Martín i Pedret, Joaquim; Milman, Mario. «Isoperimetric weights and generalized uncertainty inequalities in metric measure spaces». Journal of Functional Analysis, Vol. 270, Issue 9 (May 2016), p. 3307-3343. DOI 10.1016/j.jfa.2016.02.016

# ISOPERIMETRIC WEIGHTS AND GENERALIZED UNCERTAINTY INEQUALITIES IN METRIC MEASURE SPACES

JOAQUIM MARTÍN AND MARIO MILMAN

*In memory of Nigel Kalton*

ABSTRACT. We extend the recent $L^1$ uncertainty inequalities obtained in [14] to the metric setting. For this purpose we introduce a new class of weights, named *isoperimetric weights*, for which the growth of the measure of their level sets $\mu(\{w \leq r\})$ can be controlled by $rI(r)$, where $I$ is the isoperimetric profile of the ambient metric space. We use isoperimetric weights, new *localized Poincaré inequalities*, and interpolation, to prove $L^p, 1 \leq p < \infty$, uncertainty inequalities on metric measure spaces. We give an alternate characterization of the class of isoperimetric weights in terms of Marcinkiewicz spaces, which combined with the sharp Sobolev inequalities of [21], and interpolation of weighted norm inequalities, give new uncertainty inequalities in the context of rearrangement invariant spaces.

## 1. Introduction

In a recent paper, Dall'ara-Trevisan [14] extended the classical uncertainty inequality (cf. [39])[1]

$$\|f\|^2_{L^2(\mathbb{R}^n)} \leq 4n^{-2} \left\| |\nabla f| \right\|_{L^2(\mathbb{R}^n)} \left\| |x| f \right\|_{L^2(\mathbb{R}^n)}, \ f \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

to a large class of homogenous spaces $M$ for a (Lie or finitely generated) group $G$ such that the isotrophy subgroups are compact and, furthermore, $M$ is endowed with an invariant measure $\mu$, an invariant distance $d$, and an invariant gradient which is compatible with $d$. To describe the weights considered in [14] let us observe that for each $r > 0$, the elements of $B(r)$, the class of balls of radius $r$ in $M$, have equal measure and, consequently, one can consider the class of weights $w : M \to \mathbb{R}^+$ that satisfy

$$\mu(\{w \leq r\}) \leq \Upsilon_M(r) := \mu(B(r)). \tag{1.2}$$

In this setting, Dall'ara-Trevisan [14] show that, for all weights that satisfy (1.2), there exists $c > 0$ such that for all $p \in [1, \infty)$,

$$\|f\|^2_{L^p(M,\mu)} \leq cp \left\| |\nabla f| \right\|_{L^p(M,\mu)} \|wf\|_{L^p(M,\mu)}, \tag{1.3}$$

1991 *Mathematics Subject Classification.*

The first named author was partially supported by Grants MTM2013-44304-P, MTM2013-40985-P, 2014SGR289.

The second named author was partially supported by a grant from the Simons Foundation (#207929 to Mario Milman).

[1] A detailed survey of the uncertainty inequality and many related inequalities can be found in [17].

for all smooth functions $f$ satisfying suitably prescribed cancellations[2].

A new feature of the result is the fact that it is crucially valid for $p=1$. Indeed, the inequalities for $L^p$, $p>1$, follow from the $L^1$ case by a familiar argument using the chain rule (cf. [14], [32]). Furthermore, as in the classical theory of Sobolev inequalities of Maz'ya (cf. [25], [26]), the $L^1$ uncertainty inequalities are naturally connected with isoperimetry. For example, when $M$ is compact, the "weak isoperimetric inequality" property used in [14] asserts the existence of a constant $C>0$, such that for all Borel sets $A$ and $E$, with $\mu(A)\leq\mu(M)/2$, and $\mu(E)\leq\Upsilon_M(r)$, we have[3]

$$\mu(A\cap E)\leq Cr\mu^{+}(A), \tag{1.4}$$

where $\mu^{+}$ is a suitable notion of perimeter[4] (cf. [14], and also [13]). The proof of (1.3) in [14] uses (1.2) and (1.4) combined with Poincaré's inequality; furthermore, the group structure associated with $M$ also plays a rôle.

The purpose of this paper is to extend (1.3) to the more general context of metric measure spaces. In particular, we will eliminate the dependence on any type of group structure as well as the requirement that the measure of a ball depends only on its radius. In particular, our uncertainty inequalities are also valid in the Gaussian setting. We are also able to extend (1.3) to rearrangement invariant norms and establish a principle that allows the transference of uncertainty inequalities between different geometries, under the assumption that the underlying isoperimetric profiles can be compared pointwise.

To explain in somewhat more detail the motivation behind the results, and the methods we shall develop in this paper, we need to introduce some notation.

Let $(\Omega,\mu,d)$ be a connected metric measure space[5], such that $\mu(K)<\infty$, for compact sets $K\subset\Omega$. The modulus of the gradient of a Lip function $f$ is defined by

$$|\nabla f(x)|=\limsup_{d(x,y)\to 0}\frac{|f(x)-f(y)|}{d(x,y)}.$$

The perimeter or Minkowski content of a Borel set $A\subset\Omega$, is defined by

$$\mu^{+}(A)=\lim\inf_{h\to 0}\frac{\mu(A_h)-\mu(A)}{h},$$

where $A_h=\{x\in\Omega: d(x,A)<h\}$. The isoperimetric profile[6] $I:=I_{(\Omega,\mu,d)}$ associated with $(\Omega,\mu,d)$, is the function $I:[0,\mu(\Omega))\to\mathbb{R}_{+}$, defined by

$$I(t)=\inf_{\mu(A)=t}\{\mu^{+}(A)\}.$$

We shall always assume that $I$ is concave, continuous, and zero at zero. In the case of finite measure spaces, i.e. when $\mu(\Omega)<\infty$, we shall also assume that $I$ is symmetric around $\mu(\Omega)/2$. In particular, in this case $I$ will be increasing on

[2]For example, if $M$ is compact, a natural normalization condition is $\int f d\mu=0$, and in the non-compact case (cf. [14]) it is natural to require that the functions have compact support. In this paper $Lip_0(\Omega)$, will always denote the set of Lip functions with compact support.

[3]For more on this we refer to [14] and Section 4 below.

[4]Roughly speaking "$\mu^{+}(A)=|||\nabla(\chi_A)|||_{L^1}$".

[5]We shall list further assumptions on $(\Omega,\mu,d)$ as needed.

[6]While isoperimetric profiles are very hard to compute exactly, most of the estimates in this paper hold true if replace $I$ by a lower bound estimator function, usually refered to as *an isoperimetric estimator* (cf. [21]).

$(0, \mu(\Omega)/2)$, and decreasing on $(\mu(\Omega)/2, \mu(\Omega))$. Moreover, if $\mu(\Omega) = \infty$, we assume that $I$ is increasing.

In what follows metric measure spaces having all the properties listed above shall be simply referred to as "**measure metric spaces**".

Our main focus will be on the validity of $L^1$ inequalities of the form[7]

$$\|f\|^2_{L^1(\Omega,\mu)} \leq c \, \||\nabla f|\|_{L^1(\Omega,\mu)} \, \|wf\|_{L^1(\Omega,\mu)} \, ,$$

for all smooth functions $f$ that satisfy suitably prescribed cancellations. As in [14] one of our main tools will be local Poincaré inequalities, but in our case, they are formulated using the isoperimetric profile, and are valid for arbitrary measurable sets, rather than balls. For example, if $\mu(\Omega) < \infty$, we localize the usual Poincaré inequality (cf. [8], [21]),

$$\int_\Omega |f - m(f)| \, d\mu \leq \frac{\mu(\Omega)}{2I(\mu(\Omega)/2)} \int_\Omega |\nabla f| \, d\mu, \tag{1.5}$$

as follows: For all $f \in Lip(\Omega)$ and for all measurable $A \subset \Omega$ we have (cf. Theorem 1 below)

$$\int_A |f - m(f)| \, d\mu \leq \frac{\min\{\mu(A), \mu(\Omega)/2\}}{I(\min\{\mu(A), \mu(\Omega)/2\})} \int_\Omega |\nabla f| \, d\mu, \tag{1.6}$$

where $m(f)$ is a median of $f$ (cf. Section 2).

A key role in our analysis is played by a class of weights, which we call *isoperimetric* weights. A positive measurable function $w : \Omega \to \mathbb{R}_+$ will be called an *isoperimetric weight* if there exists a constant $C = C(w)$ such that

$$\frac{\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\})} \leq Cr, \quad r > 0. \tag{1.7}$$

In particular, when $\mu(\Omega) = \infty$, the condition (1.7) takes the simpler form[8]

$$\mu(\{w \leq r\}) \leq CrI(\mu(\{w \leq r\})), \quad r > 0. \tag{1.8}$$

As a consequence, the growth of the measure of the level sets of isoperimetric weights is controlled by the isoperimetric profile associated with the geometry. To get some insight on the difference between (1.8) and (1.2) we shall now briefly compare them in the context of $\mathbb{R}^n$. The classical weight used for Euclidean uncertainty inequalities (cf. (1.1) above) is $w(x) = |x|$. For this weight, both conditions, (1.8) and (1.2), are satisfied, but the calculations needed for their verifications are different. Indeed, let $\mu_{R^n}$ denote the Lebesgue measure on $\mathbb{R}^n$, and let $w(x) = |x|$ , then we have

$$\mu_{\mathbb{R}^n}(\{w \leq r\}) = \mu_{\mathbb{R}^n}(\{|x| \leq r\}) = \beta_n r^n,$$

where $\beta_n$ is the measure of the unit ball; on the other hand, since $I_{\mathbb{R}^n}(r) = n\,(\beta_n)^{1/n}\, r^{(1-1/n)}$, we also have

$$\begin{aligned} rI_{\mathbb{R}^n}(\mu(\{w \leq r\})) &= rn\,(\beta_n)^{1/n}\,(\beta_n r^n)^{(1-1/n)} \\ &= n\beta_n r^n. \end{aligned}$$

[7]Below we will also develop methods to treat uncertantity inequalities for rather general rearrangement invariant norms.

[8]To understand the reason why condition (1.7) is slightly more complicated when $\mu(\Omega) < \infty$, note that if $\mu(\{w \leq r\}) = 1$, then, since $I(1) = 0$, $I(\mu(\{w \leq r\})) = 0$ and (1.8) has no meaning. A comparable phenomenon occurs with condition (1.2), which has no meaning when $r >$diameter of $M$.

Thus, $w(x) = |x|$ satisfies both (1.8) and (1.2). In fact, more generally, for geometries that satisfy the assumptions of [14] and, moreover, have concave isoperimetric profiles, we will show that if a weight $w$ satisfies (1.2) then it is an *isoperimetric weight* in our sense (cf. Section 5, Theorem 6).

For isoperimetric weights we will show (cf. Theorem 3 below) that there exists a universal constant $c = c(w)$ such that, for all suitably normalized Lipschitz functions $f$, the following uncertainty inequality holds

$$\|f\|_{L^1(\Omega,\mu)}^2 \leq c \, \||\nabla f|\|_{L^1(\Omega,\mu)} \|wf\|_{L^1(\Omega,\mu)} . \tag{1.9}$$

Moreover, a weak converse holds. Namely, if (1.9) holds for a given weight $w$, then it is easy to see that the growth of the measure of the level sets of $w$ must be controlled in some fashion by their corresponding perimeters. More precisely, we have (cf. Remark 1 below),

$$\mu(\{w \leq r\}) \leq cr\mu^+(\{w \leq r\}) .$$

From a technical point of view, the class of isoperimetric weights is useful for our development in this paper since these weights are directly related to the local Poincaré inequalities described above (cf. (1.6)). In fact, with these tools at hand, combined with interpolation, we are able to adapt the main argument of [14] to prove $L^1$ uncertainty inequalities in our setting. As it turns out, there is still a different characterization of the class of isoperimetric weights through the use of rearrangements. Indeed, we will show that isoperimetric weights are functions that belong to a Marcinkiewicz space whose fundamental function behaves essentially like $\frac{t}{I(t)}$. We then observe that, in view of a classical inequality of Hardy-Littlewood, the usual self improvements of Sobolev inequalities can be formulated as weighted norm inequalities, where the weights are precisely the isoperimetric weights! At this point we use interpolation theory to derive new uncertainty inequalities for rearrangement invariant norms.

