The Axial Vector Vertex in Spinor Electrodynamics and the Analytic Regularization Method.

H. FANCHIOTTI, C. A. GARCÍA CANAL (*), H. O. GIROTTI (**) and H. VUCETICH
Departamento de Física, Universidad Nacional de La Plata - La Plata

(ricevuto il 19 Novembre 1971)

Our purpose here is to re-examine some specific features of the axial vector vertex in spinor electrodynamics (1) in the light of the analytic regularization method (2,3) in its gauge-invariant version given by BREITENLOHNER and MITTER (4) to deal with closed fermion loops. As a result of our work we have found that:

i) Although the integral defining the VVA triangle graph (***) (see Fig. 1 *a*)) appears to be superficially linearly divergent by power counting, it is, within the spirit of analytic regularization, a *convergent integral.*

ii) Since, when analytic regularization is used, it is not dangerous to shift the integration variable in a divergent graph and no mass-independent terms are lost in the process of regularization, we conclude that the result obtained by ROSENBERG (6) for the VVA triangle graph is unique. By this we mean that the value of the graph is not ambiguous and it does not depend on the labeling convention or on the method of evaluation of the integral defining it (****).

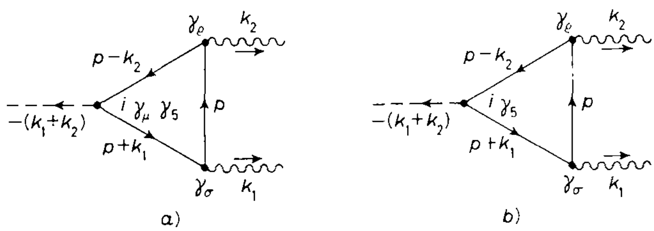

Fig. 1. – *a*) The VVA triangle graph. *b*) The VVP triangle graph.

(*) Member of Consejo Nacional de Investigaciones Científicas y Técnicas de Argentina.
(**) Member of Comisión de Investigaciones Científicas de la Provincia de Buenos Aires, Buenos Aires.
(1) S. ADLER: *Phys. Rev.*, **177**, 2426 (1969).
(2) C. G. BOLLINI, J. J. GIAMBIAGI and A. GONZÁLEZ DOMÍNGUEZ: *Nuovo Cimento*, **31**, 550 (1964).
(3) E. SPEER: *Journ. Mah. Phys.*, **9**, 1404 (1968).
(4) P. BREITENLOHNER and H. MITTER: *Nucl. Phys.*, **7** B, 443 (1968).
(***) In connection with this nomenclature for the triangle graphs, see ref. (5).
(5) S. ADLER: *Perturbation Theory Anomalies*, in *1970 Brandeis Universiy Summer Institute in Theoretical Physics*, Vol. **1**.
(6) L. ROSENBERG: *Phys. Rev.*, **129**, 2786 (1963).
(****) As is known, the analytic regularization method preserves unitarity at all stages of the calculation.

iii) If $R_{\varrho\sigma\mu}(k_1, k_2, \lambda)$ and $R_{\varrho\sigma}(k_1, k_2, \lambda)$ are the amplitudes associated with diagrams VVA and VVP (see Fig. 1 *a*) and *b*)) and the corresponding diagrams with thet photon indices interchanged, respectively, the axial vector Ward identity turns ou to be anomalous for *any* value of the complex regularizing parameter λ. This outcome of the analytic regularization method, for the present problem, is to be compared with the following statement of ADLER (*), made up in connection with the Pauli-Villars regulator technique: As far as the regulator mass M_0 is kept finite, the triangle diagrams satisfy the normal axial vector Ward identity.

It is clear that the results we have just summarised contradict some of the arguments raised by ADLER [1] to justify the failure of the axial vector Ward identity in the case of the triangle graph (**).

