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THE BAND-GAP STRUCTURE OF THE SPECTRUM IN A PERIODIC MEDIUM OF MASONRY TYPE

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Abstract. We consider the spectrum of a class of positive, second-order elliptic systems of partial differential equations defined in the plane $\mathbb{R}^2$. The coefficients of the equation are assumed to have a special form, namely, they are doubly periodic and of high contrast. More precisely, the plane $\mathbb{R}^2$ is decomposed into an infinite union of the translates of the rectangular periodicity cell Ω^0, and this in turn is divided into two components, on each of which the coefficients have different, constant values. Moreover, the second component of Ω^0 consist of a neighborhood of the boundary of the cell of the width h and thus has an area comparable to h, where $h > 0$ is a small parameter.

Using the methods of asymptotic analysis we study the position of the spectral bands as $h \to 0$ and in particular show that the spectrum has at least a given, arbitrarily large number of gaps, provided h is small enough.

1. Introduction.

1.1. Statement of the problem. Let $\Omega^0 = (-l_1, l_1) \times (-l_2, l_2)$ be a rectangle containing another, deformed rectangle Ω^h with curved sides, Fig. 1, a,

$$\Omega^h = \{x \in \Omega^0 : -l_j + hH_j^-(x_{3-j}) < x_j < l_j - hH_j^+(x_{3-j}), j = 1, 2\} \tag{1}$$

where $h > 0$ is a small parameter and $H_j^\pm$ are positive profile functions, which are smooth in the variable $x_{3-j} \in [-l_j, l_j]$. We will treat the plane $\mathbb{R}^2$ as paved by the shifts

$$\begin{gathered}\Omega^0(\theta) = \{x = (x_1, x_2) : (x_1 - l_1\theta_1, x_2 - l_2\theta_2) \in \Omega^0\}, \\ \theta = (\theta_1, \theta_2) \in \mathbb{Z}^2, \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}\end{gathered} \tag{2}$$

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of the periodicity cell Ω^0. In the plane, we consider the spectral problem for an elliptic system of second-order differential equations

$$L^h(x,\nabla)u^h(x) = \lambda^h B^h(x)u^h(x), \quad x \in \mathbb{R}^2, \tag{3}$$

and its variational form

$$a^h(u^h, v^h; \mathbb{R}^2) = \lambda^h b^h(u^h, v^h; \mathbb{R}^2) \quad \forall v^h \in H^1(\mathbb{R}^2)^J. \tag{4}$$

Let us explain the notation. The number λ^h is a spectral parameter and $u^h = (u_1^h, \dots, u_J^h)^\top$ is a column of functions so that $\top$ stands for the transposition. In (3), $L^h(x,\nabla)$ is a $J \times J$-matrix of differential operators,

$$L^h(x,\nabla) = D(\nabla)^* A^h(x) D(\nabla), \tag{5}$$

where $D(\nabla)$ is a $N \times J$-matrix of first-order homogeneous differential operators with constant (complex) coefficients and $D(\nabla)^* = \overline{D(-\nabla)}^\top$ is the adjoint of $D(\nabla)$, while A^h and B^h are Hermitian matrix functions of sizes $N \times N$ and $J \times J$, respectively. These matrices are assumed piecewise constant and of high contrast, depending on the small parameter h and the subdomain Ω^h,

$$\begin{gathered} A^h(x) = A^\bullet,\ B^h(x) = B^\bullet \text{ for } x \in \Omega^h, \\ A^h(x) = h^{\alpha_A} A^\circ,\ B^h(x) = h^{\alpha_B} B^\circ \text{ for } x \in \Gamma^h = \Omega^0 \setminus \overline{\Omega^h}. \end{gathered} \tag{6}$$

where $A^\bullet$, A° and $B^\bullet$, B° are positive definite, constant matrices. Moreover, A^h and B^h are extended periodically to the plane $\mathbb{R}^2$:

$$A^h(x_1 - l_1\theta_1, x_2 - l_2\theta_2) = A^h(x), \quad B^h(x_1 - l_1\theta_1, x_2 - l_2\theta_2) = B^h(x)$$

for all $x \in \Omega^0$ and $\theta \in \mathbb{Z}^2$.

Furthermore, in (4) a^h and b^h are Hermitian sesquilinear forms,

$$\begin{gathered} a^h(u^h, v^h; \Xi) = (A^h D(\nabla)u^h, D(\nabla)v^h)_\Xi, \\ b^h(u^h, v^h; \Xi) = (B^h u^h, v^h)_\Xi, \end{gathered} \tag{7}$$

where $(\,,)_\Xi$ is the natural scalar product in the Lebesgue space $L^2(\Xi)^m$, which is either scalar ($m = 1$), or vectorial ($m > 1$). In (4), $H^1(\Xi^h)$ stands for the standard Sobolev space, while the superscript J indicates the number of the components of the test vector functions v^h; this superscript is omitted in the notation of norms and scalar products.

As a consequence of our assumptions, the operator (5) is formally self-adjoint and the forms (7) are positive.

The main goal of our paper is to describe the asymptotic behavior of the spectrum of the problem (4) when $h \to +0$. For the formal asymptotic procedure, we assume that the exponents in (6) satisfy

$$\alpha_B + 2 > \alpha_A > 1. \tag{8}$$

In the framework of elastic materials, see Example 1.2, this means that the material in Ω^h is much harder and much heavier than in Γ^h, provided $\alpha_B \in (-1, 0)$. This kind of structure appears in many natural and man-made elastic composites, e.g. quartzite and brick masonry.

We restrict ourselves to treating only the main asymptotic term in the expansion of eigenvalues. Thus, the formal asymptotic structures, which are derived in Section

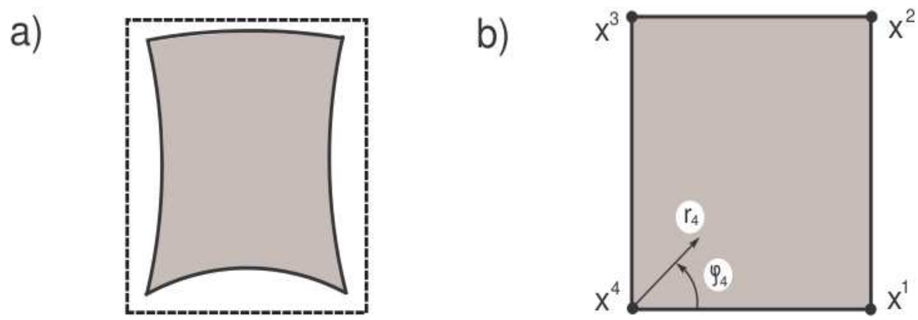

FIGURE 1. The original (a) and limit (b) periodicity cell.

2, are sufficient for the justification of the asymptotics only under the additional assumptions

$$\alpha_A < 2, \quad \alpha_B + 1 > \alpha_A. \tag{9}$$

These will be accepted in Section 3 in order to avoid the construction of higher order asymptotic terms and to simplify the proofs (see Section 2.4 for a generalization of the presented results).

1.2. **Spectrum.** We assume that the matrix $D(\xi)$ is *algebraically complete* [24]: there exists a positive integer $\rho_D \in \mathbb{N} = \{1, 2, 3, \dots\}$ such that, for any row $p(\xi) = (p_1(\xi), \dots, p_J(\xi))$ of homogeneous polynomials of degree $\rho \geqslant \rho_D$, one can find a row $q(\xi) = (q_1(\xi), \dots, q_N(\xi))$ of polynomials satisfying

$$p(\xi) = q(\xi)D(\xi) \quad \forall \xi \in \mathbb{R}^2. \tag{10}$$

Furthermore, the form a^h in (7) possesses the *polynomial property* [17], namely, there exists a finite dimensional space $\mathcal{P}$ of polynomials in x such that for any domain $\Xi \subset \mathbb{R}^2$ there holds the equivalence

$$u \in H^1(\Xi)^J, \ D(\nabla)u(x) = 0, \ x \in \Xi \quad \Leftrightarrow \quad u \in \mathcal{P}\big|_{\Xi}. \tag{11}$$

In other words, the quadratic energy form a^h degenerates only for some polynomials. From (10) it follows that all polynomials in $\mathcal{P}$ are of degree at most $\rho_D - 1$.

As proved in [24, Thm 3.7.7], the property (10) assures the Korn inequality

$$\|u^h; H^1(\Omega^0)\|^2 \leqslant c_D \left(\|D(\nabla)u^h; L^2(\Omega^0)\|^2 + \|u^h; L^2(\Omega^0)\|^2 \right),$$

where the constant c_D is independent of $u \in H^1(\Omega^0)$, so that the sum

$$\langle u^h, v^h \rangle_h = a^h(u^h, v^h; \mathbb{R}^2) + b^h(u^h, v^h; \mathbb{R}^2) \tag{12}$$

is a scalar product in the Hilbert space $\mathcal{H} = H^1(\mathbb{R}^2)^J$. We introduce the positive definite, symmetric and continuous, therefore, self-adjoint, operator $\mathcal{T}^h$ by the identity

$$\langle \mathcal{T}^h u^h, v^h \rangle_h = b^h(u^h, v^h; \mathbb{R}^2) \quad \forall u^h, v^h \in \mathcal{H}, \tag{13}$$

which reduces the problem (4) to the abstract equation

$$\mathcal{T}^h u^h = \tau^h u^h \text{ in } \mathcal{H} \tag{14}$$

with the new spectral parameter

$$\tau^h = (1 + \lambda^h)^{-1}. \tag{15}$$

The spectrum $\mathcal{S}^h$ of the operator $\mathcal{T}^h$ is contained to the segment $[0,1]$ where $t^h = 1$ is the norm of $\mathcal{T}^h$, see [20, Remark 1]. The set

$$\sigma^h = \{\lambda^h : (1+\lambda^h)^{-1} \in \mathcal{S}^h\} \tag{16}$$

is regarded as the spectrum of the problem (4).

The structure of the spectrum (16) is described by the Floquet–Bloch–Gelfand-(FBG-) theory, see, e.g. [11, 25, 12], which yields for it the band-gap structure

$$\sigma^h = \bigcup_{n\in\mathbb{N}} \beta_m^h. \tag{17}$$

Here, the bands

$$\beta_m^h = [\beta_{m-}^h, \beta_{m+}^h] = \{\Lambda_m^h(\eta) \,|\, \eta = (\eta_1,\eta_2) \in \mathbb{Y}\} \tag{18}$$

are formed by the eigenvalue sequence

$$0 \leqslant \Lambda_1^h(\eta) \leqslant \Lambda_2^h(\eta) \leqslant \cdots \leqslant \Lambda_m^h(\eta) \leqslant \dots \ \to\ +\infty \tag{19}$$

of the model problem in the periodicity cell

$$\begin{gathered} a^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^0) = \Lambda^h(\eta) b^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^0) \\ \forall\ V^h(\cdot;\eta) \in \mathcal{H}(\eta) = H^1_\eta(\Omega^0)^J \end{gathered} \tag{20}$$

depending on the Floquet parameter $\eta = (\eta_1,\eta_2)$, which is the Gelfand dual variable belonging to the rectangle

$$\mathbb{Y} = [0,1/l_1) \times [0,1/l_2).$$

In (20), $H^1_\eta(\Omega^0)$ is the subspace of functions $U^h \in H^1(\Omega^0)$ subject to the quasi-periodicity conditions

$$\begin{aligned} U^h(l_1,x_2;\eta) &= e^{2\pi i \eta_1 l_1} U^h(-l_1,x_2;\eta), \quad |x_2| < l_2, \\ U^h(x_1,l_2;\eta) &= e^{2\pi i \eta_2 l_2} U^h(x_1,-l_2;\eta), \quad |x_1| < l_1. \end{aligned} \tag{21}$$

Using an argument similar to (13)–(15) and recalling the compactness of the embedding $H^1(\Omega^0) \subset L^2(\Omega^0)$, we conclude that the spectrum of the model problem (20) is discrete and consists of the positive monotone unbounded sequence (19), cf. [3, Thm 10.1.5, 10.2.2], while the corresponding eigenvectors $U^h_{(m)}(\cdot;\eta) \in \mathcal{H}(\eta)$ can be subject to the normalization and orthogonality conditions

$$b^h(U^h_{(m)}, U^h_{(n)}) = \delta_{m,n}, \tag{22}$$

where $\delta_{m,n}$ is the Kroneker symbol. Moreover, the functions $\mathbb{Y} \ni \eta \mapsto \Lambda^h_m(\eta)$ are continuous and periodic with the periods $1/l_j$ in η_j so that the bands (18) indeed are compact intervals.

