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On the Stabilization of a Network of a Class of SISO Coupled Hybrid Linear Subsystems via Static Linear Output Feedback

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Abstract: This paper deals with the closed-loop stabilization of a network which consists of a set of coupled hybrid single-input single-output (SISO) subsystems. Each hybrid subsystem involves a continuous-time subsystem together with a digital (or, eventually, discrete-time) one being subject to eventual mutual couplings of dynamics and also to discrete delayed dynamics. The stabilizing controller is static and based on linear output feedback. The controller synthesis method is of algebraic type and based on the use of a linear algebraic system, whose unknown is a vector equivalent form of the controller gain matrix, which is obtained from a previous algebraic problem version which is based on the ad hoc use of the matrix Kronecker product of matrices. As a first step of the stabilization, an extended discrete-time system is built by discretizing the continuous parts of the hybrid system and to unify them together with its digital/discrete-time ones. The stabilization study via static linear output feedback contains several parts as follows: (a) stabilizing controller existence and controller synthesis for a predefined targeted closed-loop dynamics, (b) stabilizing controller existence and its synthesis under necessary and sufficient conditions based on the statement of an ad hoc algebraic matrix equation for this problem, (c) achievement of the stabilization objective under either partial or total decentralized control so that the whole controller has only a partial or null information about couplings between the various subsystems and (d) achievement of the objective under small coupling dynamics between subsystems.

Keywords: hybrid dynamic systems; decentralized control; stabilization; output feedback; static output feedback controller

MSC: 93C05; 93D20; 93C55

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1. Introduction

The stabilization of dynamic systems via feedback is a very important topic in Control Theory since a necessary minimum requirement for any controlled system is that it operates in a stable way. Therefore, the stabilization theory is relevant in continuous-time systems, discrete-time systems and the hybrid ones which have mixed continuous-time and discrete-time parts. See, for instance, [1–13] and some related references therein. The discretization of continuous-time systems can be performed to constant sampling rates or to non-uniform ones [3,6] so as to take the sampling rate as an extra design function which can be accommodated to the rate of variation of the signals of interest in the system under study. The works in [1,2] focus on the stabilization of saturated discrete-time switching systems. On the other hand, the works in [1,4] are focused on the stabilization of multirate control systems, so on those which have signals being sampled at different sampling rates, also with the objective of facilitating the accommodation of signals in the system that, because of their different nature, evolve at different variation rates or which are needed to be sampled at different rates. A useful design technique for stabilization purposes is the use of Lyapunov functions, which can involve the structure of the closed-loop system parameterization (that is, the one involved the incorporation of the feedback control law), so as to allow the appropriate synthesis of the feedback controller, [7]. Some stabilization problems also incorporate the extra effort of needing to follow the behavior of a certain prescribed model which is known as the "model-matching" or "model-following" objective. In this case, it is not only needed to stabilize the closed-loop modes (stabilization problem) but also to prescribe the values of both the zeros and poles of the closed-loop transfer function to prescribed values defined by the reference model [8,9]. Different devices and design techniques which should be examined to decide on combining discretization tools with continuous-time analysis in complex dynamic systems are the use of appropriate sampling and hold devices [8–10] to update discretized signal information for control purposes, the eventual influence of delays either in the input or output, or in the sates, and also the possible stabilization via state or output feedback involving either centralized control, i.e. involving all the available output information, or decentralized control, i.e. involving only local information or a partial information on the whole system. See [9,11–38] and references therein.

The main objective of this work is to deal with hybrid dynamic systems. Such systems that combine the involvement of both continuous-time signals and digital signals in an integrated way have received important attention [4,39–46]. They involve modeling tools which are very versatile allowing to describe the whole system in a discrete-time way for a certain sampling period as a first modeling stage due to the combination of the discretization of the continuous-time subsystem with the either discrete or digital subsystem. In particular, the optimization of inputs and the fundamental properties of such systems have received attention in [39] and their multirate sampling modeling tools to accommodate the various signals in the system and its control performance concerns have been studied in [4,40,45]. The main importance of hybrid dynamic systems arises from the fact that continuous-time and digital subsystems usually operate in a combined and integrated fashion in many real world situations. A second reason to establish such hybrid models is their suitability, for technological implementation reasons, for describing the use of either discrete-time or digital controllers to either stabilize or control continuous-time plants. For that purpose, a wide class of linear hybrid systems proposed in [39], and also dealt with in [40–44], have been considered for model-following purposes. The whole state of the hybrid dynamic systems studied in the above approaches is described by its continuous-time substate being forced by both the current input in continuous time and its sampled value at the last preceding sampling instant as well, while the discrete-time or, eventually, digital subsystem is driven by the sampled control at sampling instants. In general, there are dynamical couplings between both substates.

In this paper, we focus on the closed-loop stabilization of a hybrid dynamic system which is a network consisting of a tandem of q subsystems, each of them being described by a continuous-time substate together with a discrete-time one. In the most general case, there are mutual couplings between both continuous and discrete substates of each subsystem, couplings between the dynamics of the various subsystems and delayed point dynamics in the whole system also with couplings between subsystems. The closed loop stabilization of such a network is investigated though linear output feedback by synthesizing a static controller. The possibility of partial or total lack of information of couplings between subsystems available for the synthesis of the controller is also studied. This leads to designs based on partial or total decentralized linear output feedback stabilizing control, [14–26], which is of interest as a design technique to reduce the amount of online information to be processed to control the total system, especially, in the case of complex high-dimensional systems.

The paper is organized as follows. Section 2 presents the proposed hybrid system which consists of a set of continuous-time systems with mutual dynamic couplings on a set of digital subsystems, both sets being integrated in a system network. Each subsystem is assumed of single-input single-output (SISO) type. The whole dynamics can be also eventually affected by discrete-time delayed dynamics for a finite number of point delays, and it is driven, in general, by a combined action of the continuous-time input along the intersample time interval together with its sampled values at the sampling instants. This section also contains two descriptions of an extended discretized system, built with the discretization of the continuous parts of the whole hybrid system being eventually coupled with the digital ones, whose stabilization objective is the first and main intended step for the stabilization of the whole hybrid system. Section 3 deals with the stabilization through linear static output feedback of the modified extended discrete system with zero input–output direct interconnection gains, what implies basically that the relative degree, or pole-zero excess in the transfer function, is greater than one. The mechanism for designing the controller gain is of algebraic type and based on converting the set of equations to solve in a linear algebraic system of equations based on a vector form version of them being obtained from the use of ad hoc Kronecker products of matrices [31,32] in the original synthesis problem. In general, the algebraic problem can be: (a) non-compatible, so that it has no solution for a pre-defined suited stable closed-loop dynamics of the extended discrete system being defined by a convergent matrix of closed-loop dynamics, or (b) it can be algebraically compatible with either one (compatible determinate) or infinitely many (compatible indeterminate) solutions for the controller to be synthesized.

In short, remember that a simple linear algebraic system of equations $y = Ax$ is solvable in x, or compatible if and only if $y \in Im(A)$. This holds if and only if $rank(A) = rank(A, b)$ (Rouché–Capelli theorem). The solution x is unique if and only if A is non-singular so that the algebraic system is compatible determinate. Otherwise, if the Rouché–Capelli theorem still holds, there are infinitely many solutions, and the algebraic system is compatible indeterminate. If $rank(A) < rank(A, b)$, then $y \notin Im(A)$ and the algebraic system has no solution so that it is incompatible. In the case of incompatible systems, it can be found the best approximate solution x, which minimizes $\|y - Ax\|$ by involving the pseudoinverse matrix techniques on the singular matrix A. The whole sets of compatible (either determinate or indeterminate) solutions, if they exist, or the best approximate solution (if no exact solution exists) can be calculated by pseudoinverse matrix techniques applied on the algebraic system. In our case, the solutions consist of finding a linear output feedback stabilizing controller gain, it does exist, so that the closed-loop dynamics equalizes some prescribed stability matrix.

A technical concern is that the algebraic test for linear output feedback stabilizability cannot be performed generically for some convergent closed-loop matrix but only for given targeted convergent matrices of closed-loop dynamics. On the other hand, Section 4 relies on linking the existence of some static linear output feedback stabilization control law of the modified extended discrete system with special Riccati matrix algebraic equalities. Section 5 is devoted to the characterization of keeping the stabilization under a total of partial degree of decentralized control. Such a decentralization consists of the achievement of the closed-loop stabilization under either a total or a partial lack of information about the couplings of mutual dynamics between couples of subsystems being transmitted to the overall controller. In this way, each subsystem controller operates just with local information about its own subsystem with eventually a minimum of available information taken about the mutual dynamical couplings between the various subsystems able to achieve the closed-loop stabilization. The final part of the article addresses, in Section 6, the particular cases of small influences of the delayed discrete dynamics and that of the couplings between the pairs of subsystems in the whole dynamics of the hybrid system. In those cases, the main controller synthesis process is performed on the nominal part of the system (that is, the one being free of uncertainties) with a sufficient stability degree so as to fight against the influence of the uncertainties while keeping the closed-loop stability of the whole system. Finally, conclusions end the paper.

Notation

$$\overline{n} = \{1, 2, \cdots, n\}$$

$\mathbf{Z}$, $\mathbf{Z}_{0+}$ and $\mathbf{Z}_{+}$ are, respectively, the sets of integer numbers, non-negative integer numbers and positive integer numbers.

