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Polynomial Chaos Approximation of the Quadratic Performance of Uncertain Time-Varying Linear Systems*

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***Abstract*— This paper presents a novel approach to robustness analysis based on quadratic performance metrics of uncertain time-varying systems. The considered time-varying systems are assumed to be linear and defined over a finite time horizon. The uncertainties are described in the form of real-valued random variables with a known probability distribution. The quadratic performance problem for this class of systems can be posed as a parametric Riccati differential equation (RDE). A new approach based on polynomial chaos expansion is proposed that can approximately solve the resulting parametric RDE and, thus, provide an approximation of the quadratic performance. Moreover, it is shown that for a zeroth order expansion this approximation is in fact a lower bound to the actual quadratic performance. The effectiveness of the approach is demonstrated on the example of a worst-case performance analysis of a space launcher during its atmospheric ascent.**

I. Introduction

The present paper develops a theoretical and computational approach for the performance analysis of parametric uncertain linear time-varying (LTV) systems over a finite horizon. A wide range of engineering applications where the primary incentive is tracking of a predefined trajectory fall in this category, e.g., robotic systems [1], space launch vehicles [2], or aircraft [3]. Of common interest to assess the performance of such systems are quadratic metrics, e.g., in form of the finite-horizon induced $\mathcal{L}_2$-norm. In a nominal setting, i.e., without uncertainties, these metrics have well-defined solutions based on Riccati differential equations (RDEs), see Section II.

For uncertain time-varying systems the nominal results can be extended resulting in performance conditions based on the solution of parametric RDEs, i.e., RDEs that explicitly depend on the uncertainty. Existing solutions to tackle this problem are mostly based on finding upper bounds on the quadratic performance with roots in the integral quadratic constraint framework, see [4], [5]. These methods are computationally expensive either relying on a nonlinear optimization [4] or an iterative solving of a RDE and a gridded differential linear matrix inequality [5]. In addition, they only provide upper bounds with unknown conservatism and can only deal with deterministic norm-bounded uncertainties.

If the nature of the uncertainty affecting the system is probabilistic, as in this paper, techniques exploiting concepts from probability theory become inevitable. Early results in the field of probabilistic robust control focused on Monte Carlo (MC) sampling techniques [6]. More recently, the theory of polynomial chaos has attracted a significant amount of interest. Originating from the work of Norbert Wiener [7], polynomial chaos is built on mathematics similar to Fourier series expansions for periodic time signals of finite energy. Example applications within the field of probabilistic robustness include linear quadratic regulation [8], [9], linear parameter-varying [10] and time-varying systems [11], or model predictive control [12].

The main contribution of this paper is to apply the theory of polynomial chaos expansions (PCEs) in order to quantify the quadratic performance of uncertain LTV systems. This is achieved by approximately solving the random parameter-dependent RDE via Galerkin projection on polynomial basis functions in Section IV. The approach allows computationally efficient performance analysis of stochastic uncertain finite-horizon LTV systems, as is demonstrated on a worst-case analysis of an uncertain space launcher during atmospheric ascent in Section VI. A second contribution is the study of the zeroth order PCE as a special case of the proposed approach. It can be shown that by using a zeroth order expansion, the method not only approximates the parameter-dependent RDE but in fact provides a lower bound on the quadratic performance metric, see Section V.

II. Quadratic Performance

A. Nominal LTV Systems

Consider a nominal (certain) LTV system G defined on $t \in [0,T]$

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t), \quad x(0) = x_0 \in \mathbb{R}^{n_x} \\ y(t) &= C(t)x(t) + D(t)w(t) \end{aligned} \tag{1}$$

with state vector $x(t) \in \mathbb{R}^{n_x}$, input $w(t) \in \mathbb{R}^{n_w}$, output $y(t) \in \mathbb{R}^{n_y}$, and finite horizon $T < \infty$. Each $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are piecewise-continuous, real-valued, and bounded matrix functions of time and appropriate dimension. It is assumed that all signals in (1) lie in the Hilbert space $\mathcal{L}_2[0,T]$ defined by

$$\mathcal{L}_2[0,T] := \left\{ f : [0,T] \mapsto \mathbb{R}^n \,\middle|\, f \text{ measurable and } \|f\|_{[0,T]} < \infty \right\} \tag{2}$$

with inner product $\langle f, g\rangle_{[0,T]} = \int_0^T f^T(t) g(t)\,\mathrm{d}t$ and induced norm $\|f\|_{[0,T]} = \sqrt{\langle f, f\rangle_{[0,T]}}$. Hereinafter, function arguments will be omitted for notational convenience, when clear from the context.

