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Completeness of uniformly accelerated observers in Galilean spacetimes

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Abstract

We analyze the concept of uniformly accelerated observer in Galilean spacetimes in the context of Newton–Cartan theory and find natural geometric assumptions to ensure that an inextensible uniformly accelerated observer in a Galilean spacetime does not disappear in a finite proper time.

Keywords Newton–Cartan theory · Galilean spacetime · Accelerated observer · Geodesic completeness

Mathematics Subject Classification 53Z05 · 53C80 · 53B50

1 Introduction

Uniformly accelerated motion is known since the fourteenth century, when Nicole Oresme stated the mean speed theorem (also known as Merton rule of uniform acceleration) [12]. Later, the works of Galileo and Newton in the seventeenth century contributed to the development of this notion and settled it as a crucial one in classical mechanics and gravitation theory. In the twentieth century, Cartan began the geometrization of Newton's theory of gravity [3, 4], which continued to be developed throughout the whole century and enabled us to express these classical concepts

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(observer, velocity, acceleration ...) in the language of differential geometry (see for instance [8–10, 13, 14, 19, 20]).

Currently, Newton–Cartan theory continues to arouse the interest of physicists and mathematicians despite the fact that the general theory of relativity constitutes the best framework to describe the universe thus far. Among other reasons, this is due to the similarities between both theories. Indeed, Newtonian gravitational theory can also be formulated as a covariant theory where gravity emerges as a manifestation of the spacetime's curvature. Thus, the structure of the spacetime is dynamical in the sense that instead of being a fixed background, it participates in the unfolding of physics. In addition, this geometric approach also enables us to clarify the vision of this theory as a limit of general relativity [15]. Moreover, the introduction of Newtonian models satisfying the cosmological principle [16] or certain symmetry properties [5, 7] can be useful to deal with similar stationary Newtonian spacetimes that appear in simple models of quantum collapse [18] and fractional quantum Hall effect [11].

The aim of this paper is to study uniformly accelerated motion in the Newton–Cartan framework (see [19] for a previous definition of uniformly accelerated motion in a Newtonian spacetime). In particular, our main goal will be to analyze the conditions that guarantee the completeness of inextensible trajectories of uniformly accelerated observers. Physically, this completeness means that inextensible uniformly accelerated observers live forever. Let us recall that in the relativistic setting, uniformly accelerated observers have been defined via the Fermi-Walker connection in [6], where the authors also study the completeness of their trajectories. Therefore, our results for the Newtonian case can be compared with the relativistic ones in order to better understand the analogies and differences between both theories. Also, note that our results can be particularized for the case where the acceleration is equal to zero, obtaining conditions that guarantee the completeness of free falling observers in these ambient spacetimes.

This article is organized as follows. In Sect. 2 we briefly describe the elements in Newton–Cartan theory that will be used in the rest of the article to obtain our main results. Concretely, in Sect. 2.1 we define the concept of uniformly accelerated observer in the language of our geometric theory of Galilean spacetimes as well as characterize them via a Cauchy problem in Proposition 3. Moreover, we also recall the concept of spatially conformally Leibnizian spacetime in Sect. 2.2, which will be the ambient spacetime where our main result is obtained. To conclude, we devote Sect. 3 to obtain our main result on the completeness of inextensible uniformly accelerated observers (Theorem 6), which is particularized for different models of Galilean spacetimes in Corollaries 7 and 8.

2 Preliminaries

A *Leibnizian* spacetime is defined as the triad (M, Ω, g), being M is a $\mathcal{C}^\infty$-differentiable connected manifold of arbitrary dimension $n+1 \geq 2$ endowed with a Leibnizian structure (Ω, g), where $\Omega \in \Lambda^1(M)$ is a differential 1-form such that $\Omega_p \neq 0$, $\forall p \in M$ and g is an Euclidean metric on the kernel of Ω. Indeed, if we denote by $\mathrm{An}(\Omega) = \{v \in TM\,,\ \Omega(v) = 0\}$ the smooth n-distribution induced on M

by Ω, and by $\Gamma(TM)$ the set of smooth vector fields on M, we have the subset of section $\Gamma(\mathrm{An}(\Omega)) = \{V \in \Gamma(TM) \,/\, V_q \in \mathrm{An}(\Omega),\ \forall p \in M\}$ that allows us to define the smooth, bilinear, symmetric and positive definite map (see [1] and [2])

$$g : \Gamma(\mathrm{An}(\Omega)) \times \Gamma(\mathrm{And}(\Omega)) \longrightarrow C^\infty(M),\ \ (V, W) \mapsto g(V, W).$$