Let $\Phi(t) := \Phi_I(t) = \frac{\min\{t,\mu(\Omega)/2)\}}{I(\min\{t,\mu(\Omega)/2)\})}$, $t \in (0, \mu(\Omega))$, then, since we assume that $I(t)$ is concave, the function $\Phi$ is non-decreasing. It can be readily seen (cf. Lemma 1 in Section 2.3 below) that $w$ is an isoperimetric weight if and only if $W := \frac{1}{w}$ belongs to the Marcinkiewicz space $M(\Phi) = M(\Phi)(\Omega, \mu)$, of functions on $\Omega$ such that

$$\|f\|_{M(\Phi)} = \sup_{t>0} t\Phi(\mu\{|f| > t\}) < \infty.$$

In terms of rearrangements (cf. [34] and Section 2.3 below) we can also write

$$\|f\|_{M(\Phi)} = \sup_{t>0} f_\mu^*(t)\Phi(t).$$

In other words, $w$ is an *isoperimetric weight* if and only $W := \frac{1}{w}$ satisfies

$$\|W\|_{M(\Phi)} = \sup_{t>0} W_\mu^*(t)\Phi(t) = \sup_{t>0} W_\mu^*(t)\frac{t}{I(t)} < \infty. \tag{1.10}$$

To set the stage for the more general developments we shall present in Section 6, let us briefly develop, in the more familiar Euclidean setting, the connection of uncertainty inequalities with sharp Sobolev inequalities, interpolation of weighted norm inequalities, and explain the rôle of the Marcinkiewicz space $M(\Phi)$. The key idea here is that the classical Gagliardo-Nirenberg inequality[9]

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{n\,(\beta_n)^{1/n}} \,\||\nabla f|\|_{L^1(\mathbb{R}^n)}\,,\; f \in Lip_0(\mathbb{R}^n),$$

self improves[10] to (cf. [36])

(1.11)

$$\|f\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} := \int_0^\infty f^*(s) s^{1-1/n} \frac{ds}{s} \leq \frac{n'}{(\beta_n)^{1/n}} \,\||\nabla f|\|_{L^1(\mathbb{R}^n)}\,,\; f \in Lip_0(\mathbb{R}^n).$$

This self improvement can be re-interpreted as an $L^1(\Omega, \mu)$ weighted inequality. Indeed, suppose that $W \in M(\Phi)(\mathbb{R}^n, \mu_{\mathbb{R}^n})$, where $\Phi(t) = t^{1/n}$, and $d\mu_{\mathbb{R}^n}(x) = dx$ is the Lebesgue measure. Then, for $f \in Lip_0(\mathbb{R}^n)$, we have,

$$\begin{aligned}
\|fW\|_{L^1(R^n)} &= \int_{\mathbb{R}^n} |f(x)|\, W(x) dx \\
&\leq \int_{R^n} f^*(t) W^*(t) dt \text{ (by the Hardy-Littlewood inequality)} \\
&\leq \|W\|_{M(\Phi)} \int_{\mathbb{R}^n} f^*(t) t^{1-1/n} \frac{dt}{t} \quad \text{(recall (1.10))} \\
&\leq \frac{n'}{(\beta_n)^{1/n}} \|W\|_{M(\Phi)} \,\||\nabla f|\|_{L^1(R^n)}\,,\; f \in Lip_0(\mathbb{R}^n) \text{ (by (1.11))}. \qquad (1.12)
\end{aligned}$$

A simple interpolation[11] between the inequality (1.12) and the trivial inequality

$$\int_{\mathbb{R}^n} |f(x)|\, w(x) dx \leq \int_{\mathbb{R}^n} |f(x)|\, w(x) dx,$$

yields[12]

$$\int_{\mathbb{R}^n} |f(x)|\, dx \leq c_n \|W\|_{M(\Phi)}^{1/2} \,\||\nabla f|\|_{L^1(\mathbb{R}^n)}^{1/2} \|fw\|_{L^1(\mathbb{R}^n)}^{1/2}\,.$$

The weighted norm inequality (1.12) appears already in [15], and corresponds to one of the end points of the Strichartz inequalities (cf. [35, Sec II, Theorem 3.6, page 1049]),

$$\|fW\|_{L^p(\mathbb{R}^n)} \leq c_n(p) \|W\|_{L^{n,\infty}} \,\||\nabla f|\|_{L^p(\mathbb{R}^n)}\,, \quad 1 < p < n, \quad f \in Lip_0(\mathbb{R}^n). \qquad (1.13)$$

---

[9] Here and in what follows we let

$$Lip_0(\Omega) = \{f \in Lip(\Omega) : f \text{ has compact support}\}.$$

[10] Note that

$$\|f\|_{L^{\frac{n}{n-1}}} \leq \frac{n}{n-1} \|f\|_{L^{\frac{n}{n-1},1}}\,.$$

[11] We always provide direct proofs of the interpolation argument using explicit decompositions. However, since the abstract method can be used to prove more general results and establish connections with other inequalities, we also briefly indicate a proof using abstract interpolation theory in Section 5.5 below.

[12] Recall that we are interpolating between $w$ and $w^{-1} = W$. Interpolation with a parameter $\theta$, yields a weighted norm inequality and the weight $w^{1-\theta}(\frac{1}{w})^\theta$ appears on $L^1$ norm on the left hand side. This explains the choice $\theta = \frac{1}{2}$, which eliminates the weight.

In fact, taking as a starting point the sharp Sobolev inequality of Hardy-Littlewood-O'Neil[13] [30]

$$\|f\|_{L^{\bar{p},p}(\mathbb{R}^n)} \leq c_n \, \||\nabla f|\|_{L^p(\mathbb{R}^n)}\,, \; \frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n},\; 1 \leq p < n, \tag{1.14}$$

and following the argument that led us to (1.12) gives a proof of (1.13) (cf. Section 5.6 below). More generally, it is perhaps a new observation that the corresponding weighted norm inequalities implied by the sharp Sobolev inequalities of [21] extend the Strichartz [35] and Faris [15] inequalities to the setting of r.i. norms (cf. Section 5.6 below). Finally, let us remark that this discussion gives another proof of the Euclidean space version of the uncertainty inequality of Dall'ara-Trevisan [14]. Indeed, for $w(x) = |x|$, then $W(x) = |x|^{-1}$, and[14]

$$c(W) = \left\| |x|^{-1} \right\|_{M(\Phi)} \approx \left\| |x|^{-1} \right\|_{L^{n,\infty}(\mathbb{R}^n)} = \beta^{1/n}.$$

As a bonus, this approach to (1.3) allow us to also replace the $L^1$ norms by rearrangement invariant norms. In fact, the argument can be also adapted to deal Besov type conditions (cf. Section 5.7 below). The connection with Marcinkiewicz spaces makes it also easy to actually construct isoperimetric weights for given geometries where we have a lower bound on the corresponding isoperimetric profiles, as we show with examples in Section 5 below. In particular, we show how to construct *isoperimetric weights* for Gaussian or more generally log concave measures (cf. Section 5.3).

For perspective, the connection between (the classical) Sobolev inequalities and Lorentz-Marcinkiewicz spaces $L^{p,\infty}$, has been known for a long time, and already appears, albeit implicitly, in the work of Hardy-Littlewood, and is already fully developed and exploited in the celebrated work of O'Neil on convolution inequalities (cf. [30]). It is in [30] that one finds explicitly the idea of using Marcinkiewicz spaces in order to treat abstractly convolution with potentials of the form $w = f(d(x, x_0))$. In our context, the "good weights" belong to Marcinkiewiecz spaces whose very definition is given in terms of the underlying isoperimetric profile. In the metric setting the use of weight functions of the form $w = f(d(x, x_0))$, where $d$ is the underlying metric, is classical (cf. [29] and the references therein). In particular, we remark that for geometries where the measure of a ball is independent of the radius[15], the functions of the form $w(x) = d(x_0, x)$, where $x_0$ is a fixed element of $\Omega$, trivially satisfy the condition (1.2) with equality.

We should also mention that the characterization of *isoperimetric weights* using Marcinkiewicz spaces also readily leads to a transference result for uncertainty inequalities which we formulate in Section 5.4 below.

Finally, and without any claim to completeness, we give a sample of recent references that treat uncertainty inequalities in different contexts and with different levels of generality, where the reader may find further references to the large literature in this field (cf. [10], [11], [24], [29], [31]), we also should mention the classical papers by Fefferman [16] and Beckner [3], [4].

[13] Comparing methods, in [15], (1.13) is proved directly and then (1.14) is obtained as a corollary

[14] Here the notation $f \approx g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq c\,f$.

[15] This condition fails for Gaussian measure (cf. [18], [38, Proposition 5.1, page 52.]).

## 2. Preliminaries

In this section we establish some further notation and background information and we provide more details about isoperimetric weights. Let $(\Omega, \mu, d)$ be a metric measure space as described in the Introduction.

### 2.1. Medians.

In this subsection we assume that $\mu(\Omega) < \infty$.

**Definition 1.** *Let $f : \Omega \to \mathbb{R}$ be an integrable function. We say that $m(f)$ is a median of $f$ if*

$$\mu\{f \geq m(f)\} \geq \mu(\Omega)/2; \text{ and } \mu\{f \leq m(f)\} \geq \mu(\Omega)/2.$$

For later use we record the following elementary estimate of the median, and provide the easy proof for the sake of completeness,

$$|m(f)| \leq \frac{2}{\mu(\Omega)} \int_{\Omega} |f|\, d\mu. \tag{2.1}$$

*Proof.* We use Chebyshev's inequality and the definition of median as follows. On the one hand,

$$\begin{aligned} m(f) &= m(f)\mu\{f \geq m(f)\} \frac{1}{\mu\{f \geq m(f)\}} \\ &\leq \frac{1}{\mu\{f \geq m(f)\}} \int_{\{f \geq m(f)\}} f d\mu \\ &\leq \frac{2}{\mu(\Omega)} \int_{\Omega} |f|\, d\mu. \end{aligned}$$

On the other hand, we similarly have

$$\begin{aligned} m(f) &= m(f)\mu\{f \leq m(f)\} \frac{1}{\mu\{f \leq m(f)\}} \\ &= \frac{m(f)}{\mu\{f \leq m(f)\}} \int_{\{f \leq m(f)\}} d\mu \\ &\geq \frac{1}{\mu\{f \leq m(f)\}} \int_{\{f \leq m(f)\}} f d\mu. \end{aligned}$$

Hence,

$$\begin{aligned} -m(f) &\leq \frac{1}{\mu\{f \leq m(f)\}} \int_{\{f \leq m(f)\}} -f d\mu \\ &\leq \frac{1}{\mu\{f \leq m(f)\}} \int_{\Omega} |f|\, d\mu \\ &\leq \frac{2}{\mu(\Omega)} \int_{\Omega} |f|\, d\mu. \end{aligned}$$

Combining estimates (2.1) follows. □

### 2.2. Rearrangement invariant spaces.

Let $u : \Omega \to \mathbb{R}$, be a measurable function. The **distribution function** of $u$ is given by

$$\mu_u(t) = \mu\{x \in \Omega : |u(x)| > t\} \qquad (t \geq 0).$$

The **decreasing rearrangement** of a function $u$ is the right-continuous non-increasing function from $[0,\mu(\Omega))$ into $\mathbb{R}^{+}$ which is equimeasurable with $u$. It can be defined by the formula

$$u_{\mu}^{*}(s)=\inf\{t\geq 0:\mu_{u}(t)\leq s\},\quad s\in[0,\mu(\Omega)),$$

and satisfies

$$\mu_{u}(t)=\mu\{x\in\Omega:|u(x)|>t\}=m\left\{s\in[0,\mu(\Omega)):u_{\mu}^{*}(s)>t\right\},\ \ t\geq 0,$$

(where $m$ denotes the Lebesgue measure on $[0,\mu(\Omega))$. It follows from the definition that

$$(u+v)_{\mu}^{*}(s)\leq u_{\mu}^{*}(s/2)+v_{\mu}^{*}(s/2). \tag{2.2}$$

The maximal average $u_{\mu}^{**}(t)$ is defined by

$$u_{\mu}^{**}(t)=\frac{1}{t}\int_{0}^{t}u_{\mu}^{*}(s)ds=\frac{1}{t}\sup\left\{\int_{E}|u(s)|\,d\mu:\mu(E)=t\right\},\ t>0.$$

The operation $u\rightarrow u_{\mu}^{**}$ is sub-additive, i.e.

$$(u+v)_{\mu}^{**}(s)\leq u_{\mu}^{**}(s)+v_{\mu}^{**}(s), \tag{2.3}$$

and moreover,

$$\int_{0}^{t}(uv)_{\mu}^{*}(s)ds\leq\int_{0}^{t}u_{\mu}^{*}(s)v_{\mu}^{*}(s)ds,\ t>0. \tag{2.4}$$

On occasion, when rearrangements are taken with respect to the Lebesgue measure or when the measure is clear from the context, we may omit the measure and simply write $u^{*}$ and $u^{**}$, etc.

We now recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces and refer the reader to [5] for a complete treatment. We say that a Banach function space[16] $X=X(\Omega)$, on $(\Omega,d,\mu)$, is a rearrangement-invariant (r.i.) space, if $g\in X$ implies that all $\mu-$measurable functions $f$ with the same rearrangement with respect to the measure $\mu$, i.e. such that $f_{\mu}^{*}=g_{\mu}^{*}$, also belong to $X$, and, moreover, $\|f\|_{X}=\|g\|_{X}$.

When dealing with r.i. spaces we will always assume that $(\Omega,d,\mu)$ is resonant in the sense of [5, Definition 2.3, pag 45].

For any r.i. space $X(\Omega)$ we have

$$L^{\infty}(\Omega)\cap L^{1}(\Omega)\subset X(\Omega)\subset L^{1}(\Omega)+L^{\infty}(\Omega),$$

with continuous embeddings. In particular, if $\mu$ is finite, then

$$L^{\infty}(\Omega)\subset X(\Omega)\subset L^{1}(\Omega).$$

An r.i. space $X(\Omega)$ can be represented by a r.i. space on the interval $(0,\mu(\Omega))$, with Lebesgue measure, $\bar{X}=\bar{X}(0,\mu(\Omega))$, such that (see [5, Theorem 4.10 and subsequent remarks])

$$\|f\|_{X}=\|f_{\mu}^{*}\|_{\bar{X}},$$

for every $f\in X$. Typical examples of r.i. spaces are the $L^{p}$-spaces, Lorentz spaces, Marcinkiewicz spaces and Orlicz spaces.

[16]We use the definition of Banach function space that one can find in [?] which, in particular, assumes that the spaces have the Fatou property.

A useful property of r.i. spaces states that if

$$\int_0^t |f|_\mu^* (s)ds \leq \int_0^t |g|_\mu^* (s)ds,$$

holds for all $t > 0$, and $X$ is a r.i. space, then,

$$\|f\|_X \leq \|g\|_X \,. \tag{2.5}$$

2.3. **Isoperimetric weights.** We give a formal discussion of the notion of isoperimetric weight and its characterization in terms of Marcinkiewicz spaces.