$\boldsymbol{R}$, $\boldsymbol{R}_{0+}$ and $\boldsymbol{R}_{+}$ are, respectively, the sets of real, non-negative real numbers and positive real numbers.

$\boldsymbol{C}$ is the set of complex numbers, $\boldsymbol{C}_{\alpha} = \{z \in \boldsymbol{C} : |z| \geq \alpha\}$ and $\boldsymbol{C}_{\alpha+} = \{z \in \boldsymbol{C} : |z| > \alpha\}$ for any real constant $\alpha \in \boldsymbol{R}_{0+}$.

I_n is the n-th identity matrix and $0_{n \times m}$ is a zero $n \times m$-matrix.

For any square real matrix M, $sp(M)$ is its spectrum, that is, the set of its eigenvalues, $det(M)$ is its determinant, and $adj(M)$ is its adjoint matrix and M^T is the transpose of M.

Let us denote $M = \left(M_{ij}\right) \preceq N = \left(N_{ij}\right)$ for any two $n \times m$ real matrices N, M if $M_{ij} \leq N_{ij}; \forall (i,j) \in \overline{n} \times \overline{m}$, and denote as $\rho(M)$ the spectral radius of any given square matrix M. In the same way, $M \prec N$ if N, $M \neq N$ if $M_{ij} \leq N_{ij}; \forall (i,j) \in \overline{n} \times \overline{m}$. Note that, at least one pair of corresponding matrix entries, the associated inequality is strict, and $M \prec\prec N$ if $M_{ij} < N_{ij}; \forall (i,j) \in \overline{n} \times \overline{m}$. Particular cases are related to comparisons with the zero matrix so that $M \in \boldsymbol{R}_{0+}^{n \times n}$, or $M \succeq 0$, denotes a *non-negative matrix*, that is, $M_{ij} \geq 0$; $\forall (i,j) \in \overline{n} \times \overline{m}$; $M(\neq 0) \in \boldsymbol{R}_{0+}^{n \times n}$, or $M \succ 0$, denotes a *positive matrix*, that is, $M_{ij} \geq 0$; $\forall (i,j) \in \overline{n} \times \overline{m}$ with $M \neq 0$; $M \in \boldsymbol{R}_{+}^{n \times n}$, or $M \succ\succ 0$, denotes a *strictly positive matrix*, that is, $M_{ij} > 0; \forall (i,j) \in \overline{n} \times \overline{m}$.

If M is a square real matrix, then $M \geq 0$ and $M > 0$ denote that it is, respectively, positive semidefinite and positive definite. $M \leq 0$ and $M < 0$ denote that the matrix is negative semidefinite and negative definite, respectively.

A square matrix M is a stability matrix if its spectral abscissa is negative, i.e., if $max\, Re\, \lambda_i < 0$ for $\lambda_i \in sp(M)$. where $sp(M)$ is the set of eigenvalues, or spectrum, of M. A real or complex square matrix M is convergent if and only if its spectral radius $\rho(M) = \{max|\lambda_i| : \lambda_i \in sp(M)\} < 1$.

$\boldsymbol{i} = \sqrt{-1}$ is the imaginary complex unit.

$\sigma[M(\boldsymbol{i}\omega)]$, for $\omega \in \boldsymbol{R}$, stands for the singular values of the complex-valued rational matrix $M : \boldsymbol{C} \to \boldsymbol{C}^{n \times m}$.

The H_∞-norm of such a matrix, which is the supremum of its singular values on the boundary of the unit circle centered at the origin of the complex plane, provided that it exists and is finite, is denoted by $\|M\|_{H_\infty}$. If $M \in \boldsymbol{R}^{n \times m}$, then the symbols $\|M\|_\infty$, $\|M\|_1$ and $\|M\|_2$ stand, respectively, for the ∞, 1 and 2 matrix norms, that is, for the maximum absolute sum of its rows, that of its columns and for its maximum singular value.

$M \otimes N = \left(\left[M_{ij}N\right]_{kl} \right)$ is the Kronecker product of the real matrices of any orders $M = \left(M_{ij}\right)$ and $N = \left(N_{ij}\right)$. In particular, if M and N are $m \times n$ and $p \times q$, then its Kronecker product is $mp \times nq$ defined by

$$M \otimes N = \begin{bmatrix} M_{11}N & \cdots & M_{1n}N \\ \vdots & \ddots & \vdots \\ M_{m1}N & \cdots & M_{mn}N \end{bmatrix}$$

$vec(M)$ is a real vector formed by the entries of the real matrix M ordered in the order of its rows.

$A^\dagger \in \boldsymbol{R}^{r \times m}$ is the Moore–Penrose generalized inverse, or Moore-Penrose pseudoinverse, of $A \in \boldsymbol{R}^{m \times r}$ which satisfies $A = AA^\dagger A$ and $A^\dagger = A^\dagger A A^\dagger$. If $A \in \boldsymbol{R}^{m \times r}$ is of rank p is, in general non-uniquely, factorized as $A = FG$ with $rankP = rankG = p$ and $F \in \boldsymbol{R}^{m \times p}$ and $G \in \boldsymbol{R}^{p \times r}$ (thus being, respectively, full column rank and full row rank), then $A^\dagger = G^T\left(GG^T\right)^{-1}\left(F^T F\right)^{-1}F^T$. Note that such F and G always exist for a given A of rank p.

The continuous time states or signals are denoted under the argument "t"in parenthesis, say $x(t)$, (running on the non-negative real set) while the discrete-time ones or the digital ones are denoted with the argument "k" in brackets, say $x[k]$, (running on the set of non-negative integer numbers).

2. Hybrid Continuous-Time and Digital System

2.1. Simple Motivation Example

To fix some ideas, we discuss a simple motivating example for purely continuous-time or discrete-time systems so that a family of static linear output feedback controllers exist defined by an open ball around a linear output feedback stabilizing controller:

Example 1. *Consider the either unstable or critically stable discrete characteristic polynomial $p(z) = det(zI_n - A)$ which describes the open-loop (i.e., uncontrolled) dynamics of the discrete continuous-time unstable n-th order system of state $x[k] \in \boldsymbol{R}^n$, control $u[k] \in \boldsymbol{R}^r$ and output $y[k] \in \boldsymbol{R}^m$for $k \in \boldsymbol{Z}_{0+}$, given by $x[k+1] = Ax[k] + Bu[k]$, $y[k] = Cx[k]$, $x(0) = x_0$; $k \in \boldsymbol{Z}_{0+}$, with $B \in \boldsymbol{R}^{n\times r}$ and $\boldsymbol{C} \in \boldsymbol{R}^{m\times n}$ with $max(r,m) \le n$. Note that the system is stabilizable by static linear feedback state control if all the unstable open-loop modes are controllable, that is, if and only if $rank[zI_n - A, B]_{s\in \boldsymbol{C}_1 \cap sp(A)} = n$. Assume that it is intended to stabilize it by a static output feedback control law $u[k] = Ky[k] = kCx[k]$; $k \in \boldsymbol{Z}_{0+}$, for some constant control gain $K \in \boldsymbol{R}^{r\times m}$, so that the closed-loop system becomes $x[k+1] = (A + BKC)x[k]$; $k \in \boldsymbol{Z}_{0+}$. Taking z-transforms in the closed-loop equation under zero initial conditions yields that the closed-loop characteristic polynomial is:*

$$zI_n - A - BKC = zI_n - A - A_m + (A_m - BKC)$$

for any given $A_m \in \boldsymbol{R}^{n\times n}$ supposed to be convergent (that is, a convergent matrix in the discrete context), i.e., with eigenvalues of modulus less than unity so that it exists $(zI_n - A_m)^{-1}$, $\forall z \in \boldsymbol{C}_1$. Then,

$$(zI_n - A - BKC)^{-1} = \left[I_n - (zI_n - A_m)^{-1}(A + BKC - A_m)\right](zI_n - A_m)^{-1}; \forall z \in \boldsymbol{C}_1$$

exists for all $z \in \boldsymbol{C}_1$ if the H_∞- norm of $(zI_n - A_m)^{-1}(A_m - A - BKC)$ is less than one which occurs if $\|A_m - A - BKC\|$ is sufficiently small to guarantee that

$$\left\|\frac{1}{det(zI_n - A_m)}[adj(zI_n - A_m)(A + BKC - A_m)]\right\|_{H_\infty} = \sup_{\omega\in \boldsymbol{R}_{0+}}\left\|\frac{adj(e^{i\omega} I_n - A_m)(A + BKC - A_m)}{det(e^{i\omega} I_n - A_m)}\right\|_{H_\infty} < 1$$

Note that the test is performed on the boundary of the complex unity circle centered at zero. In addition, in this case, the closed-loop eigenvalues are stable, which is, in particular, guaranteed if

$$\|A + BKC - A_m\|_2 \le \bar{\varepsilon} < 1/\left\|(zI_n - A_m)^{-1}\right\|_{H_\infty}$$

Now, the problem reduces to find if it exists, a triple(A_m, K, Δ), such that $A + BKC - A_m = \Delta$ with $K \in \boldsymbol{R}^{r\times m}$ and $A_m, \Delta \in \boldsymbol{R}^{n\times n}$ such that A_m is a convergent matrix with $\|\Delta\|_2 = \varepsilon$ for any real constant $\varepsilon \in [0, \bar{\varepsilon}) \equiv \left[0, 1/\left\|(zI_n - A_m)^{-1}\right\|_{H_\infty}\right)$. Since it also has to be fulfilled for $\varepsilon = 0$, one concludes that a necessary condition is that $A + BK_0C = A_m$ for some convergent matrix A_m so that (A, B) has to be stabilizable and (C, A) detectable, that is, $rank[zI_n - A, B]_{z\in \boldsymbol{C}_1 \cap sp(A)} = rank\left[zI_n - A^T, C^T\right]_{z\in \boldsymbol{C}_1 \cap sp(A)} = n$. Then, any other static linear output feedback controller of gain K stabilizes the closed-loop system if $\Delta = A + BKC - A_m$ has a norm $\varepsilon \in [0, \bar{\varepsilon})$. Thus, $A + BKC$ is a convergent matrix for any $K \in K_d = \left\{K : \|K - K_0\| < \frac{1}{\|B\|\|C\|\left\|(zI_n - A_m)^{-1}\right\|_{H_\infty}}\right\}$.