Following the relativistic terminology, a point $p \in M$ is called an *event*, the linear form Ω_p is known as the *absolute clock* at p and the Euclidean vector space $(\mathrm{An}(\Omega_p)\,, g_p)$ is called the *absolute space* at $p \in M$. Furthermore, we will say that a tangent vector $v \in T_pM$ is *spacelike* if $\Omega_p(v) = 0$ and *timelike* otherwise. For the timelike case, v is said to be *future* pointing if $\Omega_p(v) > 0$ and *past* pointing if $\Omega_p(v) < 0$.

In this article, one of the key notions will be the concept of *observer* in a Leibnizian spacetime, which is a smooth curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ whose velocity γ' is a unitary future pointing timelike vector field (i.e., $\Omega(\gamma'(t)) = 1$ for all $t \in I$). The *proper time* of the observer γ is given by the parameter t. In addition, a *field of observers* or reference frame is a vector field $Z \in \Gamma(TM)$ with $\Omega(Z) = 1$, that is, its integral curves are observers.

It is well known that the smooth distribution $\mathrm{An}(\Omega)$ is integrable if and only if the absolute clock Ω satisfies $\Omega \wedge d\Omega = 0$ (see [21, Chap. 2.73]). In this case, the Leibnizian spacetime (M, Ω, g) is called *locally synchronizable*. In this case, Frobenius theorem (see [21]) ensures that the spacetime can be foliated by a family of hypersurfaces $\{\mathcal{F}_\lambda\}$ whose tangent space at each point is the absolute space.

If $\Omega \wedge d\Omega = 0$, it is not difficult to see that each $p \in M$ has a neighborhood where $\Omega = \beta\, dT$, for certain smooth functions $\beta > 0,\ T$, such that any hypersurface $\{T = \text{constant}\}$ locally coincides with a leaf of the foliation $\mathcal{F}$. As a consequence, the observers can rescale their proper time to be synchronized with the "compromise time" T. In addition, if $d\Omega = 0$ the Leibnizian spacetime is called *proper time locally synchronizable* since observers are synchronized directly by their proper time (up to a constant) and, locally, $\Omega = dT$. When this smooth function $T \in C^\infty(M)$ such that $\Omega = dT$ is globally defined, any observer can be assumed to be parameterized by this *absolute time function* T.

In this setup, a vector field $K \in \mathfrak{X}(M)$ that maintains the Leibnizian structure of the spacetime is called a *Leibnizian* vector field [2]. Specifically, the stages Φ_s of the local flows of a Leibnizian vector field are *Leibnizian diffeomorphisms*, that is,

$$\Phi_s^*\Omega = \Omega, \qquad \text{and} \qquad \Phi_s^* g = g.$$

These two conditions are equivalent to the following ones,

(i) $\Omega([K, X]) = K(\Omega(X)), \quad \forall X \in \Gamma(TM)$.
(ii) $K(g(V, W)) = g([K, V], W) + g(V, [K, W]), \quad \forall V, W \in \Gamma(\mathrm{An}(\Omega))$.

Observe that (ii) is well defined, since $[K, V],\ [K, W] \in \Gamma\big(\mathrm{An}(\Omega)\big)$ by (i).

In a Leibnizian spacetime the inertia principle must be codified via an affine connection. Nevertheless, the nonexistence of a canonical affine connection associated with a Leibnizian structure makes it necessary to introduce a compatible connection with the absolute clock Ω and the space metric g, i.e., a connection ∇ satisfying

(a) $\nabla\Omega = 0$ (equivalently, $\Omega(\nabla_X Y) = X(\Omega(Y))$ for any $X, Y \in \Gamma(TM)$).
(b) $\nabla g = 0$ (i.e., $Z(g(V, W)) = g(\nabla_Z V, W) + g(\nabla_Z W, V)$ for any $Z \in \Gamma(TM)$ and V, W spacelike vector fields).