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Among the various system theoretic criteria available for measuring the nominal performance of (1), this paper focuses on the quadratic index $J : \mathcal{L}_2[0,T] \mapsto \mathbb{R}$

$$J(w) = x^T(T)Fx(T) + \int_0^T \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \mathrm{d}t \tag{3}$$

subject to (1). Therein, $Q = Q^T : [0,T] \mapsto \mathbb{R}^{n_x \times n_x}$, $R = R^T : [0,T] \mapsto \mathbb{R}^{n_w \times n_w}$, $S : [0,T] \mapsto \mathbb{R}^{n_x \times n_w}$ are piecewise-continuous (bounded) matrix functions and $F = F^T \in \mathbb{R}^{n_x \times n_x}$. For instance, setting $Q(t) = C^T(t)C(t)$, $S(t) = C^T(t)D(t)$, $R(t) = D^T(t)D(t) - \gamma^2 I_{n_w}$, and $F = 0$, the finite-horizon induced $\mathcal{L}_2$-gain of G

$$\|G\|_{[0,T]} = \sup_{w \in \{\mathcal{L}_2[0,T] \setminus 0\}} \frac{\|y\|_{[0,T]}}{\|w\|_{[0,T]}} \tag{4}$$

can be bounded by $\|G\|_{[0,T]} \leq \gamma$ if and only if $J(w) \leq 0$ $\forall w \in \mathcal{L}_2[0,T]$ [5].

As in many robust and optimal control problems, evaluating the worst-case $J(w)$ essentially breaks down to the solution of a Riccati differential equation. The relationship between quadratic performance (3) and RDEs is stated in the following theorem, taken from [5].

Theorem 1 (Bounded Real Lemma [5])**.** *Let the quadratic cost $J(w)$ be parametrized by given $Q(t)$, $S(t)$, $R(t)$, and F; with $R(t) \prec 0$ $\forall t \in [0,T]$. The solution $X(t)$ of the RDE*

$$\begin{aligned} \dot{X} &= -A^T X - XA - Q + (XB+S)R^{-1}(XB+S)^T \\ X(T) &= F \end{aligned} \tag{5}$$

exists on $[0,T]$ if and only if $\exists \epsilon > 0 : J(w) \leq -\epsilon \|w\|^2_{[0,T]}$ $\forall w \in \mathcal{L}_2[0,T]$.

B. Uncertain LTV Systems

This paper focuses on uncertain LTV systems, i.e., systems where the state-space matrices A, B, C, and D not only depend on time t but also on parametric uncertainty δ. Generally, the initial condition x_0 can also be uncertain. The considered system has the following form

$$\begin{aligned} \dot{x}(t,\delta) &= A(t,\delta)x(t,\delta) + B(t,\delta)w(t), \quad x(0,\delta) = x_0(\delta) \\ y(t,\delta) &= C(t,\delta)x(t,\delta) + D(t,\delta)w(t) \end{aligned} \tag{6}$$

where $\delta(\omega)$ is a $\mathbb{R}^{n_\delta}$-valued random vector containing n_δ independent random variables with sample space $\Omega^{n_\delta} = \Omega_1 \times ... \times \Omega_{n_\delta}$ and *bounded variance*. This type of uncertainty arises frequently in the modeling of various physical processes. Assuming deterministic $w(t) \in \mathcal{L}_2[0,T]$, this paper views the system response as a stochastic process, i.e., functions $x : [0,T] \times \Omega^{n_\delta} \mapsto \mathbb{R}^{n_x}$ and $y : [0,T] \times \Omega^{n_\delta} \mapsto \mathbb{R}^{n_y}$, such that for any $t \in [0,T]$ each $x(t,\cdot)$ and $y(t,\cdot)$ are random vectors.

For each realization of δ, the quadratic performance of an uncertain LTV system can be measured by generalizing (3) to $J : \mathcal{L}_2[0,T] \times \Omega^{n_\delta} \mapsto \mathbb{R}$

$$\begin{aligned} J(w,\delta) &= x^T(T,\delta)F(\delta)x(T,\delta) + \\ &\int_0^T \begin{bmatrix} x(t,\delta) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q(t,\delta) & S(t,\delta) \\ S^T(t,\delta) & R(t,\delta) \end{bmatrix} \begin{bmatrix} x(t,\delta) \\ w(t) \end{bmatrix} \mathrm{d}t \end{aligned} \tag{7}$$

subject to (6). Therein, Q, S, R, and F are mappings from $[0,T] \times \Omega^{n_\delta}$ to a real matrix of corresponding dimension, e.g., $S : [0,T] \times \Omega^{n_\delta} \mapsto \mathbb{R}^{n_x \times n_w}$. Define $E(t,\delta)$ by

$$E := \begin{bmatrix} Q & A^T - SR^{-1}B^T \\ A - BR^{-1}S^T & -SR^{-1}S^T \end{bmatrix} \tag{8}$$

where, generally, all matrix functions have the same arguments. This allows stating a robust analog of Theorem 1.