The variational problem (20) is obtained from (4) by the FBG-transform [5]. All objects related to the model problem are denoted by capital letters; in particular, $\{\Lambda^h(\eta), U^h(x;\eta)\}$ is a new notation for eigenpairs.

The spectral bands (18) may overlap, but between them there can also exist *gaps*, i.e., nonempty open intervals $\gamma_m^h =]\beta_m^h, \beta_{m+1}^h[$ which are free of the spectrum. In the paper [20] it was proved that the spectrum of the problem (4) has at least one open gap of width $O(1)$. In what follows, we will describe the asymptotic structure of the low-frequency range $\{\lambda^h \in \sigma^h : \lambda^h \leqslant const\}$ of the spectrum (17). The results imply the existence of a large number of open gaps, the geometric characteristics of which will also be described asymptotically.

1.3. **Special cases.** Let us list some concrete problems in mathematical physics which have the properties assumed above. Other important examples will be discussed in Sections 4.4 and 4.5.

Example 1.1. Let $J = 1$ and $N = 2$. Then $D(\nabla) = \nabla$ and (5) is a scalar elliptic second-order differential operator in the divergence form. Clearly, $\mathcal{P} = \mathbb{R}$ in (11). The problem (3) describes e.g. a heterogeneous acoustic medium with thin high-conductive streaks. ⊠

Example 1.2. Let $J = 2$, $N = 3$ and

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 \end{pmatrix}. \tag{23}$$

It is known that in this case $\rho_D = 2$ in the property (10), cf. [24, § 3.7], [17, Example 1.12]. In the Voigt–Mandel notation of elasticity, $u = (u_1, u_2)^\top$ is the displacement vector, $D(\nabla)u$ and $AD(\nabla)u$ are the strain and stress columns and A is a real, symmetric and positive definite 3×3-matrix of elastic moduli. Furthermore, $B = \operatorname{diag}\{b, b\}$ and $b > 0$ is the mass density of the elastic material. The space of polynomials (11)

$$\mathcal{P} = \{u :\ u_1(x) = c_1 - c_0 x_2, u_2(x) = c_2 + c_0 x_1, c_p \in \mathbb{R}\} \tag{24}$$

consists of rigid motions.

The problem (3) describes elastic composites, some of which were already mentioned in Section 1.1. ⊠

1.4. **State of the art and the architecture of the paper.** The band-gap structure of the spectrum of an elliptic equation

$$-\nabla^\top A^\varepsilon(x)\nabla u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \mathbb{R}^d, \quad d \geqslant 2,$$

with highly contrasting coefficients

$$A^\varepsilon(x) = \begin{cases} \varepsilon^{-1}A^\bullet, & x \in \omega \\ A^\circ, & x \in \Omega \setminus \overline{\omega} \end{cases} \tag{25}$$

which are 1-periodic in all coordinates x_j, was first investigated in the paper [6], where the existence of a non-empty spectral gap was proved; see also [7]. Notice that the coefficients of $d \times d$-matrix (25) become large in an interior subdomain ω of the unit open cube Ω, $\overline{\omega} \subset \Omega$ ($\varepsilon > 0$ is a small parameter). A similar problem was considered in [28], and there it was in addition shown that the number of open gaps grows unboundedly when $\varepsilon \to +0$.

The subdomain ω is fixed in the definition (25), but the papers [20, 1] deal with the situation described in Section 1.1: the subdomain (1) of the high contrast covers the whole periodicity cell Ω^h in the limit $h \to +0$.

In the present paper we employ an asymptotic method which differs quite significantly from the analysis used in [6, 7, 28, 22, 2] and [20, 1].

2. Formal asymptotic analysis.

2.1. **The limit problem.** First of all, we rewrite the problem (20) in differential form. In view of (5) and (6), it consists of two systems

$$\overline{D(-\nabla)}^\top A^\bullet D(\nabla) U^{\bullet\, h}(x;\eta) = \Lambda^h(\eta) B^\bullet U^{\bullet\, h}(x;\eta), \quad x \in \Omega^h, \tag{26}$$

$$h^{\alpha_A}\overline{D(-\nabla)}^\top A^\circ D(\nabla) U^{\circ\, h}(x;\eta) = \Lambda^h(\eta) h^{\alpha_B} B^\circ U^{\circ\, h}(x;\eta), \quad x \in \Gamma^h, \tag{27}$$

coupled by the transmission conditions

$$U^{\bullet\, h}(x;\eta) = U^{\circ\, h}(x;\eta), \quad x \in \Sigma^h = \partial\Omega^h, \tag{28}$$

$$\overline{D(\nu^h(x))}^\top A^\bullet D(\nabla) U^{\bullet\, h}(x;\eta) = h^{\alpha_A} \overline{D(\nu^h(x))}^\top A^\circ D(\nabla) U^{\circ\, h}(x;\eta), \quad x \in \Sigma^h, \tag{29}$$

where $U^{\bullet\, h}$ and $U^{\circ\, h}$ are the restrictions of U^h onto Ω^h and Γ^h, respectively, and $\nu^h(x)$ is the exterior normal unit vector on the boundary Σ^h of Ω^h, i.e. the interface. The sides of the rectangle Ω^0 are supplied, in the terminology of [14], with the stable quasi-periodicity conditions (21) as well as the intrinsic conditions

$$\begin{aligned} \partial_1 U^h(l_1, x_2;\eta) &= e^{2\pi i \eta_1 l_1} \partial_1 U^h(-l_1, x_2;\eta), \quad |x_2| < l_2, \\ \partial_2 U^h(x_1, l_2;\eta) &= e^{2\pi i \eta_2 l_2} \partial_2 U^h(x_1, -l_2;\eta), \quad |x_1| < l_1, \end{aligned} \tag{30}$$

where $\partial_j = \partial/\partial x_j$, $j = 1, 2$.

Since the right-hand-side of (29) includes the small coefficient h^{α_A}, the passing to the limit $h \to 0$ turns Σ^h into $\Sigma^0 = \partial\Omega^0$ (see (1)) and leads to the limit problem

$$\overline{D(-\nabla)}^\top A^\bullet D(\nabla) U^0(x) = \Lambda^0 B^\bullet U^0(x), \quad x \in \Omega^0, \tag{31}$$

$$\overline{D(\nu^0(x))}^\top A^\bullet D(\nabla) U^0(x) = 0, \quad x \in \Sigma^0 = \partial\Omega^0. \tag{32}$$

Notice that the Floquet parameter η does not appear in this problem because the quasi-periodicity conditions (21), (30) are isolated from the interior part Ω^h by a thin "shim" Γ^h. The variational formulation of this problem is written as the integral identity [14]

$$a^\bullet(U^0, V^0; \Omega^0) = \Lambda^0 b^\bullet(U^0, V^0; \Omega^0) \quad \forall\, V^0 \in H^1_\eta(\Omega^0)^J. \tag{33}$$

The spectrum of the problem (31), (32) or (33) is discrete, as it consists of the eigenvalues

$$0 \leqslant \Lambda^0_1 \leqslant \Lambda^0_2 \leqslant \cdots \leqslant \Lambda^0_m \leqslant \ldots \ \to\ +\infty, \tag{34}$$

while the corresponding vector eigenfunctions $U^0_{(m)} \in H^1(\Omega^0)^J$ can be subject to the orthogonality and normalization conditions

$$(B^\bullet U^0_{(m)}, U^0_{(p)})_{\Omega^0} = \delta_{m,p}, \quad m, p \in \mathbb{N}. \tag{35}$$

We emphasize that the multiplicity of the null eigenvalue in (34) equals $\dim \mathcal{P}$, by (11).

2.2. **Formal asymptotics of eigenvalues.** Let $\{\Lambda^0, U^0\}$ be an eigenpair of the limit problem (31), (32). Due to the boundary condition (32), the eigenfunction U^0 leaves only a small discrepancy in the intrinsic transmission conditions (29), but the discrepancy in the stable transmission condition (28) is of order $1 = h^0$. To compensate the latter, we need to construct a boundary layer W in the thin bordering Γ^h. We consider two vertical curved strips

$$\Gamma^h_{1\pm} = \{x : x_2 \in (-l_2, l_2), l_1 > \pm x_1 > l_1 - hH^\pm_1(x_2)\}$$

and denote by W^1 the restriction of W onto $\Gamma^h_1 = \Gamma^h_{1-} \cup \Gamma^h_{1+}$. Notice that due to the periodicity, the set Γ^h_1 can be identified with

$$\{x : |x_2| < l_2, l_1 - hH^+(x_2) < x_1 < l_1 - hH^-(x_2)\}. \tag{36}$$

As always in the asymptotic theory of elliptic problems in thin domains, we introduce the rapid variable

$$\zeta = h^{-1}(x_1 - l_1) \text{ in } \Gamma^h_{1+},$$
$$\zeta = h^{-1}(x_1 + l_1) \text{ in } \Gamma^h_{1-}, \tag{37}$$

so that, according to (1) and (36),

$$\zeta \in \Upsilon_1(y) = (-H_1^+(y), H_1^-(y))$$

while $y = x_2$ is still a slow variable. The boundary layer term W^1 depends on the coordinate couple (ζ, y). Inside the set $\Gamma_2^h = \Gamma^h_{2-} \cup \Gamma^h_{2+}$ we define the corresponding coordinates and the boundary layer term W^2 analogously.

We have

$$D(\nabla) = h^{-1} D(e_{(1)}\partial_\zeta) + D(e_{(2)}\partial_y), \quad e_{(j)} = (\delta_{1,j}, \delta_{2,j})^\top.$$

Thus, the left-hand side of (27) takes the form

$$h^{\alpha_A - 2}\overline{D(-e_{(1)}\partial_\zeta)}^\top A^\circ D(e_{(1)}\partial_\zeta) W^1(\zeta, y) + \dots, \tag{38}$$

where dots stand for higher-order terms that are inessential in our asymptotic analysis, and the right-hand side is

$$\Lambda^0 h^{\alpha_B} B^\circ W^1(\zeta, y) + \dots = \dots \tag{39}$$

In other words, by (8), the expression (39) is much smaller than (38). Thus, the boundary layer term W^1 satisfies the problem

$$\overline{D(-e_{(1)}\partial_\zeta)}^\top A^\circ D(e_{(1)}\partial_\zeta) W^1(\zeta, y) = 0, \quad \zeta \in \Upsilon(y) \setminus \{0\}, \tag{40}$$

$$W^1(H_1^-(y), y) = U^0(-l_1, y), \; W^1(-H_1^+(y), y) = U^0(l_1, y), \tag{41}$$

together with the following transmission conditions at the point $\zeta = 0$

$$W^1(-0, y) = e^{2\pi i \eta_1 l_1} W^1(+0, y), \; \partial_\zeta W^1(-0, y) = e^{2\pi i \eta_1 l_1} \partial_\zeta W^1(+0, y) \tag{42}$$

coming from (30) and (37).

It follows from (40)–(42) that

$$W^1(\zeta, y) = \begin{cases} C^0(y) + C^1(y)\zeta & \text{for} \quad \zeta > 0, \\ e^{2\pi i \eta_1 l_1}(C^0(y) + C^1(y)\zeta) & \text{for} \quad \zeta < 0, \end{cases} \tag{43}$$

where the coefficient columns $C^0(y)$ and $C^1(y)$ can be found from the linear system

$$C^0(y) + C^1(y) H_1^-(y) = U^0(-l_1, y),$$
$$C^0(y) - C^1(y) H_1^+(y) = e^{-2\pi i \eta_1 l_1} U^0(-l_1, y).$$

Thus,

$$C^1(y) = H_1(y)^{-1} \left[U^0\right]_1 (y; \eta_1), \tag{44}$$

where

$$H_j(x_{3-j}) = H_j^+(x_{3-j}) + H_j^-(x_{3-j})$$
$$\left[U^0\right]_j (x_{3-j}; \eta_j) = U^0(x)\Big|_{x_j = -l_j} - e^{-2\pi i \eta_j l_j} U^0(x)\Big|_{x_j = l_j}. \tag{45}$$

We will not need an explicit expression for the coefficient $C^0(y)$.