**Definition 2.** *We will say that a locally integrable function $w : \Omega \to \mathbb{R}_+$ is an* ***isoperimetric weight****, if $w > 0$ a.e., and there exists a constant $C := C(w) > 0$ such that*

$$\frac{\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\})} \leq Cr, \quad r > 0. \tag{2.6}$$

*It is easy to see that (2.6) is equivalent to*

$$\frac{\min\{\mu(\{w < r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w < r\}), \mu(\Omega)/2)\})} \leq Cr, \quad r > 0. \tag{2.7}$$

In what follows we write

$$C(w) := \sup_{r>0} \frac{1}{r} \frac{\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\})}.$$

Let $\Phi(t) = \frac{\min\{t,\mu(\Omega)/2\}}{I(\min\{t,\mu(\Omega)/2)\})}$, $t \in (0, \mu(\Omega))$. The **Marcinkiewicz** $M(\Phi)(\Omega)$ is defined by the condition

$$\|f\|_{M(\Phi)} = \sup_{t>0} t\Phi(\mu_f(t)) < \infty.$$

Since $f_\mu^*$ and $\mu_f$ are generalized inverses of each other a simple argument (cf. [34]) shows that $f \in M(\Phi)(\Omega)$ if and only if

$$\|f\|_{M(\Phi)} = \sup_{t>0} f_\mu^*(t)\Phi(t) < \infty.$$

The previous discussion leads to the following

**Lemma 1.** *$w$ is an isoperimetric weight if and only if $W := \frac{1}{w} \in M(\Phi)(\Omega)$.*

*Proof.* From (2.7) and the fact that $\mu(\{w < r\}) = \mu_W(\frac{1}{r})$, we have

$$\begin{aligned} C(w) &= \sup_{r>0} \frac{1}{r} \frac{\min\{\mu(\{w < r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w < r\}), \mu(\Omega)/2)\})} \\ &= \sup_{r>0} \frac{1}{r} \frac{\min\{\mu_W(\frac{1}{r}), \mu(\Omega)/2\}}{I(\min\{\mu_W(\frac{1}{r}), \mu(\Omega)/2)\})} \\ &= \sup_{r>0} r\Phi\mu_W(r) \\ &= \|W\|_{M(\Phi)} \,. \end{aligned}$$

□

**Remark 1.** *In some sense we don't have too many choices of weights in order for uncertainty inequalities to be true. For example, suppose that* $\mu(\Omega)=\infty$*, and* $w$ *is a weight such that*

$$\|f\|_{L^1(\Omega)}^2 \leq c \, \||\nabla f|\|_{L^1(\Omega)} \, \|wf\|_{L^1(\Omega)} \tag{2.8}$$

*holds for all* $f \in Lip_0(\Omega)$*. Suppose that* $\mu(\{w \leq t\}) < \infty$*, then,*

$$\mu(\{w \leq t\}) \leq ct\mu^+(\{w \leq t\}). \tag{2.9}$$

*Proof.* We can select $f_n \in Lip_0(\Omega)$ such that

$$\||\nabla f_n|\|_{L^1(\Omega)} \to \mu^+(\{w \leq t\})$$

while

$$\|wf_n\|_{L^1(\Omega)} \to \int_{\{w \leq t\}} w d\mu$$

and

$$\|f_n\|_{L^1(\Omega)}^2 \to \mu^2(\{w \leq t\}).$$

Inserting this information back in (2.8), we have

$$\begin{aligned} \mu^2(\{w \leq t\}) &\leq c\mu^+(\{w \leq t\}) \int_{\{w \leq t\}} w d\mu \\ &\leq c\mu^+(\{w \leq t\}) t\mu(\{w \leq t\}), \end{aligned}$$

and (2.9) follows. □

We now prove the localized Poincaré inequality described in the Introduction.

**Theorem 1.** *Suppose that* $\mu(\Omega) < \infty$*. Then, for all* $f \in Lip(\Omega)$*, and for all measurable* $A \subset \Omega$*, we have*

$$\int_A |f - m(f)| \, d\mu \leq \frac{\min\{\mu(A), \mu(\Omega)/2\}}{I(\min\{\mu(A), \mu(\Omega)/2\})} \int_\Omega |\nabla f| \, d\mu. \tag{2.10}$$

Before giving the simple proof we recall the following important consequence of the co-area formula and the definition of isoperimetric profile (cf. [8], [21]).

**Theorem 2.** *(i) Suppose that* $\mu(\Omega) < \infty$ *(resp.* $\mu(\Omega) = \infty$*). Then, for all* $f \in Lip(\Omega)$ *(resp. for all* $f \in Lip_0(\Omega)$*), we have*

$$\int_{-\infty}^{\infty} I(\mu(\{f > s\}))ds \leq \int_\Omega |\nabla f| \, d\mu. \tag{2.11}$$

*Proof.* **(of Theorem 1)** Let $f \in Lip(\Omega)$, and let $A \subset \Omega$ be a measurable set. We compute

$$\begin{aligned} \int_A |f - m(f)| \, d\mu &= \int_{A \cap \{f \geq m(f)\}} (f - m(f)) \, d\mu + \int_{A \cap \{f < m(f)\}} (m(f) - f) \, d\mu \\ &= \int_0^\infty \mu(\{f - m(f) > s\} \cap A)ds + \int_0^\infty \mu(\{m(f) - f \geq s\} \cap A)ds \\ &= \int_{m(f)}^\infty \mu(\{f > s\} \cap A)ds + \int_{-\infty}^{m(f)} \mu(\{f \leq s\} \cap A)ds. \\ &= (I) + (II). \end{aligned} \tag{2.12}$$

We estimate $(I)$. Suppose that $s > m(f)$. We claim that

$$\mu(\{f > s\} \cap A) \leq \min\{\mu(A), \mu(\Omega)/2\}. \tag{2.13}$$

It is plain that (2.13) will follow if can show that $\mu(\{f>s\}) \leq \mu(\Omega)/2$. Suppose, to the contrary, that $\mu(\{f>s\}) > \mu(\Omega)/2$. Then, since $\{f>s\}$ and $\{f \leq m(f)\}$ are disjoint sets, $\mu(\{f>s\})+\mu(\{f \leq m(f)\}) \leq \mu(\Omega)$. It follows that $\mu(\{f \leq m(f)\}) < \mu(\Omega)/2$, which is impossible since $m(f)$ is a median of $f$.

From (2.13), and the fact that $t/I(t)$ increases, we get

$$\mu(\{f>s\}\cap A\}) \leq I(\mu(\{f>s\}\cap A\}))\frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}. \tag{2.14}$$

Since $I$ is increasing on $(0,\mu(\Omega)/2]$, and $\mu(\{f>s\}) \leq \mu(\Omega)/2$, we see that

$$I(\mu(\{f>s\}\cap A\})) \leq I(\mu(\{f>s\})).$$

Updating (2.14) we have

$$\mu(\{f>s\}\cap A\}) \leq I(\mu(\{f>s\}))\frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2)\})}.$$

Integrating we obtain

$$(I) \leq \frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}\int_{m(f)}^{\infty} I(\mu(\{f>s\}))ds. \tag{2.15}$$

In a similar way we can estimate $(II)$

$$(II) \leq \frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}\int_{-\infty}^{m(f)} I(\mu(\{f \leq s\}))ds. \tag{2.16}$$

Inserting the estimates (2.15) and (2.16) in (2.12) we obtain

$$\int_A |f-m(f)|\,d\mu \leq \frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}\left(\int_{m(f)}^{\infty} I(\mu(\{f>s\}))ds + \int_{-\infty}^{m(f)} I(\mu(\{f\leq s\}))ds\right). \tag{2.17}$$

We now show that the integrals inside the parenthesis can be combined. Indeed, when $s < m(f)$ we have $\mu(\{f \leq s\}) < \mu(\Omega)/2$, therefore, by the symmetry of $I$ around the point $\mu(\Omega)/2$, we find that

$$I(\mu(\{f\leq s\})) = I(\mu(\Omega)\diagdown\mu(\{f\leq s\})) = I(\mu(\Omega\diagdown\{f\leq s\})) = I(\mu(\{f>s\})).$$

Whence,

$$\int_{-\infty}^{m(f)} I(\mu(\{f\leq s\}))ds = \int_{-\infty}^{m(f)} I(\mu(\{f>s\}))ds.$$

Inserting the last equality in (2.17) yields

$$\begin{aligned}\int_A |f-m(f)|\,d\mu &\leq \frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}\int_{-\infty}^{\infty} I(\mu(\{f>s\}))ds \\ &\leq \frac{\min\{\mu(A),\mu(\Omega)/2\}}{I(\min\{\mu(A),\mu(\Omega)/2\})}\int_\Omega |\nabla f|\,d\mu \quad \text{(by (2.11))},\end{aligned}$$

as we wished to show. □

**Remark 2.** *Note that (2.10) reduces to (1.5) when* $A=\Omega$. *Moreover, since*

$$\frac{1}{2}\int_\Omega \left| f - \frac{1}{\mu(\Omega)}\int_\Omega f d\mu\right| d\mu \leq \int_\Omega |f-m(f)|\,d\mu,$$

*we also get*

$$\int_\Omega \left| f - \frac{1}{\mu(\Omega)}\int_\Omega f d\mu\right| d\mu \leq \frac{\mu(\Omega)}{I(\mu(\Omega)/2)}\int_\Omega |\nabla f|\,d\mu.$$

## 3. $L^1$ Uncertainty Inequalities via local Poincaré inequalities

Let $(\Omega, \mu, d)$ be a metric measure space. In this section we prove our main result concerning $L^1$ uncertainty inequalities.

**Theorem 3.** *Let $w$ be an isoperimetric weight, and let $\alpha > 0$. Suppose that $\mu(\Omega) < \infty$ (resp. $\mu(\Omega) = \infty$), then, for all $f \in Lip(\Omega)$, with $m(f) = 0$ or $\int_\Omega f d\mu = 0$ (resp. for all $f \in Lip_0(\Omega)$), we have*

$$\|f\|_1 \leq 2C(w)r \, \||\nabla f|\|_1 + r^{-\alpha} \|w^\alpha f\|_1 \,, \text{ for all } r > 0. \tag{3.1}$$

*Proof.* **Case of finite measure.**

Suppose that $f \in Lip(\Omega)$. We consider each normalization separately.

(i) Suppose $m(f) = 0$. Then, for all $r > 0$, we have

$$\begin{aligned}
\int_\Omega |f| \, d\mu &= \int_{\{w \leq r\}} |f| \, d\mu + \int_{\{r<w\}} |f| \, d\mu \\
&= \int_{\{w \leq r\}} |f - m(f)| \, d\mu + \int_{\{w>r\}} |f| \, d\mu \\
&\leq \frac{\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\}}{I(\min\{\mu(\{w \leq r\}), \mu(\Omega)/2)\})} \int_\Omega |\nabla f| \, d\mu + \int_{\{w>r\}} |f| \, d\mu \quad \text{(by (2.10))} \\
&\leq C(w)r \int_\Omega |\nabla f| \, d\mu + \int_{\{w>r\}} |f| \, d\mu \text{ (since } w \text{ is an isoperimetric weight)} \\
&= C(w)r \int_\Omega |\nabla f| \, d\mu + \int_{\{w>r\}} \left(\frac{w}{w}\right)^\alpha |f| \, d\mu \\
&\leq C(w)r \int_\Omega |\nabla f| \, d\mu + r^{-\alpha} \int_\Omega w^\alpha |f| \, d\mu,
\end{aligned}$$

as desired.

(ii) Suppose that $\int_\Omega f d\mu = 0$. Let $r > 0$, and write

$$\begin{aligned}
\int_\Omega |f| \, d\mu &= \int_{\{w \leq r\}} |f| \, d\mu + \int_{\{w>r\}} |f| \, d\mu \\
&\leq \int_{\{w \leq r\}} |f - m(f)| + |m(f)| \, \mu(\{w \leq r\}) + \int_{\{w>r\}} |f| \\
&= A(r) + B(r) + C(r).
\end{aligned}$$

The terms $A(r)$ and $C(r)$ can be estimated as the corresponding terms in (i), above. To estimate the remaining term we proceeds as follows:

$$\begin{aligned}
B(r) = |m(f)| \, \mu(\{w \leq r\}) &= \left| m(f) - \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right| \mu(\{w \leq r\}) \\
&\leq \frac{\mu(\{w \leq r\})}{\mu(\Omega)} \int_\Omega |f - m(f)| \, d\mu \\
&\leq \frac{\mu(\{w \leq r\})}{2I(\mu(\Omega)/2)} \int_\Omega |\nabla f| \, d\mu \quad \text{(by (1.5))}.
\end{aligned}$$

Since $I$ increases on $(0, \mu(\Omega)/2)$ and decreases on $(\mu(\Omega)/2, \mu(\Omega))$, we have

$$I(\mu(\{w \leq r\})) \leq I(\mu(\Omega)/2).$$

Consequently,

$$\begin{aligned}\frac{\mu(\{w\leq r\})}{2I(\mu(\Omega)/2)} &\leq \frac{\mu(\{w\leq r\})}{2I(\mu(\{w\leq r\})} \\ &\leq C(w)r \text{ (since } w \text{ is an isoperimetric weight).}\end{aligned}$$

It follows that

$$B(r)\leq C(w)r\int_{\Omega}|\nabla f|\,d\mu.$$

Combining the previous estimates the desired result follows.