Theorem 2 (Robust Bounded Real Lemma)**.** *Let the generic quadratic cost $J(w,\delta)$ be specified by $Q(t,\delta)$, $S(t,\delta)$, $R(t,\delta)$, and $F(\delta)$; with $R(t,\delta) \prec 0$ for all $[0,T] \times \Omega^{n_\delta}$. There exists an $\epsilon > 0$ such that $J(w,\delta) \leq -\epsilon \|w\|^2_{[0,T]}$ for all $w \in \mathcal{L}_2[0,T]$ and realizations of δ if and only if the solution of the random parameter-dependent RDE*

$$\begin{aligned} \dot{X}(t,\delta) &= -\begin{bmatrix} I_{n_x} \\ X(t,\delta) \end{bmatrix}^T E(t,\delta) \begin{bmatrix} I_{n_x} \\ X(t,\delta) \end{bmatrix} \\ X(T,\delta) &= F(\delta) \end{aligned} \tag{9}$$

exists for $t \in [0,T]$ and all realizations of δ.

Generally, nominal quadratic performance according to Theorem 1 is straightforward to evaluate by numerical integration of the RDE (5). However, certifying robust performance as in Theorem 2 is computationally much more challenging, since existence of the solution to (9) needs to be checked for all values of δ. To the best of the authors' knowledge, only approximate techniques exist to solve this problem. Typical analysis conditions within robust control theory often involve some conservatism, e.g., due to overbounding of the uncertainty by integral quadratic constraints [4], [5]. The chief idea within this paper is to apply the theory of polynomial chaos to the random RDE (9), in order to approximate the quadratic performance of uncertain LTV systems.

III. Mathematical Background

A. Polynomial Chaos Expansions

In this paper, each entry of δ is considered as a $\mathbb{R}$-valued random variable, i.e., a measurable function f with probability density function (pdf) ρ and sample space Ω. Herein, f is supposed to be square integrable, i.e., belong to the Hilbert function space

$$L^2_\rho(\Omega) := \left\{ f : \Omega \mapsto \mathbb{R} \;\middle|\; f \text{ measurable and } \|f\|_{L^2_\rho} < \infty \right\} \tag{10}$$

with inner product

$$\langle f, g \rangle_{L^2_\rho} = \int_\Omega f(\omega)g(\omega)\rho(\omega)\,\mathrm{d}\omega = \mathbb{E}_\rho[fg] \tag{11}$$

and induced norm $\|f\|_{L^2_\rho} = \sqrt{\langle f, f \rangle_{L^2_\rho}}$. In order to highlight the association with orthogonality, the expectation $\mathbb{E}_\rho[\cdot]$ w.r.t. ρ is written as $\langle \cdot \rangle$ when appropriate, e.g., $\langle f, g \rangle_{L^2_\rho} = \langle fg \rangle$. We shall further consider the Sobolev space

$$H^k_\rho(\Omega) := \left\{ f \in L^2_\rho(\Omega) \;\middle|\; \frac{\partial^j f}{\partial \omega^j} \in L^2_\rho(\Omega) \;\forall j = 0, ..., k \right\} \tag{12}$$

equipped with inner product $\langle f,g\rangle_{H^k_\rho} = \sum_{j=0}^{k}\langle \frac{\partial^j f}{\partial \omega^j}, \frac{\partial^j g}{\partial \omega^j}\rangle_{L^2_\rho}$ and norm $\|f\|_{H^k_\rho} = \sqrt{\langle f,f\rangle_{H^k_\rho}}$.

The idea of generalized polynomial chaos (gPC) expansions [13] is grounded in the observation that any random process $f(\omega) \in L^2_\rho(\Omega)$ can be developed as an orthogonal generalized Fourier series

$$f(\omega) = \sum_{\alpha=0}^{\infty} f_\alpha \psi_\alpha(\omega) \tag{13}$$

with deterministic expansion coefficients

$$f_\alpha = \frac{\langle f, \psi_\alpha\rangle}{\langle \psi_\alpha, \psi_\alpha\rangle} = \frac{\int_\Omega f(\omega)\psi_\alpha(\omega)\rho(\omega)\,\mathrm{d}\omega}{\int_\Omega \psi_\alpha^2(\omega)\rho(\omega)\,\mathrm{d}\omega}. \tag{14}$$

Assuming $\rho(\omega)$ is continuous, suitable orthogonal basis polynomials $\{\psi_\alpha\}_{\alpha=0}^{\infty}$ of $L^2_\rho(\Omega)$ can be constructed for arbitrary distributions of δ [14]. Henceforth, Greek characters are used in order to index within the basis for $L^2_\rho(\Omega)$, whereas Latin characters are used as spatial indices, e.g., for the entries of matrices or vectors. The orthogonal basis polynomials ψ_α fulfill

$$\begin{aligned} \psi_0 &= 1 \\ \langle \psi_\alpha, \psi_\beta\rangle &= \delta_{\alpha\beta}\mathbb{E}_\rho[\psi_k^2] \quad \forall \alpha,\beta \in \mathbb{N}_0 \end{aligned} \tag{15}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