Let us return to the transmission conditions. By (1), the normal vector on $\Sigma^h_{1\pm}$ equals

$$\nu^h(x) = \left(1 + h^2 |\partial_2 H_1^\pm(x_2)|^2\right)^{-1/2} (\pm 1, h\partial_2 H_1^\pm(x_2)) = \pm e_{(1)} + O(h), \tag{46}$$

hence, the main asymptotic term of the right-hand side of (29), calculated for the boundary layer term $W^1(\zeta, x_2)$, reads as

$$h^{\alpha_A - 1}\overline{D(\pm e_{(1)})}^\top A^\circ D(e_{(1)})\partial_\zeta W^1(\zeta, x_2)\Big|_{\zeta = \mp H_1^\pm(x_2)} = \\ = h^{\alpha_A-1} A^\circ_{(1)} \frac{1}{H_1(x_2)} \left\{ \begin{array}{ll} e^{2\pi i \eta_1 l_1}\left[U^0\right]_1(x_2;\eta_1) & \text{at} \quad \Sigma^h_{1+}, \\ -\left[U^0\right]_1(x_2;\eta_1) & \text{at} \quad \Sigma^h_{1-}, \end{array}\right. \tag{47}$$

where $\Sigma^h_{j\pm} = \Sigma^h \cap \partial\Gamma^h_{j\pm}$ and $A^\circ_{(1)}$ is a Hermitian and positive definite $J \times J$- matrix; we denote

$$A^\circ_{(j)} = \overline{D(e_{(j)})}^\top A^\circ D(e_{(j)}), \quad j = 1, 2. \tag{48}$$

The coefficient h^{α_A-1} is small, since its exponent is positive in view of our assumption (8). We thus readily accept the asymptotic ansätze

$$U^{\bullet\, h}(x;\eta) = U^0(x) + h^{\alpha_A-1}U'(x;\eta) + \dots, \tag{49}$$

$$\Lambda^h(\eta) = \Lambda^0 + h^{\alpha_A-1}\Lambda'(\eta) + \dots \tag{50}$$

The correction term satisfies the problem

$$\overline{D(-\nabla)}^\top A^\bullet D(\nabla)U'(x;\eta) - \Lambda^0 B^\bullet U'(x;\eta) = \Lambda'(\eta)B^\bullet U^0(x), \quad x \in \Omega^0, \tag{51}$$

$$\overline{D(\pm e_{(j)})}^\top A^\bullet D(\nabla)U'(x;\eta) = F^{j\pm}(x;\eta), \quad x_j = \pm l_j, \ |x_{3-j}| < l_{3-j}, \tag{52}$$

where the data of the boundary conditions is taken from (47) and a similar formula for $j = 2$ so that

$$F^{j+}(x_{3-j};\eta) = -A^\circ_{(j)}\frac{1}{H_j(x_{3-j})}\, e^{2\pi i\eta_j l_j}\left[U^0\right]_j(x_{3-j};\eta_j) \\ F^{j-}(x_{3-j};\eta) = A^\circ_{(j)}\frac{1}{H_j(x_{3-j})}\left[U^0\right]_j(x_{3-j};\eta_j), \quad j = 1,2. \tag{53}$$

The variational formulation of the problem (49), (52) reads as

$$a^\bullet(U', V;\Omega^0) - \Lambda^0 b^\bullet(U', V;\Omega^0) = \Lambda'(\eta)(B^\bullet U', V)_{\Omega^0} + \sum_{j=1,2}\sum_\pm (F^{j\pm}, V)_{\Sigma^0_{j\pm}}$$

$$\forall\, V^0 \in \mathcal{H}(\eta) = H^1_\eta(\Omega^0)^J,$$

where $\Sigma^0_{j\pm}$ denote the sides of the rectangle Ω^0.

First, we assume that $\Lambda^0 = \Lambda^0_m$ is a simple eigenvalue of the problem (31) and hence, by the Fredholm alternative, the problem (51), (52) gets one compatibility condition, namely

$$\Lambda'_m(\eta) = \Lambda'_m(\eta)(B^\bullet U^0_{(m)}, U^0_{(m)})_{\Omega^0} = (L^\bullet(\nabla)U' - \Lambda^0_m B^\bullet U', U^0_{(m)})_{\Omega^0} =$$

$$= \sum_{j=1,2}\sum_\pm \int\limits_{-l_{3-j}}^{l_{3-j}} \overline{U^0_{(m)}(x)}^\top \overline{D(\pm e_{(j)})}^\top A^\bullet D(\nabla)U'(x;\eta)\Big|_{x_j=\pm l_j} dx_{3-j} =$$

$$= -\sum_{j=1,2}\int\limits_{-l_{3-j}}^{l_{3-j}} H_j(x_{3-j})^{-1}\left(e^{2\pi i\eta_j l_j}\overline{U^0_{(m)}(x)}\Big|_{x_j=l_j} - \overline{U^0_{(m)}(x)}\Big|_{x_j=-l_j}\right)^\top \times$$

$$\times A^\circ_{(j)}\left[U^0_{(m)}\right]_j(x_{3-j};\eta_j)dx_{3-j} =$$

$$= \sum_{j=1,2} \int_{-l_{3-j}}^{l_{3-j}} H_j(y)^{-1} \overline{[U^0_{(m)}]_j(y;\eta_j)}^\top A^\circ_{(j)} \left[U^0_{(m)}\right]_j (y;\eta_j) dy,$$

and, therefore,

$$\Lambda'_m(\eta) = \mathcal{J}(U^0_{(m)}, U^0_{(m)}; \eta) \tag{54}$$

where the Hermitian sesquilinear form $\mathcal{J}$ is defined by

$$\mathcal{J}(U^0, V^0; \eta) = \sum_{j=1,2} \int_{-l_{3-j}}^{l_{3-j}} H_j(y)^{-1} \overline{[V^0]_j(y;\eta_j)}^\top A^\circ_{(j)} \left[U^0\right]_j (y;\eta_j) dy, \tag{55}$$

see the notation in (45).

Second, if $\Lambda^0 = \Lambda^0_m$ is an eigenvalue of multiplicity $\varkappa_m$ and $U^0_{(m)}, \dots, U^0_{(m+\varkappa_m-1)}$ are the related, orthonormalized eigenfunctions, similar calculations show that the correction terms in the ansatz (50) for the eigenvalues $\Lambda^0_m(\eta), \dots, \Lambda^0_{m+\varkappa_m-1}(\eta)$ are nothing but eigenvalues of the Hermitian $\varkappa_m \times \varkappa_m$-matrix M^m with entries

$$M^m_{pq}(\eta) = \mathcal{J}(U^0_{(p)}, U^0_{(q)}; \eta), \quad p, q, = m, \dots, m + \varkappa_m - 1. \tag{56}$$

Moreover, the eigenvectors $U^0_{(m)}, \dots, U^0_{(m+\varkappa_m-1)}$ can be fixed such that the matrix $M(\eta)$ with the entries (56) becomes diagonal with eigenvalues

$$\Lambda'_m(\eta) \leqslant \Lambda'_{m+1}(\eta) \leqslant \dots \leqslant \Lambda'_{m+\varkappa_m-1}(\eta). \tag{57}$$

2.3. **Theorem on asymptotics and its consequences.** In Section 3, we will prove the following error estimates for the asymptotics constructed above. However, one additional albeit reasonable assumption, (68), on the eigenfunction $U^0_{(m)} \in H^2(\Omega^0)$ has to be made; see Section 3.1 for details.

Theorem 2.1. *Let the assumptions (8), (9) and (68) hold true. Then, for any eigenvalue Λ^0_m of multiplicity $\varkappa_m$ of the limit problem (33),*

$$\Lambda^0_{m-1} < \Lambda^0_m = \dots = \Lambda^0_{m+\varkappa_m-1} < \Lambda^0_{m+\varkappa_m}, \tag{58}$$

there exist positive h_m and c_m such that, for $h \in (0, h_m]$, the sequence (19) contains the eigenvalues $\Lambda^h_n(\eta), \dots, \Lambda^h_{n+\varkappa_m-1}(\eta)$ of the problem (20) satisfying the estimates

$$|\Lambda^h_{n+l}(\eta) - \Lambda^0_{m+l} - h^{\alpha_A-1}\Lambda'_{m+l}(\eta)| \leqslant c_m h^{\alpha_A-1+\delta_{AB}}, \quad l = 0, \dots, \varkappa_m - 1, \tag{59}$$

where

$$\delta_{AB} = \min\{1 - \alpha_A/2, \alpha_B + 2 - 3\alpha_A/2, (\alpha_A - 1)/2, (\alpha_B + 1)/2, 2 - \alpha_A\} > 0 \tag{60}$$

and $\Lambda'_{m+l}(\eta)$ are the corrections terms constructed in Section 2.4, see (54) and (57). Moreover, $n = m$, i.e. formula (59) includes the asymptotic relationship between the corresponding entries of the eigenvalue sequencies (19) and (34).

The number (60) is positive because of the restrictions (8) and (9) so that Theorem 2.1 indeed confirms the asymptotic form (50) of the eigenvalues, including the formulas (54)–(57) for the correction term.

Since the $J \times J$-matrix (48) is positive definite, the matrix M^m is positive. The latter means that the numbers (54) and (57) are non-negative. Thus, formulas (50) and (59) imply that, for a small $h > 0$, the inequalities

$$\Lambda^h_m(\eta) \geqslant \Lambda^0_m(\eta), \quad m \in \mathbb{N}, \ \eta \in \mathbb{Y},$$

are valid. In other words, the appearence of the flexible thin frame Γ^h shifts the dispersion surfaces upwards.

The following description of the spectral bands is an important consequence of Theorem 2.1, and it will have further applications in Section 4.

Theorem 2.2. *If the hypotheses of Theorem 2.1 are true, then the endpoints of the spectral band (18) have the asymptotic form*

$$|\beta^h_{m+l\pm} - \Lambda^0_m - h^{\alpha_A-1}\beta'_{m+l\pm}| \leqslant c_m h^{\delta_{AB}}, \quad l = 0, \dots, \varkappa_m - 1, \tag{61}$$

where $h \in (0, h_m]$,

$$\beta'_{m+l-} = \min_{\eta\in\mathbb{Y}} \Lambda'_{m+l}(\eta), \quad \beta'_{m+l+} = \max_{\eta\in\mathbb{Y}} \Lambda'_{m+l}(\eta)$$

and the correction terms in (61) come from the formulas (54), (55) and (57).

Relation (61) shows in particular that in the situation (58) there appears an open spectral gap of width

$$\Lambda^0_m - \Lambda^0_{m-1} + h^{\alpha_A-1}(\beta'_{m-} - \beta'_{m-l+}) + O(h^{\delta_{AB}}).$$

between the bands β^h_{m-1} and β^h_m. To make conclusions on gaps between the bands $\beta^h_m, \dots, \beta^h_{m+\varkappa-1}$ we need to put the formulas (56)–(57) into a more conrete form, see Section 3.3.

3. Justification of the asymptotics.

3.1. Additional assumptions on singularities at the corners.

The Kondratiev theory of elliptic problems in domains with corners and conical points of the boundary (see the key works [8, 15, 16, 17] and, e.g., the monographs [21, 10]) shows that the solution of the problem (31), (32) is of the form

$$U^0(x) = \sum_{j=1}^{4} \chi_j(x) \left(P^j(x - x^j) + \sum_{k=1}^{K^j} C_{kj} \Psi^{kj}(r_j, \varphi_j) \right) + \widetilde{U}^0(x), \tag{62}$$

where P^{jk} are some vector polynomials, (r_j, φ_j) are polar coordinates centered at the corner points x^j (see Fig. 1.b), $\varphi_j \in (0, \pi/2)$ in Ω^0,

$$x^1 = (l_1, -l_2),\ x^2 = (l_1, l_2),\ x^3 = (-l_1, l_2),\ x^4 = (-l_1, -l_2), \tag{63}$$

and χ_j is a smooth cut-off function supported and equal to 1 in a small neighborhood of x^j. Moreover, C_{kj} is a constant coefficient and Ψ^{kj} is a power-logarithmic solution,

$$\Psi^{kj}(r_j, \varphi_j) = r_j^{\mu_{kj}} \psi^{kj}(\varphi_j; \ln r_j) \tag{64}$$

where ψ^{kj} is a polynomial of degree deg ψ^{kj} in $\ln r_j$ with coefficients, which are smooth functions in the angular variable $\varphi_j \in [0, \pi/2]$. The number K^j in (62) can be fixed such that $\widetilde{U}^0 \in H^2(\Omega^0)^J$ and in this way the exponents of power-law solutions (64) satisfy the inequalities

$$0 < \mu_{kj} < 1 + \delta_0 \text{ for some } \delta_0 > 0. \tag{65}$$

In view of (65), the polynomials P^j can be reduced to linear vector functions, i.e. (64) with $\mu_{kj} = 1$ and $\deg \psi^{kj} = 0$

