Universitat Autònoma de Barcelona

Dipòsit digital de documents de la UAB

This is the **accepted version** of the journal article:

Martín i Pedret, Joaquim; Milman, Mario. «Integral isoperimetric transference and dimensionless Sobolev inequalities». Revista Matematica Complutense, Vol. 28 (May 2015), p. 359-392. DOI 10.1007/s13163-014-0153-7

This version is available at https://ddd.uab.cat/record/268403

under the terms of the IN COPYRIGHT license

INTEGRAL ISOPERIMETRIC TRANSFERENCE AND DIMENSIONLESS SOBOLEV INEQUALITIES

JOAQUIM MARTÍN* AND MARIO MILMAN**

Abstract. We introduce the concept of Gaussian integral isoperimetric transferеnce and show how it can be applied to obtain a new class of sharp Sobolev-Poincaré inequalities with constants independent of the dimension. In the special case of L^q spaces on the unit $n-$dimensional cube our results extend the recent inequalities that were obtained in [12] using extrapolation.

1. Introduction

Let Q_n be the open unit cube in $\mathbb{R}^n$, let $1 \leq q < n$ be fixed, and let $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$; a special case of the classical (homogeneous) Sobolev inequality states that for all $f \in C_0^\infty(Q_n)$,

$$\|f\|_{L^p(Q_n)} \leq c(n,q) \|\nabla f\|_{L^q(Q_n)}.$$

It is well known (cf. [1], [29]) that $c(n,q) \asymp c(q)n^{-\frac{1}{2}}$, with $c(q)$ independent of n. Triebel [30], [31], recently suggested the problem of finding dimension free[1] Sobolev inequalities, at least in what concerns the constants involved, by means of replacing the power gain of integrability on the left hand side by a (smaller) logarithmic gain. This, indeed, can be achieved by different methods (e.g. by transference from Gaussian inequalities via isoperimetry and symmetrization (cf. [19]), by direct transference from Log Sobolev inequalities [17], by extrapolation, using weighted norms inequalities (cf. [17], [18]), etc.). The sharper known results in this direction give inequalities of the form

$$\|f\|_{L^q(LogL)^{q/2}(Q_n)} \leq C(q) \|\nabla f\|_{L^q(Q_n)}, \; f \in C_0^\infty(Q_n), \tag{1.1}$$

with $C(q)$ independent of the dimension, and a corresponding non-homogeneous version: for $f \in W^{1,1}(Q_n)$, we have

$$\|f\|_{L^q(LogL)^{q/2}(Q_n)} \leq C(q) \left(\|f\|_{L^q(Q_n)} + \|\nabla f\|_{L^q(Q_n)} \right). \tag{1.2}$$

The proof via Gaussian isoperimetric transference obtained in [19] explains the presence of the factor $\frac{1}{2}$ in the logarithmic exponent. To summarize, sacrificing the

---

1991 *Mathematics Subject Classification.* 2000 Mathematics Subject Classification Primary: 46E30, 26D10.

Key words and phrases. Sobolev inequalities, symmetrization, isoperimetric inequalities, extrapolation.

*Partially supported in part by Grants MTM2010-14946, MTM-2010-16232.

**This work was partially supported by a grant from the Simons Foundation (#207929 to Mario Milman).

This paper is in final form and no version of it will be submitted for publication elsewhere.

[1] Dealing with a large number of variables (high dimensionality) gives rise to a set of specific problems and issues in analysis, e.g. in approximation theory, numerical analysis, optimization.. (cf. [14]).

power gain of integrability of the classical Sobolev inequality for the log Sobolev gain of integrability associated with Gaussian measure, we obtain Sobolev inequalities on the unit cube Q_n with constants independent of the dimension.

It is not hard to see that the logarithmic inequalities (1.2), (1.1), give the optimal results within the class of $L^q(LogL)^r$ spaces. However, Fiorenza-Krbec-Schmeisser [12] have recently shown that, on the larger class of rearrangement invariant spaces, the optimal inequality with dimensionless constants corresponding to (1.1) is given by

$$\|f\|_{L_{(q,\frac{q'}{2}}(Q_n)} \leq c(q)\,\|\nabla f\|_{L^q(Q_n)}\,,\ f\in C_0^\infty(Q_n), \tag{1.3}$$

where the space $L_{(q,\frac{q'}{2}}(Q_n)$ is the so called 'small' Lebesgue space introduced[2] by Fiorenza [10]. The space $L_{(q,\frac{q'}{2}}(Q_n)$ is defined by means of the following norm

$$\|f\|_{L_{(q,q'}(Q_n)} = \inf_{f=\sum f_j}\left(\sum_j \inf_{0<\varepsilon<q'-1}\varepsilon^{-\frac{q'/2}{q'-\varepsilon}}\,\|f_j\|_{L^{(q'-\varepsilon)'}(Q_n)}\right), \tag{1.4}$$

where as usual $q'=q/(q-1)$. Moreover, $L_{(q,\frac{q'}{2}}(Q_n)$ can be characterized as an extrapolation space in the sense of Karadzhov-Milman [16]; therefore, its norm can be computed explicitly (cf. [11]),

$$\|f\|_{L_{(q,q'}(Q_n)} \approx \int_0^1\left(\int_0^t f^*(s)^q ds\right)^{1/q}\frac{dt}{t(\log\frac{1}{t})^{\frac{1}{2}}}. \tag{1.5}$$

As usual, the symbol $f\approx g$ will indicate the existence of a universal constant $C>0$ (independent of all parameters involved) so that $(1/C)f\leq g\leq C\,f$, while the symbol $f\preceq g$ means that for a suitable constant C, $f\leq C\,g$, and likewise $f\succeq g$ means that $f\geq Cg$.

It is not difficult to see that (cf. [12], see also (1.11) below)

$$L_{(q,q'}(Q_n)\subsetneq L^q(LogL)^{q/2}(Q_n).$$

The proof of the inequality (1.3) given in [12] depends on extrapolation and is accomplished using (1.4) or (1.5), and the explicit form of the Sobolev embedding constant.

In this paper we investigate the connection of inequalities of the form (1.3) to the isoperimetric inequality and show a new associated transference principle. We work on metric probability spaces and study a class of Sobolev inequalities which include (1.3), which are valid if the isoperimetric profile satisfies a suitable integrability condition. In particular, inequalities of the form (1.3) are connected with what could be termed a Gaussian transfer condition, as we now explain.

Underlying our method are certain pointwise rearrangement inequalities for Sobolev functions which are associated with the isoperimetric profile of a given geometry (cf. [19], [20]). Using these pointwise inequalities we will obtain, by integral transference, inequalities that are stronger than the usual transferred (log) Sobolev inequalities, while still preserving the dimensionless of the constants involved. More generally, we are able to give a unified approach to a class of dimensionless inequalities for different geometries, that hold within the class of general rearrangement

[2]As the dual of the 'grand' Lebesgue spaces, introduced by Iwaniec and Sbordone [15].

invariant spaces, and as we shall show elsewhere, encompass fractional[3] inequalities as well.

To discuss the results of the paper in more detail it will be useful to recall first a version of the transference principle developed in [19]. It was shown in [19] that, for a large class of connected metric probability spaces (Ω, d, μ), with associated isoperimetric profile[4] $I = I_{(\Omega,\mu)}$ and for rearrangement invariant spaces $\bar{X} = \bar{X}(0,1)$ on $(0,1)$, that in a suitable technical sense are 'away' from L^1, we have the following Sobolev-Poincaré inequality: For all Lipchitz functions on Ω,

$$\left\| (f^{**}(t) - f^*(t)) \frac{I(t)}{t} \right\|_{\bar{X}} \leq c \left\| |\nabla f|^* \right\|_{\bar{X}}, \tag{1.6}$$

where f^* is the non-increasing rearrangement[5] of f, $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, and c is a universal constant that depends only on $\bar{X}$.

The inequality (1.6) is best possible[6], but the isoperimetric profile involved may depend on the dimension, and thus there could be a dimensional dependency in the constants. A natural method to obtain dimensionless inequalities is to weaken the norm inequality (1.6) via transference. For example, if (Ω, d, μ) is of "Gaussian isoperimetric type", i.e. if for some universal constant independent of the dimension, it holds

$$I_{(\Omega,\mu)}(t) \succeq t\left(\log\frac{1}{t}\right)^{\frac{1}{2}}, \text{ on } \left(0, \frac{1}{2}\right); \tag{1.7}$$

then the Gaussian log Sobolev inequalities can be transferred to (Ω, d, μ) with constants independent of the dimension. The method of [19] simply expresses the fact that from (1.6) and (1.7) we can see that for all r.i. spaces away from L^1, we have

$$\left\| (f^{**}(t) - f^*(t)) \left(\log\frac{1}{t}\right)^{\frac{1}{2}} \right\|_{\bar{X}} \leq c \left\| |\nabla f|^* \right\|_{\bar{X}}. \tag{1.8}$$

In particular, the Euclidean unit $n-$dimensional cube Q_n is of Gaussian type, with constant 1 (cf. [28]), therefore (1.8) for $\bar{X} = L^q$ gives (1.2). More generally, these inequalities can be reformulated as

$$\|(f^{**}(t) - f^*(t))\, G_\infty(t)\|_{\bar{X}} \leq c \left\| G_\infty \frac{t}{I(t)} \right\|_{L^\infty(0,\frac{1}{2})} \left\| |\nabla f|^* \right\|_{\bar{X}},$$

where (1.8) corresponds to the choice $G_\infty(t) = (\log\frac{1}{t})^{\frac{1}{2}}$.

The inequality (1.6) was proved under the assumption that $\frac{I(t)}{t}$ decreases, an assumption we shall keep in this paper. To proceed further we note that the left

[3]The basic fractional inequalities that underlie our analysis were obtained in [20].

[4]See Section 2 below; in particular we assume that $I(t)$ is concave and symmetric about 1/2.