This paper studies δ as a $\mathbb{R}^{n_\delta}$-valued random vector containing n_δ independent random variables with joint pdf $\rho(\omega) = \prod_{i=1}^{n_\delta} \rho_i(\omega_i)$ and sample space $\Omega^{n_\delta} = \Omega_1 \times \ldots \times \Omega_{n_\delta}$. In this multivariate setting, an orthogonal basis for $L^2_\rho(\Omega^{n_\delta})$ is obtained simply by taking products of the respective univariate orthogonal polynomials [15], [16]

$$\mathcal{P}_d^{n_\delta} = \operatorname{span}\Big\{\psi_\alpha(\omega) = \prod_{i=1}^{n_\delta}\psi_{\alpha_i}(\omega_i) \;\Big|\; |\alpha| = \sum_{i=1}^{n_\delta}\alpha_i \le d\Big\}. \tag{16}$$

Therein, α is a n_δ-dimensional multi-index $\alpha \in \mathbb{N}_0^{n_\delta}$. For simplicity, the space of polynomials $\mathcal{P}_d^{n_\delta}$ with maximum total degree d is considered. It is remarked that it is also possible to construct a polynomial basis with different polynomial degrees, both spatially and w.r.t. $\delta_i(\omega_i)$ [14].

According to the Cameron-Martin theorem [17], the convergence of (13) holds in the mean-square sense, i.e., w.r.t. $\|\cdot\|_{L^2_\rho}$. Theoretically, polynomial chaos expansions can show superior convergence compared to Monte Carlo based methods. The related conditions and issues are discussed briefly in the subsequent section.

B. Spectral Convergence

An important property of expansions by orthogonal polynomials is that the quality of the function approximation improves exponentially as the stochastic regularity (i.e. smoothness) of the random process to be approximated increases [16]. This property, widely referred to as spectral convergence, is stated formally in the next theorem. It is shown here for the family of Legendre polynomials, i.e., the orthogonal polynomials for the uniform measure. A similar result can be proven for classical orthogonal polynomials, i.e., for the beta, gamma, and Gaussian distribution, see e.g. [15] or [16].

Theorem 3 (Spectral convergence of Legendre polynomial expansions)**.** *For all $f \in H^k_\rho(\Omega^{n_\delta})$ depending on n_δ independent uniformly distributed random variables, there exists a constant $C_\pi \ge 0$ such that*

$$\|f - \Pi_d^{n_\delta} f\|_{L^2_\rho} := \Big\| f - \sum_{|\alpha|\le d} \frac{\langle f,\psi_\alpha\rangle}{\langle \psi_\alpha^2\rangle}\psi_\alpha \Big\|_{L^2_\rho} \le C_\pi n_\delta d^{-k}\|f\|_{H^k_\rho}. \tag{17}$$

Thus, the smoother the random process, the better the convergence rate of the PCE in mean square. However, discontinuities and poor regularity may result in convergence issues. A classical example is Gibbs' Phenomenon [16]. Luckily, within the scope of this paper, solutions of Riccati differential equations exhibit rather appealing smoothness and monotonicity properties w.r.t. the initial data and coefficients.

C. Riccati Differential Equations

Solutions of Riccati differential equations depend monotonically on the initial condition and the coefficients [18]. In order to elaborate, define the matrix $E_i(t)$, for $i = 1, 2$, similar to (8)

$$E_i(t) := \begin{bmatrix} Q_i(t) & A_i^T(t) - S_i(t)R_i^{-1}(t)B_i^T(t) \\ A_i(t) - B_i(t)R_i^{-1}(t)S_i^T(t) & S_i(t)R_i^{-1}(t)S_i^T(t) \end{bmatrix}. \tag{18}$$

This allows to state the following theorem.

Theorem 4 (Comparison Theorem [18])**.** *Let $[0,T] \subset \mathbb{R}$ be a given interval and let the piecewise-continuous bounded matrix functions $E_i : [0,T] \mapsto \mathbb{R}^{2n_x \times 2n_x}$, $i = 1,2$, be defined by* (18)*. If $X_i(t)$, $i = 1,2$, are on $[0,T]$ solutions of*

$$\dot{X}_i(t) = -\begin{bmatrix} I_{n_x} \\ X_i(t)\end{bmatrix}^T E_i(t) \begin{bmatrix} I_{n_x} \\ X_i(t)\end{bmatrix} \tag{19}$$

with terminal constraints $X_1(T) \preceq X_2(T)$, then

$$E_1(t) \preceq E_2(t) \quad \forall t \in [0,T] \tag{20}$$

implies $X_1(t) \preceq X_2(t)$ for $t \in [0,T]$.

The comparison theorem can be used to prove in an elegant manner existence conditions for hermitian Riccati differential equations and also for generalized, perturbed, and coupled RDEs, see [18]. We will use a similar idea in Section V.

In addition to monotonicity properties, solutions of RDEs are also smooth functions of the data. In order to specify this statement, the concept of a Fréchet-derivative needs to be introduced, see [18]. Roughly speaking, it is a generalization of the derivative of a scalar $\mathbb{R}$-valued function of a single variable to multivariate functions of a multiple variables. The following theorem is taken from [18].