Lemma 3.1. *In (62) we have*

$$D(\nabla)P^j = 0 \quad \Leftrightarrow \quad P^j \in \mathcal{P}. \tag{66}$$

Proof. By the above conclusion, we can write

$$P^j(x) = P^j_{(1)}x_1 + P^j_{(2)}x_2 + P^j_{(0)}, \tag{67}$$

and here the constant term $P^j_{(0)}$ satisfies (66). The boundary conditions (32) on the sides of the rectangle Ω^0 imply

$$0 = \overline{D(e_{(k)})}^\top A^\bullet \left(D(e_{(1)})P^j_{(1)} + D(e_{(2)})P^j_{(2)} \right), \quad k = 1, 2,$$

and, hence,

$$0 = \overline{(D(\nabla)P^j(x))}^\top A^\bullet D(\nabla)P^j(x).$$

We obtain the formula (66), since $A^\bullet$ is Hermitian and positive definite. □

To simplify the justification scheme, which would otherwise become too cumbersome, we assume that for some $\delta_0 > 0$ we have in (62)

$$K^j = 0. \tag{68}$$

In other words, non-constant and non-linear power-logarithmic terms (64) are assumed not to exist in the representation (62). Consequently, we have $U^0 \in H^2(\Omega^0)^J$. Moreover, Hölder estimates in domains with corner and conical points on the boundary (see the original paper [16] and also the monographs [21, 10]) yield the bounds

$$|U^0(x)| \leqslant c_0, \quad |\nabla U^0(x)| \leqslant c_1, \quad |\nabla^n U^0(x)| \leqslant c_n r^{\delta_0 - n + 1}, \quad n = 2, 3, \ldots, \tag{69}$$

where $r = \min\{r_j \,|\, j = 1, \ldots, 4\}$ is the distance of the rectangle Ω^0 to the vertices and $\delta_0 > 0$ is the number in (65).

In Section 4.2, we will consider examples of scalar equations and elasticity systems which meet all the assumptions made above.

Let us then turn to the equations for the correction term U'. The right-hand side of the system (51) is sufficiently smooth but the data (53) of the Neumann conditions (52) does not vanish at the corner points (63) of Ω^0. Again, the Kondratiev theory provides for the solution U' the decomposition

$$U'(x) = \sum_{j=1}^{4} \chi_j(x) \left(P^{j\prime}(x - x^j) + C'_j \Psi^j(r_j, \varphi_j) \right) + \widetilde{U}'(x), \tag{70}$$

where $\widetilde{U}' \in H^2(\Omega^2)^J$, the term $P^{j\prime}$ is a linear vector function in x, and Ψ^j is of the form (64), where $\mu_j = 1$ and ψ^j is a linear function of $\ln r_j$. Therefore, $\chi_j \Psi^j \notin H^2(\Omega^2)^J$, if a logarithmic term exists. Furthermore, $\Psi^j(r_j, \varphi_j)$ is a polynomial in x if and only if, in the Neumann conditions (52), the data (53) frozen at sides of the rectangle Ω^0 in the corner point x^j can be compensated by a linear vector function (67). This remark allows us to verify the following property.

Lemma 3.2. *The terms $\Psi^j(r_j, \varphi_j)$ are always of the form (67) with some $P^j_{(k)} \in \mathbb{C}^j$, $k = 0, 1, 2$. Moreover, $P^j_{(0)} = 0$ if and only if $\mathcal{P} = \mathbb{C}^J$.*

Proof. Considering the vertex x^1 and searching for the linear vector (67), we need to solve the system of $2J$ algebraic equations

$$\begin{aligned} \overline{D(e^1)}^\top A^\bullet D(e^1)P^1_{(1)} + \overline{D(e^1)}^\top A^\bullet D(e^2)P^1_{(2)} &= F^{1+}(-l_2; \eta), \\ \overline{D(e^2)}^\top A^\bullet D(e^1)P^1_{(1)} + \overline{D(e^2)}^\top A^\bullet D(e^2)P^1_{(2)} &= F^{2-}(l_1; \eta). \end{aligned} \tag{71}$$

The $(2J \times 2J)$-matrix of this system is Hermitian and, therefore, the system is uniquely solvable provided the homogeneous system (71) has only the trivial solution. According to Lemma 3.1, the latter is true provided the polynomial subspace $\mathcal{P}$ in (11) does not contain a non-trivial linear vector function. This completes the proof, since $\mathcal{P}$ is invariant with respect to the coordinate translation $x \mapsto x+x^0$. $\square$

Using the notation of (69), we derive the following estimates for the decomposion (70) :

$$|U'(x)| \leqslant c_0, \quad |\nabla U'(x)| \leqslant c_1(1+|\ln r|), \quad |\nabla^2 U'(x)| \leqslant c_2 r^{-1}. \tag{72}$$

In spite of the singularities these will be sufficient to justify our asymptotic formulas for the eigenvalues (19).

3.2. **The operator formulation of the model problem in the periodicity cell.** Similarly to (12), we introduce the scalar product

$$\langle U^h, V^h\rangle_{h,\eta} = a^h(U^h, V^h; \Omega^h) + b^h(U^h, V^h; \Omega^h) \tag{73}$$

in the space

$$\mathcal{H}(\eta) = \{U^h(\cdot;\eta) \in H^1(\Omega^0)^J : (21) \text{ is satisfied}\}$$

and define the self-adjoint and positive operator $\mathcal{T}^h(\eta)$ in $\mathcal{H}(\eta)$ by the identity

$$\langle \mathcal{T}^h(\eta)U^h(\cdot;\eta), V^h(\cdot;\eta)\rangle_{h,\eta} = a^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^h) + b^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^h)$$
$$\forall U^h(\cdot;\eta), V^h(\cdot;\eta) \in \mathcal{H}(\eta).$$

In this way, the variational formulation

$$a^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^h) = \Lambda^h(\eta) b^h(U^h(\cdot;\eta), V^h(\cdot;\eta);\Omega^h) \quad \forall U^h(\cdot;\eta) \in \mathcal{H}(\eta)$$

of the problem (26)–(30), (21) turns into the abstract equation in $\mathcal{H}(\eta)$,

$$\mathcal{T}^h(\eta)U^h(\cdot;\eta) = \tau^h(\eta)U^h(\cdot;\eta)$$

where

$$\tau^h(\eta) = (1+\Lambda^h(\eta))^{-1}. \tag{74}$$

The operator $\mathcal{T}^h(\eta)$ is compact because of the compact embedding $H^1(\Omega^0) \subset L^2(\Omega^0)$ in the bounded domain Ω^0, and its spectrum forms the monotone positive sequence (see, e.g., [3, Theorem 10.1.5, 10.2.2])

$$\tau_1^h(\eta) \geqslant \tau_2^h(\eta) \geqslant \cdots \geqslant \tau_n^h(\eta) \geqslant \cdots \to +0,$$

which turns into the sequence (19) by the formula $\Lambda^h(\eta) = \tau^h(\eta)^{-1} - 1$, cf. (74).

The following assertion is known as lemma on "near eigenvalues and eigenvectors", cf. [27], and it follows immediately from the spectral decomposition of resolvent, see [3, Ch. 6].

Lemma 3.3. *Let $\mathcal{U}^h(\eta) \in \mathcal{H}(\eta)$ and $t^h(\eta) \in (0,+\infty)$ be such that*

$$\|\mathcal{U}^h(\eta);\mathcal{H}(\eta)\| = 1, \quad \|\mathcal{T}^h(\eta)\mathcal{U}^h(\eta) - t^h(\eta)\mathcal{U}^h(\eta);\mathcal{H}(\eta)\| =: \epsilon \in (0, t^h(\eta)). \tag{75}$$

Then at least one eigenvalue $\tau_n^h(\eta)$ of the operator $\mathcal{T}^h(\eta)$ satisfies the inequality

$$|t^h(\eta) - \tau_n^h(\eta)| \leqslant \epsilon.$$

Moreover, for every $\epsilon_* \in (\epsilon, t^h(\eta))$ one can find coefficients $a_M^h(\eta), \dots, a_{M+X-1}^h(\eta)$ such that

$$\left\|\mathcal{U}^h(\eta) - \sum_{q=M}^{M+X-1} a_q^h(\eta) U_{(q)}^h(\eta);\mathcal{H}(\eta)\right\| \leqslant 2\frac{\epsilon}{\epsilon_*}, \qquad \sum_{q=M}^{M+X-1} |a_q^h(\eta)|^2 = 1 \tag{76}$$

where $\tau^h_M(\eta), \dots, \tau^h_{M+X-1}(\eta)$ *are all the eigenvalues of* $\mathcal{T}^h(\eta)$ *contained in the interval* $[t^h(\eta) - \epsilon_*, t^h(\eta) + \epsilon_*]$ *(the numbers* M *and* X *may dependent on* h*), and* $U^h_{(M)}(\eta), \dots, U^h_{(M+X-1)}(\eta) \in \mathcal{H}(\eta)$ *are the corresponding eigenvectors subject to the orthogonality and normalization conditions*

$$\langle U^h_{(p)}(\eta), U^h_{(q)}(\eta)\rangle_{h,\eta} = \delta_{p,q}. \tag{77}$$

Naturally, our next task is the construction of a proper approximate eigenpair $\{t^h(\eta), \mathcal{U}^h(\eta)\}$. This will be based on the formal asymptotic analysis in Section 2.

3.3. **The global approximation of an eigenpair.** Let Λ^0_m be an eigenvalue of the limit problem (31), (32) with multiplicity $\varkappa = \varkappa_m$ and let the corresponding eigenvectors $U^0_{(m)}, \dots, U^0_{(m+\varkappa-1)}$ be subject to the normalization and orthogonality conditions (35). We take

$$t^h_m(\eta) = \big(1 + \Lambda^0_m + h^{\alpha_A - 1}\Lambda'_\ell(\eta)\big)^{-1} \tag{78}$$

as an approximate eigenvalue of the operator $\mathcal{T}^h(\eta)$ in (74). Here, $\ell = m, \dots, m + \varkappa - 1$ and $\Lambda'_\ell(\eta)$ is the correction term constructed in Section 2.1.

The approximate eigenvectors are defined in Ω^h as

$$u^{h\bullet}_{(\ell)} = U^0_{(\ell)}(x) + h^{\alpha_A - 1}U'_{(\ell)}(x). \tag{79}$$

However, the definition becomes much more complicated inside the thin frame Γ^h and it involves several smooth cut-off functions. First of all, we select a cut-off function whose support covers almost the whole cell:

$$\mathbf{X}^h = 1 \text{ in } \Omega^h, \quad \mathbf{X}^h = 0 \text{ in a neighborhood of } \partial\Omega^0; \tag{80}$$

$$0 \leqslant \mathbf{X}^h \leqslant 1, \quad |\nabla \mathbf{X}^h| \leqslant c_{\mathbf{X}} h^{-1}. \tag{81}$$

Then, the function $\chi^h_{j\pm} \in C^\infty(\overline{\Omega^0})$ is defined such that the support is contained in $\overline{\Gamma^h_{j\pm}}$ and

$$\chi^h_{j\pm}(x) = 1 \text{ for } |x_j| < l_j - 2\varrho h, \quad \chi^h_{j\pm}(x) = 0 \text{ for } |x_j| > l_j - \varrho h,$$

where $\varrho > 0$ is chosen such that $\mathrm{supp}\chi^h_{j\pm} \cap (\Gamma^h_{3-j-} \cup \Gamma^h_{3-j+}) = \varnothing$. Finally, $\mathcal{X}^h_1$ is supported in a small neighborhood $\mathcal{V}_1$ of the corner point x^1 of the rectangle Ω^0, vanishes in the vicinity of Σ^0 and equals $1 - \chi^h_{2-} - \chi^h_{2+}$ near the curve $\Sigma^h \cap \mathcal{V}_1$. The relations (81) hold true for all these cut-off-functions.

Completing the definition (78) of $u^h_{(\ell)}$, we set in Γ^h_{1+}

$$\begin{gathered} u^{h\circ}_{(\ell)}(x) = w^{h\circ}_{(\ell)}(x) + h^{\alpha_A - 1}u^{h\prime}_{(\ell)}(x), \\ w^{h\circ}_{(\ell)}(x) = (\mathcal{X}^h_1(x) + \mathcal{X}^h_2(x))U^0_{(\ell)}(x) + \chi^h_{j+}(x)W^1(h^{-1}(x_1 - l_1), x_2), \\ u^{h\prime}_{(\ell)}(x) = \mathbf{X}^h(x)U'_{(\ell)}(x). \end{gathered} \tag{82}$$

where $U'_{(\ell)}$ is the correction term in the ansatz (49), i.e. the solution of the problem (51), (52) with data (53). The definitions (82) can clearly be extended to Γ^h_{1-}, $\Gamma^h_{2\pm}$ and thus to the whole frame Γ^h.