[5]Precise definitions and properties of rearrangements and related topics coming into play in this section are contained in Section 2

[6]as far as the condition on the left hand side (or target space).

hand side of (1.6) can be minorized as follows,

$$\begin{aligned}\left\|(f^{**}(\cdot)-f^{*}(\cdot))\,\frac{I(\cdot)}{(\cdot)}\right\|_{\bar{X}} &\geq \left\|(f^{**}(\cdot)-f^{*}(\cdot))\,\chi_{(0,t)}(\cdot)\frac{I(\cdot)}{(\cdot)}\right\|_{\bar{X}} \\ &\geq \left\|(f^{**}(\cdot)-f^{*}(\cdot))\,\chi_{(0,t)}(\cdot)\right\|_{\bar{X}}\,\frac{I(t)}{t}.\end{aligned}$$

Therefore, we have

$$\left\|(f^{**}(\cdot)-f^{*}(\cdot))\,\chi_{(0,t)}(\cdot)\right\|_{\bar{X}} \leq c\frac{t}{I(t)}\left\||\nabla f|^{*}\right\|_{\bar{X}}. \tag{1.9}$$

Now, if G is such that $\left(\int_0^1 G(t)\frac{t}{I(t)}dt\right) < \infty$, it follows immediately from (1.9) that

$$\int_0^1 \left\|(f^{**}(\cdot)-f^{*}(\cdot))\,\chi_{(0,t)}(\cdot)\right\|_{\bar{X}} G(t)dt \leq C\left(\int_0^1 G(t)\frac{t}{I(t)}dt\right)\left\||\nabla f|^{*}\right\|_{\bar{X}}.$$

For example, for $G(t)=\frac{1}{t\left(\log\frac{1}{t}\right)^{\frac{1}{2}}}$, we are able to control the functional

$$\int_0^1 \left\|(f^{**}(s)-f^{*}(s))\,\chi_{(0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\log\frac{1}{t}\right)^{\frac{1}{2}}},$$

as long as the following (stronger) Gaussian isoperimetric transference condition is satisfied,

$$\int_0^1 \frac{dt}{I(t)(\log\frac{1}{t})^{\frac{1}{2}}} < \infty. \tag{1.10}$$

To compare the different results let us note that by the triangle inequality we have

$$\begin{aligned} I &= \int_0^1 \left\|(f^{**}(\cdot)-f^{*}(\cdot))\chi_{(0,t)}(\cdot)\right\|_{\bar{X}}\frac{dt}{t\left(\log\frac{1}{t}\right)^{1/2}} \\ &= \int_0^1 \left\|(f^{**}(\cdot)-f^{*}(\cdot))\chi_{(0,t)}(\cdot)\frac{1}{t\left(\log\frac{1}{t}\right)^{1/2}}\right\|_{\bar{X}} dt \\ &= \int_0^1 \|h(\cdot,t)\|_{\bar{X}}\,dt,\ \text{ where } h(s,t)=(f^{**}(s)-f^{*}(s))\chi_{(0,t)}(s)\frac{1}{t\left(\log\frac{1}{t}\right)} \\ &\geq \left\|\int_0^1 h(s,t)dt\right\|_{\bar{X}} \\ &= \left\|(f^{**}(\cdot)-f^{*}(\cdot))\int_s^1 \frac{1}{t\left(\log\frac{1}{t}\right)^{1/2}}dt\right\|_{\bar{X}} \\ &\geq 2\left\|(f^{**}(\cdot)-f^{*}(\cdot))\left(\log(\frac{1}{\cdot})\right)^{1/2}\right\|_{\bar{X}} \end{aligned}$$

Thus, if (1.10) holds, we have[7]

$$\left\|(f^{**}(\cdot)-f^{*}(\cdot))\left(\log(\frac{1}{\cdot})\right)^{1/2}\right\|_{\bar{X}} \leq c\int_0^1 \left\|(f^{**}(s)-f^{*}(s))\,\chi_{(0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\log\frac{1}{t}\right)^{\frac{1}{2}}}$$

$$\leq c\left(\int_0^1\frac{dt}{I(t)(\log\frac{1}{t})^{\frac{1}{2}}}\right)\left\||\nabla f|^{*}\right\|_{\bar{X}}. \tag{1.11}$$

At this point we can explain what could be gained from our efforts. For suitable domains, e.g. for the Euclidean unit cubes Q_n, or, more generally, for some other classes of $n-$Euclidean domains, the corresponding integral conditions can be estimated by a constant independent of the dimension as follows

$$\sup_n\int_0^1\frac{dt}{I_n(t)(\log\frac{1}{t})^{\frac{1}{2}}}<\infty. \tag{1.12}$$

When (1.12) holds the resulting inequalities we have thus obtained are both dimensionless and stronger than the ones that could be derived via the 'pointwise' Gaussian transference condition.

The restriction that the space $\bar{X}$ must be 'away from L^1' can be removed using the generalized Pólya-Szegö principle of [19]: Under the assumption that $\bar{X}$ is 'away from L^∞', we can replace (1.9) by (cf. Theorem 2 below),

$$\left\|(f^{*}(\cdot)-f^{*}(t))\,\chi_{[0,t)}(\cdot)\right\|_{\bar{X}}\leq c\frac{t}{I(t)}\left\|Q\right\|_{\bar{X}\to\bar{X}}\left\||\nabla f|^{*}\right\|_{\bar{X}},$$

where $Qg(t)=\int_t^1 g(s)\frac{ds}{s}$, and from this point the analysis can proceed along the lines outlined above.

We will also show a partial converse of this result, for example, the inequality

$$\int_0^1\left\|(f^{*}(\cdot)-f^{*}(t))\,\chi_{[0,t)}(\cdot)\right\|_{L^1}\frac{dt}{t(\log\frac{1}{t})^{1/2}}\leq C\left\||\nabla f|^{*}\right\|_{L^1}, \tag{1.13}$$

implies that the profile of (Ω,d,μ) must satisfy a Gaussian type condition (cf. Corollary 1 below)

$$\frac{t}{I(t)}\int_t^1\frac{dt}{t(\log\frac{1}{t})^{\frac{1}{2}}}\leq C.$$

The paper is organized as follows. In Section 2 we collect the necessary information concerning symmetrizations, isoperimetric profiles and function spaces considered in this paper; the main inequalities of this paper are proved in Section 3, while Section 4 contains applications to different geometries[8]; in particular, in Subsection 4.1 we show in detail how our approach, in the special case of the unit cubes Q_n and L^q spaces, yields (1.3).

2. Background

2.1. **Rearrangements.** Let (Ω,d,μ) be a Borel probability metric space. For measurable functions $u:\Omega\to\mathbb{R}$, the distribution function of u is given by

$$\mu_u(t)=\mu\{x\in\Omega:u(x)>t\}\qquad(t\in\mathbb{R}).$$

[7]which of course is still weaker than the optimal inequality (1.6).

[8]For further possible metric measure spaces were one could consider applications of our method we refer to [24] and the very recent [25].

The **decreasing rearrangement**[9] of a function u is the right-continuous non-increasing function from $[0,1)$ into $\mathbb{R}$ which is equimeasurable with u. It can be defined by the formula

$$u_\mu^*(s)=\inf\{t\geq 0:\mu_u(t)\leq s\},\quad s\in[0,1),$$

and satisfies

$$\mu_u(t)=\mu\{x\in\Omega:u(x)>t\}=m\left\{s\in[0,1):u_\mu^*(s)>t\right\}\ ,\ t\in\mathbb{R},$$

where m denotes the Lebesgue measure on $[0,1)$.

The maximal average $u_\mu^{**}(t)$ is defined by

$$u_\mu^{**}(t)=\frac{1}{t}\int_0^t u_\mu^*(s)ds=\frac{1}{t}\sup\left\{\int_E u(s)d\mu:\mu(E)=t\right\},\ t>0.$$

It follows directly from the definition that $(u+v)_\mu^{**}(t)\leq u_\mu^{**}(t)+v_\mu^{**}(t)$, moreover, since u_μ^* is decreasing, it follows that u_μ^{**} is also decreasing, and $u_\mu^*\leq u_\mu^{**}$.

When the probability we are working with is clear from the context, or when we are dealing with Lebesgue measure, we may simply write u^* and u^{**}, etc.

2.2. **Isoperimetry.** In what follows we always assume that we work with connected Borel probability metric spaces (Ω,d,μ), which we shall simply refer to as "measure probability metric spaces".

Recall that for a Borel set $A\subset\Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$P(A;\Omega)=\liminf_{h\to 0}\frac{\mu\left(A_h\right)-\mu\left(A\right)}{h},$$

where $A_h=\{x\in\Omega:d(x,A)<h\}$.

The **isoperimetric profile** is defined by

$$I_\Omega(s)=I_{(\Omega,d,\mu)}(s)=\inf\left\{P(A;\Omega):\ \mu(A)=s\right\},$$

i.e. $I_{(\Omega,d,\mu)}:[0,1]\to[0,\infty)$ is the pointwise maximal function such that

$$P(A;\Omega)\geq I_\Omega(\mu(A)),$$

holds for all Borel sets A. Again, when no confusion arises, we shall drop the subindex Ω and simply write I.

We will always assume that, for the probability metric spaces (Ω,d,μ) under consideration, the associated isoperimetric profile I_Ω satisfies that, $I(0)=0$, I is continuous, concave and symmetric about $\frac{1}{2}$. In many cases it is enough to control an 'isoperimetric estimator', i.e. a function $J:[0,\frac{1}{2}]\to[0,\infty)$ with the same properties as I and such that

$$I_\Omega(t)\geq J(t),\quad t\in(0,1/2).$$

For a Lipschitz function f on Ω (briefly $f\in Lip(\Omega)$) we define the **modulus of the gradient** by[10]

$$|\nabla f(x)|=\limsup_{d(x,y)\to 0}\frac{|f(x)-f(y)|}{d(x,y)}.$$

[9] Note that this notation is somewhat unconventional. In the literature it is common to denote the decreasing rearrangement of $|u|$ by u_μ^*, while here it is denoted by $|u_\mu|^*$ since we need to distinguish between the rearrangements of u and $|u|$. In particular, the rearrangement of u can be negative. We refer the reader to [27] and the references quoted therein for a complete treatment.

[10] In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω,d) (cf. [6, pp. 2, 3] for more details).

We shall now summarize some useful inequalities that relate the isoperimetry with the rearrangements of Lipschitz functions; for more details we refer to [19] and [21].

Theorem 1. *The following statements are equivalent*

(1) *Isoperimetric inequality:* $\forall A\subset\Omega$, *Borel set with*

$$P(A;\Omega)\geq I(\mu(A)),$$

(2) *Oscillation inequality:* $\forall f\in Lip(\Omega)$,

$$(f_{\mu}^{**}(t)-f_{\mu}^{*}(t))\frac{I(t)}{t}\leq\frac{1}{t}\int_{0}^{t}\left|\nabla f\right|_{\mu}^{*}(s)ds,\quad 0<t<1. \tag{2.1}$$

(3) *Pólya-Szegö inequality:* $\forall f\in Lip(\Omega)$, f_{μ}^{*} *is locally absolutely continuous and satisfies*

$$\int_{0}^{t}\left(\left(-f_{\mu}^{*}\right)'(\cdot)I(\cdot)\right)^{*}(s)ds\leq\int_{0}^{t}\left|\nabla f\right|_{\mu}^{*}(s)ds,\qquad 0<t<1. \tag{2.2}$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure on $[0,1)$*).*

Remark 1. *Note that* $f\in Lip(\Omega)$ *implies that* $|f|\in Lip(\Omega)$ *and* $|\nabla|f||\leq|\nabla f|$, *consequently, (2.1) and (2.2) hold for* $|f|$.

2.3. **Rearrangement invariant spaces.** We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces, and refer the reader to [5] for a complete treatment.