Using the gauge-invariant formalism for closed loops, developed in ref. [4], we write for the sum of the VVA triangle diagram illustrated in Fig. 1 *a*) and the corresponding diagram with the two photons interchanged the following explicit expression (***):

$$(1)\qquad R_{\varrho\sigma\mu}(k_1, k_2, \lambda) = -2\int d^4p\left(\prod_{i=1}^{3}\int_0^1 d\xi_i\right)\delta\left(1-\sum_{i=1}^{3}\xi_i\right) m_0^{2\lambda} f(\lambda)\left[\left(\frac{d}{dz}\right)^2 \frac{Q_{\varrho\sigma\mu}(z)}{(m_0^2 - z - i0)^{1+\lambda}}\right]_{z=z_0},$$

$$z_0 = \xi_3 p^2 + \xi_1(p+k_1)^2 + \xi_2(p-k_2)^2,$$

where

$$Q_{\varrho\sigma\mu}(z) = \frac{1}{2K}\left[(m_0+K)P_{\varrho\sigma\mu}(K) - (m_0-K)P_{\varrho\sigma\mu}(-K)\right]_{K^2=z},$$

$$P_{\varrho\sigma\mu}(m_0) = \mathrm{Tr}\left\{\gamma_\varrho(m_0+\gamma\cdot p)\gamma_\sigma[m_0+\gamma\cdot(p+k_1)]\, i\gamma_\mu\gamma_5[m_0+\gamma\cdot(p-k_2)]\right\},$$

m_0 is the unrenormalized fermion mass and $f(\lambda)$ is an arbitrary, regular function of the regularizing parameter λ with $f(0)=1$.

From eq. (1) and the definition of $Q_{\varrho\sigma\mu}(z)$ it follows that for sufficiently positive values of Re λ the four-momentum integral in d^4p becomes convergent. We can perform the integral in d^4p keeping the complex parameter λ within the region of convergence and then the result can be analytically extended to the entire λ-complex plane in accordance with Gel'fand's methods [9]. Thus we obtain (****)

$$(2)\qquad R_{\varrho\sigma\mu}(k_1, k_2, \lambda) = m_0^{2\lambda} f(\lambda)\int_0^1 d\xi_1\int_0^{1-\xi_1} d\xi_2\left\{-8\pi^2\varepsilon_{\varrho\sigma\mu\alpha}[k_1^\alpha(1-\xi_1) - k_2^\alpha(1-\xi_2)]\cdot\right.$$

$$\cdot\frac{R}{(m_0^2 - R - i0)^{1+\lambda}} + \frac{\pi^2}{\lambda}\mathsf{C}_{\varrho\alpha\sigma\beta\mu\delta}\frac{1}{(m_0^2 - R - i0)^{\lambda}}(g^{\alpha\beta}u^\delta + g^{\alpha\delta}u^\beta + g^{\beta\delta}u^\alpha - k_1^\beta g^{\alpha\delta} + k_2^\delta g^{\alpha\beta}) -$$

$$\left. - 2\pi^2\mathsf{C}_{\varrho\alpha\sigma\beta\mu\delta}\frac{1}{(m_0^2 - R - i0)^{1+\lambda}}(u^\alpha u^\beta u^\delta - u^\alpha u^\delta k_1^\beta + u^\alpha u^\beta k_2^\delta - u^\alpha k_1^\beta k_2^\delta)\right\},$$

(*) See ref. [5], p. 94.

(**) Field-theoretical derivations of the anomalous axial-vector-current divergence have been given in ref. [7].

[7] J. SCHWINGER: *Phys. Rev.*, **82**, 664 (1951); C. R. HAGEN: *Phys. Rev.*, **177**, 2622 (1969); R. JACKIW and K. JOHNSON: *Phys. Rev.*, **182**, 1457 (1969); B. ZUMINO: *Proceedings of the Topical Conference on Weak Interactions* (Geneva, 1969), p. 361.

(***) Our metric convention is that of ref. [8].

[8] S. SCHWEBER: *An Introduction to Relativistic Quantum Field Theory* (Evanston, Ill., 1961).

[9] I. M. GEL'FAND and G. E. SHILOV: *Le distribution*, Vol. 1 (Paris, 1962).