Owing to the boundary conditions (41), relations (42) and the definitions of the cut-off functions, we conclude that (82) satisfies the quasi-periodicity conditions (21) and has the same trace on $\Sigma^h = \partial\Omega^h$ as the approximate solution (79).

We proceed to evaluate the scalar product $\langle u^h_{(\ell)}, u^h_{(n)}\rangle_{h,\eta}$, see (73). The inequality

$$|a^\bullet(u^{h\bullet}_{(\ell)}, u^{h\bullet}_{(n)}; \Omega^h) + b^\bullet(u^{h\bullet}_{(\ell)}, u^{h\bullet}_{(n)}; \Omega^h) - \delta_{\ell,n}(1+\Lambda^0_m)| \leqslant ch \tag{83}$$

follows from the formulas (79) for $u^{h\bullet}_{(\ell)}$, (35), (33) and (69) for the eigenfunction $U^0_{(\ell)}$ and (72) for the correction term $U'_{(\ell)}$ as well as the fact that area of Γ^h is of order h.

Now, to derive the formulas

$$\begin{gathered}|\langle u^{h\bullet}_{(\ell)}, u^{h\bullet}_{(n)}\rangle_{h,\eta} - \delta_{\ell,n}(1+\Lambda^0_m)| \leqslant ch^{\min\{1,\alpha_A-1\}},\\ \|u^{h\bullet}_{(\ell)}; \mathcal{H}(\eta)\| \geqslant c_m > 0,\end{gathered} \tag{84}$$

we recall (6) and write the estimate

$$|h^{\alpha_A} a^\circ(u^{h\circ}_{(\ell)}, u^{h\circ}_{(n)}; \Gamma^h) + h^{\alpha_B} b^\bullet(u^{h\circ}_{(\ell)}, u^{h\circ}_{(n)}; \Gamma^h)| \leqslant$$

$$\leqslant c(h^{\alpha_A}(1+(1+h^{\alpha_A-1}|\ln h|)h^{-1})^2 h^1 + h^{\alpha_B}h^1) \leqslant ch^{\alpha_A-1}.$$

Here, in the central expression, the factor h^{-1} comes from the differentiation of the cut-off functions and the boundary layer term (cf. (81) and (37)). The factor h^1 is the order of the area of Γ^h and $h^{\alpha_A-1}|\ln h|$ results from the gradient estimate (72) of $h^{\alpha_A-1}U'_{(\ell)}$. Notice that the exponent of h in the bound (84) equals $\alpha_A - 1$, owing to (9).

Let us estimate the value $\epsilon = \epsilon_\ell$ in (75) for the number (78) and the vector

$$\mathcal{U}^h_{(\ell)} = \|u^h_{(\ell)}; \mathcal{H}(\eta)\|^{-1} u^h_{(\ell)} \in \mathcal{H}(\eta). \tag{85}$$

We have

$$\begin{aligned}\epsilon_\ell &= \|\mathcal{T}^h\mathcal{U}^h_{(\ell)} - t^h_{(\ell)}\mathcal{U}^h_{(\ell)}; \mathcal{H}(\eta)\| = \sup|\langle \mathcal{T}^h\mathcal{U}^h_{(\ell)} - t^h_{(\ell)}\mathcal{U}^h_{(\ell)}, v^h\rangle_{h,\eta}| = \\ &= \|u^h_{(\ell)}; \mathcal{H}(\eta)\|^{-1} t^h_{(\ell)} \sup|a^h(u^h_{(\ell)}, v^h; \Omega^h) - (\Lambda^0_m + h^{\alpha_A-1}\Lambda'_\ell)b^h(u^h_{(\ell)}, v^h; \Omega^h)|\end{aligned} \tag{86}$$

where the supremum is calculated over the unit ball in $\mathcal{H}(\eta)$. Since the first two factors on the right-hand side of (86) are uniformly bounded in h, see (68) and (84), it suffices to consider the expression

$$\begin{aligned}a^h(u^h_{(\ell)}, v^h; \Omega^h) - (\Lambda^0_m + h^{\alpha_A-1}\Lambda'_\ell(\eta))b^h(u^h_{(\ell)}, v^h; \Omega^h)\\ = I^h_\bullet + I^h_A + I^h_B + I^{h\prime}_A + I^{h\prime}_B + I^h_\Sigma\end{aligned} \tag{87}$$

where

$$\begin{gathered}I^h_\bullet = ((L^\bullet - (\Lambda^0_m + h^{\alpha_A-1}\Lambda'_\ell(\eta)))(U^0_m + h^{\alpha_A-1}U'_{(\ell)}), v^{h\bullet})_{\Omega^h}\\ I^h_A = h^{\alpha_A}(L^\circ w^{h\circ}_{(\ell)}, v^{h\circ})_{\Gamma^h}, \quad I^h_B = -h^{\alpha_B}(\Lambda^0_m + h^{\alpha_A-1}\Lambda'_\ell(\eta)(B^\circ w^{h\circ}_{(\ell)}, v^{h\circ})_{\Gamma^h}\\ I^{h\prime}_A = h^{2\alpha_A-1}a^\circ(u^{h\prime}_{(\ell)}, v^{h\circ}; \Gamma^h),\ I^{h\prime}_B = h^{\alpha_B+\alpha_A-1}(\Lambda^0_m + h^{\alpha_A-1}\Lambda'_\ell(\eta))b^\circ(u^{h\prime}_{(\ell)}, v^{h\circ}; \Gamma^h)\\ I^h_\Sigma = (D(\nu^h)^\top(A^\bullet D(\nabla)u^{h\bullet}_{(\ell)} - h^{\alpha_A}A^\circ D(\nabla)w^{h\circ}_{(\ell)}), v^h)_{\Sigma^h}.\end{gathered} \tag{88}$$

For the first term $I^h_\bullet$ we have, according to (31) and (52),

$$|I^h_\bullet| = h^{2(\alpha_A-1)}|\Lambda'_\ell(\eta)(U'_{(\ell)}), v^{h\bullet})_{\Omega^h}| \leqslant ch^{2(\alpha_A-1)}. \tag{89}$$

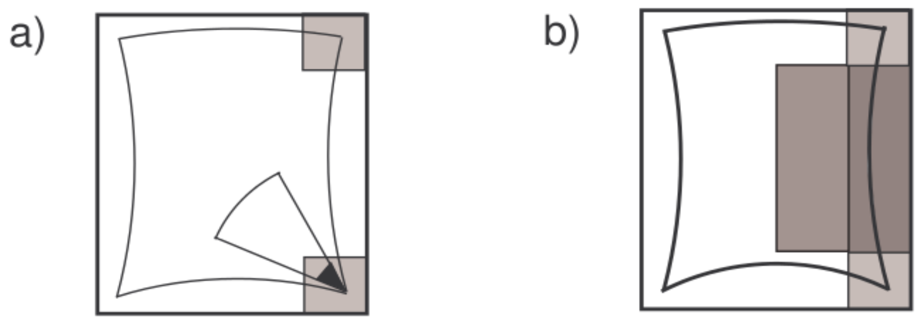

FIGURE 2. Additional geometric objects.

We will need some additional inequalities for the processing of the other terms in (88). To this end, we introduce eight geometric figures depicted in Fig. 3, namely four squares $\square_k^h$ attached to the vertices x^k $(k = 1, \dots, 4)$ having side lengths ϱh, and four rectangles $\Pi_{j\pm}^h$ with short and long sides of lengths ρh and $l_{3-j} - 2\varrho h$, respectively. Here, the coefficient $\varrho > 0$ of the parameter h is chosen such that the union of the eight sets contains Γ^h and a ch-neighborhood of $\Sigma^h = \partial\Omega^h$ for some $c > 0$ which we fix now. In the following we will use the symmetry to replace some estimates over the set Γ^h by corresponding estimates only over the rectangle Π_{1+}^h and the squares $\square_1^h$ and $\square_2^h$. We denote $\square_{1,2}^h = \Pi_{1+}^h \cup \square_1^h \cup \square_2^h$.

Lemma 3.4. *If $v^h \in \mathcal{H}(\eta)$, then there holds the inequalities*

$$\|v^h; L^2(\square_k^h)\| \leqslant ch\, h^{-\alpha_A/2}\|v^h; \mathcal{H}(\eta)\|, \tag{90}$$

$$\|v^h; L^2(\Pi_{j\pm}^h)\| \leqslant ch^{1/2}\, h^{-\alpha_A/2}\|v^h; \mathcal{H}(\eta)\|, \tag{91}$$

where c is independent of v^h and $h \in (0, h_\circ]$.

Proof. Let $\sphericalangle_1^h = \{x : \mathbf{r} \in (0, d_1), \phi \in (0, \phi_1)\} \subset \Omega^h$ be the sector which is shown in Fig. 2.a and which contains the small triangle $\blacktriangle_1^h$ (marked with black in Fig. 3.a) inside the square $\square_1^h$. The estimate

$$\|\mathbf{r}^{-1}|\ln\mathbf{r}|^{-1}v^h; L^2(\sphericalangle_1^h)\| \leqslant c\|v^h; H^1(\sphericalangle_1^h)\| \tag{92}$$

is a consequence of the classical one-dimensional Hardy inequality

$$\int_0^{\mathbf{d}} \mathbf{r}^{-2}|V^h(\mathbf{r})|^2 d\mathbf{r} \leqslant c \int_0^{\mathbf{d}} \left(\left|\frac{dV^h}{d\mathbf{r}}(\mathbf{r})\right| + |V^h(\mathbf{r})|^2\right)\mathbf{r}d\mathbf{r}$$

integrated in the angular variable ϕ. Taking into account the weight factor on the left-hand side of (92), we can write

$$h^{-2}\|v^h; L^2(\blacktriangle_1^h)\|^2 \leqslant c|\ln h|^2\|v^h; H^1(\Omega^h)\|^2$$

Then, we apply the Poincaré ineqality

$$h^{-2}\|v^h; L^2(\square_1^h)\|^2 \leqslant c(\|\nabla v^h; L^2(\square_1^h)\|^2 + h^{-2}\|v^h; L^2(\blacktriangle_1^h)\|^2)$$

which can be easily derived by a coordinate dilation, and obtain the desired inequality (90) as follows:

$$h^{-2}\|v^h; L^2(\square_1^h)\|^2 \leqslant c(a^\circ(v^h, v^h; \Gamma^h) + |\ln h|^2(a^\bullet(v^h, v^h; \Omega^h) + b^\bullet(v^h, v^h; \Omega^h)) \leqslant$$
$$\leqslant c\max\{h^{-\alpha_A}, |\ln h|^2\}\,\|v^h; \mathcal{H}(\eta)\|^2 = ch^{-\alpha_A}\|v^h; \mathcal{H}(\eta)\|^2.$$

To derive (91), we employ the Newton–Leibnitz formula and write

$$v^h(l_1-\varrho h+t,x_2)=v^h(0,x_2)+\int_0^{l_1-\varrho h+t}\frac{\partial v^h}{\partial x_1}(x_1,x_2)\,dx_1.$$

We estimate

$$\int_{l_1-\varrho h}^{l_1}|v^h(t,x_2)|^2dt\leqslant 2\int_{l_1-\varrho h}^{l_1}|v^h(0,x_2)|^2dx+2l_1\int_{l_1-\varrho h}^{l_1}\int_0^{l_1}\left|\frac{\partial v^h}{\partial x_1}(x_1,x_2)\right|^2dx_1dt\leqslant$$

$$\leqslant 2\varrho h|v^h(0,x_2)|^2+2l_1\varrho h\int_0^{l_1}\left|\frac{\partial v^h}{\partial x_1}(x-1,x_2)\right|^2dx_1.$$

It suffices to integrate x_2 over the interval $(-l_2+\varrho h,l_2-\varrho h)$ and to apply the standard trace inequality. □