The above conclusion will be identical if $p(s) = det(sI_n - A)$ is the either unstable or critically stable characteristic polynomial which describes the open-loop dynamics of the linear continuous-time unstable n-th order system of state $x(t) \in \boldsymbol{R}^n$, control $u(t) \in \boldsymbol{R}^r$ and output $y(t) \in \boldsymbol{R}^m$ given by

$$\dot{x}(t) = Ax(t) + Bu(t); y(t) = Cx(t), x(0) = x_0$$

To stabilize it under a static output control law $u(t) = Ky(t)$, the open-loop system has to be stabilizable and detectable, that is, $rank[sI_n - A, B]_{s \in \mathbf{C}_0 \cap sp(A)} = rank[zI_n - A^T, C^T]_{s \in \mathbf{C}_0 \cap sp(A)} = n$ so that a closed-loop equation $A + BK_0C = A_m$ is achieved for some convergent matrix A_m and some stabilizing static output linear feedback controller of gain K_0 and any other static linear output feedback controller of gain K stabilizes also the closed-loop system if

$$\Delta = A + BKC - A_m = A + B(K - K_0)C + BK_0C - A_m = B(K - K_0)C$$

has a norm $\varepsilon \in [0, \bar{\varepsilon})$ with $\bar{\varepsilon} = \left[0, 1/\left\|(sI_n - A_m)^{-1}\right\|_\infty\right)$ so that $A + BKC$ is a convergent matrix for any $K \in K_c = \left\{K : \|K - K_0\| < \frac{1}{\|B\|\|C\|\left\|(sI_n - A_m)^{-1}\right\|_\infty}\right\}$.

2.2. System Structure

Consider the subsequent single-input single-output hybrid linear system which consists of q coupled subsystems:

$$x_c(t) = \left[x_{c1}^T(t), x_{c2}^T(t), \cdots, x_{cq}^T(t)\right]^T; x_d[k] = \left[x_{d1}^T[k], x_{d2}^T[k], \cdots, x_{dq}^T[k]\right]^T \tag{1}$$

$$u(t) = [u_1(t), u_2(t), \cdots, u_q(t)]^T y(t) = [y_1(t), y_2(t), \cdots, y_q(t)]^T \tag{2}$$

$$\dot{x}_{ci}(t) = \sum_{j=1}^{q} \sum_{\ell=0}^{p} \left(A_{cij}\, x_{cj}(t) + A_{cs\ell ij}\, x_{cj}[k-\ell] + A_{cd\ell ij}\, x_{dj}[k-\ell]\right) + b_{ci}\, u_i(t) + b_{csi}\, u_i[k] \tag{3}$$

$$x_{di}[k+1] = \sum_{j=1}^{q} \sum_{\ell=0}^{p} \left(A_{d\ell ij}\, x_{dj}[k-\ell] + A_{dc\ell ij}\, x_{cj}[k-\ell]\right) + b_{di}\, u_i[k] \tag{4}$$

$$y_i(t) = c_{ci}^T x_{ci}(t) + c_{csi}^T x_{c_i}[k] + c_{di}^T x_{di}[k] + d_{ci}\, u_i(t) + d_{di}\, u_i[k] \tag{5}$$

for all $t \in [kT, (k+1)T)$ for any integer $k \geq 0$ with T being the sampling period, where $x_{ci}(t) \in \mathbf{R}^{n_{ci}}$ and $x_{di}[k] \in \mathbf{R}^{n_{di}}$; $\forall i \in \bar{q}$ are, respectively, the dimensions of the i-th continuous and digital subsystems, respectively, whose scalar input and output are $u_i(.)$ and $y_i(.)$, respectively, for $i \in \bar{q}$, and p is the number of discrete internal delays. Thus, $n_c = \sum_{i=1}^{q} n_{ci}$ and $n_d = \sum_{i=1}^{q} n_{di}$ are the continuous and digital dimensions of the whole system integrated by the various subsystem. The parameterization of (1)–(5) is given by matrices $A_{cij}, A_{cs\ell ij} \in \mathbf{R}^{n_{ci} \times n_{cj}}$, $A_{d\ell ij} \in \mathbf{R}^{n_{di} \times n_{dj}}$, $A_{cd\ell ij} \in \mathbf{R}^{n_{ci} \times n_{dj}}$, $A_{dc\ell ij} \in \mathbf{R}^{n_{di} \times n_{cj}}$; $\forall i, j \in \bar{q}$, $\forall l \in \bar{p} \cup \{0\}$, which are matrices of continuous and digital dynamics; $b_{ci}, b_{csi}, c_{ci}, c_{csi} \in \mathbf{R}^{n_{ci}}$; $\forall i \in \bar{q}$, which are control and output vectors of the i-th subsystem; $b_{di}, c_{di} \in \mathbf{R}^{n_{di}}$, $\forall i \in \bar{q}$, which are control and output vectors of the i-th digital subsystem; and d_{ci}, $d_{di} \in \mathbf{R}$, which are the continuous and digital direct input–output interconnection gains of the i-th continuous and digital subsystem, respectively; $\forall i \in \bar{q}$. The continuous-time argument is denoted by '(t)' while the discrete-time argument is denoted by '$[k]$' and the associated continuous and digital variables are denoted correspondingly. Thus, a continuous variable at sampling instants is denoted in the same way as a digital variable so that $x_c[k] = x_c(kT)$, $u[k] = u(kT)$ and $y[k] = y(kT)$ for any integer $k \geq 0$. Similar notations with brackets and parenthesis are for the time arguments of the discrete and continuous variables of the subsystems. In this way, there is no distinction in the treatment of digital and time-discretized variables at sampling instants. The orders of all the real constant matrices in (1) agree with the dimensions of the substates and scalar input and output. It can be pointed out that a digital system within the whole hybrid structure could instead be a dynamic system being discretized from a continuous one at a certain sampling period in a situation such that the original continuous-time structure has no specific interest in the analysis since the associated signals are only relevant at the sampling instants. This kind of system can be treated in the same way within the proposed hybrid continuous/discrete structure. It can be also pointed out that a typical structure of a hybrid dynamic system can be found in cases when a continuous system is in operation under a discretized controller so that the whole structure has a hybrid continuous-time/discrete-time nature consisting of a minimum of two subsystems.

The above system can be described in a compact form as follows:

$$\dot{x}_c(t) = A_c x_c(t) + \sum_{\ell=0}^{p}(A_{cs\ell} x_c[k-\ell] + A_{cd\ell} x_d[k-\ell]) + B_c u(t) + B_{cs} u[k] \quad (6)$$

$$x_d\,[k+1] = \sum_{\ell=0}^{p}\Big(A_{dc\ell}\, x_c\,[k-\ell] + A_{d\ell}\, x_{dj}[k-\ell]\Big) + B_d\, u\,[k] \quad (7)$$

$$y(t) = C_c x_c(t) + C_{cs} x_{cs}[k] + C_d x_d[k] + D_c u(t) + D_d u[k] \quad (8)$$

where $A_c \in \boldsymbol{R}^{n_c \times n_c}$, $A_{cs\ell}, A_{cd\ell} \in \boldsymbol{R}^{n_c \times n_d}$, $B_c, B_{cs} \in \boldsymbol{R}^{n_c \times q}$, $A_{dc} \in \boldsymbol{R}^{n_d \times n_c}$, $A_d \in \boldsymbol{R}^{n_d \times n_d}$,$B_d \in \boldsymbol{R}^{n_d \times q}$, C_c , $C_{cs} \in \boldsymbol{R}^{q \times n_c}$, $C_d \in \boldsymbol{R}^{q \times n_d}$, D_c , $D_d \in \boldsymbol{R}^{q \times q}$; $\ell = 0, 1, \ldots, p$, with $n_c = \sum_{i=1}^{p} n_{ci}$ and $n_d = \sum_{i=1}^{p} n_{di}$ are defined by:

$$A_c = \begin{bmatrix} A_{c11} & \cdots & A_{c1q} \\ \vdots & \vdots & \vdots \\ A_{cq1} & \cdots & A_{cqq} \end{bmatrix} \quad (9)$$