Such a connection is called *Galilean* and a Leibnizian spacetime endowed with such a connection is known as a *Galilean spacetime* (M, Ω, g, ∇). When the torsion tensor of the connection vanishes identically ($\mathrm{Tor}_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$), ∇ is said to be *symmetric*.

On the other hand, the existence of a symmetric Galilean connection for a Leibnizian spacetime ensures its proper time local synchronizability [2, Lemma 13]. In this case, it is clear that the restriction of the Galilean connection to the spacelike leaves of the foliation $\{\mathcal{F}_\lambda\}$ coincides with the Levi-Civita connection associated with the metric g.

For any field of observers Z on a Galilean spacetime (M, Ω, g, ∇), we can also define the *gravitational field* induced by ∇ in Z as the spacelike vector field $\mathcal{G} = \nabla_Z Z$. Moreover, the *vorticity* or *Coriolis field* of Z is the 2−form $\omega(Z) = \frac{1}{2}\mathrm{Rot}(Z)$, given by

$$\omega(Z)(V, W) = \frac{1}{2}\Big(g(\nabla_V Z, W) - g(\nabla_W Z, V)\Big) \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)).$$

For proper time locally synchronizable spacetimes, the gravitational field and the vorticity of a field of observers become of great importance, since they determine a unique symmetric Galilean connection [2, Cor. 28]. In fact, that symmetric Galilean connection admits a formula 'à la Koszul' for the field of observers Z given by

$$\nabla_X Y = P^Z(\nabla_X Y) + X(\Omega(Y))Z, \qquad \forall X, Y \in \Gamma(TM), \tag{1}$$

where $P^Z(X) = X - \Omega(X)Z$ is the natural spacelike projection for Z and, for each $V \in \Gamma(\mathrm{An}(\Omega))$,

$$\begin{aligned} &2g(P^Z(\nabla_X Y), V) \\ &\quad = X(g(P^Z(Y), V)) + Y(g(P^Z(X), V)) - V(g(P^Z(X), P^Z(Y)) \\ &\qquad +2\,\Omega(X)\Omega(Y)g(\mathcal{G}, V) + 2\,\Omega(X)\omega(Z)(P^Z(Y), V) \\ &\qquad +2\,\Omega(Y)\omega(Z)(P^Z(X), V) + \Omega(X)\Big(g([Z, P^Z(Y)], V) \\ &\qquad - g([Z, V], P^Z(Y))\Big) - \Omega(Y)\Big(g([Z, P^Z(X)], V) + g([Z, V], P^Z(X))\Big) \\ &\qquad +g([P^Z(X), P^Z(Y)], V) - g([P^Z(Y), V], P^Z(X)) \\ &\qquad -g([P^Z(X), V], P^Z(Y)). \end{aligned} \tag{2}$$

2.1 Uniformly accelerated observers

Let us now introduce the notion of accelerated observer in the language of our geometric theory of Galilean spacetimes as well as describe their properties.

Given an observer γ in a Galilean spacetime (M, Ω, g, ∇), the covariant derivative $\frac{D\gamma'}{dt}$ is called the (proper) *acceleration* of γ. This vector field along γ is always a spacelike vector field, indeed:

$$\Omega\Big(\frac{D\gamma'}{dt}\Big) = \gamma'\Big(\Omega(\gamma')\Big) = 0. \tag{3}$$

Definition 1 An observer $\gamma : I \subset \mathbb{R} \longrightarrow M$ in a Galilean spacetime (M, Ω, g, ∇) is called *uniformly accelerated* if its acceleration is ∇-parallel, i.e.,

$$\frac{D^2\gamma'}{dt^2} = 0, \qquad \forall t \in I. \tag{4}$$

Remark 2 (i) Each free falling (geodesic) observer is, in particular, a uniformly accelerated observer.
(ii) The modulus of the acceleration of a uniformly accelerated observer is constant. Indeed,

$$\frac{d}{dt}\left(g\Big(\frac{D\gamma'}{dt}, \frac{D\gamma'}{dt}\Big)\right) = 2\,g\left(\frac{D^2\gamma'}{dt^2}, \frac{D\gamma'}{dt}\right) = 0.$$

Taking into account the Galilean character of the connection ∇ as well as the spatial character along γ of observer's acceleration (3), it is easy to see how the next result characterizes uniformly accelerated observers by means of a Cauchy problem.