Since $\|v^h;\mathcal{H}(\eta)\|=1$ in (86), we can use in the following calculations the fact that the left-hand sides of (90) and (91) are uniformly bounded in h. Moreover, we denote by $I_A^h(\square)$ and $I_B^h(\square)$ the quantities where the inner products of the expressions I_A^h and I_B^h, respectively, are taken over the set $\square_{1,2}^h\cap\Gamma^h$ instead of Γ^h. For $I_A^h(\square)$ we obtain the bound

$$\begin{gathered}|I_A^h(\square)|\leqslant ch^{\alpha_A}((\|\nabla^2U^0_{(m)};L^2(\square^h_{1,2})\|+h^{-1}\|\nabla U^0_{(m)};L^2(\square^h_{1,2})\|+\\+h^{-2}\|U^0_{(m)};L^2(\square^h_{1,2})\|)\,\|v^h;L^2(\square^h_{1,2})\|+\|L^0(\chi_{1+}W^1);L^2(\Pi^h_{1+})\|\,\|v^h;L^2(\gamma^h_{1+})\|)\leqslant\\\leqslant ch^{\alpha_A}((h^{\delta-1}+h^{-1}+h^{-2})h^1hh^{-\alpha_A/2}+(h^{1/2}h^{-1}h^{1/2}+h^1h^{-2}h^1h)h^{-\alpha_A/2})\\\leqslant ch^{\alpha_A/2}.\end{gathered}\tag{93}$$

Here, the norms $h^{2-p}\|\nabla^pU^0_{(m)};L^2(\square^h_{1,2})\|$, $p=0,1,2$, came from the differentiation of the product $\mathcal{X}_k^hU^0_{(\ell)}$ and (81); they were estimated by using (69) and taking into account the area $O(h^2)$ of $\square_k^h$. In a similar way we used (40) and (43), (44), (69) to get

$$\begin{gathered}L^0(\chi_{1+}W^1)=\chi_{1+}L^0W^1+[L^0,\chi_{1+}]W^1,\\|\chi_{1+}L^0W^1|\leqslant ch^{-1},\ \operatorname{supp}(\chi_{1+}L^0W^1)\subset\gamma^h_{1+}\cup\square^h_{1,2},\\|[L^0,\chi_{1+}]W^1|\leqslant ch^{-2},\ \operatorname{supp}([L^0,\chi_{1+}]W^1)\subset\square^h_{1,2}.\end{gathered}$$

Finally, inequalities (90) and (91) for v^h together with bounds for the areas of the above-mentioned supports were used to complete the estimate (93). A much simpler consideration yields the estimate

$$\begin{gathered}|I_B^h(\square)|\leqslant ch^{\alpha_B}(\|U^0_{(m)};L^2(\square^h_{1,2})\|\,\|v^h;L^2(\square^h_{1,2})\|+\|W^1;L^2(\gamma^h_{1+}\|\,\|v^h;L^2(\gamma^h_{1+})\|)\leqslant\\\leqslant ch^{\alpha_B}(h^1h+h^{1/2}h^{1/2})h^{-\alpha_A/2}\leqslant ch^{1+\alpha_B-\alpha_A/2}.\end{gathered}\tag{94}$$

Notice that the exponents of the bounds in (93) and (94) are included in the formula (60).

In view of symmetry, the same estimates hold when $\square^h_{1,2}$ is replaced by the unions of the other rectangles $\square^h_k$ and $\Pi^h_{j\pm}$ (see the explanation above Lemma 3.4). Since the frame Γ^h is covered by these sets, we obtain

$$|I_A^h|\leqslant ch^{\alpha_A/2}\ ,\quad |I_B^h|\leqslant ch^{1+\alpha_B-\alpha_A/2}.\tag{95}$$

To treat the fourth and fifth terms $I_A^{h\prime}$ and $I_B^{h\prime}$ on the right-hand side of (87), we apply formulas (72) and the basic estimates following directly from definitions (73) and (6),

$$\|\nabla v^h; L^2(\Gamma^h)\| \leqslant ch^{-\alpha_A/2}, \quad \|v^h; L^2(\Gamma^h)\| \leqslant ch^{-\alpha_B/2}.$$

Accordingly, we obtain

$$|I_A'| \leqslant ch^{2\alpha_A-1}(h^{-1}\|U_{(\ell)}'; L^2(\Gamma^h)\| + \|\nabla U_{(\ell)}'; L^2(\Gamma^h)\|)\,\|\nabla v^h; L^2(\Gamma^h)\| \leqslant$$

$$\leqslant ch^{2\alpha_A-1}((h^{-1}h^{1/2} + |\ln h|h^{1/2})h^{-\alpha_A/2} \leqslant ch^{3(\alpha_A-1)/2}, \tag{96}$$

$$|I_B'| \leqslant ch^{\alpha_B+\alpha_A-1}h^{1/2}h^{-\alpha_B/2} \leqslant ch^{\alpha_A-1+(\alpha_B+1)/2},$$

Exponents in both bounds are included in (60).

Let us consider the last term I_Σ^h in (88). Here, our assumption on the smoothness properties in Section 3.1 plays an important role. Since the vector function $U_{(\ell)}^0 \in C^2(\Omega^0)$ satisfies the homogeneous Neumann conditions (32), we have

$$\begin{gathered} |D(\nabla)U_{(\ell)}^0(x)| \leqslant ch,\ x \in \Sigma^h, \\ |(D(\nu^h)^\top A^\bullet D(\nabla)U_{(\ell)}^0, v^h)_{\Sigma^h}| \leqslant ch\|v^h; L^2(\Sigma^h)\| \leqslant ch\|v^h; H^1(\Omega^h)\| \\ \leqslant ch\|v^h; \mathcal{H}(\eta)\| \leqslant ch. \end{gathered} \tag{97}$$

Furthermore, the boundary condition (52), (53), formula (46) for the normal vector ν^h and the relations (72) for $U_{(\ell)}'$ imply that, for $x \in \Sigma^h$,

$$|D(\nu^h)^\top A^\bullet D(\nabla)U_{(\ell)}'(x) - D(\nu^h)^\top A^\circ D(\nabla)W^1(H^+(x_2), x_2)| \leqslant chr^{-1} \tag{98}$$

where the last singular factor r^{-1} is caused by the second derivatives of $U_{(\ell)}'$. Now we use the known weighted trace inequality

$$\|\mathbf{r}^{-1/2}|\ln \mathbf{r}|^{-1}v^h; L^2(\Sigma^h)\| \leqslant c(\|\nabla v^h; L^2(\Omega^h)\| + \|\mathbf{r}^{-1}|\ln \mathbf{r}|^{-1}v^h; L^2(\Omega^h)\|),$$

cf. (92) and (97), (98), to obtain

$$|I_\Sigma^h| \leqslant c\Big(h + h^{\alpha_A-1}h\int\limits_{\Sigma^h}\mathbf{r}^{-1}|v^h(x)|\,ds\Big) \leqslant ch(1 + h^{\alpha_A-1}\|(\mathbf{r}+h)^{-1}v^h; L^2(\Sigma^h)\|) \leqslant$$

$$\leqslant ch(1 + h^{\alpha_A-1}|\ln h|) \leqslant ch^{3(\alpha_A-1)/2}, \tag{99}$$

where we also applied the inequalities $r \geqslant \mathbf{r} + ch$, $c > 0$ on Σ^h, see Fig. 2.a, and $h|\ln h| \leqslant ch^{(\alpha_A-1)/2}$, due to (9).

Combining the estimates (89), (95), (96), (99) and recalling the definition (60) yield for the number (86) the estimates

$$\epsilon_\ell \leqslant c_m h^{\alpha_A-1+\delta_{AB}}, \quad \ell = m, \ldots, m + \varkappa_m - 1, \tag{100}$$

which according to Lemma 3.3 means that the operator $\mathcal{T}^h(\eta)$ has an eigenvalue $\tau_{n(\ell)}^h(\eta)$ related to the "almost eigenvalues" (78) by

$$|\tau_{n(\ell)}^h(\eta) - t_\ell^h(\eta)| \leqslant c_\ell h^{\alpha_A-1+\delta_{AB}}.$$

If all eigenvalues (57) of the matrix $M^n(\eta)$ are simple, in particular, Λ_m^0 is a simple eigenvalue in the sequence (34), then the distance of any two points

$t^h_m(\eta), \dots, t^h_{m+\varkappa_m-1}(\eta)$ is at least $Ch^{\alpha_A-1+\delta_{AB}}$, and thus there exist $\varkappa_m$ different eigenvalues $\tau^h_{n(m)}(\eta), \dots, \tau^h_{n(m+\varkappa_m-1)}(\eta)$. However, in the case

$$\Lambda'_{\mathbf{m}-1}(\eta) < \Lambda'_{\mathbf{m}}(\eta) = \cdots = \Lambda'_{\mathbf{m}+\mathbf{x}-1}(\eta) < \Lambda'_{\mathbf{m}+\mathbf{x}}(\eta)$$

with $\mathbf{x} > 1$, Lemma 3.3 would not prevent that some of the eigenvalues $\tau^h_{n(\mathbf{m})}(\eta), \dots, \tau^h_{n(\mathbf{m}+\mathbf{x}-1)}(\eta)$ might coincide. We show that they can be chosen to be different from each other.

Taking into account the second assertion in Lemma 3.3, we set $\epsilon = \max\{\epsilon_{\mathbf{m}}, \dots, \epsilon_{\mathbf{m}+\mathbf{x}-1}\}$ and

$$\epsilon_* = T\epsilon, \tag{101}$$

where T is fixed to be large enough, and denote by $\tau^h_M, \dots, \tau^h_{M+X-1}$ all eigenvalues of the operator $\mathcal{T}^h$ in the interval

$$\upsilon^h_m = [t^h_{\mathbf{m}}(\eta) - \epsilon_*, t^h_{\mathbf{m}}(\eta) + \epsilon_*]. \tag{102}$$

For each $\ell = \mathbf{m}, \dots, \mathbf{m}+\mathbf{x}-1$, Lemma 3.3 gives a column of coefficients $a^h_{(\ell)} = (a^h_{M\ell}, \dots, a^h_{M+X-1\ell})$ such that the relations (76) are valid for the vector function (85). We denote by $\mathcal{S}^h_\ell$ the linear combination

$$\mathcal{S}^h_\ell = a^h_{M\ell} U^h_M + \cdots + a^h_{M+X-1\ell} U^h_{M+X-1}$$

and write

$$\langle \mathcal{U}^h_\ell, \mathcal{U}^h_p \rangle - \delta_{\ell,p} = \langle \mathcal{U}^h_\ell - \mathcal{S}^h_\ell, \mathcal{U}^h_p \rangle + \langle \mathcal{S}^h_\ell, \mathcal{U}^h_p - \mathcal{S}^h_p \rangle + \langle \mathcal{S}^h_\ell, \mathcal{S}^h_p \rangle - \delta_{\ell,p}.$$

Inequalities (84), (76), (100) and conditions (77) yield

$$\begin{aligned} |\overline{a^h_{(p)}} \cdot a^h_{(\ell)} - \delta_{\ell,p}| &= |\langle \mathcal{S}^h_\ell, \mathcal{S}^h_p \rangle - \delta_{\ell,p}| \leqslant c(\epsilon + \epsilon + \epsilon\epsilon_*^{-1}) \\ &\leqslant c(h^{\alpha_A-1}\max\{c_{\mathbf{m}}, \dots, c_{\mathbf{m}+\mathbf{x}-1}\} + T^{-1}). \end{aligned}$$

In other words, the columns $a^h_{\mathbf{m}}, \dots, a^h_{\mathbf{m}+\mathbf{x}-1} \in \mathbb{C}^X$ are almost orthonormalized for small h and big T. The latter situation can only happen in the case $X \geqslant \mathbf{x}$, when the interval (102) contains at least $\mathbf{x}$ eigenvalues of the operator $\mathcal{T}^h(\eta)$ which meet the estimate

$$|\tau^h_{n+q}(\eta) - t^h_{\mathbf{m}}(\eta)| \leqslant T c_m h^{\alpha_A-1+\delta_{AB}}, \quad q = 1, \dots, \mathbf{x}, \tag{103}$$

see (101), (102), (100). Definition (78) and relation (74) between the spectral parameters, turn formula (103) into the desired inequality (59).

Finally, the assertion on the equality of n and m in Theorem 2.1 follows from a standard convergence theorem which we formulate as the next lemma and prove in the next section.

Lemma 3.5. *The entries of the eigenvalue sequencies* (19) *and* (34) *are related by* $\Lambda^h_m \to \Lambda^0_m$, *as* $h \to +0$.