Let (Ω,μ) be a probability measure space. Let $X=X(\Omega)$ be a Banach function space on (Ω,μ), with the Fatou property[11]. We shall say that X is a **rearrangement-invariant** (r.i.) space, if $g\in X$ implies that all $\mu-$measurable functions f with $|f|_{\mu}^{*}=|g|_{\mu}^{*}$ also belong to X and, moreover, $\|f\|_{X}=\|g\|_{X}$. For any r.i. space $X(\Omega)$ we have

$$L^{\infty}(\Omega)\subset X(\Omega)\subset L^{1}(\Omega),$$

with continuous embeddings. Typical examples of r.i. spaces are the L^{p}-spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, etc.

The associated space $X'(\Omega)$ is the r.i. space defined by the following norm

$$\|h\|_{X'(\Omega)}=\sup_{g\neq0}\frac{\int_{\Omega}|g(x)h(x)|\,d\mu}{\|g\|_{X(\Omega)}}=\sup_{g\neq0}\frac{\int_{0}^{\mu(\Omega)}|g|_{\mu}^{*}(s)\,|h|_{\mu}^{*}(s)ds}{\|g\|_{X(\Omega)}}.$$

In particular, the following **generalized Hölder's inequality** holds

$$\int_{\Omega}|g(x)h(x)|\,d\mu\leq\|g\|_{X(\Omega)}\,\|h\|_{X'(\Omega)}\,.$$

Let $X(\Omega)$ be a r.i. space, then there exists a **unique** r.i. space (the **representation space** of $X(\Omega)$), $\bar{X}=\bar{X}(0,1)$ on $((0,1),m)$, (where m denotes the Lebesgue measure on the interval $(0,1)$) such that

$$\|f\|_{X(\Omega)}=\|\,|f|_{\mu}^{*}\,\|_{\bar{X}(0,1)};$$

[11]This means that if $f_n\geq0$, and $f_n\uparrow f$, then $\|f_n\|_X\uparrow\|f\|_X$ (i.e. Fatou's Lemma holds in the X norm).

and
$$X'(\Omega)=\bar{X}'(0,1).$$
For example, for $1\leq p<\infty$,
$$\|f\|_{L^p(\Omega)}=\left(\int_\Omega |f|^p(x)d\mu\right)^{1/p}=\left(\int_0^1\left(|f|_\mu^*(s)\right)^p ds\right)^{1/p}=\|\,|f|_\mu^*\,\|_{\bar{L}^p(0,1)},$$
and
$$\|f\|_{L^\infty(\Omega)}=ess\sup|f(z)|=|f|_\mu^*(0^+)=\|\,|f|_\mu^{**}\,\|_{\bar{L}^\infty(0,1)}.$$
If $\bar{Y}(0,1)$ is a r.i. space $((0,\mu(\Omega)),m)$, then defining
$$\|f\|_{Y(\Omega)}:=\|\,|f_\mu|^*\,\|_{\bar{Y}(0,1)}$$
we obtain a r.i. space on (Ω,μ), in fact there is a one-to-one correspondence between r.i. spaces on (Ω,μ) and those over $((0,1),m)$. In what follows if there is no possible confusion we shall use X or $\bar{X}$ without warning.

The following majorization principle holds for r.i. spaces: if
$$\int_0^r |f|_\mu^*(s)ds\leq\int_0^r|g|_\mu^*(s)ds, \tag{2.3}$$
holds for all $r>0$, then, for any r.i. space $\bar{X}$,
$$\left\||f|_\mu^*\right\|_{\bar{X}}\leq\left\||g|_\mu^*\right\|_{\bar{X}}.$$
Remark 2. *The following variant of the majorization principle holds. Suppose that (2.3) holds, then for all $t>0$,*
$$\|\,|f|_\mu^*(\cdot)\chi_{[0,t)}(\cdot)\|_{\bar{X}}\leq\|\,|g|_\mu^*(\cdot)\chi_{[0,t)}(\cdot)\|_{\bar{X}}.$$
In fact,
$$\int_0^r|f|_\mu^*(s)\chi_{[0,t)}(s)ds=\int_0^{\min\{t,r\}}|f|_\mu^*(s)\leq\int_0^{\min\{t,r\}}|g|_\mu^*(s)ds=\int_0^r|g|_\mu^*(s)\chi_{[0,t)}(s)ds,$$
and we conclude using the majorization principle above.

The **fundamental function** of $\bar{X}$ is defined by
$$\phi_{\bar{X}}(s)=\left\|\chi_{[0,s]}\right\|_{\bar{X}},\quad 0\leq s\leq 1,$$
We can assume without loss of generality that $\phi_{\bar{X}}$ is concave (cf. [5]). Moreover, for all $s\in(0,1)$ we have
$$\phi_{\bar{X}'}(s)\phi_{\bar{X}}(s)=s.$$
For example, if $1\leq p<\infty, 1\leq q\leq\infty$, and we let $\bar{X}=L^p$ or $\bar{X}=L^{p,q}$ (Lorentz space), then $\phi_{L^p}(t)=\phi_{L^{p,q}}(t)=t^{1/p}$, moreover, $\phi_{L^\infty}(t)\equiv 1$. If N is a Young's function, then the fundamental function of the Orlicz space $\bar{X}=L_N$ is given by $\phi_{L_N}(t)=1/N^{-1}(1/t)$.

The Lorentz $\Lambda(\bar{X})$ space and the Marcinkiewicz space $M(\bar{X})$ associated with $\bar{X}$ are defined by the quasi-norms
$$\|f\|_{M(\bar{X})}=\sup_t|f|_\mu^*(t)\phi_{\bar{X}}(t),\qquad \|f\|_{\Lambda(\bar{X})}=\int_0^1|f|_\mu^*(t)d\phi_{\bar{X}}(t).$$
Notice that
$$\phi_{M(\bar{X})}(t)=\phi_{\Lambda(\bar{X})}(t)=\phi_{\bar{X}}(t),$$
and, moreover,
$$\Lambda(\bar{X})\subset\bar{X}\subset M(\bar{X}).$$

2.4. **Boyd indices and extrapolation spaces.** The Hardy operators are defined by[12]

$$Pf(t)=\frac{1}{t}\int_0^t f(s)ds;\qquad Q_af(t)=\frac{1}{t^a}\int_t^1 s^af(s)\frac{ds}{s}\quad (0\leq a<1).$$

The boundedness of the Hardy operators on a r.i. space $\bar{X}$ can be formulated in terms of conditions on the so called **Boyd indices**[13]

$$\bar{\alpha}_{\bar{X}}=\inf_{r>1}\frac{\ln h_{\bar{X}}(r)}{\ln r}\quad\text{and}\quad \underline{\alpha}_{\bar{X}}=\sup_{r<1}\frac{\ln h_{\bar{X}}(r)}{\ln r},$$

where $h_{\bar{X}}(r)$ denotes the norm of the compression/dilation operator E_s on $\bar{X}$, defined for $s>0$, by

$$E_rf(t)=\begin{cases} f^*(\frac{t}{r}) & 0<t<r,\\ 0 & r\leq t.\end{cases}\tag{2.4}$$

The operator E_s is bounded on every r.i. space $\bar{X}$, moreover,

$$h_X(r)\leq\max\{1,r\},\text{ for all } s>0.$$

For example, if $\bar{X}=L^p$, then $\overline{\alpha}_{L^p}=\underline{\alpha}_{L^p}=\frac{1}{p}$.

We have the following well known fact (cf. [7]):

$$\begin{array}{c} P \text{ is bounded on } \bar{X}\ \Leftrightarrow \overline{\alpha}_{\bar{X}}<1,\\ Q_a \text{ is bounded on } \bar{X}\ \Leftrightarrow \underline{\alpha}_{\bar{X}}>a.\end{array}\tag{2.5}$$

Moreover,

$$\|Q_a\|:=\|Q_a\|_{\bar{X}\to\bar{X}}\leq\int_1^\infty h_{\bar{X}}(\frac{1}{s})s^{\frac{1}{a}-1}ds<\infty.\tag{2.6}$$

The following extrapolation spaces, introduced by Fiorenza [10] and Fiorenza and Karadzhov [11] in the special case of L^p spaces[14], will play an important role in this paper.

Definition 1. *Let $\bar{X}$ be a r.i. space, and $k\in\mathbb{N}$. We let $\bar{X}_{k,\log}$ be the r.i. space defined by*

$$\bar{X}_{k,\log}=\left\{f:\|f\|_{\bar{X}_{k,\log}}:=\int_0^1\left\||f|^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{1-k/2}}<\infty\right\}.$$

It can be easily verified that

$$\|f\|_{\bar{X}_{k,\log}}\approx\int_0^1\left\||f|^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}}.$$

We now briefly indicate how the $\bar{X}_{k,\log}$ spaces can be identified with real interpolation/extrapolation spaces of the form (cf. [4])

$$(\bar{X},L^\infty)_{w_k,1}=\left\{f:\|f\|_{(\bar{X},L^\infty)_{w_k,1}}=\int_0^1K(t,f;\bar{X},L^\infty)w_k(t)dt<\infty\right\},$$

---

[12] where if $a=0$ we simply let $Q:=Q_0$

[13] Introduced by D.W. Boyd in [7].

[14] For a discussion of the extrapolation properties of a more general class of spaces we refer to [2].

where the $K-$functional, $K(t,f;\bar{X},L^{\infty})$, is defined (cf. [4]) by

$$K(t,f;\bar{X},L^{\infty})=\inf_{f=f_0+f_1}\{\|f_0\|_{\bar{X}}+t\,\|f_1\|_{L^{\infty}}\},$$

and

$$w_k(t)=\frac{\left(\phi_{\bar{X}}^{-1}(t)\right)'}{\phi_{\bar{X}}^{-1}(t)\left(1+\ln(\frac{1}{\phi_{\bar{X}}^{-1}(t)})\right)^{1-k/2}}.$$

This identification follows readily from the well known formula (cf.[26])

$$K(t,f;\bar{X},L^{\infty})\approx\left\||f|^{*}\chi_{(0,\phi_{\bar{X}}^{-1}(t))}\right\|_{\bar{X}}.$$

For example, if $\bar{X}=L^{p}$, and $k=1$, then

$$(L^{p},L^{\infty})_{w_1,1}=L_{(p,p'}.$$

This characterization simplifies a number of calculations with these spaces.

Proposition 1. *Let $\bar{X}$ be a r.i. space on Ω, and $k\in\mathbb{N}$. Then,*
(i)

$$\bar{X}_{k+1,\log}\subset\bar{X}_{k,\log}\subset\bar{X}. \tag{2.7}$$

(ii) If $\overline{\alpha}_{\bar{X}}<1$, then $\overline{\alpha}_{\bar{X}_{k,\log}}<1$.
(ii) If for some $r>0$, we have $\underline{\alpha}_{\bar{X}}>r\Rightarrow\underline{\alpha}_{\bar{X}_{k,\log}}>r$.