(****) Needless to say, the integration variable can be shifted at pleasure in the region of convergence; in the rest, analytic continuation defines the result.

where

$$R = R(\xi_1, \xi_2, k_1, k_2) = \xi_1(1-\xi_1)k_1^2 + \xi_2(1-\xi_2)k_2^2 + 2\xi_1\xi_2 k_1\cdot k_2\,,$$

$$u^\alpha = u^\alpha(\xi_1, \xi_2, k_1, k_2) = \xi_1 k_1^\alpha - \xi_2 k_2^\alpha\,,$$

$$\mathbf{C}_{\varrho\alpha\sigma\beta\mu\delta} = \mathrm{Tr}\,(\gamma_\varrho\gamma_\alpha\gamma_\sigma\gamma_\beta\gamma_\mu\gamma_5\gamma_\delta)$$

and

$\varepsilon_{\varrho\sigma\mu\alpha}$ is the completely antisymmetric tensor.

Now we want to investigate more closely the behaviour of $R_{\varrho\sigma\mu}(\lambda)$ as a complex function of λ in the neighborhood of the «physical value» of the parameter, that is $\lambda=0$. Since the second term of the right-hand side of eq. (2) presents a pole at $\lambda=0$, we make a Laurent expansion of $R_{\varrho\sigma\mu}(\lambda)$ around this pole:

$$R_{\varrho\sigma\mu}(\lambda) = \frac{K_{\varrho\sigma\mu}}{\lambda} + R^{(f)}_{\varrho\sigma\mu} + O(\lambda)\,, \tag{3}$$

where $K_{\varrho\sigma\mu}$ is the residue of $R_{\varrho\sigma\mu}(\lambda)$ at the pole $\lambda=0$ and $R^{(f)}_{\varrho\sigma\mu}$ its finite part. $K_{\varrho\sigma\mu}$ can be readily calculated from

$$K_{\varrho\sigma\mu} = \lim_{\lambda\to 0}\,[\lambda R_{\varrho\sigma\mu}(\lambda)]$$

with the result

$$K_{\varrho\sigma\mu} = \pi^2\,\mathbf{C}_{\varrho\alpha\sigma\beta\mu\delta}\int_0^1 \mathrm{d}\xi_1\int_0^{1-\xi_1}\mathrm{d}\xi_2(g^{\alpha\beta}u^\delta + g^{\alpha\delta}u^\beta + g^{\beta\delta}u^\alpha - k_1^\beta g^{\alpha\delta} + k_2^\delta g^{\alpha\beta})\,.$$

Replacing into our last equation the relations

$$g^{\alpha\beta}\,\mathbf{C}_{\varrho\alpha\sigma\beta\mu\delta} = 8\varepsilon_{\varrho\sigma\mu\delta}\,,\qquad g^{\alpha\delta}\,\mathbf{C}_{\varrho\alpha\sigma\beta\mu\delta} = 8\varepsilon_{\varrho\sigma\mu\beta}\qquad\text{and}\qquad g^{\beta\delta}\,\mathbf{C}_{\varrho\alpha\sigma\beta\mu\delta} = 8\varepsilon_{\varrho\sigma\mu\alpha}\,,$$

we obtain

$$K_{\varrho\sigma\mu} = 0\,. \tag{4}$$

This is our first result. $R_{\varrho\sigma\mu}(\lambda)$ is an analytic function of λ which is regular in a neighborhood of $\lambda=0$. Therefore, from the viewpoint of analytic regularization, the integral defining the VVA triangle graph is well defined. Furthermore, since we have only dealt with convergent integrals and no mass-independent terms have been lost during the process of regularization, we can also conclude that the expression for $R^{(f)}_{\varrho\sigma\mu}(k_1, k_2)$, to be obtained, is unique; in the sense of being independent of any particular labeling of the triangle graph.

From the definition of finite part ([2])

$$R^{(f)}_{\varrho\sigma\mu} = \lim_{\lambda\to 0}\left\{\frac{\partial}{\partial\lambda}\,[\lambda R_{\varrho\sigma\mu}(\lambda)]\right\},$$

we get, after some algebraic rearrangements,

$$R^{(f)}_{\varrho\sigma\mu}(k_1, k_2) = -\{A_1(k_1, k_2)\, k_1^\alpha \varepsilon_{\alpha\varrho\sigma\mu} + A_2(k_1, k_2)\, k_2^\alpha \varepsilon_{\alpha\varrho\sigma\mu} + \tag{5}$$

$$+ A_3(k_1, k_2)\, k_{1\varrho} k_1^\beta k_2^\delta \varepsilon_{\beta\delta\sigma\mu} + A_4(k_1, k_2)\, k_{2\varrho} k_1^\beta k_2^\delta \varepsilon_{\beta\delta\sigma\mu} +$$