Proposition 3 *Let (M, Ω, g, ∇) be a Galilean spacetime and $\gamma : I \longrightarrow M$, $0 \in I$, a curve satisfying equation (4) such that*

$$\gamma'(0) = v \quad \text{and} \quad \frac{D\gamma'}{dt}(0) = a, \quad \text{with} \quad \Omega(v) = 1, \quad \Omega(a) = 0. \tag{5}$$

Then, γ is a uniformly accelerated observer.

2.2 Spatially conformally Leibnizian spacetimes

Let us recall the concept of spatially conformally Leibnizian vector field, which appears in several class of cosmological models in the in the context of the generalized Newton–Cartan theory (see [7]) and will be key to obtain our main result.

Definition 4 Let (M, Ω, g) be a Leibnizian spacetime and K a vector field satisfying

$$\Omega([K, V]) = 0 \quad \text{for all} \quad V \in \Gamma(\mathrm{An}(\Omega)). \tag{6}$$

The vector field K is called *spatially conformally Leibnizian* vector field if the Lie derivative of the absolute space metric satisfies

$$\mathcal{L}_K\, g = 2\rho\, g, \tag{7}$$

for some smooth function $\rho \in C^\infty(M)$. The function ρ is called conformal factor of K.

Observe that condition (6) ensures that the previous notion is well defined. As a direct consequence we have,

$$K(g(V, W)) \\ = 2\rho\, g(V, W) + g([K, V], W) + g([K, W], V), \;\; \forall V, W \in \Gamma(\mathrm{An}(\Omega)) \quad (8)$$

In fact, the next result provides another condition to guarantee that (6) holds for a vector field.

Proposition 5 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection ∇. Then, a vector field K satisfies equation (6) if and only if the function $\Omega(K)$ is spatially invariant, i.e., $V(\Omega(K)) = 0$, $\forall V \in \Gamma\big(\mathrm{An}(\Omega)\big)$.*

Proof The symmetry of the connection ensures $d\Omega = 0$. Thus,

$$\mathcal{L}_K\Omega(Y) = K(\Omega(Y)) - \Omega([K, Y]) = Y(\Omega(K)), \;\; \forall Y \in \mathfrak{X}(M).$$

Moreover, if $\Omega([K, V]) = 0$, $\forall V \in \Gamma\big(\mathrm{An}(\Omega)\big)$, then

$$\mathcal{L}_K\Omega(V) = 0 \;\text{ and }\; V(\Omega(K)) = 0, \;\; \forall V \in \Gamma\big(\mathrm{An}(\Omega)\big).$$

Conversely, if $V(\Omega(K)) = 0$, then $\mathcal{L}_K\Omega(V) = 0$ and, as direct consequence, $\Omega([K, V]) = 0$. □

3 Main result

In this section we will analyze certain natural geometric conditions under which the inextensible trajectories of uniformly accelerated observers in a Galilean spacetime are complete. From a physical viewpoint, we are studying the hypotheses on the spacetime that ensure that uniformly accelerated observers live forever.

Consider a Galilean spacetime (M, Ω, g, ∇) with ∇ symmetric admitting a timelike vector field $K \in \Gamma(TM)$ and let us introduce the following auxiliary Riemannian metric on M:

$$g_R = \Omega \otimes \Omega + g(P^K\cdot, P^K\cdot), \quad (9)$$

where $P^K(X) := X - \frac{\Omega(X)}{\Omega(K)}K, \forall X \in \mathfrak{X}(M)$. Notice that, with respect to g_R, $\mathrm{An}(\Omega_p)$ is the orthogonal subspace to K_p for each $p \in M$. This metric will allow us to prove our main result for uniformly accelerated observers.

Theorem 6 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection ∇, and a timelike spatially conformally Leibnizian vector field K with conformal factor*

ρ. Suppose that $\frac{\rho}{\Omega(K)}$ is bounded from below. If the auxiliary metric g_R is complete and the gravitational field $\mathcal{G}$ associated with $K/\Omega(K)$ is bounded, i.e.,

$$\sup_M(g(\mathcal{G},\mathcal{G})) \leq L^2 \qquad L > 0,$$

then each inextensible solution of equation (4) with initial conditions (5) is complete.