**Case of infinite measure.** $f\in Lip_0(\Omega)$. For all $r>0$ we write

$$\begin{aligned}\|f\|_1 &= \int_{\{w\leq r\}}|f|\,d\mu+\int_{\{w>r\}}|f|\,d\mu \\ &= (I)+(II). \end{aligned}\tag{3.2}$$

The term $(II)$ can be estimated exactly as in the previous case. It remains to estimate

$$(I)=\int_0^{\infty}\mu(\{|f|>s\}\cap\{w\leq r\})ds.$$

The integrand can be estimated as follows,

$$\begin{aligned}\mu(\{|f|>s\}\cap\{w\leq r\}) &\leq I(\mu(\{|f|>s\}\cap\{w\leq r\}))\frac{\mu(\{w\leq r\})}{I(\mu(\{w\leq r\}))} \text{ (since } \frac{s}{I(s)} \text{ increases)} \\ &\leq C(w)rI(\mu(\{|f|>s\}\cap\{w\leq r\})) \text{ (since } w \text{ is an isoperimetric weight)} \\ &\leq C(w)rI(\mu(\{|f|>s\})) \text{ (since } I \text{ increases).}\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\{w\leq r\}}|f|\,d\mu &\leq C(w)r\int_0^{\infty}I(\mu(\{|f|>s\}))ds \\ &\leq C(w)r\int_{\Omega}|\nabla|f||\,d\mu \quad \text{(by (2.11))} \\ &\leq C(w)r\int_{\Omega}|\nabla f|\,d\mu.\end{aligned}$$

Inserting the estimates that we have obtained for $(I)$ and $(II)$ into (3.2) gives the desired result. □

**Remark 3.** *Selecting the value* $r=\left(\frac{\|w^{\alpha}f\|_1}{2C(w)\||\nabla f|\|_1}\right)^{\frac{1}{1+\alpha}}$ *to compute (3.1) balances the two terms and we obtain the multiplicative inequality*

$$\|f\|_1\leq(2C(w))^{\frac{\alpha}{\alpha+1}}\||\nabla f|\|_1^{\frac{\alpha}{\alpha+1}}\|w^{\alpha}f\|_1^{\frac{1}{\alpha+1}},$$

*for all* $f\in Lip(\Omega)$ *such that* $m(f)=0$, *or* $\int_{\Omega}fd\mu=0$ *if* $\mu(\Omega)<\infty$ *(or for all* $f\in Lip_0(\Omega)$, *if* $\mu(\Omega)=\infty$*).*

Following closely the chain rule argument used in [14, Section 6.5] we now show that Theorem 3 implies the corresponding $L^p$ version of itself. More precisely, we have

**Theorem 4.** *Let $w$ be an isoperimetric weight, let $p>1$, and let $\alpha>0$. Suppose that $\mu(\Omega)<\infty$ (resp. $\mu(\Omega)=\infty$), then, there is a constant $D=D(w,p,\mu(\Omega))$ (resp. $D=D(w,p)$ in case that $\mu(\Omega)=\infty$) such that, for all $f\in Lip(\Omega)$ with $m(f)=0$, or $\int_\Omega f d\mu=0$ (resp. for all $f\in Lip_0(\Omega)$), we have*

$$\|f\|_p^p \leq Dr \left\||\nabla f|\right\|_p + r^{-\alpha} \left\|w^\alpha |f|\right\|_p, \qquad r>0.$$

*Proof.* The main technical difficulty to prove the Theorem is that, as we attempt to apply Theorem 3 by replacing $f$ with powers of $f$, we may loose the required normalizations in the process. Let us first record the following well known elementary version of the chain rule for power functions, which is valid in the metric setting,

$$|\nabla |f|^p (x)| \leq 2p\, |f(x)|^{p-1} |\nabla |f| (x)| \leq 2p\, |f(x)|^{p-1} |\nabla f(x)|\,,\ f\in Lip(\Omega),$$

and

$$\left|\nabla \left(f\, |f|^{p-1}\right)(x)\right| \leq 2p\, |f(x)|^{p-1} |\nabla f(x)|\,,\ f\in Lip(\Omega).$$

Applying Hölder's inequality we find

$$\left\||\nabla |f|^p|\right\|_1 \leq 2p \left\||\nabla f|\, |f|^{p-1}\right\|_1 \leq 2p \left\||\nabla f|\right\|_p \|f\|_p^{p-1}, \tag{3.3}$$

and

$$\left\|\nabla \left(f\, |f|^{p-1}\right)\right\|_1 \leq 2p \left\||\nabla f|\right\|_p \|f\|_p^{p-1}. \tag{3.4}$$

Let us also note here that Hölder's inequality gives us,

$$\left\|w^\alpha |f|^p\right\|_1 = \left\|w^\alpha |f|\, |f|^{p-1}\right\|_1 \leq \left\|w^\alpha |f|\right\|_p \|f\|_p^{p-1}. \tag{3.5}$$

We now divide the proof in several cases. The easiest case to deal with the normalizations issue is when $\mu(\Omega)=\infty$. Indeed, if $f\in Lip_0(\Omega)$, we also have $|f|^p\in Lip_0(\Omega)$. Thus, by (3.1) we have,

$$\left\||f|^p\right\|_1 \leq 2C(w)r \left\||\nabla |f|^p|\right\|_1 + r^{-\alpha} \left\|w^\alpha |f|^p\right\|_1,\ \text{for all } r>0.$$

Then, from (3.3) and (3.5), we get,

$$\|f\|_p^p \leq 4C(w)p \left\||\nabla f|\right\|_p \|f\|_p^{p-1} + \left\|w^\alpha |f|\right\|_p \|f\|_p^{p-1},\ \text{for all } r>0,$$

and the desired result follows.

Suppose that $\mu(\Omega)<\infty$. Let $f\in Lip(\Omega)$. It is not difficult to verify that $m(f)\, |m(f)|^{p-1}$ is a median of $f\, |f|^{p-1}$ (cf. [14]). To take advantage of this fact we estimate $\left\||f|^p\right\|_1$ as follows

$$\begin{aligned}
\int_\Omega |f|^p\, d\mu &= \int_{\{w\leq r\}} |f|^p\, d\mu + \int_{\{r<w\}} |f|^p\, d\mu \\
&\leq \int_{\{w\leq r\}} \left|f\, |f|^{p-1} - m(f)\, |m(f)|^{p-1}\right| + \left|\int_{\{w\leq r\}} (m(f)-f)\, d\mu\right| |m(f)|^{p-1} \mu(\{w\leq r\}) \\
&+ \int_{\{w>r\}} |f|^p \\
&= A(r)+B(r)+C(r).
\end{aligned}$$

$A(r)$ can be estimated using the local Poincaré inequality, the fact that $w$ is an isoperimetric weight and (3.3):

$$\begin{aligned} A(r) &\leq C(w)r \left|\left|\left|\nabla\left(f\left|f\right|^{p-1}\right)\right|\right|\right|_1 \\ &\leq 2pC(w)r \left\|\left|\nabla f\right|\right\|_p \left\|f\right\|_p^{p-1} \text{ (by (3.4)).} \end{aligned}$$

The term $C(r)$ can be readily estimated in a familiar fashion:

$$\begin{aligned} C(r) &\leq r^{-\alpha} \left\|w^\alpha \left|f\right|^p\right\|_1 \\ &\leq \left\|w^\alpha \left|f\right|\right\|_p \left\|f\right\|_p^{p-1} \text{ (by (3.5)).} \end{aligned}$$

It remains to estimate $B(r)$, and for this purpose we consider two cases. If $m(f) = 0$, then $B(r) = 0$ and we are done. Suppose now that $\int_\Omega f d\mu = 0$. We estimate each of the factors of $B(r)$. Using (1.5) and Hölder's inequality, we get

$$\begin{aligned} \left|\int_{\{w\leq r\}} (m(f) - f)\, d\mu\right| &\leq \int_\Omega \left|m(f) - f\right| d\mu \\ &\leq \frac{\mu(\Omega)}{2I(\mu(\Omega)/2)} \int_\Omega \left|\nabla f\right| d\mu \\ &\leq \frac{\mu(\Omega)}{2I(\mu(\Omega)/2)} \left(\int_\Omega \left|\nabla f\right|^p d\mu\right)^{1/p} \mu\left(\Omega\right)^{\frac{1}{p'}}. \end{aligned}$$

To estimate $\left|m(f)\right|^{p-1}$ we observe that $m(f)\left|m(f)\right|^{p-2}$ is a median of $f\left|f\right|^{p-2}$. Therefore, by (2.1) and Hölder's inequality,

$$\begin{aligned} \left|m(f)\right|^{p-1} &= \left|m(f)\left|m(f)\right|^{p-2}\right| \\ &\leq \frac{2}{\mu(\Omega)} \int_\Omega \left|f\left|f\right|^{p-2}\right| d\mu \\ &= \frac{2}{\mu(\Omega)} \int_\Omega \left|f\right|^{p-1} d\mu \\ &\leq \frac{2}{\mu(\Omega)} \left(\int_\Omega \left|f\right|^p d\mu\right)^{1/p'} \mu(\Omega)^{1/p}. \end{aligned}$$

Inserting these estimates we find

$$\begin{aligned} B(r) &\leq \frac{\mu(\Omega)}{2I(\mu(\Omega)/2)} \left(\int_\Omega \left|\nabla f\right|^p d\mu\right)^{1/p} \frac{2}{\mu(\Omega)} \left(\int_\Omega \left|f\right|^p d\mu\right)^{1/p'} \mu(\Omega)^{1/p} \mu\left(\Omega\right)^{\frac{1}{p'}} \mu(\{w \leq r\}) \\ &\leq C(w)\mu\left(\Omega\right) r \left\|\left|\nabla f\right|\right\|_p \left\|f\right\|_p^{p-1}. \end{aligned}$$

Combining estimates we see that

$$\left\|f\right\|_p^p \leq C(w)(2p + \mu\left(\Omega\right))r \left\|\left|\nabla f\right|\right\|_p \left\|f\right\|_p^{p-1} + r^{-\alpha} \left\|w^\alpha \left|f\right|\right\|_p \left\|f\right\|_p^{p-1},$$

as desired. □

**Remark 4.** *If we select* $r = \left(\frac{\|w^\alpha f\|_p}{D\||\nabla f|\|_p}\right)^{\frac{1}{1+\alpha}}$ *to compute (3.1), then we obtain the multiplicative inequality*

$$\left\|f\right\|_p \leq D^{\frac{p\alpha p}{p\alpha + p}} \left\|\left|\nabla f\right|\right\|_p^{\frac{\alpha}{\alpha+1}} \left\|w^\alpha f\right\|_p^{\frac{1}{\alpha+1}},$$

*for all* $f \in Lip(\Omega)$ *such that* $m(f) = 0$ *or* $\int_\Omega f d\mu = 0$ *if* $\mu(\Omega) < \infty$ *(or for all* $f \in Lip_0(\Omega)$ *if* $\mu(\Omega) = \infty$*).*

## 4. Isoperimetric weights vs Dall'Ara-Trevisan weights

Dall'Ara-Trevisan [14] proved versions of Theorems 3 and 4, for homogeneous spaces[17] $M$, and for weights $w : M \to \mathbb{R}^+$ that satisfy the growth condition

$$\mu(\{w \leq r\}) \leq \Upsilon_M(r) := \mu(B(r)). \tag{4.1}$$

In this section we compare the weights in the Dall'Ara-Trevisan class with the corresponding isoperimetric weights defined on $M$. In preparation for this task let us introduce some notation and recall useful information.

In this section we shall consider homogeneous spaces $M$ that satisfy the assumptions of [14] and, moreover, are metric measure spaces in the sense of the present paper[18]. We shall simply refer to these spaces as *homogeneous metric measure spaces*.

We now recall the weak isoperimetric inequality used in [14] (cf. also [13]).

**Theorem 5.** *Let $M$ be an homogeneous space satisfying the assumptions of* [14]*. Then the following statements hold.*

(1) *Suppose that $M$ is non-compact. Then, there exists $C > 0$ such that for all $r > 0$, and for all Borel sets $A \subset M$ such that $\mu(A) \leq \Upsilon_M(r)$, we have*

$$\mu(A) \leq Cr\mu^+(A). \tag{4.2}$$

(2) *Suppose that $M$ is compact. Then,*

*(i) There exists $C > 0$ such that for $r > 0$, and all Borel sets $E$ with $\min\{\mu(E), \mu(E^c)\} \leq \frac{\Upsilon_M(r)}{2}$, we have*

$$\min\{\mu(E), \mu(E^c)\} \leq Cr\mu^+(E).$$

*(ii) If $\mu(E) \leq \Upsilon_M(r)$, then, for all $r > 0$, and for all Borel sets $A \subset M$ such that $\mu(A) \leq \mu(M)/2$,*

$$\mu(A \cap E) \leq Cr\mu^+(A). \tag{4.3}$$

**Theorem 6.** *Let $M$ be an homogeneous metric measure space. Then, the class of isoperimetric weights contains the class of weights satisfying the growth condition (4.1).*

The next result will be useful in the proof.

**Lemma 2.** *(i) Let $M$ be an homogeneous metric measure space. Then, there exists $C > 0$ such that, for all $r > 0$, with $\Upsilon_M(r) \leq \frac{\mu(M)}{2}$, it holds that*

$$\Upsilon_M(r) \leq CrI(\Upsilon_M(r)). \tag{4.4}$$

*In particular, if $\Upsilon_M(r) \leq \frac{\mu(M)}{2}$, then, for any Borel set $E$ such that $\mu(E) \leq \Upsilon_M(r)$,*

$$\mu(E) \leq CrI(\mu(E)). \tag{4.5}$$

*Proof.* Let $r > 0$ be fixed. Suppose that $\Upsilon_M(r) \leq \frac{\mu(M)}{2}$, and let $A \subset M$ be a Borel set such that

$$\mu(A) = \Upsilon_M(r).$$

[17] We refer to the Introduction for the basic assumptions on $M$, and [14] for complete details.

[18] In particular, we assume that the isoperimetric profile $I_{(M,\mu,d)} := I$ satisfies the usual assumptions.