Theorem 5 (Smoothness of RDE Solutions [18]). *For each $\bar{X} = \bar{X}^T \in \mathbb{R}^{n_x \times n_x}$, $\bar{E}(t) = \bar{E}^T(t) \in \mathbb{R}^{2n_x \times 2n_x}$, and $t_0 \in [0, T]$ there exists an interval $I = (t_0 - a, t_0 + a) \subset [0, T]$ and a neighborhood $V(t) = V\big(\bar{X}, \bar{E}(t)\big)$ defined on I where the unique solution, denoted $X(X_0, E)$ of*

$$\begin{aligned}\dot{X}(t) &= -\begin{bmatrix} I_{n_x} \\ X(t) \end{bmatrix}^T E(t) \begin{bmatrix} I_{n_x} \\ X(t) \end{bmatrix} \\ X(t_0) &= X_0\end{aligned} \tag{21}$$

is a unique infinitely Fréchet-differentiable function $X : V \mapsto \mathcal{C}^1$.

These smoothness and monotonicity properties of solutions of RDEs provide optimism for the application of polynomial chaos expansions to approximate (9).

IV. Stochastic Galerkin Projection of Random RDEs

Assume a unique solution $X(t, \delta)$ to the random parameter-dependent RDE (9) exists for all realizations of δ and $t \in [0, T]$. Stochastic Galerkin projection (SGP) approximates the solution $X(t, \delta)$ by projecting the parameter-dependent RDE on a finite number of polynomial basis functions ψ_α. Thus, for a given maximum total polynomial degree d, the solution is sought for within the finite-dimensional space (16)

$$\begin{aligned}&\forall \psi_\alpha \in \mathcal{P}_d^{n_\delta} : \\ &\langle \dot{Y}(t, \cdot), \psi_\alpha \rangle = -\Big\langle \begin{bmatrix} I_{n_x} \\ Y(t, \cdot) \end{bmatrix}^T E(t, \cdot) \begin{bmatrix} I_{n_x} \\ Y(t, \cdot) \end{bmatrix},\ \psi_\alpha \Big\rangle \\ &\langle Y(T, \cdot), \psi_\alpha \rangle = \langle F,\ \psi_\alpha \rangle.\end{aligned} \tag{22}$$

where the inner product $\langle \cdot, \cdot \rangle_{L^2_\rho}$ defined via the expectation (11) is evaluated component-wise. For every fixed time instant $t \in [0, T]$, the solution is developed by a finite version of (13), i.e., the truncated series expansion

$$Y(t, \delta) = \sum_{|\alpha| \le d} Y_\alpha(t) \psi_\alpha(\delta). \tag{23}$$

Therein, the expansion coefficients $Y_\alpha(t)$, also known as stochastic modes of $Y(t, \delta)$, are orthogonal w.r.t. $\langle \cdot, \cdot \rangle_{L^2_\rho}$. Thus, plugging (23) into (22) yields

$$\begin{aligned}&\forall |\alpha| \le d : \\ &\dot{Y}_\alpha(t) = -\frac{1}{\langle \psi_\alpha^2 \rangle} \Big\langle \psi_\alpha, \begin{bmatrix} I_{n_x} \\ \sum_{|\beta| \le d} Y_\beta \psi_\beta \end{bmatrix}^T E(t, \cdot) \begin{bmatrix} I_{n_x} \\ \sum_{|\xi| \le d} Y_\xi \psi_\xi \end{bmatrix} \Big\rangle \\ &Y_\alpha(T) = \frac{\langle F, \psi_\alpha \rangle}{\langle \psi_\alpha^2 \rangle}.\end{aligned} \tag{24}$$

Denoting the number of basis polynomials spanning $\mathcal{P}_d^{n_\delta}$ by $n_\psi + 1$, notice that (24) is a coupled system of $n_\psi + 1$ deterministic RDEs. This system has the same form as the original differential equation (9) evaluated for specific realizations of δ. To see this, denote by E_{ij}, $i, j = 1, 2$, the $\mathbb{R}^{n_x, n_x}$-valued blocks of E given in (8), i.e.,

$$E = \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right]. \tag{25}$$

Define further the symmetric $\mathbb{R}^{(n_\psi+2)n_x \times (n_\psi+2)n_x}$-valued matrix $\mathcal{E}(t, \delta)$ by

$$\begin{aligned}&\mathcal{E}(t, \delta) := \\ &\begin{bmatrix} E_{11} & E_{12}\psi_0 & E_{12}\psi_1 & \dots & E_{12}\psi_{n_\psi} \\ E_{21}\psi_0 & E_{22}\psi_0\psi_0 & E_{22}\psi_0\psi_1 & \dots & E_{22}\psi_0\psi_{n_\psi} \\ \vdots & \vdots & & & \vdots \\ E_{21}\psi_{n_\psi} & E_{22}\psi_{n_\psi}\psi_0 & E_{22}\psi_{n_\psi}\psi_1 & \dots & E_{22}\psi_{n_\psi}\psi_{n_\psi} \end{bmatrix}.\end{aligned} \tag{26}$$