$$A_{cs\ell} = \begin{bmatrix} A_{cs\ell 11} & \cdots & A_{c\ell s1q} \\ \vdots & \vdots & \vdots \\ A_{cs\ell q1} & \cdots & A_{cs\ell qq} \end{bmatrix};\ A_{cd\ell} = \begin{bmatrix} A_{cd\ell 11} & \cdots & A_{cd\ell 1q} \\ \vdots & \vdots & \vdots \\ A_{cd\ell q1} & \cdots & A_{cd\ell qq} \end{bmatrix};\ \ell = 0, 1, \ldots, p \quad (10)$$

$$A_{dc\ell} = \begin{bmatrix} A_{dc\ell 11} & \cdots & A_{dc\ell 1q} \\ \vdots & \vdots & \vdots \\ A_{dc\ell q1} & \cdots & A_{dc\ell qq} \end{bmatrix}; A_{d\ell} = \begin{bmatrix} A_{d\ell 11} & \cdots & A_{d\ell 1q} \\ \vdots & \vdots & \vdots \\ A_{d\ell q1} & \cdots & A_{d\ell qq} \end{bmatrix};\ \ell = 0, 1, \ldots, p \quad (11)$$

$$\begin{gathered} B_c = block\,diag\big[b_{c1} \cdots b_{cq}\big]; \\ B_{cs} = block\,diag\big[b_{cs1} \cdots b_{csq}\big];\ B_d = block\,diag\Big[b_{d1} \cdots b_{dq}\Big] \end{gathered} \quad (12)$$

$$\begin{gathered} C_c = block\,diag\Big[c_{c1}^T \cdots c_{cq}^T\Big]; \\ C_{cs} = block\,diag\Big[c_{cs1}^T \cdots c_{csq}^T\Big];\ C_d = block\,diag\Big[c_{d1}^T \cdots c_{dq}^T\Big] \end{gathered} \quad (13)$$

$$D_c = diag\big[d_{c1} \cdots d_{cq}\big]\, D_d = diag\Big[d_{d1} \cdots d_{dq}\Big] \quad (14)$$

The continuous-time substate evolves through time according to the following solution equation obtained from (6):

$$\begin{aligned} x_c\,(kT+\sigma) = {} & e^{A_c \sigma}\left(I_{n_c} + \left(\int_0^\sigma e^{-A_c \tau}\, d\tau\right) A_{cs0}\right) x_c\,[k] \\ & + \left(\int_0^\sigma e^{A_c(\sigma-\tau)}\, d\tau\right)\left(\sum_{\ell=1}^{p} A_{cs\ell}\, x_c\,[k-\ell] + \sum_{\ell=0}^{p} A_{cd\ell}\, x_d\,[k-\ell]\right) \\ & + \left(\int_0^\sigma e^{A_c(\sigma-\tau)}\, d\tau\right) B_{cs}\, u\,[k] \\ & + \int_0^\sigma e^{A_c(\sigma-\tau)}\, B_c\, u(kT+\tau)\, d\tau \geq 0;\ \forall k \in \boldsymbol{Z}_{0+},\ \forall \sigma \in (0, T] \end{aligned} \quad (15)$$

which can be compacted as follows at the sampling times:

$$\begin{gathered} x_c\,[k+1] = \overline{\Psi}_c[k]\overline{x}[k] + \Gamma_{cs}(T)\, u\,[k] + \int_0^T e^{A_c(T-\tau)}\, B_c\, u(kT+\tau)\, d\tau \geq 0;\ \forall k \in \boldsymbol{Z}_{0+} \\ ,\ \forall \sigma \in (0, T] \end{gathered} \quad (16)$$

where

$$\overline{x}[k] = \left(\overline{x}_c^T[k], \overline{x}_d^T[k]\right)^T \in \boldsymbol{R}^{(p+1)n},\ \ \overline{\Psi}_c[k] = \big[\overline{\Psi}_{cc}[k],\, \overline{\Psi}_{cd}[k]\big] \in \boldsymbol{R}^{n_c \times (p+1)n} \quad (17)$$

$$\overline{x}_c^T[k] = \left(x_c^T[k], x_c^T[k-1], \cdots, x_c^T[k-p] \right) \in \boldsymbol{R}^{(p+1)n_c} \tag{18}$$

$$\overline{x}_d^T[k] = \left(x_d^T[k], \overline{x}_d^T[k-1], \cdots, \overline{x}_d^T[k-p] \right) \in \boldsymbol{R}^{(p+1)n_d} \tag{19}$$

$$\overline{\Psi}_{cc}[k] = \left[\Phi_c(T) + \Gamma_{cs0}(T), \widetilde{\overline{\Psi}}_{cc}[k] \right] = \left[\Phi_c(T) + \Gamma_{cs0}(T), \Gamma_{cs1}(T), \cdots, \Gamma_{csp}(T) \right] \in \boldsymbol{R}^{n_c \times (p+1)n_c} \tag{20}$$

$$\overline{\Psi}_{cd}[k] = \left[\Gamma_{cd0}(T), \widetilde{\overline{\Psi}}_{cd}(T) \right] = \left[\Gamma_{cd0}(T), \Gamma_{cd1}(T), \cdots, \Gamma_{cdp}(T) \right] \in \boldsymbol{R}^{n_c \times (p+1)n_d} \tag{21}$$

$$\Phi_c(T) = e^{A_c T}, \Gamma_{cs\ell}(T) = \left(\int_0^T e^{A_c(T-\tau)} d\tau \right) A_{cs\ell}, \Gamma_{cd\ell}(T) = \left(\int_0^T e^{A_c(T-\tau)} d\tau \right) A_{cdl}; \quad l = 0, 1, \ldots, p \tag{22}$$

$$\Gamma_{cs}(T) = \left(\int_0^T e^{A_c(T-\tau)} d\tau \right) B_{cs} \tag{23}$$

;$\forall k \in \boldsymbol{Z}_{0+}$, with $n = n_c + n_d$, $n_c = \sum_{i=1}^q n_{ci}$ and $n_d = \sum_{i=1}^q n_{di}$. The dimension of the extended discrete state $\overline{x}[k] = \left(\overline{x}_c^T[k], \overline{x}_d^T[k] \right)^T$ is $\overline{n} = (2p+1)n$. The discrete-time substate evolves through time according to the following solution equation, rewritten equivalently from (7):

$$x_d[k+1] = \overline{\Psi}_d \overline{x}[k] + B_d u[k]; \ \forall k \in \boldsymbol{Z}_{0+} \tag{24}$$

where

$$\overline{\Psi}_d(T) = \left[\overline{\Psi}_{dc}(T), \overline{\Psi}_{dd} \right] \in \boldsymbol{R}^{n_d \times (p+1)n} \tag{25}$$

$$\overline{\Psi}_{dc} = \left[A_{dc0}, \widetilde{\overline{\Psi}}_{dc} \right] = \left[A_{dc0}, A_{dc1}, \cdots, A_{dcp} \right] \in \boldsymbol{R}^{n_d \times (p+1)n_c} \tag{26}$$

$$\overline{\Psi}_{dd} = \left[A_{d0}, \widetilde{\overline{\Psi}}_{dd} \right] = \left[A_{d0}, A_{d1}, \cdots, A_{dp} \right] \in \boldsymbol{R}^{n_d \times (p+1)n_d} \tag{27}$$

2.3. Extended Discrete System

Combining (16) and (23), one concludes that the extended discrete vector, built with the sampled values of the continuous-time substate and the discrete substate, evolves according to the following extended discrete Equation:

$$\overline{x}[k+1] = \overline{A}_d \overline{x}[k] + \overline{B}_d u[k] + \left(\overline{B}_{c\tau} \right) u[k] \tag{28}$$

where

$$\overline{A}_d = \begin{bmatrix} \overline{\Psi}_{cc}[k] & \overline{\Psi}_{cd}[k] \\ I_{pn_c} & 0_{p \times (p+1)n_d} \\ \overline{\Psi}_{dc} & \overline{\Psi}_{dc} \\ 0_{p \times (p+1)n_c} & I_{pn_d} \end{bmatrix} \in \boldsymbol{R}^{\overline{n} \times \overline{n}} \tag{29}$$

$$\overline{B}_d = \begin{bmatrix} \Gamma_{cs} \\ 0_{pn_c \times 2q} \\ B_d \\ 0_{pn_d \times 2q} \end{bmatrix} \in \boldsymbol{R}^{\overline{n} \times q}; \ \left(\overline{B}_{c\tau} \right) u[k] = \begin{bmatrix} \int_0^T e^{A_c(T-\tau)} B_c u(kT+\tau) d\tau \\ 0_{pn_c \times 2q} \\ 0_{n_d \times 2q} \\ 0_{pn_d \times 2q} \end{bmatrix} \in \boldsymbol{R}^{\overline{n} \times q} \tag{30}$$

For purposes of generating the intersample input from a discrete sequence defined at the sampling instants, the following technical assumption is made which will be useful for some of the coming results:

Assumption 1. *Assume that $u(kT+\tau) = L(kT+\tau)v[k]$; $\forall k \in \boldsymbol{Z}_{0+}$, $\forall \tau \in (0, T)$ for some control sequence $\{v[k]\}_{k=0}^{\infty}$ and some matrix function $L : \boldsymbol{Z}_{0+} \times [0, T) \to \boldsymbol{R}^{q \times q}$; $\forall \tau \in [0, T)$ which satisfies the constraints:*

(1) It is periodic with period T, that is, $L(kT+\tau) = L_k(T, \tau) = L(T, \tau)$; $\forall k \in \boldsymbol{Z}_{0+}$, $\forall \tau \in [0, T)$.
(2) It has a support of nonzero Lebesgue measure on $[0, T)$.
(3) The Lebesgue integral $\overline{B}_c = \int_0^T e^{A_c(T-\tau)} B_c L(\tau) d\tau$ exists and it is finite.