Proof. (i) The first inclusion is obvious. To prove the second inclusion we observe that the identity operator maps

$$I:\bar{X}\rightarrow\bar{X},\text{ and }I:L^{\infty}\rightarrow\bar{X},$$

thus, by interpolation,

$$I:\bar{X}_{k,\log}\rightarrow(\bar{X},\bar{X})_{w_k,1}=\bar{X}.$$

(ii) By (2.5) we need to prove that if $\overline{\alpha}_{\bar{X}}<1$, then $P:\bar{X}_{k,\log}\rightarrow\bar{X}_{k,\log}$. But P is bounded on L^{∞}, consequently the result follows interpolating the estimates

$$P:\bar{X}\rightarrow\bar{X},\text{ and }P:L^{\infty}\rightarrow L^{\infty}.$$

(iii) The proof will be by direct estimation of the norm of the compression/dilation operator E_rf (cf. (2.4) above). Let $0<r<1$, then

$$\begin{aligned}\|E_rf\|_{\bar{X}_{k,\log}}&\leq c\int_0^1\left\||f|^{*}\left(\frac{s}{r}\right)\chi_{[0,r)}(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}}\\&=c\int_0^1\left\||f|^{*}\left(\frac{s}{r}\right)\chi_{[0,\min(t,r))}(s)\right\|_{\bar{X}}\frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}}\\&\leq c\left(\int_0^r\left\||f|^{*}\left(\frac{s}{r}\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}}+\int_r^1\left\||f|^{*}\left(\frac{s}{r}\right)\chi_{[0,r)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}}\right)\\&=c(A(r)+B(r)).\end{aligned}$$

We estimate each of these terms as follows

(2.8)

$$\begin{aligned}
A(r) &= \int_0^r \left\| |f|^* \left(\frac{s}{r}\right) \chi_{[0,t/r)}(\frac{s}{r}) \right\|_{\bar{X}} \frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}} \\
&\leq h_{\bar{X}}(r) \int_0^r \left\| |f|^*(s)\chi_{[0,t/r)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}} \\
&= h_{\bar{X}}(r) \int_0^1 \left\| |f|^*(s)\chi_{[0,u)}(s) \right\|_{\bar{X}} \frac{du}{u\left(1+\ln\frac{1}{ur}\right)^{1-k/2}} \\
&\leq h_{\bar{X}}(r) \sup_{0<u<1} \left(\frac{1+\ln\frac{1}{u}}{1+\ln\frac{1}{ur}}\right)^{1-k/2} \int_0^1 \left\| |f|^*(s)\chi_{[0,u)}(s) \right\|_{\bar{X}} \frac{du}{u\left(1+\ln\frac{1}{u}\right)^{1-k/2}} \\
&= h_{\bar{X}}(r) \sup_{0<u<1} \left(\frac{1+\ln\frac{1}{u}}{1+\ln\frac{1}{ur}}\right)^{1-k/2} \|f\|_{\bar{X}_{k,\log}}.
\end{aligned}$$

Now, the term containing the supremum can be easily computed. Indeed, by direct differentiation one sees that the function $\frac{1+\ln\frac{1}{u}}{1+\ln\frac{1}{ur}}$ is decreasing, therefore $\left(\frac{1+\ln\frac{1}{u}}{1+\ln\frac{1}{ur}}\right)^{1-k/2}$ decreases (resp. increases) when $1-k/2>0$ (resp. $1-k/2\leq 0$). It follows that

$$\sup_{0<u<1} \left(\frac{1+\ln\frac{1}{u}}{1+\ln\frac{1}{ur}}\right)^{1-k/2} = \begin{cases} 1 & \text{if } k=1,2; \\ \left(1+\ln\frac{1}{r}\right)^{k/2-1} & \text{if } k\geq 3. \end{cases} \tag{2.9}$$

We estimate $B(r)$:

$$\begin{aligned}
B(r) &\leq \|E_r f\|_{\bar{X}} \int_r^1 \frac{dt}{t\left(1+\ln\frac{1}{t}\right)^{1-k/2}} \\
&\leq \|E_r f\|_{\bar{X}} \frac{2}{k}\left(1+\ln\frac{1}{r}\right)^{k/2} \\
&\leq h_{\bar{X}}(r)\frac{2}{k}\left(1+\ln\frac{1}{r}\right)^{k/2} \|f\|_{\bar{X}} \\
&\leq \bar{c}(k)\left(1+\ln\frac{1}{r}\right)^{k/2} \|f\|_{\bar{X}_{k,\log}} \quad \text{(by (2.7))}.
\end{aligned} \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we see that there exists a constant $c=c(k)$, such that

$$\|E_r f\|_{\bar{X}_{k,\log}} \leq c h_{\bar{X}}(r)\left(1+\ln\frac{1}{r}\right)^{k/2} \|f\|_{\bar{X}_{k,\log}}.$$

Therefore,

$$h_{\bar{X}_{k,\log}}(r) \leq c h_{\bar{X}}(r)\left(1+\ln\frac{1}{r}\right)^{k/2}.$$

Thus, for $0<r<1$, we have

$$\frac{\ln h_{\bar{X}_{k,\log}}(r)}{\ln r} \geq \frac{\ln h_{\bar{X}}(r)}{\ln r} + \frac{\ln c\left(1+\ln\frac{1}{r}\right)^{k/2}}{\ln r}.$$

It follows that

$$
\begin{aligned}
\underline{\alpha}_{\bar{X}_{k,\log}} &= \sup_{0<r<1} \frac{\ln h_{\bar{X}_{k,\log}}(r)}{\ln r} \\
&= \lim_{r\mapsto 0} \frac{\ln h_{\bar{X}_{k,\log}}(r)}{\ln r} \\
&\geq \lim_{r\mapsto 0} \left\{ \frac{\ln h_{\bar{X}}(r)}{\ln r} + \frac{\ln c\left(1+\ln\frac{1}{r}\right)^{k/2}}{\ln r} \right\} \\
&= \underline{\alpha}_{\bar{X}} + \lim_{r\to 0} \frac{\ln c\left(1+\ln\frac{1}{r}\right)^{k/2}}{\ln r} \\
&= \underline{\alpha}_{\bar{X}},
\end{aligned}
$$

as we wished to show. □

3. The main Theorem

In this section we always work with (Ω, d, μ) probability metric spaces, as described in the previous section, and will always let J denote an isoperimetric estimator of (Ω, d, μ).

Theorem 2. *Let $\bar{X}$ be a rearrangement invariant space, and suppose that $G : (0, 1/2) \to (0, \infty)$, satisfies*

$$
\int_0^{\frac{1}{2}} \frac{t}{J(t)} G(t)dt < \infty.
$$

Then:

(1) *If Q is bounded on $\bar{X}$, then the following Sobolev inequality holds: $\forall f \in Lip(\Omega)$,*

(3.1)
$$
\int_0^{\frac{1}{2}} \left\| \left(f^*(\cdot) - f^*(t)\right) \chi_{[0,t)}(\cdot) \right\|_{\bar{X}} G(t)dt \leq \|Q\|_{\bar{X}\to\bar{X}} \left\| |\nabla f|^* \right\|_{\bar{X}} \int_0^{\frac{1}{2}} \frac{t}{J(t)} G(t)dt.
$$

(2) *If P is bounded on $\bar{X}$, then the following Sobolev inequality holds: $\forall f \in Lip(\Omega)$,*

(3.2)
$$
\int_0^{\frac{1}{2}} \left\| (f^{**}(\cdot) - f^*(\cdot)) \chi_{[0,t)}(\cdot) \right\|_{\bar{X}} G(t)dt \leq \|P\|_{\bar{X}\to\bar{X}} \left\| |\nabla f|^* \right\|_{\bar{X}} \int_0^{\frac{1}{2}} \frac{t}{J(t)} G(t)dt.
$$

Proof. Part 2: To complete the details of the proof outlined in the introduction, simply note that the inequality (1.9) above, follows directly from (2.1). The proof of first part of the theorem requires an extra argument. Suppose that $f \in Lip(\Omega)$, then f^* is locally absolutely continuous and, since f^* is decreasing, it follows that $(-f^*)' \geq 0$. By the fundamental theorem of calculus we can write

$$
f^*(s) - f^*(t) = \int_s^t (-f^*)'(z)dz, \quad 0 < s < t < \frac{1}{2}.
$$

Consequently,

$$\begin{aligned}
\left\|\left(f^*(s)-f^*(t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}} &= \left\|\int_s^t \left(-f^*\right)'(z)dz\right\|_{\bar{X}} \\
&= \left\|\left(\int_s^1 J(z)\frac{z}{J(z)}\left(-f^*\right)'(z)\chi_{[0,t)}(z)\frac{dz}{z}\right)\right\|_{\bar{X}} \\
&\leq \|Q\|_{\bar{X}\to\bar{X}}\left\|\left(\frac{z}{J(z)}\right)J(z)\left(-f^*\right)'(z)\chi_{[0,t)}(z)\right\|_{\bar{X}} \quad \text{(since } Q \text{ is bounded on } \bar{X}\text{)} \\
&= \|Q\|_{\bar{X}\to\bar{X}}\left(\frac{t}{J(t)}\right)\left\|\left(J(\cdot)\left(-f^*\right)'(\cdot)\chi_{[0,t)}(\cdot)\right)^*\right\|_{\bar{X}} \quad \left(\text{since } \frac{z}{J(z)}\uparrow\right) \\
&\leq \|Q\|_{\bar{X}\to\bar{X}}\frac{t}{J(t)}\left\||\nabla f|^*(z)\chi_{[0,t]}(z)\right\|_{\bar{X}} \quad \text{(by (2.2) and Remark 2)} \\
&\leq \|Q\|_{\bar{X}\to\bar{X}}\frac{t}{J(t)}\left\||\nabla f|^*\right\|_{\bar{X}}.
\end{aligned}$$

Thus,

$$\int_0^{\frac{1}{2}}\left\|\left(f^*(\cdot)-f^*(t)\right)\chi_{[0,t)}(\cdot)\right\|_{\bar{X}}G(t)dt \leq \|Q\|_{\bar{X}\to\bar{X}}\left\||\nabla f|^*\right\|_{\bar{X}}\int_0^{\frac{1}{2}}\frac{t}{J(t)}G(t)dt.$$

□

Remark 3. *Since $f\in Lip(\Omega)\Rightarrow |f|\in Lip(\Omega)$ with $|\nabla|f||\leq|\nabla f|$, the inequalities (3.1) and (3.2) also hold for $|f|$.*