Denoting the Hilbert space projection of (26) on ψ_α by

$$\mathcal{E}_\alpha(t) = \frac{1}{\langle \psi_\alpha^2 \rangle} \langle \psi_\alpha, \mathcal{E}(t, \cdot) \rangle \tag{27}$$

enables to write $\dot{Y}_\alpha(t)$ in (24) as

$$\dot{Y}_\alpha(t) = -\begin{bmatrix} I_{n_x} \\ Y_0 \\ Y_1 \\ \vdots \\ Y_{n_\psi} \end{bmatrix}^T \mathcal{E}_\alpha(t) \begin{bmatrix} I_{n_x} \\ Y_0 \\ Y_1 \\ \vdots \\ Y_{n_\psi} \end{bmatrix} \quad \forall |\alpha| \le d. \tag{28}$$

Thus, the computationally intractable, infinite-dimensional RDE (9) is reduced to $n_\psi + 1$ coupled RDEs (28). The approximate solution $Y(t, \delta)$ can then be obtained simply by numerical integration.

Note that the expansion (23) assumes independence of the deterministic and stochastic function spaces $\mathcal{L}_2[0, T]$ and $L^2_\rho(\Omega^{n_\delta})$. This implies that the same deterministic space is valid for all realizations of δ [19], i.e., existence of a unique solution to the RDE for all $t \in [0, T]$.

Often, analyzing robust quadratic performance of an uncertain system is converted to finding the smallest performance index γ such that the solution of the RDE (9) exists. An example is the upper bound calculation of the finite-horizon induced $\mathcal{L}_2$-norm as stated in (4). Typically, this involves a bisection over γ, where for small enough γ the RDE (9) exhibits finite escape time, i.e., the solution ceases to exist for some $t < T$ and realization of δ. It has to be emphasized that the Galerkin approximation may still provide a bounded solution when the exact solution actually grows unbounded. Intuitively, the Galerkin projection can be interpreted as a method of mean weighted residuals [16], implying that the solution to (9) need only exist on the finite-dimensional test space $\mathcal{P}_d^{n_\delta}$. Consequently, the presented Galerkin approximation approach can underestimate the actual robust performance index γ of the uncertain system.

V. Zeroth Order Polynomial Chaos Expansion

Within this section, it is proven that existence of the solution to (9) for all $t \in [0, T]$ and realizations of δ implies existence of the $d = 0$ Galerkin projection (24). This has important implications for the computation of the performance index γ. Since the exact performance index also implies the $d = 0$ approximate index but not conversely, the γ obtained by the zeroth order PCE is a lower bound to the exact γ. The remainder of the section provides the proof to this statement.

For a polynomial degree $d = 0$, the expansion (23) consists of only one term (such that $n_\psi + 1 = 1$). The stochastic Galerkin projection (24) is therefore

$$\begin{aligned} \dot{Y}_0(t) &= -\begin{bmatrix} I_{n_x} \\ Y_0(t) \end{bmatrix}^T \mathbb{E}_\rho\big[E(t,\cdot)\big] \begin{bmatrix} I_{n_x} \\ Y_0(t) \end{bmatrix} \\ Y_0(T) &= \mathbb{E}_\rho\big[F\big]. \end{aligned} \tag{29}$$

Note that, in contrast to the scalar case, the expectation of a vector or matrix-valued nonlinear functional $A(\delta)$ of a random vector δ is not necessarily a realization of A. However, the spectrum of $\mathbb{E}_\rho[A]$ can still be bounded by the spectrum of realizations of $A(\delta)$. This will be the main idea for the proof of boundedness within this section. Let $\mathrm{spect}(A)$ denote the spectrum of a symmetric matrix $A = A^T$, with minimum and maximum eigenvalue $\underline{\lambda}(A)$ and $\bar{\lambda}(A)$. This allows to state the following intermediate result.

Lemma 1. *For a symmetric random matrix $A(\delta) = A^T(\delta)$, it holds*

$$\mathrm{spect}\big(\mathbb{E}_\rho[A]\big) \subset \big[\min_\delta \underline{\lambda}\big(A(\delta)\big), \max_\delta \bar{\lambda}\big(A(\delta)\big)\big]. \tag{30}$$

Proof. See the proof of Theorem 2 in [20]. □

Thus, being able to bound the spectrum of $\mathbb{E}_\rho\big[E(t,\cdot)\big]$ in (29), we can prove boundedness of the $d = 0$ Galerkin approximation, provided the actual solution exists.