3.4. **Completion of the proof of Theorem 2.1: The convergence result.** To prove Lemma 3.5, let $\{\Lambda^h_m(\eta), U^h_m(\cdot;\eta)\}$ be an eigenpair of the problem (20) for some $\eta \in \mathbb{Y}$. We fix this Floquet parameter and suppress it in the notation from now on. In Section 3.3 it was proved that in the vicinity of each eigenvalue Λ^0_p, $p = 1, \dots, m$, of the limit problem (33) there exists an eigenvalue $\Lambda^h_{n(p)}$ of the problem (20) and $n(p_1) \neq n(p_2)$ for $p_1 \neq p_2$, and we have

$$\Lambda^h_m \leqslant \Lambda^h_{n(m)} \leqslant \Lambda^0_m + c_m h^{\alpha_A-1} \leqslant C_m.$$

We normalize the eigenvector $U^h_{(m)}$ by

$$b^h(U^h_{(m)}, U^h_{(m)}; \Omega^0) = 1.$$

The integral identity (20) and formulas (6), (7) yield the implication

$$a^h(U^h_{(m)}, U^h_{(m)}; \Omega^0) \leqslant c_m \;\Rightarrow\; \|U^h_{(m)}; H^1(\Omega(\varrho))\| \leqslant c^\bullet_m, \tag{104}$$

where the number $\varrho > 0$ is chosen such that the rectangular domain

$$\Omega(\varrho) = \{x \in \Omega^0 : |x_j| < l_j(1 - \varrho h))\}$$

is contained in the domain Ω^h.

According to (104), the vector function

$$\Omega^0 \ni x \;\mapsto\; \mathbf{U}^h(x) = U^h_{(m)}((1 - \varrho h)^{-1} x) \tag{105}$$

has a uniformly bounded $H^1(\Omega^0)$-norm and, hence, there exists a positive sequence $\{h_n\}_{n\in\mathbb{N}}$ tending to 0 such that

$$\begin{gathered} \Lambda^{h_n}_m \to \Lambda^0, \\ \mathbf{U}^{h_n} \to \mathbf{U}^0 \text{ weakly in } H^1(\Omega^0) \text{ and stronly in } L^2(\Omega^0) \text{ as } n \to +\infty. \end{gathered} \tag{106}$$

We take an arbitrary test function $\mathbf{V} \in C^\infty(\overline{\Omega^0})$, extend it smoothly outside the square Ω^0 and insert into the integral identity (20) the function

$$V^h = \mathbf{X}^h \mathbf{V}^h, \tag{107}$$

where $\mathbf{V}^h(x) = \mathbf{V}((1 - \varrho h)^{-1} x)$. Notice that the function (107) vanishes in the vicinity of $\partial\Omega^0$, cf. (80), and therefore it belongs to $\mathcal{H}(\eta)$.

We have

$$\begin{gathered} 0 = a^h(U^h_{(m)}, V^h; \Omega^0) - \Lambda^h_m b^h(U^h_{(m)}, V^h; \Omega^0) = \\ = a^\bullet(U^h_{(m)}, V^h; \Omega(\varrho)) - \Lambda^h_m b^\bullet(U^h_{(m)}, V^h; \Omega(\varrho)) + \\ + a^\bullet(U^h_{(m)}, V^h; \Omega^0 \setminus \Omega(\varrho)) - \Lambda^h_m b^\bullet(U^h_{(m)}, V^h; \Omega^0 \setminus \Omega(\varrho)). \end{gathered} \tag{108}$$

Using (105) and (107), (80) we observe that

$$a^\bullet(U^h_{(m)}, V^h; \Omega(\varrho)) = \int\limits_{\Omega(\varrho)} \overline{D(\nabla_x)\mathbf{U}^h((1 - \varrho h)^{-1} x)}^\top A^\bullet D(\nabla_x)\mathbf{V}^h((1 - \varrho h)^{-1} x)\, dx =$$

$$= a^\bullet(\mathbf{U}^h, \mathbf{V}; \Omega^0) \;\to\; a^\bullet(\mathbf{U}^0, \mathbf{V}; \Omega^0),$$

$$\Lambda^h b^\bullet(U^h, V^h; \Omega(\varrho)) = \Lambda^h(1 - \varrho h)^{-2} b^\bullet(\mathbf{U}^h, \mathbf{V}^h; \Omega^0) \;\to\; \Lambda^0 b^\bullet(\mathbf{U}^0, \mathbf{V}; \Omega^0).$$

To process the remaining terms in (108) we write

$$\begin{gathered} |a^h(U^h, V^h; \Omega^0 \setminus \Omega(\varrho))| \leqslant c(\|\nabla_x U^h; L^2(\Omega^h \setminus \Omega(\varrho))\|\,\|\nabla_x \mathbf{V}^h; L^2(\Omega^h \setminus \Omega(\varrho))\| + \\ + h^{\alpha_A}\|\nabla_x U^h; L^2(\Gamma^h)\|\,\|\nabla_x \mathbf{V}^h; L^2(\Gamma^h)\|) \leqslant c(\|\nabla_x U^h; L^2(\Omega^h \setminus \Omega(\varrho))\| h^{1/2} + \\ + h^{\alpha_A/2}\|\nabla_x U^h; L^2(\Gamma^h)\| h^{\alpha_A/2} h^{1/2}(1 + h^{-1})) \leqslant c(h^{1/2} + h^{-1+(1+\alpha_A)/2}) \;\to\; 0, \\ \Lambda^h|b^h(U^h, V^h; \Omega^0 \setminus \Omega(kh))| \leqslant c(\|U^h; L^2(\Omega^h \setminus \Omega(kh))\|\,\|\mathbf{V}^h; L^2(\Omega^h \setminus \Omega(kh))\| + \\ + h^{\alpha_B}\|U^h; L^2(\Gamma^h)\|\,\|\mathbf{V}^h; L^2(\Gamma^h)\|) \leqslant c(h^{1/2} + h^{(\alpha_B+1)/2}) \;\to\; 0. \end{gathered}$$

Thus, passing to the limit $h_n \to +0$ yields the integral identity for the limit problem,

$$a^\bullet(\mathbf{U}^0, \mathbf{V}; \Omega^0) - \Lambda^0 b^\bullet(\mathbf{U}^0, \mathbf{V}; \Omega^0) = 0 \quad \forall\, \mathbf{V} \in C^\infty(\overline{\Omega^0}).$$

By a completion argument, the test function space can be changed here to be $H^1(\Omega)^J$.

Hence, to conclude that $\{\Lambda^0, \mathbf{U}^0\}$ is an eigenpair of the problem (31), (32), it is sufficient to verify that $\mathbf{U}^0$ is non-zero. To this end we write

$$1 = b^h(U^h_{(m)}, U^h_{(m)}; \Omega^0) =$$

$$= b^\bullet(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Omega(\varrho)) + b^\bullet(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Omega^h \setminus \Omega(\varrho)) + h^{\alpha_B} b^\circ(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Gamma^h). \tag{109}$$

Lemma 3.4 gives estimates for the last two terms:

$$b^\bullet(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Omega^h \setminus \Omega(\varrho)) \leqslant c^\bullet h \|U^{h\bullet}_{(m)}; H^1(\Omega^h)\|^2 \leqslant C^\bullet h \|U^h_{(m)}; \mathcal{H}\|^2 = C^\bullet h,$$

$$h^{\alpha_B} b^\circ(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Gamma^h) \leqslant h^{\alpha_B + 1 - \alpha_A} \|U^h_{(m)}; \mathcal{H}\|^2$$

These the upper bounds tend to 0 as $h \to +0$, see (9). Recalling (105) yields

$$b^\bullet(U^{h\bullet}_{(m)}, U^{h\bullet}_{(m)}; \Omega(\varrho)) = (1 - \varrho h)^{-2} b^\bullet(\mathbf{U}^h, \mathbf{U}^h; \Omega^0),$$

and we thus obtain $b^\bullet(\mathbf{U}^0, \mathbf{U}^0; \Omega^0) = 1$ by passing to the limit $h \to +0$ in (109).

Now Lemma 3.5 can be proved in a standard way. Namely, if U^h and U^h_* are two different eigenvectors of the problem (20) and they are orthogonal in the sense that $b^h(U^h, U^h_*) = 0$, see (22), then the orthogonality $b^\bullet(\mathbf{U}^0, \mathbf{U}^0_*) = 0$ follows from the above calculations. In this way, supposing $n > m$ would contradict our way to compose the eigenvalue sequences (19) and (34). But we recall that the inequality $n \geqslant m$ was already verified in Section 3.3. Thus, the identity $n = m$ holds, and this completes the proofs of Lemma 3.5 as well as Theorem 2.1.

4. Some examples and generalizations.

4.1. Concrete formulas.

Let us derive exactly the correction terms in the eigenvalue asymptotics (59) for the problems mentioned in Examples 1.1 and 1.2.

Example 4.1. Let $L^\bullet = -\Delta$ and $L^\circ = -\Delta$, cf. Example 1.1, and let $1/2 = l_1 > l_2$. Then

$$\Lambda^0_1 = 0, \quad U^0_{(1)} = (2l_2)^{-1/2},$$

$$\Lambda^0_2 = \pi^2, \quad U^0_{(2)} = l_2^{-1/2} \sin(\pi x_1).$$

The asymptotic formulas (59), (54) and the definitions (55), (45) show that

$$\Lambda^h_1(\eta) = 0 + h^{\alpha_A - 1} \sum_{j=1,2} \frac{1}{l_j}(1 - \cos(2\pi \eta_j l_j)) \int_{-l_{3-j}}^{l_{3-j}} \frac{dy}{H_j(y)} + O(h^{\alpha_A - 1 + \delta_{AB}}),$$

$$\Lambda^h_2(\eta) = \pi^2 + h^{\alpha_A - 1} \frac{2}{l_2} \Bigg((1 + \cos(2\pi \eta_1 l_1)) \int_{-l_2}^{l_2} \frac{dx_2}{H_1(x_2)} +$$

$$+ (1 - \cos(2\pi \eta_2 l_2)) \int_{-l_1}^{l_1} \sin^2(\pi x_1) \frac{dx_1}{H_2(x_1)} \Bigg) + O(h^{\alpha_A - 1 + \delta_{AB}}).$$

Example 4.2. Assuming that the frame Γ^h is made of a homogeneous isotropic material, we have the following 3×3-matrix

$$A^\circ = \begin{pmatrix} \lambda_\circ + 2\mu_\circ & \lambda_\circ & 0 \\ \lambda_\circ & \lambda_\circ + 2\mu_\circ & 0 \\ 0 & 0 & 2\mu_\circ \end{pmatrix},$$

where $\lambda_\circ \geqslant 0$ and $\mu_\circ > 0$ are the Lamé constants, cf. Example 1.2. We choose the orthonormalized basis

$$U^0_{(j)}(x) = \frac{1}{2}\,(l_1 l_2)^{-1/2}\, e_j,\ \ j = 1,2,$$

$$U^0_3(x) = \left(\frac{3}{4}\right)^{1/2} (l_1 l_2)^{-1/2}\,(l_1^2 + l_2^2)^{-1/2}(x_2 e_1 - x_1 e_2)$$

in the polynomial space $\mathcal{P}$ of rigid motions.

We have

$$\Lambda^h_1(\theta) = 0 + \frac{h^{\alpha_A - 1}}{2l_1 l_2}\Bigg((\lambda + 2\mu)\,(1 - \cos(2\pi l_1 \eta_1) \int_{-l_2}^{l_2} \frac{dx_2}{H_1(x_2)} +$$

$$+\mu\,(1 - \cos(2\pi l_2 \eta_2) \int_{-l_1}^{l_1} \frac{dx_1}{H_2(x_1)}\Bigg) + O(h^{\alpha_A - 1 + \delta_{AB}}),$$

$$\Lambda^h_2(\theta) = 0 + \frac{h^{\alpha_A - 1}}{2l_1 l_2}\Bigg(\mu\,(1 - \cos(2\pi l_1 \eta_1) \int_{-l_2}^{l_2} \frac{dx_2}{H_1(x_2)} +$$

$$+(\lambda + 2\mu)\,(1 - \cos(2\pi l_2 \eta_2) \int_{-l_1}^{l_1} \frac{dx_1}{H_2(x_1)}\Bigg) + O(h^{\alpha_A - 1 + \delta_{AB}}),$$

$$\Lambda^h_3(\theta) = 0 + \frac{4h^{\alpha_A - 1}}{3l_1 l_2(l_1^2 + l_2^2)}\Bigg(\int_{-l_2}^{l_2} \Big((\lambda + 2\mu)\,(1 - \cos(2\pi l_1 \eta_1) x_2^2 +$$

$$+\mu\,(1 + \cos(2\pi l_1 \eta_1) l_1^2\Big) \frac{dx_2}{H_1(x_2)} +$$

$$+ \int_{-l_1}^{l_1} \Big((\lambda + 2\mu)(1 - \cos(2\pi l_2 \eta_2) x_1^2 + \mu(1 + \cos(2\pi l_2 \eta_2) l_2^2\Big) \frac{dx_1}{H_2(x_1)}\Bigg) + O(h^{\alpha_A - 1 + \delta_{AB}}).$$

4.2. **Singularities at corner points.** The next two examples show that the restrictions introduced in Section 3.1 are relevant in certain problems of mathematical physics.