Let us also note the following converse to Theorem 2

Corollary 1. *Let $r\in(0,1]$ and suppose that suppose that*

$$\int_0^r\left\|\left(f^*(\cdot)-f^*(t)\right)\chi_{[0,t)}(\cdot)\right\|_{\bar{X}}G(t)dt \leq C(X)\left\||\nabla f|^*\right\|_{\bar{X}}, \tag{3.3}$$

holds for all r.i. spaces X away from L^∞. Then, for all Borel sets $A\subset\Omega$, with $\mu(A)\leq r$, we have

$$\mu(A)\int_{\mu(A)}^r G(t)dt \leq CP(A;\Omega), \tag{3.4}$$

and consequently,

$$\frac{t}{I(t)}\int_t^r G(t)dt\leq C, \text{ for all } t\in(0,r).$$

Proof. Our assumption implies that the inequality (3.3) holds for $X=L^1$. Let A be a Borel set with $\mu(A)\leq r$. We may assume without loss of generality that $P(A;\Omega)<\infty$. By [6] we can select a sequence $\{f_n\}_{n\in N}$ of Lip functions such that $f_n \underset{L^1}{\to} \chi_A$, and

$$P(A;\Omega)=\limsup_{n\to\infty}\||\nabla f_n|\|_{L^1}.$$

Therefore, by (3.3) applied to the sequence of $f_n's$ above, we obtain

$$\limsup_{n\to\infty}\int_0^r\left(\int_0^t\left(\left(f_n\right)_\mu^*(s)-\left(f_n\right)_\mu^*(t)\right)ds\right)G(t)dt\leq CP(A;\Omega).$$

It is known that $f_n \underset{L^1}{\to} \chi_A$ implies that (cf. [13, Lemma 2.1]):

$$\left(f_n\right)_\mu^*(t)\to\left(\chi_A\right)_\mu^*(t)=\chi_{[0,\mu(A)]}(t) \text{ at all points of continuity of } \left(\chi_A\right)_\mu^*.$$

Consequently,

$$\mu(A)\int_{\mu(A)}^{r}G(t)dt\leq\limsup_{n\to\infty}\int_0^r\left(\int_0^t\left(\left(f_n\right)_\mu^*(s)-\left(f_n\right)_\mu^*(t)\right)ds\right)G(t)dt \\ \leq CP(A;\Omega).$$

□

Definition 2. *Let J be an isoperimetric estimator of (Ω,d,μ). The* ***isoperimetric Hardy operator*** *Q_J is defined by*

$$Q_Jf(t):=\frac{J(t)}{t}\int_t^{\frac{1}{2}}f(z)\frac{dz}{J(z)}.$$

Theorem 3. *Let $\bar{X}$ be a rearrangement invariant space, and let J be an isoperimetric estimator. Let $G:(0,1)\to(0,\infty)$ be such that*

$$\int_0^{\frac{1}{2}}\frac{t}{J(t)}G(t)dt<\infty.$$

Suppose that the isoperimetric operator Q_J is bounded on $\bar{X}$. Let $f\in Lip(\Omega)$ and let $med(f)$ be a median[15] *of f, then*

$$\int_0^{\frac{1}{2}}\left\|(f-med(f))^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}G(t)dt\leq\left(\|Q\|_{\bar{X}\to\bar{X}}+\|Q_J\|_{\bar{X}\to\bar{X}}\right)\left\||\nabla f|^*\right\|_{\bar{X}}\int_0^{\frac{1}{2}}\frac{t}{J(t)}G(t)dt.$$

Proof. Let us start by remarking that since $\frac{t}{J(t)}$ is increasing, for $f\geq 0$ we have

$$Q_Jf(t)=\frac{J(t)}{t}\int_t^{\frac{1}{2}}f(z)\frac{dz}{J(z)}\geq\int_t^{\frac{1}{2}}f(z)\frac{dz}{z}=Qf(t).$$

Consequently, if Q_J is bounded on $\bar{X}$ then Q is also bounded on $\bar{X}$.

Let $f\in Lip(\Omega)$, and let $0<s<t<\frac{1}{2}$. Since f^* is decreasing, we have

$$\begin{aligned}\left\|\left(f^*(s)-f^*(1/2)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}&\leq\left\|(f^*(s)-f^*(t))\chi_{[0,t)}(s)\right\|_{\bar{X}}+\left|f^*(t)-f^*(1/2)\right|\left\|\chi_{[0,t)}(s)\right\|_{\bar{X}}\\&=\left\|(f^*(s)-f^*(t))\chi_{[0,t)}(s)\right\|_{\bar{X}}+\left(f^*(t)-f^*(1/2)\right)\left\|\chi_{[0,t)}(s)\right\|_{\bar{X}}\\&=(A)+(B).\end{aligned}$$

By the proof of Theorem 2 we know that

$$(A)\leq\frac{t}{J(t)}\|Q\|_{\bar{X}\to\bar{X}}\left\||\nabla f|^*\right\|_{\bar{X}}. \tag{3.5}$$

[15]Let f be a measurable function, a real number $med(f)$ will be called a **median** of f if

$$\mu\left\{f\geq med(f)\right\}\geq 1/2\ \text{ and }\mu\left\{f\leq med(f)\right\}\geq 1/2.$$

We estimate the second term as follows:

$$
\begin{aligned}
(B) &= \left(\int_t^{\frac{1}{2}} (-f^*)'(z)dz\right)\phi_{\bar{X}}(t) \\
&= \frac{t}{J(t)}\left(\frac{J(t)}{t}\int_t^{\frac{1}{2}}\left(J(z)\,(-f^*)'(z)\right)\frac{dz}{J(z)}\right)\phi_{\bar{X}}(t) \\
&= \frac{t}{J(t)}Q_J\left(\left(J(z)\,(-f^*)'(z)\right)\right)(t)\phi_{\bar{X}}(t) \\
&\leq \frac{t}{J(t)}\sup_t\left[Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)(t)\phi_{\bar{X}}(t)\right] \\
&= \frac{t}{J(t)}\left\|Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)\right\|_{M(\bar{X})}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(f^*(t)-f^*(1/2)\right)\phi_{\bar{X}}(t) &\leq \frac{t}{J(t)}\sup_t\left[Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)(t)\phi_{\bar{X}}(t)\right] \\
&= \frac{t}{J(t)}\left\|Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)\right\|_{M(\bar{X})} \\
&\leq \frac{t}{J(t)}\left\|Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)\right\|_{\bar{X}}.
\end{aligned}
$$

Since we are assuming that Q_J is bounded on $\bar{X}$, we have

$$
\begin{aligned}
\left\|Q_J\left(J(\cdot)\,(-f^*)'(\cdot)\right)\right\|_{\bar{X}} &\leq \|Q_J\|_{\bar{X}\to\bar{X}}\left\|J(\cdot)\,(-f^*)'(\cdot)\right\|_{\bar{X}} \\
&\leq \|Q_J\|_{\bar{X}\to\bar{X}}\left\||\nabla f|^*\right\|_{\bar{X}} \quad \text{(by (2.2))}.
\end{aligned} \tag{3.6}
$$

Adding the estimates for (A) and (B) (cf. (3.5) and (3.6) above) we obtain

$$
\left\|\left(f^*(s)-f^*(1/2)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}} \leq \frac{t}{J(t)}\left(\|Q\|_{\bar{X}\to\bar{X}}+\|Q_J\|_{\bar{X}\to\bar{X}}\right)\left\||\nabla f|^*\right\|_{\bar{X}}.
$$

It is easy to see that $f^*(\frac{1}{2})$ is a median of f (cf. [20]), moreover, since for any constant a, we have $f^*(s)-a=(f-a)^*(s)$, we finally arrive at

$$
\int_0^{\frac{1}{2}}\left\|(f-med(f))^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}G(t)dt \leq \left(\|Q\|_{\bar{X}\to\bar{X}}+\|Q_J\|_{\bar{X}\to\bar{X}}\right)\left\||\nabla f|^*\right\|_{\bar{X}}\int_0^{\frac{1}{2}}\frac{t}{J(t)}G(t)dt.
$$

□

4. Applications

4.1. **Homogeneous Sobolev spaces.** In this subsection we consider bounded domains $\Omega\subset\mathbb{R}^n$ normalized so that $|\Omega|=1$. We consider the Sobolev space $W_0^{k,1}(\Omega)$ of functions $f\in L^1(\Omega)$ that are $k-$ times weakly differentiable on Ω and such that their continuation by 0 outside Ω are $k-$ times weakly differentiable functions on $\mathbb{R}^n$. For $f\in W_0^{k,1}(\Omega)$ we then have $f\in W_0^{k,1}(\mathbb{R}^n)$, with

$$
\left\||D^jf|\right\|_{L^1(\Omega)}=\left\||D^jf|\right\|_{L^1(\mathbb{R}^n)} \qquad (j=0,1,\cdots k).
$$

More generally, given $\bar{X}$ a r.i. space on $(0,1)$, the Sobolev space $W_0^{k,\bar{X}} := W_0^{k,\bar{X}}(\Omega)$, will be defined as

$$W_0^{k,\bar{X}} = \left\{ f \in W_0^{k,1}(\Omega) : \|f\|_{W_0^{1,\bar{X}}} := \sum_{j=0}^{k} \left\| \left| D^j f \right|^* \right\|_{\bar{X}} < \infty \right\}.$$

Let

$$I_n(t) = n \left(\gamma_n\right)^{1/n} t^{1-1/n},$$

where $\gamma_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the measure of the unit ball in $\mathbb{R}^n$ (i.e. $I_n(t)$ is the isoperimetric profile associated to $\mathbb{R}^n$).

Let $f \in W_0^{1,1}$ then (see [22] and [19]):

(1)

$$f^{**}(t) - f^*(t) \leq \frac{t}{I_n(t)} \frac{1}{t} \int_0^t \left| \nabla f \right|^* (s) ds,\ 0 < t < 1. \tag{4.1}$$

(2) f^* is locally absolutely continuous, and

$$\int_0^t \left| (-f^*)'(\cdot) I_n(\cdot) \right|^* (s) \leq \int_0^t \left| \nabla f \right|^* (s) ds. \tag{4.2}$$

Using (4.1) and (4.2) and the method of proof of Theorem 2 we readily obtain

Theorem 4. *Let $\bar{X}$ be a r.i. space. Let $G : (0,1) \to (0,\infty)$ be such that*

$$\int_0^1 \frac{t}{I_n(t)} G(t) dt < \infty. \tag{4.3}$$

Then,

(1) *If $\|Q\|_{\bar{X}\to\bar{X}} < \infty$, then for all $f \in W_0^{1,\bar{X}}$,*

$$\int_0^1 \left\| \left(f^*(\cdot) - f^*(t)\right) \chi_{[0,t)}(\cdot) \right\|_{\bar{X}} G(t) dt \leq \|Q\|_{\bar{X}\to\bar{X}} \left\| \left| \nabla f \right|^* \right\|_{\bar{X}} \int_0^1 \frac{t}{I_n(t)} G(t) dt.$$

(2) *If $\|P\|_{\bar{X}\to\bar{X}} < \infty$, then for all $f \in W_0^{1,\bar{X}}$,*

$$\int_0^1 \left\| (f^{**}(\cdot) - f^*(\cdot)) \chi_{[0,t)}(\cdot) \right\|_{\bar{X}} G(t) dt \leq \|P\|_{\bar{X}\to\bar{X}} \left\| \left| \nabla f \right|^* \right\|_{\bar{X}} \int_0^1 \frac{t}{I_n(t)} G(t) dt.$$