Theorem 6. *Let $E(t,\delta)$ be specified by* (8)*, with $R(t,\delta) \prec 0$ for all realizations of δ and $t \in [0,T]$. If the solution $X(t,\delta)$ to the RDE* (9) *exists for all $t \in [0,T]$ and realizations of δ, then the degree $d = 0$ approximation $Y_0(t)$ obtained via SGP* (29) *remains bounded as well and does not blow up, i.e., show finite escape time.*

Proof. The main idea of the proof is to bound the spectrum of $Y_0(t)$ by the worst-case, i.e., maximum over δ, spectral radius of the well-defined solution $X(t,\delta)$. Consider the maximum singular value $\bar{\sigma}(\cdot)$ of (29). Using $\bar{\sigma}[A\ B] \le \bar{\sigma}(A) + \bar{\sigma}(B)$ and $\bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B)$

$$\begin{aligned} \bar{\sigma}(\dot{Y}_0) &\le \big(1 + \bar{\sigma}(Y_0)\big)^2 \bar{\sigma}\big(\mathbb{E}_\rho\big[E(t,\cdot)\big]\big) \\ \bar{\sigma}\big(Y_0(T)\big) &= \bar{\sigma}\big(\mathbb{E}_\rho\big[F\big]\big). \end{aligned} \tag{31}$$

Denote $r(A)$ the spectral radius

$$r(A) := \max\big\{|\lambda| \;\big|\; \lambda \in \mathrm{spect}(A)\big\} \tag{32}$$

of a matrix A. Since all matrices in (31) are symmetric, $\sigma_i(\cdot) = |\lambda_i(\cdot)|$, and hence (31) can be rewritten in terms of the spectral radius $r(\cdot) = \bar{\sigma}(\cdot)$

$$\begin{aligned} r(\dot{Y}_0) &\le \big(1 + r(Y_0)\big)^2 r\big(\mathbb{E}_\rho\big[E(t,\cdot)\big]\big) \\ r\big(Y_0(T)\big) &= r\big(\mathbb{E}_\rho\big[F\big]\big). \end{aligned} \tag{33}$$

By assumption, the exact solution $X(t,\delta)$ exists for all $t \in [0,T]$ and realizations of δ, with $r\big(X(t,\delta)\big) < \infty$. Analogous to (33), $X(t,\delta)$ can be bounded by considering the maximum spectral radius over all realizations of δ

$$\begin{aligned} \max_\delta r(\dot{X}) &\le \big(1 + \max_\delta r(X)\big)^2 \max_\delta r\big(E(t,\delta)\big) \\ \max_\delta r\big(X(T,\delta)\big) &= \max_\delta r\big(F(\delta)\big). \end{aligned} \tag{34}$$

Due to Lemma 1,

$$r\big(Y_0(T)\big) \le \max_\delta r\big(X(T,\delta)\big) \tag{35}$$

and similarly

$$r\big(\mathbb{E}_\rho\big[E(t,\cdot)\big]\big) \le \max_\delta r\big(E(t,\delta)\big). \tag{36}$$

Therefore, comparison of (33) and (34) yields

$$r(\dot{Y}_0) \le \max_\delta r(\dot{X}) < \infty \tag{37}$$

for $t \in [0,T]$. Since the above arguments remain unchanged when substituting t by $\tau = T - t$ with $\frac{\mathrm{d}\tau}{\mathrm{d}t} = -1$, it holds

$$r\big(Y_0(t)\big) \le \max_\delta r\big(X(t,\delta)\big) < \infty \quad \forall t \in [0,T]. \tag{38}$$

Consequently, $Y_0(t)$ exists for all $t \in [0,T]$ and cannot have finite escape time. □

VI. Space Launcher Example

Consider the robustness analysis of the Vanguard space launcher also studied in [4], with origin [21]. Notable deviations from the nominal trajectory during atmospheric ascent are to be avoided in order to prevent a loss of the launcher. Therefore, the goal of the analysis is to assess the influence of external wind disturbances and parametric aerodynamic uncertainties on the launch trajectory. The LTV formulation of the launcher's first-stage rigid body pitch dynamics along a predefined gravity-turn trajectory is extended in [4] to

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{m v_d} & \frac{-g \sin\theta_d}{v_d} & 1 \\ 0 & 0 & 1 \\ \frac{M_\alpha}{J_{yy}} & 0 & \frac{M_q}{J_{yy}} \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \\ q \end{bmatrix} + \begin{bmatrix} \frac{T_d}{m v_d} & \frac{Z_\alpha}{m v_d} \\ 0 & 0 \\ \frac{T_d \xi}{J_{yy}} & \frac{M_\alpha}{J_{yy}} \end{bmatrix} \begin{bmatrix} \delta_\mu \\ \delta_\alpha \end{bmatrix}. \tag{39}$$

System states are the angle of attack α, pitch angle θ, and pitch rate q. δ_μ is a corrective gimbal input used for attitude control. External disturbances attributed to wind are simulated in form of disturbances δ_α in the angle of attack α. Nominal values of the aerodynamic stability derivatives Z_α, M_α, and M_q are given in [21] as functions of time. Similarly, the time-varying mass m and pitch inertia J_{yy} are provided. Nominal values for speed v_d and pitch angle θ_d along the reference trajectory are given for $t \in [11.35, 146.35]$ with a step size of 2.7 s. Thrust T_d, lever arm ξ of δ_μ w.r.t. the center of gravity, as well as the gravitational acceleration g are constants.