Remark 1. *Note that Assumption 1 allows a large variety of definitions for* $L : \boldsymbol{Z}_{0+} \times [0, T) \to \boldsymbol{R}^{q \times q}$ *including isolated bounded discontinuities or even a finite numbers of Dirac impulses on each interval* $[kT, (k+1)T)$. *Moreover, if* $u[k]$ *and* $v[k]$ *are designed independently of each other from, in general, distinct pre-calculated bounded sequences* $\{u[k]\}_{k=0}^{\infty}$ *and* $\{v[k]\}_{k=0}^{\infty}$, *according to some beneficial design criterion, namely without using a constraint* $L[k]v[k] = u[k]$; $k \in \boldsymbol{Z}_{0+}$, *then the control function* $u : \boldsymbol{Z}_{0+} \times [0, T) \to \boldsymbol{R}^q$ *can have bounded discontinuities at the sampling instants since* $L[k]v[k] \neq u[k]$. *However, an advantage of this situation is that the extended discrete control sequence* $\left\{ \left(u^T[k], v^T[k]\right)^T \right\}_{k=0}^{\infty} \in \boldsymbol{R}^{2q}$ *which governs (28)–(30) has a dimension* $2q$ *instead of* q *such that the potential stabilization of such a modified extended discrete system might be achievable under weaker conditions than the use of the equalizing control constraint* $L[k]v[k] = u[k]$ *at the sampling instants.*

The output at sampling instants becomes from (8):

$$y[k] = (C_c + C_{cs}, C_d) \begin{pmatrix} x_c[k] \\ x_d[k] \end{pmatrix} + (D_c + D_d)u[k] = \overline{C}\overline{x}[k] + (D_c + D_d)u[k] \tag{31}$$

where $\overline{C} = \left(C_c + C_{cs}, 0_{q \times n_c \ell}, C_d, 0_{q \times n_d \ell} \right)$.

2.4. Modified Extended Discrete System with Two Input Channels

We now generate the continuous-time control from a discrete sequence being, in general, distinct of the primary discrete control sequence $\{u[k]\}_{k=0}^{\infty}$. This strategy allows taking advantage of the use of a double dimensioned control input for the extended discrete system which will facilitate its potential stabilization. In fact, the second control channel is obtained from the generation of the intersample continuous-time input from the auxiliary discrete sequence. As a result, the extended system is controlled by a 2q dimensional discrete control sequence. Under Assumption 1, Equations (28)–(30), together with (32), take the compact form of the following modified extended discrete system of state of dimension $\overline{n} = (2p+1)n$ which describes at sampling times the joint discretized dynamics of the continuous-time system plus that of the digital one for a discrete control sequence $\{\overline{u}[k]\}_{k=0}^{\infty}$ with $\overline{u}[k] = \left(u^T[k], v^T[k]\right)^T \in \boldsymbol{R}^{2q}$:

$$\overline{x}[k+1] = \overline{A}_d \overline{x}[k] + \overline{\Gamma}_d \overline{u}[k];\ y[k] = \overline{C}\overline{x}[k] + \overline{D}\overline{u}[k] \tag{32}$$

where $\overline{u}[k] = \left(u^T[k], v^T[k]\right)^T$, and

$$\overline{\Gamma}_d = \left[\ \overline{B}_d \quad , \overline{B}_c\ \right] \in \boldsymbol{R}^{\overline{n} \times 2q};\ \overline{B}_c = \begin{bmatrix} B_c \\ 0_{pn_c \times 2q} \\ 0_{n_d \times 2q} \\ 0_{pn_d \times 2q} \end{bmatrix} \in \boldsymbol{R}^{\overline{n} \times q};\ B_c = \int_0^T e^{A_c(T-\tau)} B_c L(kT+\tau)\, d\tau$$
$$\overline{C} = \left(C_c + C_{cs}, 0_{q \times n_c p}, C_d, 0_{q \times n_d p} \right) \in \boldsymbol{R}^{q \times \overline{n}};\ \overline{D} = \left(\ D_c + D_d, \quad 0_{q \times q}\ \right) \in \boldsymbol{R}^{q \times 2q} \tag{33}$$

The following preliminary stabilizability result is of interest for subsequent results to be then obtained:

Remark 2. *The extended discrete system (32) and (33) is stabilizable by static linear state feedback* $\overline{u}[k] = \overline{K}\overline{x}[k]$ *if and only if* $rank\left(zI_{\overline{n}} - \overline{A}_{dd}, \overline{\Gamma}_d\right) = \overline{n}$ *for each* $z \in \boldsymbol{C}_1 \cap sp\left(\overline{A}_d\right)$ *[22,27,33,34]. This result follows directly from the Popov–Belevitch–Hautus stabilizability test, [40,41] for discrete systems applied to (32) and (33) by using a similarity transform on* $\overline{A}_d$ *to a triangular form* $\hat{\overline{A}}_d = \begin{bmatrix} \hat{\overline{A}}_{d11} & \hat{\overline{A}}_{d12} \\ 0 & \hat{\overline{A}}_{d22} \end{bmatrix}$, *which makes the associated control matrix* $\hat{\overline{\Gamma}}_d = \begin{bmatrix} \hat{\overline{\Gamma}}_{d1} \\ 0 \end{bmatrix}$ *and the transformed state vector becomes* $\hat{\overline{x}}(t) = \begin{bmatrix} \hat{\overline{x}}_1(t) \\ \hat{\overline{x}}_2(t) \end{bmatrix}$, *such that* $\hat{\overline{A}}_{dd22}$ *is stable describing the* *dynamics of the uncontrollable modes and* $\left[\hat{\overline{A}}_{d11}, \hat{\overline{\Gamma}}_{d1}\right]$ *is controllable and describes the dynamics of the unstable and critically stable modes. Moreover, a necessary condition for the discrete system (32) and (33) to be stabilizable by static linear output feedback* $\overline{u}[k] = \overline{K}y[k]$ *is that it be stabilizable and detectable, that is, that* $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = rank\begin{pmatrix} \overline{C} \\ zI_{\overline{n}} - \overline{A}_d \end{pmatrix} = \overline{n}$ *for each* $z \in \boldsymbol{C}_1 \cap sp\left(\overline{A}_d\right)$. *Such a condition is not sufficient, and it can be proved that, even if a stronger joint controllability and observability condition, it is not sufficient for the stabilizability of any linear system under some linear static output feedback control law [28–30,32,37,38].*

Remark 3. *Concerning the discussion in Remark 2, note that it is obvious that* $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = \overline{n}$; $z \in sp\left(\overline{A}_d\right)$ *implies and it is implied by* $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = \overline{n}$; $\forall z \in \boldsymbol{C}_1$ *since* $\left(zI_{\hat{n}} - \overline{A}_d, \overline{\Gamma}_d\right)$ *is everywhere full row rank if it is full rank in any complex subset at the eigenvalues of* $\overline{A}_d$ *since* $det\left(zI_{\overline{n}} - \overline{A}_d\right) \neq 0$; $z \in \boldsymbol{C} \backslash sp\left(\overline{A}_d\right)$. *Therefore, it is sufficient to apply the stabilizability test to the set* $\boldsymbol{C}_1 \cap sp\left(\overline{A}_d\right)$*, that is, for joint critically stable and unstable eigenvalues of* $\overline{A}_d$*, as Lemma 1 states. Since* $\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right)$ *can only be eventually rank defective at the eigenvalues of* $\overline{A}_d$*, then the Popov–Belevitch–Hautus controllability test of (32) and (33) is similar to that of Remark 2 by extending the rank test to all the eigenvalues of* $\overline{A}_d$*, that is, (32) and (33) is controllable if and only if* $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = \overline{n}$; $\forall z \in sp\left(\overline{A}_d\right)$*, which is equivalent, by the mentioned reasons of potential rank defectiveness at the eigenvalues, to* $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = \overline{n}$; $\forall z \in \boldsymbol{C}$.

Remark 4. *Note from (16) that if* $\{\overline{u}[k]\}_{k=0}^{\infty} \to 0$*, then* $\{u[k]\}_{k=0}^{\infty} \to 0$ *and* $\{v[k]\}_{k=0}^{\infty} \to 0$*, if* $\{\overline{x}[k]\}_{k=0}^{\infty} \to 0$*, then* $u(t) \to 0$*, and* $x_c(t) \to 0$ *as* $t \to \infty$ *and* $y(t) \to 0$ *as* $t \to \infty$.