In order to describe in detail the consequences of the previous result we need to compute the integral (4.3). Towards this end let us consider the function

$$G(t) = \frac{1}{t\sqrt{\ln\left(\frac{1}{t}\right)}},\ t \in (0,1).$$

Then,

$$\int_0^1 \frac{t}{tI_n(t)}G(t)dt = \frac{1}{n\left(\gamma_n\right)^{1/n}}\int_0^1 t^{1/n}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \tag{4.4}$$
$$= \frac{1}{n\left(\gamma_n\right)^{1/n}}\int_0^\infty z^{-\frac{1}{2}}e^{-z/n}dz$$
$$= \frac{\sqrt{\pi}n^{\frac{1}{2}}}{n\left(\gamma_n\right)^{1/n}}$$
$$= \frac{\Gamma(1+\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}}.$$

Consequently, we have the following

Corollary 2. *Let $\bar{X}$ be a r.i. space on $(0,1)$. Then,*

(1) *If $\underline{\alpha}_X > 0$, then, for all $f \in W_0^{1,\bar{X}}$*

$$\int_0^1 \left\|\left(f^*(s)-f^*(t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \frac{\Gamma(1+\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}}\left\|Q\right\|_{\bar{X}\to\bar{X}}\left\|\left|\nabla f\right|^*\right\|_{\bar{X}}.$$

(2) *If $\overline{\alpha}_X < 1$, then, for all $f \in W_0^{1,\bar{X}}$,*

$$\int_0^1 \left\|\left(f^{**}(s)-f^*(s)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \frac{\Gamma(1+\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}}\left\|P\right\|_{\bar{X}\to\bar{X}}\left\|\left|\nabla f\right|^*\right\|_{\bar{X}}.$$

Is easy to see that Corollary 2 gives the main result of [12] as a special case. In fact, we will now show an extension, valid for higher derivatives, which for easier comparison, we shall formulate in terms of the spaces defined in Definition 1 above.

The isoperimetric operator in this case is given by

$$Q_{I_n}f(t) := \frac{I_n(t)}{t}\int_t^1 f(z)\frac{dz}{I_n(z)} = t^{-1/n}\int_t^1 z^{1/n}f(z)\frac{dz}{z}.$$

Observe that Q_{I_n} is bounded on $\bar{X}$ if and only if $\underline{\alpha}_{\bar{X}} > 1/n$.

Theorem 5. *Let $\bar{X}$ be a r.i. space such that $\underline{\alpha}_{\bar{X}} > 0$. Let M be the smallest natural number such that*

$$\underline{\alpha}_{\bar{X}} > 1/M. \tag{4.5}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $|\Omega| = 1$, and suppose that $n \geq M$. Then for all $f \in W_0^{k,\bar{X}}$, we have

$$\|f\|_{\bar{X}_{k,\log}} \leq c(M,k,\bar{X})\left\|\left|D^k f\right|^*\right\|_{\bar{X}}, \tag{4.6}$$

where the constant $c(M,k,\bar{X})$ does not depend on the dimension.

Proof. We proceed by induction. Let $k=1$, and let $f \in W_0^{1,\bar{X}}$. For $0 < s < t < 1$, we have

$$\begin{aligned}\left\|\left|f\right|^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}} &\leq \left\|(\left|f\right|^*(s)-\left|f\right|^*(t))\chi_{[0,t)}(s)\right\|_{\bar{X}} + \left|f\right|^*(t)\left\|\chi_{[0,t)}(s)\right\|_{\bar{X}} \\ &= (A)+(B).\end{aligned}$$

By the proof of Theorem 2 we have

$$(A) \leq \frac{t^{1/n}}{n\gamma_n^{1/n}} \|Q\|_{\bar{X}\to\bar{X}} \left\| |\nabla f|^* \right\|_{\bar{X}}. \tag{4.7}$$

Now, since $f \in W_0^{1,\bar{X}}$ implies $|f| \in W_0^{1,\bar{X}}$, and moreover, since $|f|^*(1) = 0$, we can write

$$|f|^*(t) = \int_t^1 \left(-|f|^*\right)'(z)dz, \quad 0 < t < 1.$$

Consequently,

$$(B) = \left(\int_t^1 \left(-|f|^*\right)'(z)dz\right) \phi_{\bar{X}}(t).$$

From this point we follow the proof of Theorem 3 to obtain

$$\left(\int_t^1 \left(-|f|^*\right)'(z)dz\right) \phi_{\bar{X}}(t) \leq \frac{t^{1/n}}{n\gamma_n^{1/n}} \|Q_{I_n}\|_{\bar{X}\to\bar{X}} \left\| |\nabla f|^* \right\|_{\bar{X}}.$$

Adding the estimates obtained for (A) and (B) we get

$$\left\| |f|^*(s)\chi_{[0,t)}(s) \right\|_{\bar{X}} \leq \frac{t^{1/n}}{n\gamma_n^{1/n}} \left(\|Q\|_{\bar{X}\to\bar{X}} + \|Q_{I_n}\|_{\bar{X}\to\bar{X}}\right) \left\| |\nabla f|^* \right\|_{\bar{X}}.$$

Therefore, using (4.4), (2.6) and (4.5) we get

$$\begin{aligned} \|f\|_{\bar{X}_{1,\log}} &= \int_0^1 \left\| |f|^*(s)\chi_{[0,t)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\ &\leq \frac{\Gamma(1+\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}} \left(\|Q\|_{\bar{X}\to\bar{X}} + \int_1^\infty h_{\bar{X}}(\frac{1}{s}) s^{\frac{1}{M}-1} ds\right) \left\| |\nabla f|^* \right\|_{\bar{X}}. \end{aligned}$$

Recall that for $x \geq 1$, we have $\Gamma(x) \leq x^x$; consequently

$$\frac{\Gamma(1+\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}} = \left(\frac{n}{2}\right)^{1/n} \frac{\Gamma(\frac{n}{2})^{1/n}}{n^{\frac{1}{2}}} \leq \frac{1}{\sqrt{2}} \left(\frac{n}{2}\right)^{1/n} \leq c.$$

Thus,

$$\|f\|_{\bar{X}_{1,\log}} \leq c(M,1,\bar{X}) \left\| |\nabla f| \right\|_{\bar{X}},$$

where $c(M,1,\bar{X})$ is a constant that does not depend on n.

Let $k \geq 2$, and suppose that the desired inequality is valid for $k-1$. Let $f \in W_0^{k,\bar{X}}$, then, by the induction hypothesis, and the fact that $|\nabla f| \in W_0^{k-1,\bar{X}}$, we have

$$\begin{aligned} \left\| |\nabla f| \right\|_{\bar{X}_{k-1,\log}} &:= \int_0^1 \left\| |\nabla f|^*(s)\chi_{[0,t)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{1-(k-1)/2}} \\ &\leq c(M,k-1,\bar{X}) \left\| \left|D^{k-1} |\nabla f|\right|^* \right\|_{\bar{X}}. \end{aligned} \tag{4.8}$$

By Proposition 1 (part 3), the r.i. space $\bar{X}_{k-1,\log}$ satisfies $\underline{\alpha}_{\bar{X}_{k-1,\log}} > r$. Consequently we may apply the result obtained in the first step of the proof to the space

$\bar{X}_{k-1,\log}$, and we obtain

$$\begin{aligned}
\|f\|_{(\bar{X}_{k-1,\log})_{1,\log}} &\leq c(M,1,\bar{X}) \left\| |\nabla f|^* \right\|_{\bar{X}_{k-1,\log}} \\
&\leq c(M,1,\bar{X}) c(M,k-1,\bar{X}) \left\| \left| D^{k-1} |\nabla f| \right|^* \right\|_{\bar{X}} \quad \text{(by (4.8))} \\
&\leq c(M,1,\bar{X}) c(M,k-1,\bar{X}) \left\| \left| D^{k} f \right|^* \right\|_{\bar{X}} . \qquad (4.9)
\end{aligned}$$

We will show in a moment that

$$\|f\|_{(\bar{X}_{k-1,\log})_{1,\log}} = \frac{2k}{k-1} \|f\|_{\bar{X}_{k,\log}} . \qquad (4.10)$$

Assuming (4.10) and combining it with (4.9) we see that

$$\|f\|_{\bar{X}_{k,\log}} \leq c(M,k,\bar{X}) \left\| \left| D^{k} f \right|^* \right\|_{\bar{X}} .$$

It thus remains to prove (4.10). For this purpose we write

$$\begin{aligned}
\|f\|_{(\bar{X}_{k-1,\log})_{1,\log}} &= \int_0^1 \left\| |f|^*(s) \chi_{[0,t)}(s) \right\|_{\bar{X}_{k-1,\log}} \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= \int_0^1 \left\| \left(|f|^* \chi_{[0,t)} \right)^* (\cdot) \chi_{[0,s)}(\cdot) \right\|_{\bar{X}_{k-1,\log}} \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= \int_0^1 \left(\int_0^1 \left\| |f|^*(\cdot) \chi_{[0,t)}(\cdot) \chi_{[0,s)}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= \int_0^1 \left(\int_0^1 \left\| |f|^*(\cdot) \chi_{[0,\min(s,t))}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= \int_0^1 \left(\int_0^t \left\| |f|^*(\cdot) \chi_{[0,s))}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&+ \int_0^1 \left(\int_t^1 \left\| |f|^*(\cdot) \chi_{[0,t))}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= A + B.
\end{aligned}$$

By Fubini's Theorem we have

$$\begin{aligned}
A &= \int_0^1 \left(\int_0^t \left\| |f|^*(\cdot) \chi_{[0,s)}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\
&= \int_0^1 \left\| |f|^*(\cdot) \chi_{[0,s)}(\cdot) \right\|_{\bar{X}} \frac{1}{s \left(\ln \frac{1}{s}\right)^{1-(k-1)/2}} \left(\int_s^1 \frac{dt}{t \left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \right) ds \\
&= 2 \int_0^1 \left\| |f|^*(\cdot) \chi_{[0,s)}(\cdot) \right\|_{\bar{X}} \frac{ds}{s \left(\ln \frac{1}{s}\right)^{1-k/2}} \\
&= 2 \|f\|_{\bar{X}_{k,\log}} .
\end{aligned}$$

We also have,

$$\begin{aligned} B &= \int_0^1 \left(\int_t^1 \left\| |f|^* (\cdot)\chi_{[0,t))}(\cdot)\right\|_{\bar{X}} \frac{ds}{s\left(\ln\frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \\ &= \int_0^1 \left\| |f|^* (\cdot)\chi_{[0,t))}(\cdot)\right\|_{\bar{X}} \left(\int_t^1 \frac{ds}{s\left(\ln\frac{1}{s}\right)^{1-(k-1)/2}} \right) \frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \\ &= \int_0^1 \left\| |f|^* (\cdot)\chi_{[0,s)}(\cdot)\right\|_{\bar{X}} \frac{2}{k-1} \left(\ln\frac{1}{t}\right)^{(k-1)/2} \frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \text{ (since } k\geq 2) \\ &= \frac{2}{k-1}\int_0^1 \left\| |f|^* (\cdot)\chi_{[0,s)}(\cdot)\right\|_{\bar{X}} \frac{dt}{t\left(\ln\frac{1}{t}\right)^{1-k/2}} \\ &= \frac{2}{k-1}\|f\|_{\bar{X}_{k,\log}}. \end{aligned}$$