Using Theorem 5 we see that

$$\Upsilon_M(r) \leq rC\mu^+(A).$$

Indeed, if $M$ is compact this follows directly from (4.3), while in the non compact case we can use (4.2), with $E = A$, to arrive to same conclusion. Taking infimum we obtain,

$$\begin{aligned}\Upsilon_M(r) &\leq rC \inf\{\mu^+(A) : \mu(A) = \Upsilon_M(r)\} \\ &= rCI(\Upsilon_M(r)),\end{aligned}$$

and therefore (4.4) holds.

Suppose now that $0 < \mu(E) \leq \Upsilon_M(r) \leq \frac{\mu(M)}{2}$. Then, since $t/I(t)$ increases, we have

$$\frac{\mu(E)}{I(\mu(E))} \leq \frac{\Upsilon_M(r)}{I(\Upsilon_M(r))} \leq Cr,$$

as desired. □

We can now proceed with the proof of Theorem 6.

*Proof.* We assume that $w$ is a weight such that $\mu(\{w \leq r\}) \leq \Upsilon_M(r)$. We are aiming to prove

$$\min\left\{\mu(\{w \leq r\}), \frac{\mu(M)}{2}\right\} \leq CrI\left(\min\{\mu(\{w \leq r\}), \frac{\mu(M)}{2}\}\right). \tag{4.6}$$

**Case I**: Compact case. (i) Suppose that $\Upsilon_M(r) \leq \frac{\mu(M)}{2}$. Then by (4.5)

$$\mu(\{w \leq r\}) \leq CrI(\{\mu(\{w \leq r\}).$$

Since our assumptions on $w$ and $\Upsilon_M(r)$ force $\mu(\{w \leq r\}) \leq \frac{\mu(M)}{2}$, we see that (4.6) holds.

(ii) Suppose that $\Upsilon_M(r) > \frac{\mu(M)}{2}$. Suppose also that $\mu(\{w \leq r\}) > \frac{\mu(M)}{2}$. Then, since $t/I(t)$ increases,

$$\begin{aligned}\frac{\frac{\mu(M)}{2}}{I(\frac{\mu(M)}{2})} &\leq \frac{\mu(\{w \leq r\})}{I(\mu(\{w \leq r\})} \\ &\leq \frac{CrI(\mu(\{w \leq r\}))}{I(\mu(\{w \leq r\})} \text{ (by (4.5))} \\ &= Cr.\end{aligned}$$

In other words, (4.6) holds in this case as well.

(iii) It remains to consider the case $\Upsilon_M(r) > \frac{\mu(M)}{2}$, $\mu(\{w \leq r\}) \leq \frac{\mu(M)}{2}$. By [14] the function $\Upsilon_M(s)$ is continuous and increasing. Let $r'$ be such that $\Upsilon_M(r') = \frac{\mu(M)}{2}$. Since $t/I(t)$ increases,

$$\begin{aligned}\frac{\mu(\{w \leq r\})}{I(\mu(\{w \leq r\}))} &\leq \frac{\Upsilon_M(r')}{I(\Upsilon_M(r'))} \\ &\leq \frac{Cr'I(\Upsilon_M(r'))}{I(\Upsilon_M(r'))} \text{ (by (4.4))} \\ &\leq Cr \text{ (since } r' \leq r).\end{aligned}$$

Therefore, (4.6) holds in this case as well, concluding the proof of Case I.

**Case II**: Non compact case. In this case we must have $\mu(M)=\infty$ (cf Remark 5 below), then $\Upsilon_M(r)\leq\frac{\mu(M)}{2}$, and $\mu(\{w\leq r\})\leq\frac{\mu(M)}{2}$, therefore we see that (4.6) holds by (4.5). $\square$

**Remark 5.** *In* [14] *the dichotomy for the normalization conditions is given in terms of whether the space $M$ is compact or not. On the other hand, the assumptions of* [14] *force $\mu(M)<\infty$, when $M$ is compact and $\mu(M)=\infty$, if $M$ is not compact. Indeed, in Section 4.1 of* [14] *the authors show that for $M$ non compact, $\Upsilon_M(r)<\infty$, for all $r>0$, and that $\Upsilon_M(r)+\Upsilon_M(s)\leq\Upsilon_M(r+s)$, for all $r,s>0$. In particular, $n\Upsilon_M(1)\leq\Upsilon_M(n)$, for all $n\in\mathbb{N}$, and therefore we have $\Upsilon_M(n)\rightarrow\infty$.*

## 5. Examples and applications

Let $(\Omega,\mu,d)$ be a metric measure space. We will use the following general scheme to construct isoperimetric weights in different settings. Let $g:[0,\mu(\Omega)]\rightarrow[0,\infty)$, be such that $g>0$ a.e., and

$$\sup_{0<t<\mu(\Omega)}g^{*}(t)\frac{\min\{t,\mu(\Omega)/2\}}{I(\min\{t,\mu(\Omega)/2\}}<\infty, \tag{5.1}$$

where the rearrangement is taken with respect to the Lebesgue measure on $[0,\mu(\Omega)]$. It is known (cf. [5, Corollary 7.8 pag. 86]) that there exists a measure-preserving transformation $\sigma:\Omega\rightarrow[0,\mu(\Omega)]$ such that $g^{*}\circ\sigma$ and $g^{*}$are equimeasurable. In particular,

$$\left(g^{*}\circ\sigma\right)_{\mu}^{*}(t)=g^{*}(t)\qquad t\in[0,\mu(\Omega)].$$

It follows that the function

$$w(x)=\frac{1}{g^{*}(\sigma(x))},\quad x\in\Omega,$$

is an isoperimetric weight. Indeed, by Lemma 1, $W(x)=\frac{1}{w(x)}=g^{*}(\sigma(x))\in M(\Phi)(\Omega)$. Consequently,

$$\left\Vert f\right\Vert_{p}\leq D^{\frac{p\alpha p}{p\alpha+p}}\left\Vert\left\vert\nabla f\right\vert\right\Vert_{p}^{\frac{\alpha}{\alpha+1}}\left\Vert\left(\frac{1}{g^{*}\circ\sigma}\right)^{\alpha}f\right\Vert_{p}^{\frac{1}{\alpha+1}},$$

for all $f\in Lip(\Omega)$, such that $m(f)=0$ or $\int_{\Omega}fd\mu=0$ if $\mu(\Omega)<\infty$ (or for all $f\in Lip_{0}(\Omega)$ if $\mu(\Omega)=\infty$).

In the examples below we consider specific metric measure spaces and make explicit calculations following the above scheme.

### 5.1. Euclidean case.

5.1.1. $\mathbb{R}^{n}$ *with Lebesgue measure.* The isoperimetric profile is given by

$$I_{\mathbb{R}^{n}}(r)=n\left(\beta_{n}\right)^{1/n}r^{(1-1/n)}.$$

**Theorem 7.** *Let $g:[0,\infty)\rightarrow[0,\infty)$ be such that $g>0$ a.e., and, moreover, suppose that*

$$\sup_{0<t<\infty}g^{*}(t)t^{1/n}<\infty.$$

*Let $w:\mathbb{R}^{n}\rightarrow\mathbb{R}^{+}$, be defined by*

$$w(x)=\frac{1}{g^{*}(\beta_{n}\left\vert x\right\vert^{n})}.$$

*Then, $w$ is an isoperimetric weight.*

*Proof.* Let $W = \frac{1}{w}$. Then, $W(x) = g^*(\beta_n |x|^n)$ and $W^*(t) = g^*(t)$ (cf. [37]). Consequently, $W \in M(\Phi)(\mathbb{R}^n)$, and the result follows. □

5.1.2. *The closed upper half of Euclidean space $\mathbb{R}^n$.* For simplicity we assume that $n = 2$.

Let $H_2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. By reflection across the boundary of $H_2$, combined with the classical isoperimetric inequality in $\mathbb{R}^2$, it follows that the corresponding isoperimetric profile, $I_{H_2}$, is given by (cf. [37])

$$I_{H_2}(t) = \beta_2^{1/2} t^{1/2}.$$

**Theorem 8.** *Let $g : [0, \infty) \to [0, \infty)$ be such that $g > 0$ a.e., and*

$$\sup_{0<t<\infty} g^*(t) t^{1/2} < \infty.$$

*Let $k > 0$, and let $w : H_2 \to \mathbb{R}^+$ be defined by*

$$w(x) = \frac{1}{g^*\left(\frac{1}{k+1} B\left(\frac{k+1}{2}, \frac{1}{2}\right) (x^2 + y^2)^{k/2+1}\right)},$$

*where $B$ denotes the Euler beta function. Then, $w$ is an isoperimetric weight.*

*Proof.* Let $W = \frac{1}{w}$. Then, $W(x) = g^*\left(\frac{1}{k+1} B\left(\frac{k+1}{2}, \frac{1}{2}\right) \left(x^2 + y^2\right)^{k/2+1}\right)$ and $W^*(t) = g^*(t)$ (cf. [37]). It follows that $W \in M(\Phi)(\mathbb{R}^n)$. □

5.2. **The unit sphere.** Let $n \geq 2$ be an integer, and let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. For each $n \geq 2$, the $n-$dimensional Hausdorff measure of $\mathbb{S}^n$ is given by $\omega_n = 2\pi^{\frac{n+1}{2}} / \Gamma(\frac{n+1}{2})$. On $\mathbb{S}^n$ we consider the geodesic distance $d$, and the uniform probability measure $\sigma_n$. For $\theta \in [-\pi/2, \pi/2]$, let

$$\varphi_n(\theta) = \frac{\omega_{n-1}}{\omega_n} \cos^{n-1} \theta \quad \text{and} \quad \Phi_n(\theta) = \int_{-\pi/2}^{\theta} \varphi_n(s) ds.$$

It is known that the isoperimetric profile of the sphere $I_{\mathbb{S}^n}$ coincides with $I_n = \varphi_n \circ \Phi_n^{-1}$ (cf. [1]).

**Theorem 9.** *Let $g : [0, 1] \to [0, \infty)$ be such that $g > 0$ a.e., and*

$$\sup_{0<t<1} g^*(t) \frac{\min\{t, 1/2\}}{I_n(\min\{t, 1/2\})} < \infty,$$

*where the rearrangement is taken with respect to the Lebesgue measure on $[0, 1]$. Let $w : \mathbb{S}^n \to \mathbb{R}^+$, be defined by*

$$w(\theta_1, .....\theta_{n+1}) = \frac{1}{g^*(\Phi_n(\theta_1))}.$$

*Then, $w$ is an isoperimetric weight.*

*Proof.* Let

$$W(\theta) = \frac{1}{w(\theta_1, ....., \theta_{n+1})} := g^*(\Phi_n(\theta_1)), \qquad (\theta_1, .....\theta_{n+1}) \in \mathbb{S}^n.$$

We need to show that $W = \frac{1}{w} \in M(\Phi)\left(\mathbb{S}^n, \mu\right)$. Let $m_f(t)$ denote the distribution function of $f$ with respect to the Lebesgue measure on $[0,1]$, and let $\mu := \sigma_n$. Then, $W$ and $g$ are equimeasurable. Indeed,

$$\begin{aligned} \mu_W(t) &= \mu\{\theta \in \mathbb{S}^n : W(x) > t\} \\ &= \mu\{\theta \in \mathbb{S}^n : g^*(\Phi_n(\theta_1)) > t\} \\ &= \mu\{\theta \in \mathbb{S}^n : \Phi_n(\theta_1) \leq m_g(t)\} \\ &= \mu\{\theta \in \mathbb{S}^n : \theta_1 \leq \Phi_n^{-1}(m_g(t))\} \\ &= m_g(t). \end{aligned}$$

Therefore,

$$W_\mu^*(t) = g^*(t).$$

Consequently, in view of our assumptions on $g$, $W = \frac{1}{w} \in M(\Phi)\left(\mathbb{S}^n, \mu\right)$. □

5.3. **Log concave measures.** We consider product measures on $\mathbb{R}^n$ that are constructed using the measures on $\mathbb{R}$ defined by the densities

$$\mathbf{d}\mu_\Psi(x) = Z_\Psi^{-1} \exp\left(-\Psi(|x|)\right) dx = \varphi(x)dx, \qquad x \in \mathbb{R},$$

where $\Psi$ is convex, $\sqrt{\Psi}$ concave, $\Psi(0) = 0$, and such that $\Psi$ is $\mathcal{C}^2$ on $[\Psi^{-1}(1), +\infty)$, and where, moreover, $Z_\Psi^{-1}$ is chosen to ensure that $\mu_\Psi(\mathbb{R}) = 1$.