Proof Let $\gamma : [0, 1) \longrightarrow M$ be a solution of equation (4) satisfying $\gamma'(0) = v$, and $\frac{D\gamma'}{dt}(0) = a$, with $\Omega(v) = 1$ and $\Omega(a) = 0$. From [17, Lemma 1.56], it is enough to prove that the g_R-length of γ is bounded. From Proposition 3, we know that $\Omega(\gamma') = 1$ along γ. Therefore,

$$\|\gamma'\|_R^2 = g_R(\gamma', \gamma') = 1 + g(P^K\gamma', P^K\gamma'). \tag{10}$$

On the other hand,

$$\frac{d}{dt}g(P^K\gamma', P^K\gamma') = 2\,g\Big(\frac{D}{dt}P^K\gamma', P^K\gamma'\Big) = 2\,g\Big(\frac{D\gamma'}{dt} - \frac{D}{dt}\Big(\frac{K}{\alpha}\Big), P^K\gamma'\Big), \tag{11}$$

where $\alpha := \Omega(K)$.

If we consider $Y \in \Gamma(TM)$ such that $Y \circ \gamma = \gamma'$, for each $V \in \Gamma(\mathrm{An}(\Omega))$, making use of "Koszul" formula (2) of the Galilean connection ∇ and the conformally Leibnizian character of K, we compute

$$\begin{aligned}
2\,g\left(\nabla_Y\Big(\frac{K}{\alpha}\Big), V\right) &= \frac{1}{\alpha}K\left(g(P^KY, V)\right) + 2\,\Omega(Y)\,g(\mathcal{G}, V) \\
&\quad +2\,\omega(P^KY, V) - \left(g\left(\left[\frac{K}{\alpha}, P^KY\right], V\right) + g\left(\left[\frac{K}{\alpha}, V\right], P^KY\right)\right) \\
&= \frac{1}{\alpha}K\left(g(P^KY, V)\right) + 2\,\Omega(Y)\,g(\mathcal{G}, V) + 2\,\omega(P^KY, V) \\
&\quad +g\left(P^KY\left(\frac{1}{\alpha}\right)K - \frac{1}{\alpha}[K, P^KY], V\right) + g\left(V\left(\frac{1}{\alpha}\right)K - \frac{1}{\alpha}[K, V], P^KY\right),
\end{aligned}$$

where ω is the vorticity of the vector field of observers $\frac{1}{\alpha}K$. Taking into account that the function $\Omega(K)$ must be spatially invariant we obtain

$$g\left(\nabla_Y\Big(\frac{K}{\alpha}\Big), V\right) = \Omega(Y)\,g(\mathcal{G}, V) + \omega(P^KY, V) + \frac{\rho}{\alpha}\,g(P^KY, V), \tag{12}$$

where ρ is the conformal factor associated with K. Inserting this in (11) we obtain the following linear ordinary differential equation,

$$\frac{d}{dt}\left(g(P^K\gamma', P^K\gamma')\right) + 2\,\frac{\rho}{\alpha}\,g(P^K\gamma', P^K\gamma') + 2\,g\left(\mathcal{G} - \frac{D\gamma'}{dt}, P^K\gamma'\right) = 0, \tag{13}$$

If we consider the initial value $\gamma'(0) = v$, the general solution may be represented as

$$
\begin{aligned}
g(P^K\gamma', P^K\gamma') &= g(P^K v, P^K v)\exp\Big(-2\int_0^t \Big(\frac{\rho}{\alpha}\circ\gamma\Big)(s)\,ds\Big)\\
&\quad -2\int_0^t g\Big(\mathcal{G}-\frac{D\gamma'}{dt}, P^K\gamma'\Big)\exp\Big(-2\int_s^t \Big(\frac{\rho}{\alpha}\circ\gamma\Big)(u)\,du\Big)\,ds\\
&\le A + 2\,\overline{L}\int_0^t \|\gamma'\|_R \exp\Big(-2\int_s^t \Big(\frac{\rho}{\alpha}\circ\gamma\Big)(u)\,du\Big)\,ds\\
&\le A + 2\,\tilde{L}\int_0^t \|\gamma'\|_R\,ds,
\end{aligned}
\tag{14}
$$