Now, $A+B$ gives (4.10) concluding the proof of the theorem. □

In particular we have

Example 1. *(cf.* [12] *for the case $k=1$) Let $\bar{X}=L^p$, then $\underline{\alpha}_{L^p}=1/p$. Let M the smallest natural number such that*

$$\frac{1}{p}>\frac{1}{M}.$$

Let $n\geq M$, and let $\Omega\subset\mathbb{R}^n$ be a bounded domain normalized so that $|\Omega|=1$. Then, for all $f\in W_0^{k,p}(\Omega)$, we have

$$\int_0^1 \left(\int_0^s \left(|f|^*(s)\right)^p ds \right)^{1/p} \frac{dt}{t\left(\ln\frac{1}{t}\right)^{1-k/2}} \leq c(M,k,L^p)\left\| \left|D^k f\right| \right\|_{L^p}.$$

4.2. **The unit ball on $\mathbb{R}^n$.** Let $(B^n,|\cdot|,\mu)$ be the open unit ball on $\mathbb{R}^n$ endowed with Euclidean metric $|\cdot|$ and with the normalized Lebesgue measure $\mu=\frac{dx}{\gamma_n}$, where $\gamma_n=\frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the measure of B^n. We consider the Sobolev space $W^{1,1}:=W^{1,1}(B^n)$ of functions $f\in L^1(B^n)$ that are weakly differentiable on B^n and $|\nabla f|\in L^1(B^n)$. Given $\bar{X}$ a r.i. space on $(0,1)$, the Sobolev space $W^{1,\bar{X}}:=W^{1,\bar{X}}(B^n)$ is defined by

$$W^{1,\bar{X}}=\left\{ f\in W^{1,1}:\|f\|_{W^{1,\bar{X}}}:=\left\| |f|_\mu^*\right\|_{\bar{X}}+\left\| |\nabla f|_\mu^*\right\|_{\bar{X}}<\infty \right\}.$$

Let I_{B^n} be the isoperimetric profile of $(B^n,|\cdot|,\mu)$. It is known that (cf. [23, Lemma 1 pag 163])

$$I_{B^n}(t)\geq\frac{\gamma_{n-1}}{\gamma_n}2^{1-1/n}\min(t,1-t)^{1-1/n}=J_{B^n}(t),\quad 0<t<1. \tag{4.11}$$

In fact, the constant that appears on the left hand side of (4.11) is best possible. Moreover, we recall that for $f\in W^{1,1}$ the inequalities (4.1) and (4.2) hold (cf. [20]).

Let

$$G(t)=\frac{1}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}.$$

Then,

$$\begin{aligned}\int_0^{1/2} \frac{t}{J_{B^n}(t)} G(t)dt &= \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \int_0^{1/2} t^{1/n} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\ &\leq \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \int_0^{1} t^{1/n} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \\ &= \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} n^{\frac{1}{2}} \int_0^{\infty} z^{-\frac{1}{2}} e^{-z/n} dz \quad (e^{-z/n} = t^{1/n}) \\ &= \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \sqrt{\pi} n^{\frac{1}{2}}.\end{aligned}$$

The associated isoperimetric operator is given by

$$Q_{J_{B^n}} f(t) := \frac{J_{B^n}(t)}{t} \int_t^{1/2} f(z) \frac{dz}{J_{B^n}(z)} = t^{-1/n} \int_t^{1/2} z^{1/n} f(z) \frac{dz}{z} = Q_{1/n} f(t).$$

By the general theory (cf. (2.5) and (2.6) in Section 2), $Q_{1/n}$ is bounded on $\bar{X}$ if and only if $\underline{\alpha}_X > 1/n$. Moreover,

$$\left\| Q_{1/n} \right\| \leq \int_1^{\infty} h_{\bar{X}}\left(\frac{1}{s}\right) s^{\frac{1}{n}-1} ds.$$

The previous discussion, combined with Theorems 2 and 3, gives the following

Theorem 6. *Let $\bar{X}$ be a r.i. space on $(0,1)$. Then,*

(1) *If $\underline{\alpha}_X > 0$, then*[16]*, for all $f \in W^{1,\bar{X}}$*

$$\int_0^{1/2} \left\| \left(f^*(\gamma_n s) - u^*(\gamma_n t)\right) \chi_{[0,t)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}} \leq \sqrt{\pi} n^{\frac{1}{2}} \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \|Q\|_{\bar{X} \to \bar{X}} \left\| \left|\nabla f\right|^* (\gamma_n s) \right\|_{\bar{X}}.$$

(2) *If $\overline{\alpha}_X < 1$, then, for all $f \in W^{1,\bar{X}}$*

$$\int_0^{1/2} \left\| \left(\frac{1}{s} \int_0^s f^*(\gamma_n z) dz - f^*(\gamma_n s) \right) \chi_{[0,t)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}}$$
$$\leq \sqrt{\pi} n^{\frac{1}{2}} \frac{\gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \|P\|_{\bar{X} \to \bar{X}} \left\| \left|\nabla f\right|^* (\gamma_n s) \right\|_{\bar{X}}.$$

3. *Suppose that $\underline{\alpha}_X > 0$, and let M be the smallest natural number such that*

$$\underline{\alpha}_{\bar{X}} > 1/M,$$

and furthermore suppose that $n \geq M$. Then, for all $f \in W^{1,\bar{X}}$, we have

$$\int_0^{1/2} \left\| (f - med(f))^*(\gamma_n s) \chi_{[0,t)}(s) \right\|_{\bar{X}} \frac{dt}{t\left(\ln \frac{1}{t}\right)^{\frac{1}{2}}}$$
$$\leq \left(\frac{\sqrt{\pi} n^{\frac{1}{2}} \gamma_n}{\gamma_{n-1}} \left(\frac{1}{2}\right)^{1-1/n} \int_1^{\infty} h_X\left(\frac{1}{s}\right) s^{\frac{1}{M}} \frac{ds}{s} \right) \left\| \left|\nabla f\right|^* (\gamma_n s) \right\|_{\bar{X}}.$$

[16] note that $u_\mu^*(s) = u^*(\gamma_n s)$.

In particular, since

$$\lim_{n\to\infty}\sqrt{\pi}n^{\frac{1}{2}}\frac{\gamma_n}{\gamma_{n-1}}\left(\frac{1}{2}\right)^{1-1/n}=\lim_{n\to\infty}\pi\frac{n^{\frac{1}{2}}\Gamma(1+\frac{n-1}{2})}{\Gamma(1+\frac{n}{2})}\left(\frac{1}{2}\right)^{1-1/n}=\frac{\sqrt{2}}{2}\pi,$$

there exists a constant c independent of n, such that for all $f\in W^{1,\bar{X}}$

$$\int_0^{1/2}\left\|(f-med(f))^*(\gamma_n s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}\leq c\left\|\left|\nabla f\right|^*(\gamma_n s)\right\|_{\bar{X}}.$$

As a consequence we obtain the following (cf. [8])

Corollary 3. $(B^n,|\cdot|,\mu)$ *is of Gaussian isoperimetric type near zero.*

Proof. By Corollary 1 we have

$$t\left(\ln\frac{1}{t}\right)^{1/2}-2t\left(\ln 2\right)^{1/2}=\int_t^{1/2}\frac{1}{s\left(\ln\frac{1}{s}\right)^{\frac{1}{2}}}ds\leq CI_{B^n}(t),\quad 0<t<1/2.$$

Therefore, for $t\in(0,1/4)$

$$\begin{aligned}t\left(\ln\frac{1}{t}\right)^{1/2}&\leq CI_{B^n}(t)+2\left(\ln 2\right)^{1/2}\\&\leq CI_{B^n}(t)+\frac{1}{2}\left(\ln\frac{1}{t}\right)^{1/2}\end{aligned}$$

and the desired result follows. □

4.3. **The $n-$sphere.** Let $n\in\mathbb{N}$, $n\geq 2$, and let $\mathbb{S}^n$ be the unit sphere. Consider the metric space $(\mathbb{S}^n,d,\frac{dx_n}{\omega_n})$, where d is the geodesic distance, dx_n is the Lebesgue measure on $\mathbb{R}^n$ and $\omega_n=2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$.

Then, we have that for $f\in Lip(\mathbb{S}^n)$ (cf. [6, Proposition 1.5]),

$$\left(\int_{\mathbb{S}^n}\left|f(x)-\int_{\mathbb{S}^n}fdx_n\right|^{\frac{n}{n-1}}\frac{dx_n}{\omega_n}\right)^{\frac{n-1}{n}}\leq\frac{\omega_n}{2\omega_{n-1}}\int_{\mathbb{S}^n}\left|\nabla f(x)\right|\frac{dx_n}{\omega_n}.$$

It follows that (cf. [23])

$$I_{(\mathbb{S}^n,d,\frac{dx_n}{\omega_n})}(t)\geq\frac{2\omega_{n-1}}{\omega_n}\min(t,1-t)^{1-1/n}=J_{\mathbb{S}^n}(t),\quad 0<t<1.$$

Consider the function

$$G(t)=\frac{1}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}$$

then

$$\begin{aligned}\int_0^{1/2}\frac{t}{J_{\mathbb{S}^n}(t)}G(t)dt&=\frac{\omega_n}{2\omega_{n-1}}\int_0^{1/2}t^{1/n}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}\\&\leq\frac{\omega_n}{2\omega_{n-1}}\int_0^{1}t^{1/n}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}\\&=\frac{\omega_n}{\omega_{n-1}}\sqrt{\pi}n^{\frac{1}{2}}.\end{aligned}$$

The isoperimetric operator in this case is given by

$$Q_{J_{\mathbb{S}^n}}f(t) := \frac{J_{\mathbb{S}^n}(t)}{t}\int_t^{1/2} f(z)\frac{dz}{J_{\mathbb{S}^n}(z)} = t^{-1/n}\int_t^{1/2} z^{1/n}f(z)\frac{dz}{z} = Q_{1/n}f(t).$$

Therefore, $Q_{J_{\mathbb{S}^n}}$ is bounded on $\bar{X}$ if and only if $\underline{\alpha}_{\bar{X}} > 1/n$. Moreover,

$$\left\|Q_{1/n}\right\|_{\bar{X}\to\bar{X}} \leq \int_1^\infty h_{\bar{X}}(\frac{1}{s})s^{\frac{1}{n}-1}ds.$$