Let $H : \mathbb{R} \to (0,1)$ be the distribution function of $\mu_\Psi$, i.e.

$$H(r) = \int_{-\infty}^{r} \varphi(x)dx = \mu_\Psi(-\infty, r). \tag{5.2}$$

It is known that the isoperimetric profile for $(\mathbb{R}, d_n, \mu_\Psi)$ is given by (cf. [9] and [7])

$$I_{\mu_\Psi}(t) = \varphi\left(H^{-1}(\min\{t, 1-t\}\right) = \varphi\left(H^{-1}(t)\right), \qquad t \in [0,1].$$

We shall denote by $\mu_\Psi^{\otimes n} = \underbrace{\mu_\Psi \otimes \mu_\Psi \otimes ... \otimes \mu_\Psi}_{n \text{ times}}$, the product probability measures on $\mathbb{R}^n$. It is known that the isoperimetric profiles $I_{\mu_\Psi^{\otimes n}}$ are dimension free (cf. [2]): there exists a universal constant $c(\Psi)$ such that for all $n \in \mathbb{N}$

$$c(\Psi) I_{\mu_\Psi}(t) \leq I_{\mu_\Psi^{\otimes n}}(t) \leq I_{\mu_\Psi}(t).$$

**Theorem 10.** *Let $g : [0,1] \to [0,\infty)$ be such that $g > 0$ a.e., and*

$$\sup_{0<t<1} g^*(t) \frac{\min\{t, 1/2)\}}{I_{\mu_\Psi}(\min\{t, 1/2\})} < \infty, \tag{5.3}$$

*where the rearrangement is taken with respect to the Lebesgue measure on $[0,1]$. Let $w : \mathbb{R}^n \to \mathbb{R}^+$ be defined by,*

$$w(x_1, .....x_n) = \frac{1}{g^*(H(x_1))}.$$

*Then, $w$ is an isoperimetric weight.*

*Proof.* Let

$$W(x) = \frac{1}{w(x_1, .....x_n)} := g^*(H(x_1)), \qquad x \in \mathbb{R}^n.$$

A calculation, similar to the one given during the course of the proof of Theorem 9, shows that $W$ and $g$ are equimeasurable. Thus,

$$W_\mu^*(t) = g^*(t),$$

and we see that $W \in M(\Phi)(\mathbb{R}^n)$, as we wished to show. $\square$

**Example 1.** *The prototype function $g$ that satisfies (5.3) is given by $g(t) = \frac{I_{\mu_\Psi}(\min\{t,1/2\})}{\min\{t,1/2\}}$. In fact, since $\frac{I_{\mu_\Psi}(t)}{t}$ decreases, we see that $g^*(t) = g(t)$. In particular, for $\Psi(|x|) = \frac{|x|^p}{p}$, $p \in [1,2]$, the isoperimetric profile $I_{\mu_\Psi}(t)$ satisfies (cf.* [37]*)*

$$I_{\mu_\Psi}(t) \simeq t\left(\log\frac{1}{t}\right)^{1-1/p}, \quad 0 < t \leq 1/2.$$

*Thus, if we let $g : [0,1] \to [0,\infty)$ be such that $g > 0$ a.e., and suppose that*

$$\sup_{0<t<1/2} g^*(t)\frac{1}{\left(\log\frac{1}{t}\right)^{1-1/p}} < \infty,$$

*it follows that the function*

$$W(x_1, .....x_n) = \frac{1}{w(x_1, .....x_n)} := g^*\left(Z_\Psi^{-1}\int_{-\infty}^{x_1}\exp\left(\frac{-|x_1|^p}{p}\right)dx_1\right)$$

*is an isoperimetric weight.*

5.4. **Transference.** We indicate a simple transference result that follows directly from the characterization of isoperimetric weights in terms of Marcinkiewicz spaces. Suppose that $(\Omega_i, \mu_i, d_i), i = 1, 2$, are two measure metric spaces as above, such that, moreover with $\mu_1(\Omega_1) = \mu_2(\Omega_2)$, and $I_1 := I_{(\Omega_1,\mu_1,d_1)} \geq I_2 := I_{(\Omega_2,\mu_2,d_2)}$. Then, the corresponding uncertainty inequalities for $(\Omega_2, \mu_2, d_2)$ can be transferred to $(\Omega_1, \mu_1, d_1)$ in the sense that, for all $f \in Lip(\Omega_1)$ that satisfy suitably prescribed cancellations[19],

(5.4)

$$\|f\|_{L^1(\Omega_1,\mu_1)} \leq C\|w\|_{M(\Phi_2)}^{1/2}\||\nabla f|\|_{L^1(\Omega_1,\mu_1)}^{1/2}\|wf\|_{L^1(\Omega_1,\mu_1)}^{1/2}, w \in M(\Phi_2)(\Omega_1, \mu_1),$$

where $\Phi_i(t) = \frac{t}{I_i(t)}$.

Indeed, if $I_1 \geq I_2$, then $M(\Phi_2) \subset M(\Phi_1)$. Therefore, the result follows from Lemma 1 and Remark 3.

5.5. **A connection with interpolation theory.** While many of the arguments we have used in this paper have been inspired by real interpolation theory, we have avoided making explicit use of the general theory. In this subsection we give a brief, somewhat informal, discussion of the formal connection with interpolation theory, since the method is useful for other related inequalities and generalizations. In particular, as another application of the method we discuss briefly a recent result by E. Milman [27].

Our general reference for all unexplained notation and background will be [6].

We consider a pair of weights, i.e. positive functions $w_i : \Omega \to [0,\infty)$, and consider the $K-$functional

[19] The key point of this transfer is that, by abuse of notation, we have "switched" the isoperimetric weights of $(\Omega_1, \mu_1)$ by the "isoperimetric weights" of $(\Omega_2, \mu_2)$. In this sense the transfer is apparently connected with the construction of representations of the space $M(\Phi)$ for different metric measure spaces. It would be interesting to study the connection of our transference result with the recent results on the transport of weighted Poincaré inequalities (cf. [12]).

$$\begin{aligned}K(t,f;L^1_{w_0},L^1_{w_1}) &= \inf_{f=f_0+f_1,f_i\in L^1_{w_i}}\{\|f_0\|_{L^1_{w_0}}+t\|f_1\|_{L^1_{w_1}}\}\\ &= \inf_{f=f_0+f_1,f_i\in L^1_{w_i}}\left\{\int_\Omega |f_0|\,w_0d\mu+t\int_\Omega |f_1|\,w_1d\mu\right\}.\end{aligned}$$

It is well known and easy to see that (cf. [6])

$$K(t,f;L^1_{w_0},L^1_{w_1})=\int_\Omega |f|\min\{w_0,tw_1\}d\mu. \tag{5.5}$$

Indeed, for any decomposition $f=f_0+f_1,\ f_i\in L^1_{w_i},\ i=0,1$, and for all $t>0$, we have

$$\begin{aligned}\int_\Omega |f|\min\{w_0,tw_1\}d\mu &\leq \int_\Omega |f_0|\min\{w_0,tw_1\}d\mu+\int_\Omega |f_1|\min\{w_0,tw_1\}d\mu\\ &\leq \int_\Omega |f_0|\,w_0d\mu+t\int_\Omega |f_1|\,w_1d\mu\\ &= \|f_0\|_{L^1_{w_0}}+t\|f_1\|_{L^1_{w_1}}.\end{aligned}$$

Consequently

$$\begin{aligned}\int_\Omega |f|\min\{w_0,tw_1\}d\mu &\leq \inf_{f=f_0+f_1}\{\|f_0\|_{L^1_{w_0}}+t\|f_1\|_{L^1_{w_1}}\}\\ &= K(t,f;L^1_{w_0},L^1_{w_1}).\end{aligned}$$

The converse inequality follows using the decomposition

$$f=f\chi_{\{w_0\leq tw_1\}}+f\chi_{\{w_0>tw_1\}}. \tag{5.6}$$

Let us define

$$\|f\|_{(L^1_{w_0},L^1_{w_1})_{1/2,1,K}}:=\int_0^\infty K(t,f;L^1_{w_0},L^1_{w_1})t^{-1/2}\frac{dt}{t}$$

and

$$(L^1_{w_0},L^1_{w_1})_{1/2,1,K}=\{f:\|f\|_{(L^1_{w_0},L^1_{w_1})_{1/2,1,K}}<\infty\}.$$

Using (5.5) and Fubini's theorem, we readily see that

$$\|f\|_{(L^1_{w_0},L^1_{w_1})_{1/2,1,K}}\approx\int_\Omega |f|\,w_0^{1/2}w_1^{1/2}d\mu.$$

In other words (cf. [6]) [20]

$$(L^1_{w_0},L^1_{w_1})_{1/2,1,K}=L^1_{w_0^{1/2}w_1^{1/2}}.$$

In particular, if we let $w_0=\frac{1}{w},w_1=w$, we have

$$\left(L^1_{\frac{1}{w}},L^1_w\right)_{1/2,1,K}=L^1. \tag{5.7}$$

A different class of Lions-Peetre interpolation functors can be constructed using the so called $J-$functional leading, in particular, to the $\left(L^1_{\frac{1}{w}},L^1_w\right)_{1/2,1,J}$ spaces.

[20] This is part of a family of interpolation inequalities

$$(L^1_{w_0},L^1_{w_1})_{\alpha,1,K}=L^1(w_0^{1-\alpha}w_1^{\alpha}),\ \alpha\in(0,1).$$

Without entering into the specifics of this classical construction we mention two important consequences for our example. Firstly, we have with norm equivalence that (cf. [6])

$$\left(L^1_{\frac{1}{w}}, L^1_w\right)_{1/2,1,K} = \left(L^1_{\frac{1}{w}}, L^1_w\right)_{1/2,1,J}. \tag{5.8}$$

The interpolation result we use in this paper is not (5.7) proper but the following consequence of (5.8). The $\left(L^1_{\frac{1}{w}}, L^1_w\right)_{1/2,1,J}$ spaces have the special embedding property

$$\left(L^1_{\frac{1}{w}}, L^1_w\right)_{1/2,1,J} \subset L^1$$

is equivalent to (cf. [6, page 49 equation 1])

$$\|f\|_{L^1} \leq c \|f\|^{1/2}_{L^1_{\frac{1}{w}}} \|f\|^{1/2}_{L^1_w}, \text{ for all } f \in L^1_{\frac{1}{w}} \cap L^1_w, \tag{5.9}$$

where $c$ is an absolute constant.

Consequently, from (5.7) and (5.8) we see that (5.9) holds. Therefore, if we know that for all suitably normalized smooth $f$, with $\||\nabla f|\|_{L^1} < \infty$, we have

$$\|f\|_{L^1_{\frac{1}{w}}} \leq C \||\nabla f|\|_{L^1},$$

we arrive to the following uncertainty inequality: for all suitably normalized smooth $f$, with $\||\nabla f|\|_{L^1} < \infty, f \in L^1_w$ we have

$$\|f\|_{L^1} \leq \tilde{c} \||\nabla f|\|^{1/2}_{L^1} \|f\|^{1/2}_{L^1_w}.$$

The analysis described above can be extended in different directions. In particular, let us note that for any Banach lattice of functions $X$ on $(\Omega, \mu)$ we have the following extension of (5.5)

$$K(t, f; X_{w_0}, X_{w_1}) = \||f| \min\{w_0, tw_1\}\|_X,$$

where

$$\|f\|_{X_w} = \||f| w\|_X.$$

In retrospect one can see that the weighted norm estimates in this paper were all based on the use of decompositions of the type (5.6).

We give another illustration of the interpolation method outlining briefly, and rather informally, an interpolation approach to some of the recent new functional inequalities of E. Milman [27].

**Example 2.** *In this illustration we shall proceed formally and focus on the derivation of a functional inequality presented in* [27]*, ignoring the background and the underlying assumptions. Let* $\alpha \in (0,1)$, *and suppose that the inequality*

$$\|f\|_{L^{\alpha,1}(\Omega)} \leq c \||\nabla f|\|_{L^1(\Omega)} \tag{5.10}$$

*holds for a class of suitably normalized Lip functions. Then (cf.* [27]*),*

$$\|f\|_{L^1} \leq C \||\nabla f|\|^{\alpha}_{L^1} \|f\|^{1-\alpha}_{L^\infty}. \tag{5.11}$$

*Proof.* One can approach the result using explicit decompositions to compute the corresponding $K-$functional

$$K(t, f; L^{\alpha,1}, L^\infty) = \inf_{f = f_0 + f_1, f_0 \in L^{1,\alpha}, f_1 \in L^\infty} \{\|f_0\|_{L^{\alpha,1}} + t \|f_1\|_{L^\infty}\}.$$

The actual computation is again known. We have (cf. [6])

$$K(t,f;L^{\alpha,1},L^{\infty})\approx\int_0^{t^{\alpha}}f_{\mu}^{*}(s)s^{1/\alpha}\frac{ds}{s}.$$

Then, if we let

$$\|f\|_{(L^{\alpha,1},L^{\infty})_{1-\alpha.1;K}}:=\int_0^{\infty}K(t,f;L^{\alpha,1},L^{\infty})t^{-(1-\alpha)}\frac{dt}{t},$$

it follows readily by Fubini that we have, with norm equivalence,

$$\begin{aligned}\left(L^{\alpha,1},L^{\infty}\right)_{1-\alpha.1;K}&=\{f:\|f\|_{(L^{\alpha,1},L^{\infty})_{1-\alpha.1;K}}<\infty\}\\&=L^{1}.\end{aligned}$$

Moreover, since we also have (with norm equivalence that depends on $\alpha$)

$$\left(L^{\alpha,1},L^{\infty}\right)_{1-\alpha.1;J}=\left(L^{\alpha,1},L^{\infty}\right)_{1-\alpha.1;K}=L^{1}$$

we see that

$$\|f\|_{L^{1}(\Omega)}\leq\|f\|_{L^{\alpha,1}(\Omega)}^{\alpha}\|f\|_{L^{\infty}(\Omega)}^{1-\alpha}.$$

Therefore, from the validity of (5.10) we can deduce the validity of (5.11) for a suitable class of smooth normalized functions. □

One can, of course, deal with more general spaces, in particular replacing the $L^{\alpha,1}$ spaces by more general quasi Banch spaces, and use slightly more general versions of the real method of interpolation to obtain estimates of the form $\|f\|_{L^{\infty}(\Omega)}^{1}\Psi(\frac{\|f\|_{X(\Omega)}}{\|f\|_{L^{\infty}(\Omega)}})$, with $\Psi$ concave, on the right hand side. We shall leave a fuller discussion of such matters for another ocassion.