3. Stabilization by Linear Static Output Feedback of the Modified Extended Discrete System with Zero Input–Output Direct Interconnection Gains

The closed-loop asymptotic stabilization of the extended discrete system is a first basic design step to stabilize the hybrid system since the state and output sequences at the sampling instants are bounded for any given finite initial conditions and they converge asymptotically to zero at the sampling instants. This does not imply that the state and output signals converge also asymptotically to zero as time tends to infinity without extra conditions.

Through this section, it is assumed that $D_c + D_d = 0$, which implies that $\overline{D} = 0$ and which includes the case when $D_c = 0$ and $D_d = 0$, that is, the particular case of zero direct input–output interconnection gains. The stabilization of the extended discrete system (32) and (33) under linear static output feedback is now discussed. Assume a control law of the form:

$$\overline{u}[k] = \overline{K}y[k] = \overline{KC}\overline{x}[k];\ k \in \boldsymbol{Z}_{0+} \tag{34}$$

The following result gives a simple algebraic necessary and sufficient condition for the existence of a static linear output feedback stabilizing controller for (32) and (33) as well as further conditions related to the eventual stabilization by a class of controllers which are perturbations of a nominal stabilizing one.

Theorem 1. *Assume that* $D_c + D_d = 0$. *Then, the following properties hold:*

(i) *The modified extended system (32) and (33) is stabilized by some linear control law of the form (34) if and only if for some convergent matrix* $\overline{A}_{cld}$ *, being of the same order as that of* $\overline{A}_d$ *, the following condition holds:*

$$rank\left[\overline{\Gamma}_d \otimes \overline{C}^T\right] = \left(rank\overline{\Gamma}_d\right)\left(rank\overline{C}\right) = rank\left[\overline{\Gamma}_d \otimes \overline{C}^T, vec\left(\overline{A}_{c\ell d} - \overline{A}_d\right)\right] \tag{35}$$

(ii) *Assume that (35) holds, so that* $\overline{A}_{cld} = \overline{A}_d + \overline{\Gamma}_d\overline{KC}$ *is a convergent matrix for some controller gain matrix* $\overline{K}$. *Assume also that* $\widetilde{\overline{A}}_{cld} = \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq \sigma\overline{A}_{cld}$ *is a perturbation of* $\overline{A}_{cld}$ *under the controller gain matrix* $\overline{K} + \widetilde{\overline{K}}$ *for some* $\sigma \in \boldsymbol{R}$ *and some incremental controller gain* $\widetilde{\overline{K}}$.

Then, a controller $\overline{K}+\widetilde{\overline{K}}$ *stabilizes also (32) and (33) provided that, for some matrix* $\Xi \succ 0$ *of the same order as* $\overline{A}_{cld}$ *, the incremental controller gain matrix* $\widetilde{\overline{K}}$ *is chosen to satisfy the subsequent equivalent vector equality:*

$$0 \preceq vec(\Xi) = \sigma\, vec\left(\overline{A}_{cld}\right) - \left(\overline{\Gamma}_d^T \otimes \overline{C}^T\right) vec\left(\widetilde{\overline{K}}\right) \tag{36}$$

with

$$\sigma \in \left(-\frac{1-\rho\left(\overline{A}_{cld}\right)}{\rho\left(\overline{A}_{cld}\right)}, \frac{1-\rho\left(\overline{A}_{cld}\right)}{\rho\left(\overline{A}_{cld}\right)} \right) \tag{37}$$

(iii) *Assume that, in Property (ii),* $\overline{A}_{cld} \succ 0$ *,* $-\sigma\overline{A}_{cld} \preceq \widetilde{\overline{A}}_{cld} = \overline{\Gamma}_d \widetilde{\overline{K}}\overline{C} \preceq \sigma\overline{A}_{cld}$ *for some* $\sigma \in \boldsymbol{R}_+$ *. Then, a controller* $\overline{K}+\widetilde{\overline{K}}$ *stabilizes also (32) and (33) provided that, for some real scalars,* $\lambda_1 \in [0,1]$ *and* $\lambda_2 \geq min(0, 2\lambda_1 - 1)$ *, either the incremental controller gain matrix* $\widetilde{\overline{K}} \succeq 0$ *is chosen to satisfy the vector equality:*

$$\left(\overline{\Gamma}_d \otimes \overline{C}^T\right) vec\left(\widetilde{\overline{K}}\right) - \left(1 - \frac{2\lambda_1}{1+\lambda_2}\right)\sigma\ vec\overline{A}_{cld} \preceq 0 \tag{38}$$

subject to

$$\lambda_2 \geq min(0, 2\lambda_1 - 1) \tag{39}$$

or, $\widetilde{\overline{K}} \prec 0$ *is chosen to satisfy the vector equality:*

$$-\left(\overline{\Gamma}_d \otimes \overline{C}^T\right) vec\left(\widetilde{\overline{K}}\right) + \frac{2\lambda_1\lambda_2}{1+\lambda_2}\, vec\overline{A}_{cld} \succeq 0 \tag{40}$$

subject to

$$0 < \sigma < \frac{(1+\lambda_2)\left(1-\rho\left(\overline{A}_{cld}\right)\right)}{(1+\lambda_2 - 2\lambda_1)\rho\left(\overline{A}_{cld}\right)} \tag{41}$$

Proof. It is direct since the closed-loop stabilization of (32) and (33), via (34), holds if and only if $\overline{A}_{cld} = \overline{A}_d + \overline{\Gamma}_d\overline{K}\overline{C}$ is a convergent matrix for some control gain $\overline{K} \in \boldsymbol{R}^{2q\times q}$. The above matrix identity is equivalent to the following linear algebraic equation:

$$\left(\overline{\Gamma}_d \otimes \overline{C}\right) vec\left(\overline{K}\right) = vec\left(\overline{A}_{cld} - \overline{A}_d\right) \tag{42}$$

which is algebraically compatible, so that there is at least a solution $vec\overline{K} \in \boldsymbol{R}^{2q^2}$ (and thus at least a solution matrix $\overline{K}$ exists) if and only if (35) holds from Rouché–Frobenius theorem. Property (i) has been proved.

To prove Property (ii), note that, since $\overline{A}_{cld} = \overline{A}_d + \overline{\Gamma}_d\overline{K}\overline{C}$, if the control gain is perturbed from $\overline{K}$ to $\overline{K}+\widetilde{\overline{K}}$, so that $\widetilde{\overline{A}}_{cld} = \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq \sigma\overline{A}_{cld}$, then

$$\overline{A}_{cld} + \widetilde{\overline{A}}_{cld} = \overline{A}_d + \overline{\Gamma}_d\overline{K}\overline{C} + \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq (1+\sigma)\overline{A}_{cld} \tag{43}$$

Thus, $\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}$ is a convergent matrix if

$$\rho\left(\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}\right) \leq |1+\sigma|\rho\left(\overline{A}_{cld}\right) < 1 \tag{44}$$

which holds if $1 + |\sigma| < 1/\rho\left(\overline{A}_{cld}\right)$, that is, if

$$\sigma \in \left(-\frac{1-\rho\left(\overline{A}_{cld}\right)}{\rho\left(\overline{A}_{cld}\right)}, \frac{1-\rho\left(\overline{A}_{cld}\right)}{\rho\left(\overline{A}_{cld}\right)} \right) \tag{45}$$

Since $\overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} = \sigma\overline{A}_{cld} - \Xi$ for some $\Xi \succeq 0$, $\widetilde{\overline{K}}$ should be chosen to satisfy the equivalent vector equality (36), subject to (37), while noting that $\sigma = 0$ implies that $\widetilde{\overline{K}} = 0$ and $\Xi = 0$, and then $vec\left(\widetilde{\overline{K}}\right) = 0$ and $vec\,(\Xi) = 0$. Property (ii) has been proved.

To prove Property (iii), note from (38) that, for $\overline{A}_{cld} \succ 0$ and some $\sigma \in \boldsymbol{R}_+$, one has:

$$\overline{A}_{cld} + \widetilde{\overline{A}}_{cld} = \overline{A}_d + \overline{\Gamma}_d\overline{KC} + \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq (1+\sigma)\overline{A}_{cld} \tag{46}$$

Thus, $\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}$ is a convergent matrix if $\rho\left(\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}\right) < 1$. Then, for some $\Xi_1 \succeq 0$ and $\Xi_2 \succeq 0$, one has from (46) that:

$$-\sigma\overline{A}_{cld} \preceq -\sigma\overline{A}_{cld} + \Xi_2 \preceq \widetilde{\overline{A}}_{cld} = \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq \sigma\overline{A}_{cld} - \Xi_1 \preceq \sigma\overline{A}_{cld} \tag{47}$$

which also implies that $\Xi_1 + \Xi_2 = 2\sigma\lambda_1\overline{A}_{cld} \preceq 2\sigma\overline{A}_{cld}$ for some $\lambda_1 \in [0,1]$. Now, choose Ξ_2 via the constraint $\Xi_2 = \lambda_2\Xi_1$ for some $\lambda_2 \in \boldsymbol{R}_{0+}$. Thus, one has:

$$\Xi_1 = \frac{2\sigma\lambda_1\overline{A}_{cld}}{1+\lambda_2};\ \Xi_2 = \frac{2\sigma\lambda_1\lambda_2\overline{A}_{cld}}{1+\lambda_2} \tag{48}$$