Therefore, Theorems 2 and 3 yield

Theorem 7. *Let $\bar{X}$ be a r.i. space on $(0,1)$.*

(1) *If $\underline{\alpha}_{\bar{X}} > 0$, then*[17]*, for all $f \in Lip(\mathbb{S}^n)$*

$$\int_0^{1/2} \left\|\left(f^*(\omega_n s) - f^*(\omega_n t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}} \frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \sqrt{\pi}n^{\frac{1}{2}}\frac{\omega_{n-1}}{\omega_n}\left\|Q\right\|_{\bar{X}\to\bar{X}}\left\|\left|\nabla f\right|^*(\omega_n s)\right\|_{\bar{X}}.$$

(2) *If $\overline{\alpha}_{\bar{X}} < 1$, then for all $f \in Lip(\mathbb{S}^n)$*

$$\int_0^{1/2}\left\|\left(\frac{1}{s}\int_0^s f^*(\omega_n z)dz - f^*(\omega_n t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{\omega_n}{t}\right)^{\frac{1}{2}}}$$
$$\leq \sqrt{\pi}n^{\frac{1}{2}}\frac{\omega_{n-1}}{\omega_n}\left\|P\right\|_{\bar{X}\to\bar{X}}\left\|\left|\nabla f\right|^*(\omega_n s)\right\|_{\bar{X}}.$$

(3) *If $\underline{\alpha}_{\bar{X}} > 0$, let M be the smallest natural number such that*

$$\underline{\alpha}_X > 1/M.$$

Suppose that $n \geq M$, then for all $f \in Lip(\mathbb{S}^n)$, we have

$$\int_0^{1/2}\left\|(f - med(f))^*(\omega_n s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}}$$
$$\leq \left(\sqrt{\pi}n^{\frac{1}{2}}\frac{\omega_{n-1}}{\omega_n}\int_1^\infty h_X(\frac{1}{s})s^{\frac{1}{M}}\frac{ds}{s}\right)\left\|\left|\nabla f\right|^*(\omega_n s)\right\|_{\bar{X}}.$$

In particular, since

$$\lim_{n\to\infty}\frac{\omega_n}{\omega_{n-1}}\sqrt{\pi}n^{\frac{1}{2}} = \lim_{n\to\infty}\pi\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} = \sqrt{2}\pi,$$

there exists a constant c independent of n, such that for all $f \in Lip_X(\mathbb{S}^n)$,

$$\int_0^{1/2}\left\|(f - med(f))^*(\omega_n s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq c\left\|\left|\nabla f\right|^*(\omega_n s)\right\|_{\bar{X}}.$$

[17] Note that $u^*_{\frac{dx_n}{\omega_n}}(s) = u^*_{dx_n}(\omega_n s)$.

4.4. **Riemannian manifolds with positive curvature.** Let V^n be a compact Riemannian manifold (without boundary). Let $R(V^n)$ denote the infimum over all the unit tangent vectors of V^n of the Ricci tensor, and let I_{V^n} be the isoperimetric profile of the manifold (with respect to the normalized Riemannian measure $d\sigma_n$). If $R(V^n) \geq (n-1)k > 0$, then (cf. [6])

$$\sqrt{\frac{2k}{\pi}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\left(\int_{V^n}\left|f(x)-\int_{V^n} f d\sigma_n\right|^{\frac{n}{n-1}} d\sigma_n\right)^{\frac{n-1}{n}} \leq \int_{V^n}|\nabla f(x)|\, d\sigma_n \ \ (f \in Lip(V^n)).$$

As a consequence (cf. [23]) the following isoperimetric inequality holds

$$I_{V^n}(t) \geq \sqrt{\frac{2k}{\pi}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\min(t,1-t)^{1-1/n} = J_{V^n}(t), \quad 0<t<1.$$

Theorems 2 and 3 give:

Theorem 8. *Let $\bar{X}$ be a r.i. space.*

(1) *If $\underline{\alpha}_{\bar{X}} > 0$, then for all $f \in Lip(V^n)$,*

$$\int_0^{\frac{1}{2}}\left\|\left(f_{\sigma_n}^*(s)-f_{\sigma_n}^*(t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \frac{\pi}{\sqrt{2k}}\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\|Q\|_{\bar{X}\to\bar{X}}\left\||\nabla f|_{\sigma_n}^*\right\|_{\bar{X}}.$$

(2) *If $\overline{\alpha}_{\bar{X}} < 1$, then for all $f \in Lip_{\bar{X}}(V^n)$,*

$$\int_0^{\frac{1}{2}}\left\|\left(f_{\sigma_n}^*(s)-f_{\sigma_n}^*(t)\right)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \frac{\pi}{\sqrt{2k}}\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\|P\|_{\bar{X}\to\bar{X}}\left\||\nabla f|_{\sigma_n}^*\right\|_{\bar{X}}.$$

(3) *If $\underline{\alpha}_{\bar{X}} > 0$, let M be the smallest natural number such that*

$$\underline{\alpha}_X > 1/M.$$

Suppose that $n \geq M$. Then for all $f \in Lip(V^n)$, we have

$$\int_0^{\frac{1}{2}}\left\|(f-med(f))_{\sigma_n}^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq \left(\frac{\pi}{\sqrt{2k}}\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\int_1^\infty h_{\bar{X}}(\frac{1}{s})s^{\frac{1}{M}}\frac{ds}{s}\right)\left\||\nabla f|_{\sigma_n}^*\right\|_{\bar{X}}.$$

In particular, since

$$\lim_{n\to\infty}\frac{\pi}{\sqrt{2k}}\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} = \lim_{n\to\infty}\pi\frac{n^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} = \frac{\pi}{\sqrt{k}},$$

there exists a constant c independent of n, such that for all $f \in Lip(V^n)$

$$\int_0^{\frac{1}{2}}\left\|(f-med(f))_{\sigma_n}^*(s)\chi_{[0,t)}(s)\right\|_{\bar{X}}\frac{dt}{t\left(\ln\frac{1}{t}\right)^{\frac{1}{2}}} \leq c\left\||\nabla f|_{\sigma_n}^*\right\|_{\bar{X}}.$$

References

[1] A. Alvino, Sulla diseguaglianza di Sobolev in spazi di Lorentz, Boll. Un. Mat. Ital. **5**, 14-A (1977), 148-156.

[2] S. V. Astashkin and K. V. Lykov, *Strong extrapolation spaces and interpolation*, Siberian Math. J. **50** (2009), 199-213.

[3] F. Barthe, *Log-concave and spherical models in isoperimetry*, Geom. Funct. Anal. **12** (2002), 32-55.

[4] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, SpringerVerlag, Berlin-Heidelberg-New York, 1976.

[5] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston**,** 1988.

[6] S. G. Bobkov and C. Houdré, *Some connections between isoperimetric and Sobolev type inequalities*, Mem. Amer. Math. Soc. **129** (1997), no 616.

[7] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245–1254.

[8] Y. D. Burago and V. G. Mazja. *Certain questions of potential theory and function theory for regions with irregular boundaries*. Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 3 (1967) 152. (English translation: Potential theory and function theory for irregular regions. Seminars in Mathematics. V. A. Steklov Mathematical Institute, Leningrad, Vol. 3, Consultants Bureau, New York, 1969)

[9] D. Cordero-Erausquin, B. Nazaret and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. **182** (2004) 307-332.

[10] A. Fiorenza, *Duality and reflexivity in grand Lebesgue spaces*, Collect. Math. **51**, (2000), 131-148.

[11] A. Fiorenza and G. E. Karadzhov, *Grand and Small Lebesgue Spaces and their analogs*, Z. Anal. Anwendungen, **23** (2004), 657-681.

[12] A. Fiorenza, M. Krbec and H. J. Schmeisser, *An improvement of dimension-free Sobolev imbeddings in r.i. spaces*, preprint.

[13] A. Garsia and E. Rodemich, *Monotonicity of certain functionals under rearrangements*, Ann. Inst. Fourier (Grenoble) **24** (1974), 67-116.

[14] M. Griebel, *Sparse grids and related approximation schemes for higher dimensional problems*, in L. Pardo, A. Pinkus, E. Suli, and M. Todd, editors, Foundations of Computational Mathematics (FoCM05), Santander, pp 106-161, Cambridge University Press, 2006.

[15] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. **119** (1992), 129-143.

[16] G. E. Karadzhov and M. Milman, *Extrapolation Theory: New Results and Applications,* J. Approx. Th. **133** (2005), 38-99.

[17] M. Krbec and H. J. Schmeisser, *On dimension-free Sobolev imbeddings I*, J. Math. Anal. Appl. **387** (2012), 114-125.

[18] M. Krbec and H. J. Schmeisser, *On dimension-free Sobolev imbeddings II*, Rev. Mat. Complutense **25** (2012), 247-265.

[19] J. Martin and M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math. **225** (2010), 121-199.

[20] J. Martin and M. Milman, *Fractional Sobolev Inequalities: Symmetrization, Isoperimetry and Interpolation*, Preprint.

[21] J. Martin and M. Milman, *Sobolev inequalities, rearrangements, isoperimetry and interpolation spaces*, Contemporary Mathematics **545** (2011), pp 167-193.

[22] J. Martin, M. Milman and E. Pustylnik, *Sobolev Inequalities: Symmetrization and Self Improvement via truncation*, J. Funct.Anal. **252** (2007), 677-695.

[23] V.G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, New York, 1985.

[24] E. Milman, *A converse to the Maz'ya inequality for capacities under curvature lower bound*, in A. Laptev (ed), Around the research of Vladimir Maz'ya I: Function Spaces, Springer, 2010, pp 321-348.

[25] E. Milman and L. Rotem, *Complemented Brunn-Minkowski Inequalities and Isoperimetry for Homogeneous and Non-Homogeneous Measures* (arXiv:1308.5695)

[26] M. Milman, *The computation of the $K-$functional for couples of rearrangement invariant spaces*, Resultate Math. **5** (1982), 174-176.

[27] J. M. Rakotoson and B. Simon *Relative Rearrangement on a finite measure space spplication to the regularity of weighted monotone rearrangement (Part 1)*, Rev. R. Acad. Cienc. Exact. Fis. Nat. **91** (1997), 17-31.

[28] A. Ros, *The isoperimetric problem*, In: Global Theory of Minimal Surfaces. Clay Math. Proc., vol. 2, pp. 175-209, Am. Math. Soc., Providence, 2005.

[29] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353-372.

[30] H. Triebel, *Tractable embeddings of Besov spaces into Zygmund spaces*, Function spaces IX, 361-377, Banach Center Publ. **92**, Polish Acad. Sci. Inst. Math., Warsaw, 2011.

[31] H. Triebel, *Tractable embeddings*, preprint, University of Jena, Nov. 2012.

Department of Mathematics, Universitat Autònoma de Barcelona
E-mail address: jmartin@mat.uab.cat

E-mail address: extrapol@bellsouth.net
URL: https://sites.google.com/site/mariomilman