5.6. **Strichartz inequalities and Sobolev inequalities.** The initial step of the interpolation process that leads to uncertainty inequalities is directly connected with the following inequality that one finds in Strichartz [35]. A different proof with sharp constants [21] was later found by Faris [15]

$$\|fg\|_{L^{p}(\mathbb{R}^{n})}\leq c_{n}(p)\|g\|_{L^{n,\infty}}\||\nabla f|\|_{L^{p}(\mathbb{R}^{n})},\ f\in C_{0}^{\infty}(\mathbb{R}^{n}),\ 1\leq p<n.$$

These inequalities are connected with the classical Sobolev inequality via the sharp form of the Sobolev inequality involving Lorentz spaces (cf. [36] and the references therein),

(5.12)

$$\left\{\int_0^{\infty}(f^{*}(s)s^{1/\bar{p}})^{p}\frac{ds}{s}\right\}^{1/p}\leq C_{n}(p)\||\nabla f|\|_{L^{p}(\mathbb{R}^{n})},\ f\in C_{0}^{\infty}(\mathbb{R}^{n}),\ 1\leq p<n,\ \frac{1}{\bar{p}}=\frac{1}{p}-\frac{1}{n}.$$

[21] Here we shall not be concerned with the best value of the constants involved but we do note that $c_{n}(p)\to\infty$ as $p\to n$.

Indeed, suppose that $f \in C_0^\infty(\mathbb{R}^n), g \in L^{n,\infty}$, and let $1 \leq p < n$; then

$$\begin{aligned}
\|fg\|_{L^p(\mathbb{R}^n)}^p &\leq \int_0^\infty f^*(s)^p g^*(s)^p ds \\
&\leq \|g\|_{L^{n,\infty}}^p \int_0^\infty f^*(s)^p s^{-p/n} ds \\
&= \|g\|_{L^{n,\infty}}^p \int_0^\infty f^*(s)^p s^{p/\bar{p}} \frac{ds}{s} \\
&\leq C_n(p)^p \|g\|_{L^{n,\infty}}^p \||\nabla f|\|_{L^p(\mathbb{R}^n)}^p .
\end{aligned}$$

Faris' method is closely connected with the above presentation. Note that in the context of (5.12) the class of isoperimetric weights can be described as

$$\{w : \frac{1}{w} \in M(\Phi)\} = \{w : \frac{1}{w} \in L^{n,\infty}\} = \{w : \frac{1}{w} \text{ is a Strichartz multiplier}\}.$$

When $p \to n$ the constant $C_n(p)$ blows up. The sharp end point result for $p = n$ is provided by the Brezis-Wainger inequality. Let $\Omega$ be a domain in $\mathbb{R}^n$. Then, for all functions $f \in C_0^\infty(\Omega)$,

$$\left\{ \int_0^{|\Omega|} \frac{f^*(s)^n}{(1 + \log \frac{|\Omega|}{s})^n} \frac{ds}{s} \right\}^{1/n} \leq c_n \||\nabla f|\|_{L^n(\mathbb{R}^n)} .$$

Thus, in this case, the corresponding Strichartz inequality holds if we replace the condition "$g \in L^{n,\infty}(\Omega)$" by "$g \in L_{\log}(n, \infty)(\Omega)$", where

$$\|g\|_{L_{\log}(n,\infty)(\Omega)} = \sup_t \left\{ g^*(t) t^{1/n} (1 + \log \frac{|\Omega|}{s}) \right\}.$$

In this notation we have,

$$\|fg\|_{L^n(\mathbb{R}^n)}^n \leq C_n(n)^n \|g\|_{L_{\log}(n,\infty)(\Omega)}^n \||\nabla f|\|_{L^n(\mathbb{R}^n)}^n .$$

More generally, let us consider Sobolev inequalities on a metric measure space $(\Omega, \mu, d)$ using rearrangement invariant norms. Let $I$ be an isoperimetric estimator for $(\Omega, \mu, d)$ and on measurable functions on $(0, \mu(\Omega))$ let us define the isoperimetric Hardy operator $\tilde{Q}_I$ by

$$\tilde{Q}_I f(t) = \frac{I(t)}{t} \int_t^{\mu(\Omega)/2} f(s) \frac{ds}{I(s)}.$$

We give the details for the case $\mu(\Omega) = \infty$, the case $\mu(\Omega) < \infty$ follows mutatis mutandi. From Theorem 11, part 2, below,

$$\left\| f_\mu^*(t) \frac{I(t)}{t} \right\|_{\bar{X}} \leq \left\| \tilde{Q}_I \right\|_{\bar{X} \to \bar{X}} \||\nabla f|\|_X , \quad f \in Lip_0(\Omega).$$

We can then reinterpret the last inequality as a weighted norm inequality ("the Strichartz inequality in $X$") : for $f \in Lip_0(\Omega)$, and $g \in M(\Phi)$,

$$\begin{aligned}
\|fg\|_X &\leq \left\| f_\mu^*(t) g_\mu^* \right\|_{\bar{X}} \\
&\leq \|g\|_{M(\Phi)} \left\| f_\mu^*(t) \frac{I(t)}{t} \right\|_{\bar{X}} \\
&\leq \|g\|_{M(\Phi)} \left\| \tilde{Q}_I \right\|_{\bar{X} \to \bar{X}} \||\nabla f|\|_X .
\end{aligned}$$

5.7. **Besov Inequalities.** In this brief section we indicate how the inequalities can be extended to suitable Besov spaces. Here we are aiming to illustrate the method rather than to prove the most general results. Thus, we shall focus on $(\Omega, d, \mu) = \mathbb{R}^n$, and $L^q$ spaces.

Our starting point is the following equivalence (cf. [6]) which here we may take as a definition: Let $1 \leq q \leq \infty, 0 < s < 1$,

$$\left\| \frac{t^{-s/n}\omega_q(t^{1/n}, f)}{t^{1/q}} \right\|_{L^q(0,\infty)} \approx \|f\|_{\mathring{B}^s_{q,q}(\mathbb{R}^n)},$$

where $\omega_q$ is the $q-$modulus of continuity defined by

$$\omega_q(t, f) = \sup_{|h| \leq t} \|f(\circ + h) - f(\circ)\|_{L^q(\mathbb{R}^n)}.$$

Suppose that $f^{**}(\infty) = 0$, then following estimate is well known (cf. [5], [20] and the references therein)

$$f^{**}(t) \leq c \int_t^\infty \frac{\omega_q(s^{1/n}, f)}{s^{1/q}} \frac{ds}{s}. \tag{5.13}$$

Let $w$ be an isoperimetric weight (i.e. $\frac{1}{w} \in L^{n,\infty}$). Let $\alpha > 0$, $1 \leq q < \infty$, $0 < \theta < 1$, with $\theta < n/q$. Then the following estimate holds,

$$\|f\|_{L^q} \leq cr \|f\|_{\mathring{B}^\theta_{q,q}(\mathbb{R}^n)} + r^{-\alpha} \left\|w^{\theta\alpha} f\right\|_{L^q},$$

where $c > 0$ is an absolute constant. Indeed, following a familiar argument we have

$$\begin{aligned}
\|f\|_{L^q} &= \left\| f\left(\frac{w}{w}\right)^\theta \right\|_{L^q} \leq \left\| f\left(\frac{w}{w}\right)^\theta \chi_{\left\{w \leq r^{1/\theta}\right\}} \right\|_{L^q} + \left\| f\left(\frac{w}{w}\right)^\theta \chi_{\left\{w > r^{1/\theta}\right\}} \right\|_{L^q} \\
&\leq r \left\| f\left(\frac{1}{w}\right)^\theta \right\|_{L^q} + \left\| f\left(\frac{w}{w}\right)^{\theta\alpha} \chi_{\left\{w > r^{1/\theta}\right\}} \right\|_{L^q} \\
&\leq r \left\| f\left(\frac{1}{w}\right)^\theta \right\|_{L^q} + r^{-\alpha} \left\|w^{\alpha\theta} f\right\|_{L^q}.
\end{aligned}$$

It remains to estimate the first term,

$$\begin{aligned}
\left\| f\left(\frac{1}{w}\right)^{\theta}\right\|_{L^q} &\leq \left\| f^*(t)\left(\left(\frac{1}{w}\right)^*(t)\right)^{\theta} t^{\theta/n}t^{-\theta/n}\right\|_{L^q} \\
&\leq \left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \left\| f^*(t)t^{-\theta/n}\right\|_{L^q} \\
&\leq \left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \left\| f^{**}(t)t^{-\theta/n}\right\|_{L^q} \\
&\leq c\left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \left\| t^{-\theta/n}\int_t^{\infty}\frac{\omega_q(s^{1/n},f)}{s^{1/q}}\frac{ds}{s}\right\|_{L^q} \quad \text{(by (5.13))} \\
&= c\left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \left\| t^{-\theta/n}\int_t^{\infty}\left(\frac{s^{-\theta/n}\omega_q(s^{1/n},f)}{s^{1/q}}\right)s^{\theta/n}\frac{ds}{s}\right\|_{L^q} \\
&\leq C\left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \left\| \frac{s^{-\theta/n}\omega_q(s^{1/n},f)}{s^{1/q}}\right\|_{L^q} \quad \text{(since } \theta<n/q) \\
&= C\left\|\frac{1}{w}\right\|_{L^{n,\infty}}^{\theta} \|f\|_{\mathring{B}^{\theta}_{q,q}(\mathbb{R}^n)}\,.
\end{aligned}$$

Using the inequalities of [23] it is possible to extend these results to Besov spaces on metric spaces but the development is too long and technical, and falls outside the scope of the present paper.

## 6. Rearrangement Invariant Uncertainty Inequalities

In this section we obtain uncertainty inequalities modeled on (3.1), where $L^1$ is replaced by a suitable r.i. space.

Our approach is based on the following Sobolev inequalities for r.i. spaces (cf. [21] where results of this type were obtained with more restrictions on the ambient measure space).

**Theorem 11.** *Let $X$ be an r.i. space on $\Omega$ such that $\tilde{Q}$ is bounded on $\bar{X}$. The following statements hold*

(1) *Suppose that $\mu(\Omega)<\infty$. Then,*

$$c_{X,I}=\left\|\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\right\|_{\bar{X}}<\infty, \tag{6.1}$$

*and for all bounded functions in $f\in Lip(\Omega)$, we have*

$$\left\| f_{\mu}^*(t)\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\right\|_{\bar{X}} \leq \left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}} \left\|\left|\nabla f\right|\right\|_X + \frac{2c_{X,I}}{\mu(\Omega)}\int_{\Omega}|f(x)|\,d\mu. \tag{6.2}$$

(2) *If $\mu(\Omega)=\infty$, then*

$$\left\| f_{\mu}^*(t)\frac{I(t)}{t}\right\|_{\bar{X}} \leq \left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}} \left\|\left|\nabla f\right|\right\|_X, \qquad f\in Lip_0(\Omega). \tag{6.3}$$

The proof of this theorem will be given at the end of this section. First we consider the corresponding uncertainty inequalities.

**Theorem 12.** *Let $X$ be an r.i. space on $\Omega$ such that $\tilde{Q}$ is bounded on $\bar{X}$. Let $w$ be an isoperimetric weight and let $\alpha > 0$, Then, there exists a constant $C = C(w, X)$ such that*

$$\|f\|_X \leq rC \, \||\nabla f|\|_X + r^{-\alpha} \|w^\alpha f\|_X \,, \text{ for all } r > 0.$$

*for all bounded $f \in Lip(\Omega)$ such that $m(f) = 0$ or $\int_\Omega f d\mu = 0$ if $\mu(\Omega) < \infty$ (or for all $f \in Lip_0(\Omega)$ if $\mu(\Omega) = \infty$).*

*Proof.* a. Suppose that $\mu(\Omega) < \infty$. Let $f \in Lip(\Omega)$, and let $w$ be an isoperimetric weight. Then, for all $\alpha > 0$, we have

$$\begin{aligned} \|f\|_X = \left\| f\frac{w}{w} \right\|_X &\leq \left\| f\frac{w}{w}\chi_{\{w\leq r\}} \right\|_X + \left\| f\frac{w}{w}\chi_{\{w>r\}} \right\|_X \\ &\leq r \left\| f\frac{1}{w} \right\|_X + r^{-\alpha} \|w^\alpha f\|_X \,. \end{aligned} \tag{6.4}$$

To estimate the first term in (6.4), we write

$$\begin{aligned} \left\| f\frac{1}{w} \right\|_X &= \left\| \left( f\frac{1}{w} \right)^*_\mu \right\|_{\bar{X}} \\ &\leq \left\| f^*_\mu(s) \left( \frac{1}{w} \right)^*_\mu (s) \right\|_{\bar{X}} \quad \text{(by (2.4) and (2.5))} \\ &\leq 2 \left\| f^*_\mu(s) \left( \frac{1}{w} \right)^*_\mu (s) \chi_{(0,\mu(\Omega)/2)}(s) \right\|_{\bar{X}} \\ &= 2 \left\| f^*_\mu(s) \left( \frac{1}{w} \right)^*_\mu (s) \frac{s}{I(s)} \frac{I(s)}{s} \chi_{(0,\mu(\Omega)/2)}(s) \right\|_{\bar{X}} \\ &\leq 2 \left( \sup_{0<s<\mu(\Omega)/2} \left( \frac{1}{w} \right)^*_\mu (s) \frac{s}{I(s)} \right) \left\| f^*_\mu(s) \frac{I(s)}{s} \chi_{(0,\mu(\Omega)/2)}(s) \right\|_{\bar{X}} \\ &= 2 \|W\|_{M(\Phi)} \left\| f^*_\mu(s) \frac{I(s)}{s} \chi_{(0,\mu(\Omega)/2)}(s) \right\|_{\bar{X}} \\ &\leq 2 \|W\|_{M(\Phi)} \left( \left\| \tilde{Q}_I \right\|_{\bar{X}\to\bar{X}} \||\nabla f|\|_X + \frac{2c_{XI}}{\mu(\Omega)} \int_\Omega |f| \, d\mu \right) \text{ (by (6.2)).} \end{aligned}$$

Let $f \in Lip(\Omega)$ be such that $m(f) = 0$ or $\int_\Omega f d\mu = 0$, by Poincaré's inequality (cf. Remark 2)

$$\begin{aligned} \int_\Omega |f| \, d\mu &\leq \frac{\mu(\Omega)}{I(\mu(\Omega)/2)} \int_\Omega |\nabla f| \, d\mu \\ &\leq \frac{\mu(\Omega)}{I(\mu(\Omega)/2)} \||\nabla f|\|_X \frac{\mu(\Omega)}{\|\chi_\Omega\|_X} \text{ (by Hölder's inequality).} \end{aligned}$$

Summarizing,

$$\|f\|_X \leq rC \, \||\nabla f|\|_X + r^{-\alpha} \|w^\alpha f\|_X \,,$$

where $C = 2 \|W\|_{M(\Phi)} \left\| \tilde{Q}_I \right\|_{\bar{X}\to\bar{X}} + \frac{2c_{XI}\mu(\Omega)}{I(\mu(\Omega)/2)\|\chi_\Omega\|_X}$.

b. $\mu(\Omega) = \infty$. We follow the same steps as in the previous case, but now we use (6.3) instead (6.2). Notice that the extra $L^1-$term does not appear in this case. □

6.0.1. *The proof of Theorem 11.* In this section we prove Theorem 11. We need the following Lemma (see [23], [22]).