where $A = g(P^K v, P^K v)\exp(-2\inf\frac{\rho}{\alpha})$, $\overline{L} = \sqrt{L^2 + g(a,a)}$ and $\tilde{L} = \overline{L}\exp(-2\inf\frac{\rho}{\alpha})$. From (10), it follows

$$
\|\gamma'\|_R^2 \le 1 + A + 2\,\tilde{L}\int_0^t \|\gamma'\|_R\,ds.
$$

Integrating both sides of the inequality

$$
\frac{\|\gamma'\|_R}{\sqrt{1 + A + 2\,\tilde{L}\int_0^t \|\gamma'\|_R\,ds}} \le 1,
$$

we obtain

$$
\Big(1 + A + 2\tilde{L}\int_0^t \|\gamma'\|_R\,ds\Big)^{1/2} \le 2\tilde{L}\,t \le 2\tilde{L},
$$

and conclude that $\int_0^t \|\gamma'\|_R\,ds \le B$, for some positive constant B. □

The previous theorem is very general and can be applied, for example, in an *Irrotational Conformally Leibnizian spacetime* [7, Def.9], which is a proper time locally synchronizable Galilean spacetime (M, Ω, g, ∇) that admits a timelike vector field $K \in \Gamma(TM)$ satisfying

$$
\nabla_X K = \rho\,X, \qquad \forall X \in \Gamma(TM). \tag{15}
$$

It is not difficult to see that condition (15) directly implies condition (6) and that K is spatially conformally Leibnizian. It should be noted that there are important families of Galilean spacetimes within the class of Irrotational Conformally Leibnizian spacetimes. For example, the relevant family of cosmological models is called Galilean Generalized Robertson-Walker spacetimes (GGRWs) [7, Def. 1].

Corollary 7 *(i) Let* (M, Ω, g, ∇) *be an Irrotational Conformally Leibnizian spacetime with timelike (spatially) conformally Leibnizian vector field* K *with conformal* *factor ρ, such that $\frac{\rho}{\Omega(K)}$ is bounded from below. If the auxiliary metric g_R is complete then each inextensible uniformly accelerated observer is complete. In particular, each inextensible geodesic is complete.*

(ii) Consider a Galilean Generalized Robertson-Walker spacetime $(M = I \times_f F, \Omega, \pi_F^* h, \nabla)$ such that $f(t) \geq C \exp(-bt)$, $\forall t \in I$, for some positive constants $b, C > 0$. Then, M is (geodesically) complete if and only if $I = \mathbb{R}$ and the fiber (F, h) is complete.

Proof The first assertion follows from Theorem 6 and the fact that $\mathcal{G} = \nabla_{\frac{K}{\alpha}} \frac{K}{\alpha} = 0$, where $\alpha = \Omega(K)$. The second one is a particular case of *(i)*, taking into account that in a GGRW spacetime $\rho = f' \circ \pi_I$ and $K = \frac{1}{f}\partial_t$. □

Finally, we can also particularize Theorem 6 to the family of *stationary Galilean* spacetimes [5, Def.1]. We recall that a Galilean spacetime (M, Ω, g, ∇) is said to be stationary if it admits a future pointing timelike Leibnizian vector field K which is affine for ∇, that is, $\mathcal{L}_K \nabla = 0$, where $\mathcal{L}$ denotes the Lie derivative. This condition can also be characterized as follows:

$$[K, \nabla_Y X] = \nabla_{[K,Y]} X + \nabla_Y [K, X], \qquad \forall X, Y \in \Gamma(TM).$$

This enables us to obtain an extension of [5, Thm. 17] to uniformly accelerated observers using Theorem 6 and taking into account [5, Prop. 6].

Corollary 8 *Let (M, Ω, g, ∇) be a stationary Galilean spacetime with symmetric connection and timelike Galilean vector field K. If the auxiliary metric g_R is complete and the gravitational field $\mathcal{G}$ associated with $K/\Omega(K)$ is bounded on a spacelike leaf $\mathcal{F}_0$ of $\mathrm{An}(\Omega)$, i.e.,*

$$\sup_{\mathcal{F}_0} (g(\mathcal{G}, \mathcal{G})) \leq L^2 \qquad L > 0,$$

then each inextensible uniformly accelerated observer is complete.

In particular, inextensible uniformly accelerated observers in a compact stationary Galilean spacetime with symmetric connection are complete.

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