Then, one finds that (47) holds if:

$$\left(\frac{2\lambda_1\lambda_2}{1+\lambda_2} - 1\right)\overline{A}_{cld}\sigma \preceq \overline{\Gamma}_d\widetilde{\overline{K}}\overline{C} \preceq \left(1 - \frac{2\lambda_1}{1+\lambda_2}\right)\sigma\,\overline{A}_{cld} \tag{49}$$

In addition, since $\sigma \in \boldsymbol{R}_+$ and $\overline{A}_{cld} \succ 0$ being stable implies that, if (40) holds, then $\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}$ is stable if

$$\begin{aligned}\rho\left(\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}\right) &\le max\left(1 + \left|\left(\tfrac{2\lambda_1\lambda_2}{1+\lambda_2} - 1\right)\right|\sigma, \left[1 + \left(1 - \tfrac{2\lambda_1}{1+\lambda_2}\right)\sigma\right]\rho\left(\overline{A}_{cld}\right)\right) \\ &= \left(1 + \left(1 - \tfrac{2\lambda_1}{1+\lambda_2}\right)\sigma\right)\rho\left(\overline{A}_{cld}\right) < 1\end{aligned} \tag{50}$$

That is, if $\overline{\Gamma}_d \succ 0$, $\overline{C} \succ 0$, then (44) holds, implying that $\overline{A}_{cld} + \widetilde{\overline{A}}_{cld}$ is stable, if either $\widetilde{\overline{K}} \succeq 0$ satisfies (38), subject to (39), or $\widetilde{\overline{K}} \prec 0$ satisfies (40) subject to (41). Property (iii) has been proved. □

Remark 5. *Two necessary conditions for (35) to hold are:*

(1) *The extended system (32) and (33) is stabilizable and detectable. It is obvious that (35) holds for some convergent matrix $\overline{A}_{cld}$ only if $rank\left(zI_{\overline{n}} - \overline{A}_d, \overline{\Gamma}_d\right) = rank\begin{pmatrix} \overline{C} \\ zI_{\overline{n}} - \overline{A}_d \end{pmatrix} = \overline{n}$ for each $z \in \boldsymbol{C}_1 \cap sp\left(\overline{A}_d\right)$ (see Remark 2).*

(2) *The extended system has no critically stable or unstable fixed mode. Note that it is obvious that the matrix $\overline{A}_d + \overline{\Gamma}_d\overline{KC}$ is not a convergent matrix for some controller matrix $\overline{K}$ if, for any such a static control gain $\overline{K}$, $\overline{A}_d + \overline{\Gamma}_d\overline{KC}$ has at least a critically stable or unstable mode. The second necessary condition is weaker than the first one. If the system is stabilizable and detectable, then it has no critically stable or unstable fixed mode since, otherwise, it would not be stabilizable and detectable. However, the converse is not true in general.*

Remark 6. *The most general case when $D_c + D_d \neq 0$, which includes, in particular, the case of the two nonzero direct input–output interconnection gains D_c and D_d being nonzero can be addressed with Theorem 1 under two possible slight modifications as follows:*

(1) *Define the auxiliary output $y_0[k] = y[k] - \overline{D}\overline{u}[k] = \overline{C}\overline{x}[k]$; $k \in \mathbf{Z}_{0+}$, which does not account for the direct input–output interconnection contribution, and consider the control law $\overline{u}[k] = \overline{K}y_0[k] = \overline{KC}\overline{x}[k]$; $k \in \mathbf{Z}_{0+}$. In this case, Theorem 1 applies directly. Note that the use of the auxiliary output in the controller design keeps the closed-loop stability of the modified discrete extended system since the closed-loop state $\overline{x}[k] \to 0$ as $k \to \infty$ implies that $y_0[k] \to 0$ and $u[k] \to 0$ as $k \to \infty$ and that $y[k] \to 0$ as $k \to \infty$.*

(2) *Assume that the implicit control law $\overline{u}[k] = \overline{K}y[k] = \overline{KC}\overline{x}[k] + \overline{KD}\overline{u}[k]$ is used. This law may be explicit in the form $\overline{u}[k] = \overline{K}'\overline{x}[k] = \left(I_{2q} - \overline{KD}\right)^{-1}\overline{KC}\overline{x}[k]$ if $I_{2q} - \overline{KD}$ is non-singular and it provides a unique control law. Thus, the problem is solved by first calculating the auxiliary static control gain $\overline{K}'$ under the conditions of Theorem 1 (with the replacement $\overline{K} \to \overline{K}'$). The above matrix equation is rewritten in a vector form, via the Kronecker product of matrices, as follows*

$$\left(I_{2q^2} \otimes \left(\overline{C} + \overline{DK}'\right)^T\right) vec\left(\overline{K}\right) = vec\left(K'\right) \tag{51}$$

which is solvable in $\overline{K}$ if and only if

$$rank\left(I_{2q^2} \otimes \left(\overline{C} + \overline{DK}'\right)^T\right) = rank\left(I_{2q^2} \otimes \left(\overline{C} + \overline{DK}'\right)^T, vec\left(K'\right)\right) \tag{52}$$

It is obvious that if the algebraic system in matrix form $\overline{\Gamma}_d\overline{KC} = \overline{A}_{cld} - \overline{A}_d$ is compatible, equivalently if its equivalent vector form (42) is compatible, which holds if and only if (35) in Theorem 1 holds, then the matrix algebraic system is solvable in $\overline{K}$ through the use of the Moore–Penrose generalized inverse techniques. Thus, we have the following solvability result of the stabilizing static controller gain$\overline{K}$.

Theorem 2. *Assume that $D_c + D_d = 0$ and that (35) holds for some convergent matrix $\overline{A}_{cld}$ so that $\overline{\Gamma}_d\overline{KC} = \overline{A}_{cld} - \overline{A}_d$ is solvable for a static output linear feedback stabilizing controller of gain $\overline{K} \in \mathbf{R}^{q\times 2q}$, such that the following properties hold:*

(i)

$$\overline{\Gamma}_d\overline{\Gamma}_d^{\dagger}\left(\overline{A}_{cld} - \overline{A}_d\right)\overline{C}^{\dagger}\overline{C} = \overline{A}_{cld} - \overline{A}_d \tag{53}$$

(ii) *The set of solutions in $\overline{K}$ to $\overline{\Gamma}_d\overline{KC} = \overline{A}_{cld} - \overline{A}_d$ is given by the static stabilizing controller gains:*

$$\overline{K} = \overline{\Gamma}_d^{\dagger}\left(\overline{A}_{cld} - \overline{A}_d\right)\overline{C}^{\dagger} + X - \overline{\Gamma}_d^{\dagger}\overline{\Gamma}_d X\overline{C}\overline{C}^{\dagger} \tag{54}$$

where X is any matrix of the same order as that of $\overline{K}$. The solutions (54) are equivalent to the set of solutions in vector form of the algebraic system (42).

(iii) *Assume that $rank\left(\overline{\Gamma}_d\right) = r_1 \leq min(\overline{n}, 2q)$ and $rank\left(\overline{C}\right) = r_2 \leq min(q, \overline{n}) = q$. Then, the following factorizations exist for some matrices $C_1 \in \mathbf{R}^{\overline{n}\times r_1}$, $D_1 \in \mathbf{R}^{r_1\times 2q}$, $C_2 \in \mathbf{R}^{q\times r_2}$ and $D_2 \in \mathbf{R}^{r_2\times\overline{n}}$:*

$$\overline{\Gamma}_c = C_1D_1; \overline{C} = C_2D_2 \tag{55}$$

Then, their generalized inverses are:

$$\overline{\Gamma}_d^{\dagger} = D_1^T\left(D_1D_1^T\right)^{-1}\left(C_1^TC_1\right)^{-1}C_1^T; \overline{C}^{\dagger} = C_2\left(C_2C_2^T\right)^{-1}\left(D_2D_2^T\right)^{-1}D_2 \tag{56}$$

so that (54) becomes

$$\overline{K} = D_1^T (D_1 D_1^T)^{-1} (C_1^T C_1)^{-1} C_1^T [\overline{A}_{cld} - \overline{A}_d - \Gamma_d X \overline{C}] C_2 (C_2 C_2^T)^{-1} (D_2 D_2^T)^{-1} D_2 + X \tag{57}$$

(iv) *The vector form equivalent set of solutions (54) is:*

$$vec(\overline{K}) = \left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger T}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right) + \left[I_{2q^2} - \left(\overline{\Gamma}_d^{\dagger} \overline{\Gamma}_d \otimes \overline{C}^{\dagger T} \overline{C}^T\right)\right] vec\,(X) \tag{58}$$

where $vecX$ *is any real vector of dimension* $2q^2$.

Proof. Note that the algebraic system $\overline{\Gamma}_d \overline{KC} = \overline{A}_{cld} - \overline{A}_d$ is solvable in $\overline{K}$, equivalently (42) is solvable in $vec\overline{K}$, if and only if (35) holds. However, this implies also that such a solvability holds if and only if (53) holds [31,32]. This proves Property (i).

Property (ii) follows directly from Property (i) since (54) gives the whole set of solutions.

Property (iii) follows from Property (ii) since (in general, non-unique) factorizations (55) exist, under the given rank conditions, leading to the Moore–Penrose pseudoinverses (56) making the set of solutions (54) to take the form (57).

