+q^{\frac{1}{2}}) \tag{6}$$

Figure 1 shows both the results obtained from our Monte Carlo calculations and the self-dual prediction (6). There we have also included an accurate estimation for $q=6$. This estimation was obtained in the following way: First, long runs with mixed initial configurations [16] were used to evaluate E, C and W, for values of β belonging to the "transition interval" obtained with the thermal cycle method, labelled β_c-interval in Tables 1 and 2. Then, accurate bounds for β_c were obtained by taking into account the discontinuity of E and the change in the behavior of W.
The results obtained for the latent heat together with the values coming from the $1/q$-expansion [2]:

$$\Delta E_l=1-\frac{20/9}{q}+\frac{1412/675}{q^2}-\frac{1.239091001}{q^3}+\ldots \tag{7}$$

are shown in Fig. 2.

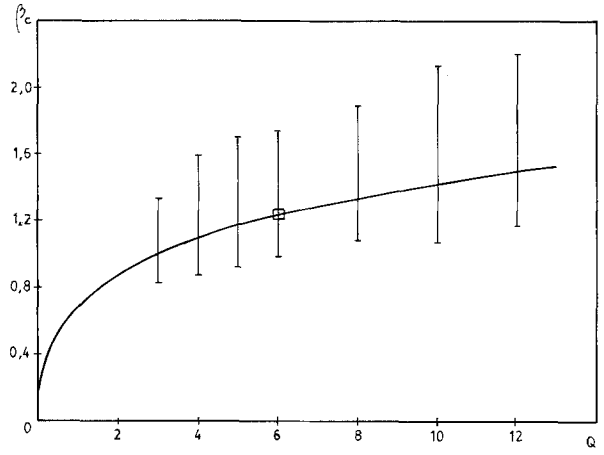

Fig. 1. Transition coupling parameter vs. q for four-dimensional systems. The vertical bars represent the thermal cycle bounds and the solid curve is the self dual prediction (6). The square dot at $q=6$ shows an accurate estimation for β_c

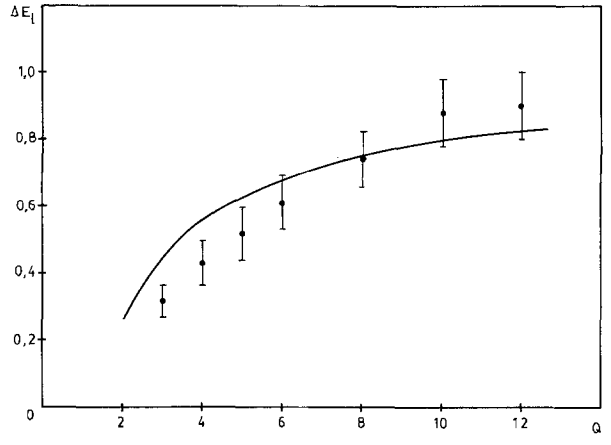

Fig. 2. Latent heat vs. q for four-dimensional systems. The circles show the values obtained from thermal cycle calculations and the solid curve is the $1/q$-expansion (7)

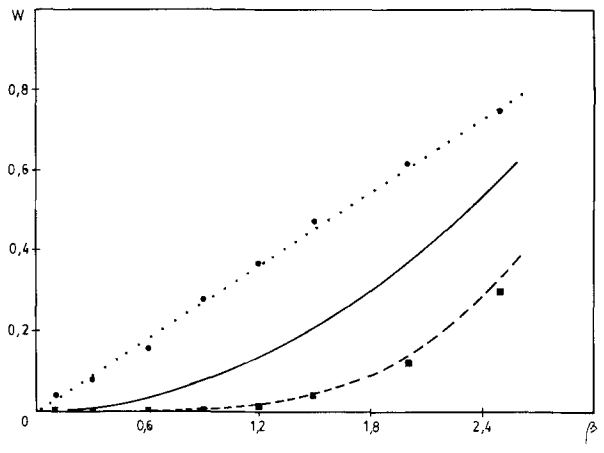

Fig. 3. Wilson loop factor vs. β for the P_4 model ($d=2$). Circular and square dots show the values of W for single and double plaquette, respectively. The solid (dashed) line represents the perimeter (area) law prediction

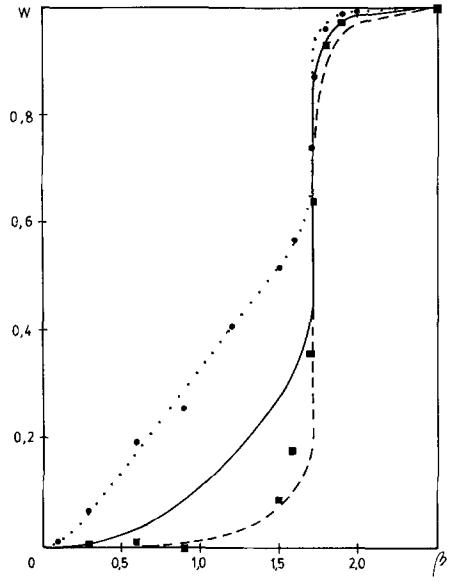

Fig. 4. As Fig. 3, but for $d=3$

Lack of computational time made impossible the calculation of a complete set of accurate estimations for β_c and ΔE_l.

Finally, we have studied in detail the behavior of the Wilson loop for the model in two and three-dimensional lattices with 36 and 9 sites per dimension respectively, and various values of q. Figures 3 and 4 show the results obtained for $q=4$. The lines identified as "perimeter law" and "area law" were obtained by plotting the 2nd and the 4th power of the corresponding single plaquette value, respectively. The square points show the values obtained for a double plaquette, that is a square "plaquette" with two links at each side. The data obtained for $d=2$ (Fig. 3) show an area law behavior. In the $d=3$ case (Fig. 4), the existence of a first-order phase transition shows up in the change in the behavior of W. Furthermore, it can be easily seen the agreement with an area law in the disordered phase (lower values of β) and with a perimeter law in the ordered one. The transition between one law to the other takes place in a small neighbourhood of the critical parameter β_c.

4. Conclusions

We have carried out an exhaustive study of the phase structure of the gauge Potts model by means

of Monte Carlo computations. The results obtained are useful to check approximate methods. If q_c is the critical q-parameter (that is, the particular value of q with the property that for any given $q > q_c$, the P_q model has a first-order phase transition), we confirmed that $q_c < 3$ for $d = 3$ and $q_c < 2$ for $d = 4$, in agreement with $1/q$-expansion predictions [2, 4]. It is important to remark that the mean field approximation [6] predicts a first-order phase transition for the P_2 model with $d = 3$, showing the limitations of that method. However, the mean field results for the latent heat are in good agreement with our data.
Studies about the influence of finite lattice size and boundary conditions as well as the inclusion of bosonic and fermionic matter fields in the action, should be developed in future works.

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H. Fanchiotti
S.J. Sciutto
H. Vucetich
Departamento de Física
Laboratorio de Física Teórica
Universidad Nacional de La Plata
C.C. 67
1900 La Plata
República Argentina