Example 4.3. An appropriate affine transform converts the problem (31), (32) into the spectral Neumann problem for the Laplace operator in a parallelogram $\lozenge$ with angles $\phi \in (0, \pi/2]$ and $\pi - \phi \in [\pi/2, \pi)$. As known for example by [21, Ch. 2], the worst singularity of an eigenfunction of this problem in $\lozenge$ is

$$r^{\pi/(\pi - \phi)} \cos\frac{\pi\varphi}{\pi - \phi}, \quad \frac{\pi\varphi}{\pi - \phi} = 1 + \delta_0, \quad \delta_0 = \frac{\phi}{\pi - \phi} > 0.$$

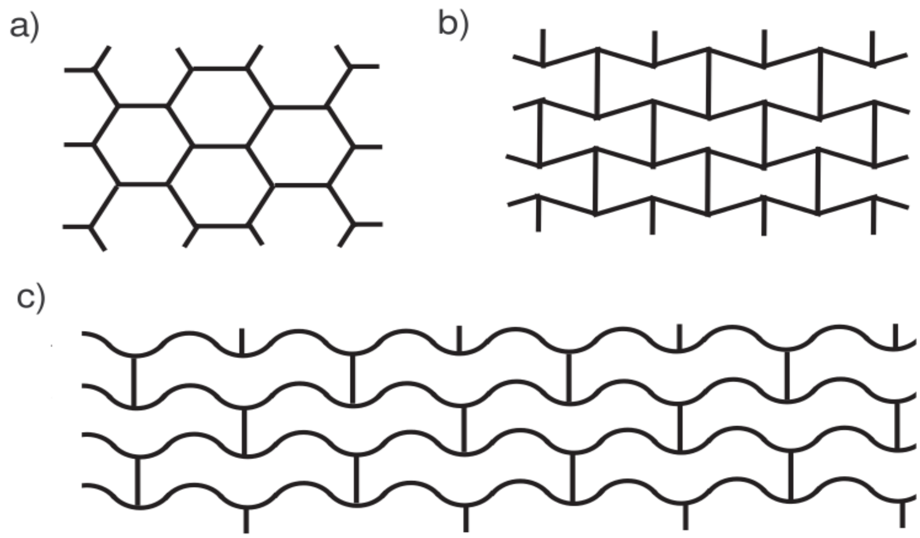

FIGURE 3. Pavements of different shapes.

Example 4.4. According to [13], an affine transform can be used to reduce the stationary elasticity problem (31), (32) with $\Lambda^0 = 0$, in Ω^0, to the particular case of an orthotropic elastic parallelogram with the elastic symmetry axis of rank 4. This means that the rigidity matrix A is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad a_{11} = a_{22}.$$

Singularities at corner points for such orthotropic materials have been computed in, e.g., [26]. However, the inequality $\mu > 1$ for the positive singularity exponents in (69) at the tops of the convex angles of the parallelogram has been proved in [9] so that the elasticity problem in Example 1.2 also satisfies our assumption (68).

4.3. **Possible geometric generalizations.** Until now we have restricted ourselves to two-dimensional problems, in order to simplify formulas and the justification scheme in Section 3. However, our formal asymptotic analysis would apply also in the multi-dimensional cases without notable changes. Namely, the periodicity cell $\Omega^0 = \{x \in \mathbb{R}^d : |x_j| < l_j, j = 1, \dots, d\}$, $d \geqslant 3$, can composed of the curved parallelepiped

$$\Omega^h = \{x \in \Omega^0 : -l_j + hH_j^-(x'_{(j)}) < x_j < l_j - hH_j^+(x'_{(j)}), j = 1, \dots, d\} \qquad (110)$$

and the surrounding box-shaped frame $\Gamma^h = \Omega^0 \setminus \Omega^h$. Here, the notation is similar to that in (1) except that $x'_{(j)} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$. On the other hand, stating smoothness properties of the eigenvectors, which we used in Section 3, would become much more complicated in higher dimensions, due to the many edges and corners of the boundary of the curved parallelepiped (110). Thus, the justification of the asymptotics might require some additional assumptions.

Other shapes of the periodicity cells like the honeycomb structure in Fig. 3.a, can be used to cover the plane, and they can be treated with the same asymptotic tools. Another example of a non-rectangular tiling with the periodicity cell in Fig. 3.b requires a modification of our asymptotic procedure, because of the inward obtuse angle. However, the main difficulty is caused in formulas (43)–(45) by the strong corner singularities on the short sides of the curved rectangle Γ_1^{h+}. The resulting strengthening of the singularities of the solutions may seriously reduce the accuracy of our asymptotic formulas (cf. Section 3.1 and the error estimates in Theorems 2.1 and 2.2).

The influence of these corner singularities may be compensated by constructing two-dimensional boundary layers (cf. [4, 18] for the Poisson equation and [19] for general elliptic problems and the elasticity system). It should be mentioned that using the same scheme as in Section 2.2 one can up to some extend find higher order asymptotic terms in Ω^h and in $\Gamma^h_{j\pm}$ outside small neighborhoods of the vertices of Ω^0, but it is not possible to specify them completely without the two-dimensional boundary layers. This was the very reason for introducing the assumption (9) which helps to avoid the problem.

It looks that near concave corner points the required, acceptable approximation can be achieved by constructing two dimensional boundary layers (cf. [19]).

There is no obstacle to treat strongly curved thin frames in the geometric situation of Fig. 3.c.

4.4. **Periodic piezoelectric composites.**

Example 4.5. Let $J = 3$, $N = 5$ and

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_2 \end{pmatrix} = \begin{pmatrix} D^{\mathsf{M}}(\nabla)^\top & \mathbb{O}_{2\times 2} \\ \mathbb{O}_{1\times 3} & \nabla^\top \end{pmatrix}, \tag{111}$$

where $D^{\mathsf{M}}(\nabla)^\top$ is the matrix (23) and $\mathbb{O}_{m\times n}$ is the null matrix of size $m \times n$. Furthermore, let

$$A^\bullet = \begin{pmatrix} A^{\mathsf{MM}} & A^{\mathsf{ME}} \\ A^{\mathsf{EM}} & -A^{\mathsf{EE}} \end{pmatrix}, \quad B^\bullet = \begin{pmatrix} b^\bullet & 0 & 0 \\ 0 & b^\bullet & 0 \\ 0 & 0 & 0 \end{pmatrix},\ b^\bullet > 0. \tag{112}$$

Here, A^{MM} and A^{EE} are symmetric and positive definite matrices of elastic and di-electric moduli, respectively, while there is no restriction on the piezoelectric matrix $A^{\mathsf{ME}} = \left(A^{\mathsf{EM}}\right)^\top$ of size 3×2. Finally, $u = (u^{\mathsf{M}}, u^{\mathsf{E}})$, $u^{\mathsf{M}} = (u_1, u_2)$ is the displacement vector and u^{E} is the electric potential.

Due to the minus sign of A^{EE}, the symmetric matrix $A^\bullet$ is not positive definite and therefore the Hermitian sesquilinear form A^h in (7) does not satisfy condition (11). However, thanks to the right lower null entry of the matrix $B^\bullet$, (112), and the Dirichlet condition

$$u^{\mathsf{E}h}(x) = 0, \quad x \in \Sigma^h, \tag{113}$$

on the insulator surface, all necessary conditions are actually satisfied and we conclude in particular that the space of polynomials $\mathcal{P}$ can be chosen to be

$$\mathcal{P} = \left\{p = (p^{\mathsf{M}}, p^{\mathsf{E}}) \,:\, p^{\mathsf{M}} \in \mathcal{P}^{\mathsf{M}},\ p^{\mathsf{E}} = 0\right\}, \tag{114}$$

where $\mathcal{P}^{\mathsf{M}}$ is the space (24) of mechanical rigid motions.

We consider the composite plane $\Omega^h_\infty \cup \Gamma^h_\infty$, where Ω^h_∞ is the union over $\theta \in \mathbb{Z}^2$ of the identical piezoelectric inclusions $\Omega^h(\theta)$ (cf. (2)), which are connected by thin paddings of pure elastic solid insulator Γ^h_∞. The boundary value problem consists of the system of differential equations

$$D(-\nabla)^\top A^\bullet D(\nabla)u^{\bullet h}(x) = \lambda^h B^\bullet u^{\bullet h}(x)\ , \quad x \in \Omega^h_\infty, \tag{115}$$

$$D^{\mathsf{M}}(-\nabla)^\top A^\circ D^{\mathsf{M}}(\nabla)u^{\circ h}(x) = \lambda^h B^\circ u^{\circ h}(x)\ , \quad x \in \Gamma^h_\infty, \tag{116}$$

the transmission conditions

$$D^{\mathsf{M}}(\nu^h(x))^\top A^\bullet D(\nabla)u^{\bullet h}(x) = D^{\mathsf{M}}(\nu^h(x))^\top A^\circ D(\nabla)u^{\circ h}(x),\ x \in \Sigma^h_\infty \tag{117}$$

and the Dirichlet condition (113) on the union Σ^h_∞ of the contours $\Sigma^h(\theta) = \partial\Omega^h(\theta)$, $\theta \in \mathbb{Z}^2$. The notation in (116) and on the right-hand side of (117) is the same as in Example 1.2. We emphasize that condition (117) means that the traction is continuous on the contact surface. Moreover, (113) describes the fact that the electric potential is constant on the surface of the insulator, and this constant can be set to zero because the set $\overline{\Gamma^h_\infty} = \cup_{\theta\in\mathbb{Z}^2}\overline{\Gamma^h(\theta)}$ is connected.

The limit problem in the rectangle $\Omega^\circ = (-\ell_1, \ell_1) \times (-\ell_2, \ell_2)$ consists of the system (31) including the matrices (111) and (112), the quasiperiodicity condition (21), (30) and the boundary conditions, cf. (32) and (117), (113), and

$$D^{\mathsf{M}}(-\nabla)^\top A^\bullet D(\nabla)U^\circ(x) = 0, \quad U^{\mathsf{E}\circ}(x) = 0, \quad x \in \Sigma^\circ. \tag{118}$$

Although the matrix $A^\bullet$ in (112) is not positive definite, the spectrum of the problem is discrete and consists of the monotone non-negative unbounded sequence (34). For example in the paper [23] one can find a procedure for reducing the weak formulation of this piezoelectricity problem to a problem with a positive self-adjoint operator in $H^1(\Omega^\circ) \ni U^{\mathsf{M}\circ}$; the reduction uses the specific structure of the matrix $B^\bullet$ in (112). The corresponding eigenfunctions $U^\circ_{(1)}, U^\circ_{(2)}, \ldots \in H^1(\Omega)^\circ)^3$ can be subject to the normalization and orthogonality conditions (35), which read as

$$\big(b^\bullet U^{\mathsf{M}\circ}_{(m)}, U^{\mathsf{M}\circ}_{(p)}\big)_{\Omega^\circ} = \delta_{m,p}, \quad m,p \in \mathbb{N},$$

with the positive constant $b^\bullet$ of (112).

Since the material in Γ^h is purely elastic, the forms of the second limit problem (40)–(42) and its solution (43) are kept unchanged. Thus, our calculation of the correction term in the eigenvalue ansatz (45) does not need modifications, and also the final formulas (56), (57) remain unchanged, if the formulas (48) and (55) are understood as

$$A^\circ_{(j)} = \overline{D^{\mathsf{M}}(e_{(j)})}^\top A^\circ D^{\mathsf{M}}(e_{(j)}),$$

$$\mathcal{J}(U^\circ, V^\circ; \eta) = \sum_{j=1,2} \int_{-\ell_{3-j}}^{\ell_{3-j}} H_j(y)^{-1} \overline{\big[V^{\mathsf{M}\circ}\big]_j(y;\eta_j)}^\top A^\circ_{(j)} \big[V^{\mathsf{M}\circ}\big]_j(y;\eta_j)dy$$

with the notation (45) preserved as such.

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