$$+ A_5(k_1, k_2)\, k_{1\sigma} k_1^\beta k_2^\delta \varepsilon_{\beta\delta\varrho\mu} + A_6(k_1, k_2)\, k_{2\sigma} k_1^\beta k_2^\delta \varepsilon_{\beta\delta\varrho\mu} + ik_1^\alpha \varepsilon_{\varrho\alpha\sigma\mu} G_1(k_1, k_2) - ik_2^\alpha \varepsilon_{\varrho\alpha\sigma\mu} G_2(k_1, k_2)\},$$

where, for comparison purposes, we have introduced the following definitions:

$$A_1(k_1, k_2) = -k_1\cdot k_2\, A_3(k_1, k_2) - k_2^2 A_4(k_1, k_2)\,,$$

$$A_2(k_1, k_2) = -k_1^2 A_5(k_1, k_2) - k_1\cdot k_2\, A_6(k_1, k_2)\,,$$

$$A_3(k_1, k_2) = -A_6(k_2, k_1) = -16\pi^2 I_{11}(k_1, k_2)\,,$$

$$A_4(k_1, k_2) = -A_5(k_2, k_1) = 16\pi^2 [I_{20}(k_1, k_2) - I_{10}(k_1, k_2)]\,,$$

$$I_{st}(k_1, k_2) = \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2\, \xi_2^s \xi_1^t [R(\xi_1, \xi_2, k_1, k_2) - m_0^2]^{-1}\,,$$

$$G_1(k_1, k_2) = \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \left[\frac{(1-2\xi_1)R + k_2^2(\xi_2^2 + \xi_1\xi_2 - \xi_2)}{m_0^2 - R} + (3\xi_1 - 1)\ln(m_0^2 - R)\right], \tag{6a}$$

$$G_2(k_1, k_2) = \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \left[\frac{(1-2\xi_2)R + k_1^2(\xi_1^2 + \xi_1\xi_2 - \xi_1)}{m_0^2 - R} + (3\xi_2 - 1)\ln(m_0^2 - R)\right]. \tag{6b}$$

Our result for $R^{(f)}_{\varrho\sigma\mu}$ does not seem to be gauge-invariant since

$$k_2^\varrho R^{(f)}_{\varrho\sigma\mu}(k_1, k_2) = -ik_2^\varrho k_1^\alpha G_1(k_1, k_2)\,\varepsilon_{\varrho\alpha\sigma\mu}$$

and

$$k_1^\sigma R^{(f)}_{\varrho\sigma\mu}(k_1, k_2) = ik_1^\sigma k_2^\alpha G_2(k_1, k_2)\,\varepsilon_{\varrho\alpha\sigma\mu}\,.$$

However, we will show now that $G_1(k_1, k_2) = G_2(k_1, k_2) = 0$. To do this we consider G_1 as a function of m_0^2 and we study dG_1/dm_0^2. From eq. (6a) we get

$$\frac{dG_1(m_0^2)}{dm_0^2} = \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \left\{-\frac{[(1-2\xi_1)R + k_2^2(\xi_2^2 + \xi_1\xi_2 - \xi_2)]}{(m_0^2 - R)^2} + \frac{3\xi_1 - 1}{m_0^2 - R}\right\}. \tag{7}$$

The integrand of the right-hand side of eq. (7) is a function of three independent variables k_1^2, k_2^2 and k_1, k_2. A Taylor expansion of it around $k_1^2 = 0$, $k_2^2 = 0$ and $k_1, k_2 = 0$ allows us to write

$$\frac{dG_1(m_0^2)}{dm_0^2} = -\sum_{N=0}^{\infty} \frac{N+1}{(m_0^2)^{N+1}} \sum_{\substack{l,m,n \\ (l+m+n=N)}} \frac{N!}{l!\,m!\,n!} (k_1^2)^l (k_2^2)^m (k_1, k_2)^n\, \mathbf{C}_{lmn}\,, \tag{8}$$

where

$$\mathbf{C}_{lmn} = \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2\, \xi_1^{l+n}(1-\xi_1)^l \xi_2^{m+n-1}(1-\xi_2)^{m-1}\{[(N-m+1) - \xi_1(2N+3)]\,\xi_2(1-\xi_2) + m\xi_1\xi_2\}.$$