**Lemma 3.** *Let $(\Omega,\mu,d)$ be a metric space and let $I$ be an isoperimetric estimator for $(\Omega,\mu,d)$. Let $h$ be a bounded Lip function on $\Omega$. Then there exists a sequence of bounded functions $(h_n)_n \subset Lip(\Omega)$, such that*

(1)

$$|\nabla h_n(x)| \leq (1+\frac{1}{n})\,|\nabla h(x)|\,,\quad x\in\Omega. \tag{6.5}$$

(2)

$$h_n \underset{n\to 0}{\to} h \text{ in } L^1. \tag{6.6}$$

(3) *The functions $(h_n)^*_\mu$ are locally absolutely continuous and for any r.i. space $X$ on $\Omega$*

$$\left\| \left(-\left|h_n\right|^*_\mu\right)'(\cdot)I(\cdot)\right\|_{\bar{X}} \leq \left\|\left|\nabla h_n\right|\right\|_X\,,\ \text{for all } n\in\mathbb{N}. \tag{6.7}$$

*Proof.* (of Theorem 11)

a. Suppose that $\mu(\Omega)<\infty$. The fact that $c_{X,I}=\left\|\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\right\|_{\bar{X}}<\infty$, follows easily from the fact that $\tilde{Q}_I$ is bounded. Indeed, for $0<t<\mu(\Omega)/4$, we have that

$$\tilde{Q}_I\chi_{(0,\mu(\Omega)/2)}(t)=\frac{I(t)}{t}\int_t^{\mu(\Omega)/2}\frac{dr}{I(r)}\geq\frac{I(t)}{t}\int_{\mu(\Omega)/4}^{\mu(\Omega)/2}\frac{dr}{I(r)},$$

and (6.1) follows.

Let $f$ be a bounded function in $Lip(\Omega)$. Let $(f_n)_n$ be the sequence associated to $f$ that is provided by Lemma 3. Since $(f_n)^*_\mu$ is locally absolutely continuous, by the fundamental theorem of calculus we have

$$\begin{aligned}
A(t) &= (f_n)^*_\mu(t)\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\\
&= \frac{I(t)}{t}\int_t^{\mu(\Omega)/2}\left(-(f_n)^*_\mu\right)'(r)dr+(f_n)^*_\mu(\mu(\Omega)/2)\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\\
&= \frac{I(t)}{t}\int_t^{\mu(\Omega)/2}\left(-(f_n)^*_\mu\right)'(r)I(r)\frac{dr}{I(r)}+(f_n)^*_\mu(\mu(\Omega)/2)\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\\
&= \tilde{Q}_I\left(\left(-(f_n)^*_\mu\right)'(\cdot)I(\cdot)\right)(t)+(f_n)^*_\mu(\mu(\Omega)/2)\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|A(t)\|_{\bar{X}} &\leq \left\|\tilde{Q}_I\left(\left(-(f_n)^*_\mu\right)'(\cdot)I(\cdot)\right)(t)\right\|_{\bar{X}}+(f_n)^*_\mu(\mu(\Omega)/2)\left\|\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\right\|_{\bar{X}}\\
&= I+II.
\end{aligned}$$

Now,

$$\begin{aligned}
I &\leq \left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}}\left\|\left(\left(-(f_n)^*_\mu\right)'(\cdot)I(\cdot)\right)(t)\right\|_{\bar{X}}\leq\left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}}\left\|\left|\nabla h_n\right|\right\|_X\quad\text{(by (6.7))}\\
&\leq \left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}}(1+\frac{1}{n})\left\|\left|\nabla f\right|\right\|_X\quad\text{(by (6.5))},
\end{aligned}$$

and

$$\begin{aligned} II &\leq \left(\frac{2}{\mu(\Omega)}\int_\Omega |f_n|(x)d\mu\right)\left(\left\|\frac{I(t)}{t}\chi_{(0,\mu(\Omega)/2)}(t)\right\|_{\bar{X}}\right) \\ &= \frac{2c}{\mu(\Omega)}\int_\Omega |f_n|(x)d\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|f_\mu^*(t)\frac{I(t)}{t}\right\| &\leq \lim\inf_{n\to\infty}\left(\left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}}(1+\frac{1}{n})\,|||\nabla f|||_X + \frac{2c}{\mu(\Omega)}\int_\Omega |f_n(x)|\,d\mu\right) \\ &= \left\|\tilde{Q}_I\right\|_{\bar{X}\to\bar{X}}|||\nabla f|||_X + \frac{2C}{\mu(\Omega)}\int_\Omega |f(x)|\,d\mu \quad \text{(by (6.6))} \\ &= D\left(|||\nabla f|||_X + \frac{2}{\mu(\Omega)}\int_\Omega |f(x)|\,d\mu\right). \end{aligned}$$

b. $\mu(\Omega)=\infty$. The proof follows the same argument. Indeed, if $f \in Lip_0(\Omega)$, then $f$ is bounded. Let $(f_n)_n$ the sequence associated to $f$ that is provided by Lemma 3. Note that $(f_n)_\mu^*$ is locally absolutely continuous, and $(f_n)_\mu^{**}(\infty)=0$. Using the fundamental theorem of calculus we find

$$\begin{aligned} (f_n)_\mu^*(t)\frac{I(t)}{t} &= \frac{I(t)}{t}\int_t^\infty \left(-(f_n)_\mu^*\right)'(r)dr \\ &= \tilde{Q}_I\left(\left(-(f_n)_\mu^*\right)'(\cdot)I(\cdot)\right)(t), \end{aligned}$$

and we conclude the proof as in the previous case. □

6.1. **Final Remarks.** a. We should mention that in the literature one can find $L^1$ uncertainty type inequalities that are not directly related to those treated in our paper. For example, in [19], sharp constants are obtained for inequalities of the following type (here $\Omega=\mathbb{R}$),

$$\|f\|_1\|f\|_2^2 \leq c\left\|\xi\hat{f}\right\|_2^2\left\|x^2 f\right\|_1$$

or

$$\|f\|_2 \leq c\left\|x^2 f\right\|_1^{2/7}\left\|\xi\hat{f}\right\|_2^{5/7}.$$

b. Multiplier inequalities have a long history. We mention two somewhat related directions of inquiry that we find intriguing. The multiplier inequalities exemplified by [33], and the long list of references therein, and the potential spaces of radial functions exemplified by [28] and the references therein.

c. In this paper we have not considered discrete inequalities. We hope to discuss the discrete world elsewhere.

**Acknowledgement 1.** *Thanks are due to Michael Cwikel for an opportune e-mail that sparked this research and, at a later stage, for a number of helpful suggestions to improve the presentation.*

## References

[1] F. Barthe, *Log-concave and spherical models in isoperimetry*, Geom. Funct. Anal. **12** (2002), 32-55.

[2] F. Barthe, P. Cattiaux and C. Roberto, *Isoperimetry between exponential and Gaussian*, Electronic Journal of Probability **12** (2007), 1212–1237.

[3] W. Beckner, *A generalized Poincaré inequality for Gaussian measures*, Proc. Amer. Math. Soc. **105** (1989), 397-400.
[4] W. Beckner, *Pitt's inequality with sharp convolution estimates*, Proc. Amer. Math. Soc. **136** (2008), 1871–1885.
[5] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston**,** 1988.
[6] J. Bergh and J. Löfstrom, *Interpolation spaces. An Introduction,* Springer-Verlag, New York, 1976.
[7] S. G. Bobkov, *Extremal properties of half-spaces for log-concave distributions*, Ann. Probab. **24** (1996), 35-48.
[8] S. G. Bobkov and C. Houdré, *Some connections between isoperimetric and Sobolev-type inequalities*, Mem. Amer. Math. Soc. **129** (1997).
[9] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207-216.
[10] P. Ciatti, M. G. Cowling and F. Ricci, *Hardy and uncertainty inequalities on stratified Lie groups*, preprint, ArXiv Math. (arXiv:1308.2373).
[11] P. Ciatti, F. Ricci and M. Sundari, *Heisenberg-Pauli-Weyl uncertainty inequalities and polynomial volume growth*, Adv. Math. **215** (2007), 616–625.
[12] D. Cordero-Erausquin and N. Gozlan, *Transport proofs of weighted Poincaré inequalities for log-concave distributions*, Arxiv Math. (arXiv:1407.3217)
[13] Th. Coulhon and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Rev. Mat. Iberoamericana **9** (1993), 293–314.
[14] G. M. Dall'ara and D. Trevisan, *Uncertainty inequalities on groups and homogeneous spaces via isoperimetric inequalities*, J. Geom. Anal. (to appear), Arxiv Math. (arXiv:1308.1069v2).
[15] W. G. Faris, *Weak Lebesgue spaces and quantum mechanical binding*, Duke Math. J. **4** (1976), 365-373.
[16] C. L. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), 129–206.
[17] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey*, J. Fourier Anal. Appl. **3** (1997), 207–238.
[18] L. Forzani, *Lemas de cubrimiento de tipo Besicovitch y su aplicación al estudio del operador maximal de Ornstein-Uhlenbeck*, Ph.D. Thesis, Univ. Nac. San Luis, Argentina, 1993.
[19] E. Laeng and C. Morpurgo, *An uncertainty inequality involving $L^1$ norms*, Proc. Amer. Math. Soc. **127** (1999), 3565–3572.
[20] J. Martin and M. Milman, *Symmetrization inequalities and Sobolev embeddings*, Proc. Amer. Math. Soc. **134** (2006), 2335–2347.
[21] J. Martin and M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math. **225** (2010), 121-199.
[22] J. Martin and M. Milman, *Sobolev inequalities, rearrangements, isoperimetry and interpolation*, Contemporary Math. **545** (2011), pp 167-193.
[23] J. Martin and M. Milman, *Fractional Sobolev Inequalities: Symmetrization, Isoperimetry and Interpolation*, Asterisqué **366** (2014).
[24] A. Martini, *Generalized uncertainty inequalities*, Math. Z. **265** (2010), 831–848.
[25] V. G. Maz'ya, *Sobolev Spaces with applications to elliptic partial differential equations.* Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **342**. Springer, Heidelberg, 2011.
[26] V. G. Maz'ya, *The p-conductivity and theorems on imbedding certain functional spaces into a C-space* (Russian), Dokl. Akad. Nauk SSSR **140** (1961), 299–302 (English translation: in Soviet Math. Dokl. 3 (1962).
[27] E. Milman, *Beyond traditional Curvature-Dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension*, Arxiv. Math (arxiv: 1409.4109v2).
[28] P. L. De Napoli and I. Drelichman, *Elementary proofs of embedding theorems for potential spaces of radial functions*, Arxiv. Math (arXiv:1404.7468).
[29] K. A. Okoudjou, L. Saloff-Coste and A. Teplyaev, *Weak uncertainty principle for fractals, graphs and metric measure spaces*, Trans. Amer. Math. Soc. **360** (2008), 3857–3873.
[30] R. O'Neil, *Convolution operators and L(p,q) spaces*, Duke Math. J. **30** (1963), 129–142.
[31] F. Ricci, *Uncertainty inequalities on spaces with polynomial volume growth*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **29** (2005), 327–337.
[32] L. Saloff-Coste, *Aspects of Sobolev-Type Inequalities*, London Mathematical Society Lecture Note Series **289**, Cambridge University Press, Cambridge, 2002.

[33] W. Sickel, *On pointwise multipliers for $F^s_{p,q}(\mathbb{R}^n)$ in case $\sigma_{p,q} < s < n/p$*, Ann. Mat. Pura App. **176** (1999), 209–250.
[34] E. M. Stein and G. Weiss, *Fourier analysis on Euclidean spaces*, Princeton Univ. Press, 1971.
[35] R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mechanics **16** (1967), 1031-1060.
[36] G. Talenti, *Inequalities in rearrangement-invariant function spaces*, in: Nonlinear Analysis, Function Spaces and Applications, vol. 5, Prometheus, Prague, 1995, pp. 177–230.
[37] G. Talenti, *A weighted version of a rearrangement inequality*, Ann. Univ Ferrara **43** (1997), 121–133.
[38] W. Urbina, *Análisis Armónico Gausiano: una visión panóramica*, Bol. Asoc. Mat. Ven.**V** (1998), 143-181.
[39] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover, NY 1949.

Department of Mathematics, Universitat Autònoma de Barcelona, jmartin@mat.uab.cat

mario.milman@gmail.com, https://sites.google.com/site/mariomilman/