A time-invariant controller solely using pitch angle θ feedback is given in [21]. It is a linear quadratic regulator in conjunction with a full-order observer designed in the region of maximum dynamic pressure. A first-order servo model with bandwidth 50 rad/s is included in the analysis. The closed-loop system matrix thus has a size of $n_x = 7$.

Due to uncertainty in the modeling process, the aerodynamic coefficients $\delta = [Z_\alpha, M_\alpha, M_q]^T$ are assumed to be distributed uniformly about their nominal values $\bar{\delta}_i$ with $\delta_i \sim U(0.75\bar{\delta}_i, 1.25\bar{\delta}_i)$. In order to assess sensitivity of the uncertain closed-loop system w.r.t. wind disturbances, the finite-horizon induced $\mathcal{L}_2$-gain (4) from $w(t) = \delta_\alpha(t)$ to $y(t) = \alpha(t)$ is to be computed for $t \in [15, 100]$. This time horizon reflects the start and end point of the gravity-turn maneuver. For all subsequent computations, a 0.5 s grid of the LTV model is generated via linear interpolation.

Fig. 1 shows approximations of the robust induced $\mathcal{L}_2$-gain obtained by polynomial chaos expansion $\hat{\gamma}_{PCE}$ and Monte Carlo sampling $\hat{\gamma}_{MC}$. Throughout this section, the minimal γ for which an associated RDE solution exists is computed by a standard bisection algorithm. A reference gain γ_{ref} is computed as the worst-case out of $N = 10^4$ MC samples. The induced $\mathcal{L}_2$ gain of the nominal system without uncertainty is $\gamma_{nom} = 3.06$.

Five Galerkin projections (24) of the RDE (9) are computed for $d \in \{0, 1, 2, 3, 4\}$ using the *PolyChaos.jl* toolbox [22] for the Julia programming environment [23]. For each expansion degree d, the minimal $\hat{\gamma}_{PCE}$ for which a solution to the RDE (24) exists is calculated. All time integrations are performed numerically using an order 5/4 explicit Runge-Kutta scheme with stiffness detection and automatic switching to TRBDF2 [24].

For the sake of neutral comparison, the average $\hat{\gamma}_{MC}$ of 10 MC trials with sample size N is computed as well. As detailed in Section IV, the Galerkin projected RDE (24) corresponds to a system of $n_\psi + 1$ coupled RDEs. Therefore, for each PCE degree d, one MC trial is calculated as the worst-case gain out of $N = n_\psi + 1$ samples of δ. Empirical mean and standard deviation are displayed w.r.t. 10 trials indicating the randomness of the MC estimator. Note that the PCE always gives the same deterministic result.

It is clear that the gain obtained via MC simulation is a lower bound to the exact robust induced $\mathcal{L}_2$-gain. Interestingly, convergence from below can be observed for the polynomial chaos approximation not only for $d = 0$, but also for $d > 0$. This observation could be reproduced in various other benchmark examples of varying complexity but could so far only be proven for the case $d = 0$.

VII. Conclusion

This paper proposed a polynomial chaos approximation of the quadratic performance of uncertain LTV systems. Possible applications of the Galerkin solution approach lie within various fields of control engineering requiring the solution of Ricatti (differential) equations where the coefficients (and initial condition) are random. For example, characterization of the robust $\mathcal{L}_2$-to-Euclidean gain [5] could be a use-case with high potential. It was further shown that the special case of a zeroth order polynomial expansion actually leads to a lower bound for the true quadratic performance. Future work aims at generalizing the lower bound result to higher order PCEs.

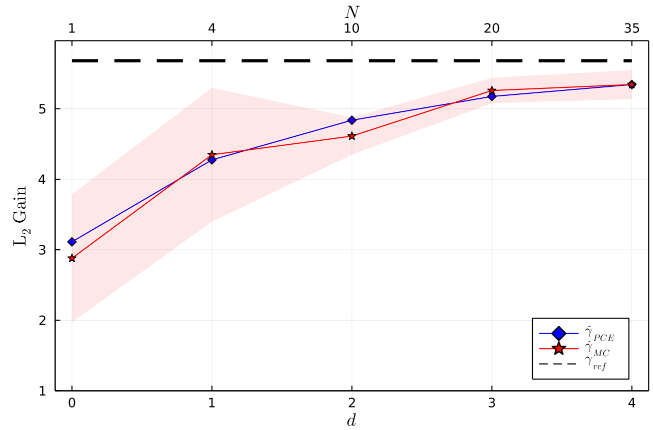

Fig. 1. Approximated Robust Induced $\mathcal{L}_2$-Gain vs. PCE Degree d and MC Sample Set Dimension N

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