The idea is to show that each coefficient $\mathbf{C}_{lmn}$ of the expansion given in eq. (8) vanishes. If in the expression defining $\mathbf{C}_{lmn}$ we perform the ξ_2-integration, we get (F being the Gauss hypergeometric function)

$$(9)\qquad \mathbf{C}_{lmn}=\int_0^1 d\xi_1(1-\xi_1)^{l+n}\xi_1^{N+1}F(m+n+1,-m;m+n+2;\xi_1)\cdot$$
$$\cdot[N-m+1-(2N+3)(1-\xi_1)+(N+m+2)(1-\xi_1)-\xi_1(l+n+1)]=0\,.$$

This result, $dG_1(m_0^2)/dm_0^2=0$, implies that $G_1(k_1,k_2)$ does not depend on the fermion mass m_0. In particular, choosing $m_0^2\to\infty$, we have from eq. (6a)

$$(10a)\qquad G_1(k_1,k_2)=\lim_{m_0^2\to\infty}\left[\ln\left\{m_0^2\int_0^1 d\xi_1\int_0^{1-\xi_1}d\xi_2(3\xi_1-1)\right\}\right]=0\,.$$

Proceeding along the same lines it can be shown that

$$(10b)\qquad G_2(k_1,k_2)=\lim_{m_0^2\to\infty}\left[\ln\left\{m_0^2\int_0^1 d\xi_1\int_0^{1-\xi_1}d\xi_2(3\xi_2-1)\right\}\right]=0\,.$$

After this proof it becomes clear that the amplitude

$$(11)\qquad R^{(f)}_{\varrho\sigma\mu}(k_1,k_2)=-\{A_1k_1^\alpha\varepsilon_{\alpha\varrho\sigma\mu}+A_2k_2^\alpha\varepsilon_{\alpha\varrho\sigma\mu}+A_3k_{1\varrho}k_1^\beta k_2^\delta\varepsilon_{\beta\delta\sigma\mu}+$$
$$+A_4k_{2\varrho}k_1^\beta k_2^\delta\varepsilon_{\beta\delta\sigma\mu}+A_5k_{1\sigma}k_1^\beta k_2^\delta\varepsilon_{\beta\delta\varrho\mu}+A_6k_{2\sigma}k_1^\beta k_2^\delta\varepsilon_{\beta\delta\varrho\mu}\}\,,$$

yielded by the analytic regularization method is finite, gauge-invariant and coincident with the result already obtained by ROSENBERG (6) for the VVA triangle.

We turn now to the analysis of the axial vector Ward identity. From eq. (1) and after a repeated use of the Bose symmetric character of our amplitudes with respect to the photon indices, we get

$$(12)\qquad (k_1+k_2)^\mu R_{\varrho\sigma\mu}(k_1,k_2,\lambda)=2m_0R_{\varrho\sigma}(k_1,k_2,\lambda)-8\pi^2\varepsilon_{\varrho\sigma\alpha\beta}k_1^\alpha k_2^\beta+O(\lambda)\,.$$

What eq. (12) says is that, for any value of the complex parameter λ the axial vector Ward identity is anomalous for the triangle graph. Since we have proved that $R_{\varrho\sigma\mu}(\lambda)$ is a regular function of λ at $\lambda=0$ and $R_{\varrho\sigma}$ obviously satisfies the same condition, the limit $\lambda\to0$ can be continuously taken in eq. (12) without destroying the anomalous character of the axial vector Ward identity. On the contrary, when the same problem is handled with the cut-off method, it is found that the regulator mass term generates the anomalous contribution of the Schwinger-Adler result (5,10).

* * *

We are gratefully indebted to Profs. C. G. BOLLINI and J. J. GIAMBIAGI for stimulating discussions and a critical reading of the manuscript.

After this manuscript was completed, we learn that P. BREITENLOHNER and H. MITTER had arrived to a result similar to ours without using a direct computation of the seagull diagrams. We are fratefully indebted to them for letting us know their results.

(10) C. R. HAGEN: *Informal Meeting on Renormalization Theory*, I.C.T.P., Trieste, IC/69/121, 58-60.