To prove Property (iv), note that the solution of (42) is of the form (58), by taking into account (54), and that the Moore–Penrose generalized inverse of the Kronecker product $\overline{\Gamma}_d \otimes \overline{C}$ is $\left(\overline{\Gamma}_d \otimes \overline{C}\right)^{\dagger} = \overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger}$ [31]. □

4. Linking Static Linear Output Feedback Stabilization of the Modified Extended Discrete System with a Riccati Equation

The following result retakes the stabilization under static linear output feedback control without the need for invoking some convergent matrix which describes a prefixed closed-loop dynamics of the modified extended discrete system as it has been formulated via Theorem 1 and Theorem 2. The study is based on the use of a particular algebraic Ricatti matrix inequality for stabilization through a static linear output feedback controller. See, for instance, [11–13,28–30,34–38].

Theorem 3. *Assume that* $D_c + D_d = 0$. *Then, the following properties hold:*

(i) *The following statements are equivalent:*

1. *The modified extended discrete system (32) and (33) is static output linear feedback stabilizable.*
2. *There exists a symmetric positive definite matrix* $\overline{P}$ and a controller gain matrix $\overline{K}$ satisfying the subsequent matrix inequality*:*

$$\left(\overline{A}_d + \overline{\Gamma}_d \overline{KC}\right)^T \overline{P}\left(\overline{A}_d + \overline{\Gamma}_d \overline{KC}\right) - \overline{P} < 0 \tag{59}$$

3. *There exist positive semidefinite symmetric matrices* $\overline{P} \in \mathbf{R}^{\overline{n} \times \overline{n}}$ *and* $\overline{R} \in \mathbf{R}^{2q \times 2q}$ *and a static controller gain matrix* $\overline{K} \in \mathbf{R}^{2q \times q}$ *satisfying the following matrix inequalities:*

$$\overline{\Phi}_d > 0; I - \overline{G}_d \overline{\Phi}_d^{-1} \overline{G}_d^T > 0 \tag{60}$$

$$\overline{\Phi}_d = -\left(\overline{A}_d^T \overline{PA}_d - \overline{P} - \overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)\right]^{-1} \overline{\Gamma}_d^T \overline{PA}_d - \overline{C}^T \overline{K}^T \overline{RKC}\right) \tag{61}$$

$$\overline{G}_d = \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1/2} \overline{\Gamma}_d^T \overline{PA}_d + \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{1/2} \overline{KC} \tag{62}$$

(ii) *The extended discrete system (32) and (33) is static output linear feedback stabilizable if and only if it is stabilizable and detectable and, furthermore, there exist real matrices $\overline{K} \in \boldsymbol{R}^{2q\times q}$, i.e., the gain of the stabilizing static linear output controller, and $\overline{G} \in \boldsymbol{R}^{2q\times \overline{n}}$ such that*

$$\overline{KC} = \overline{G} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA}_d \quad (63)$$

where $\overline{P}$ is the real symmetric non-negative definite solution of

$$\overline{A}_d^T \overline{PA}_d - \overline{P} - \overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA}_d + \overline{C}^T \overline{C} + \overline{G}^T \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right) \overline{G} = 0 \quad (64)$$

In addition, $\overline{R}$ is a real symmetric positive definite matrix of appropriate order. Furthermore, $\overline{K}$ satisfies the linear algebraic compatible equation:

$$\left(\overline{A}_d^T \overline{P\Gamma}_d \otimes \overline{C}^T\right) vec\overline{K} = -\left[\overline{A}_d^T \otimes \overline{A}_d - I_{\hat{n}^2} - \overline{A}_d^T \otimes \overline{G}^T \overline{\Gamma}_d^T + \overline{G}^T \overline{\Gamma}_d^T \otimes \overline{G}^T \overline{\Gamma}_d^T\right] vec\,\overline{P} \quad (65)$$

whose set of solutions is given by

$$\overline{K} = \overline{K}_1 \overline{C}^{\dagger} + \overline{K}_2 \left(I_q - \overline{C}\,\overline{C}^{\dagger}\right) \quad (66)$$

$$= \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} \overline{K}_3 \overline{C}^{\dagger} + \overline{K}_4 - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} G^T \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{K}_4 \overline{C}\,\overline{C}^{\dagger} \quad (67)$$

For arbitrary matrices $\overline{K}_2$, $\overline{K}_4 \in \boldsymbol{R}^{2q\times q}$, $\overline{K}_1 \in \boldsymbol{R}^{2q\times \overline{n}}$ and $\overline{K}_3 \in \boldsymbol{R}^{\overline{n}\times \overline{n}}$ given by

$$\overline{K}_1 = \overline{G} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA}_d \quad (68)$$

$$\overline{K}_3 = \overline{P} - \overline{A}_d^T \overline{PA}_d - \overline{C}^T \overline{C} + \left(\overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} + \overline{G}^T\right) \overline{\Gamma}_d^T \overline{PA}_d \quad (69)$$

(iii) *The matrix $\overline{G}$ has to satisfy the subsequent general constraint in order to be compatible with (63) and (64):*

$$\left[\overline{G} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} \left(\overline{P} - \overline{A}_d^T \overline{PA}_d - \overline{C}^T \overline{C} + \left(\overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} + \overline{G}^T\right) \overline{\Gamma}_d^T \overline{PA}_d\right)\right] \overline{C}^{\dagger} + L = 0 \quad (70)$$

where

$$L = \overline{K}_2 \left(I_q - \overline{CC}^{\dagger}\right) - \overline{K}_4 + \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} G^T \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{K}_4 \overline{CC}^{\dagger} = 0 \quad (71)$$

which is also satisfied under the simpler constraint:

$$\overline{G} = \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} \left(\overline{P} - \overline{A}_d^T \overline{PA}_d - \overline{C}^T \overline{C} + \left(\overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} + \overline{G}^T\right) \overline{\Gamma}_d^T \overline{PA}_d\right) \quad (72)$$

Thus, a set of stabilizing linear output feedback controller gains is given by:

$$\begin{aligned} \overline{K} = \Bigg[& \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{PA} - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} \left(\overline{P} - \overline{A}_d^T \overline{PA}_d - \overline{C}^T \overline{C} + \left(\overline{A}_d^T \overline{P\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} + \overline{G}^T\right) \overline{\Gamma}_d^T \overline{PA}_d\right) \\ & - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{\Gamma}_d^T \overline{P}\Bigg] \overline{C}^{\dagger} + \left[\overline{K}_4 - \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} G^{T^{\dagger}} G^T \left(\overline{\Gamma}_d^T \overline{P\Gamma}_d + \overline{R}\right)^{-1} \overline{K}_4 \overline{CC}^{\dagger}\right] \left(I_q - \overline{CC}^{\dagger}\right)^{\dagger} \left(I_q - \overline{C}\,\overline{C}^{\dagger}\right) \end{aligned} \quad (73)$$

for any arbitrary $\overline{K}_4 \in \boldsymbol{R}^{2q\times q}$.

Proof. Property (i) follows from (Theorem 5, [37]) applied to the extended discrete system (32) and (33).

Property (ii) related to the joint solvability of (63) and (64) follows from [Theorem 1, [38]] applied to the linear output feedback via stabilization through a static controller for the modified extended discrete system (32) and (33). The controller gain satisfies the equivalent conditions (65) and (66), subject to (68), since static stabilizing controller gains $\overline{K}$ exist such that (65) is an algebraic compatible system so that

$$rank\left(\overline{A}_d^T\overline{P}\overline{\Gamma}_d\otimes\overline{C}^T\right)=rank\left[\left(\overline{A}_d^T\overline{P}\overline{\Gamma}_d\otimes\overline{C}^T\right),-\left(\overline{A}_d^T\otimes\overline{A}_d-I_{\hat{n}^2}-\overline{A}_d^T\otimes\overline{G}^T\overline{\Gamma}_d^T+\overline{G}^T\overline{\Gamma}_d^T\otimes\overline{G}^T\overline{\Gamma}_d^T\right)\right] \quad (74)$$

In addition, (67), subject to (69), follows by replacing $\overline{G}$ in the last additive left-hand-side term of (64), obtained from (53) into (64), and then calculating $\overline{K}$ with the general solution based on pseudoinversion rules.

To prove Property (iii), note that one obtains by equalizing the two right-hand-sides of (66) and (67) that

$$\left[\overline{K}_1-\left(\overline{\Gamma}_d^T\overline{P}\overline{\Gamma}_d+\overline{R}\right)^{-1}G^{T^\dagger}\overline{K}_3\right]\overline{C}^\dagger+\overline{K}_2\left(I_q-\overline{C}\overline{C}^\dagger\right)-\overline{K}_4+\left(\overline{\Gamma}_d^T\overline{P}\overline{\Gamma}_d+\overline{R}\right)^{-1}G^{T^\dagger}G^T\left(\overline{\Gamma}_d^T\overline{P}\overline{\Gamma}_d+\overline{R}\right)^{-1}\overline{K}_4\overline{C}\overline{C}^\dagger=0 \quad (75)$$

which leads to (71) after replacing (68) and (69) in its left-hand side. Since $\overline{K}_2$, $\overline{K}_4$ are arbitrary, one obtains (72) by taking

$$\overline{K}_2=\left[\overline{K}_4-\left(\overline{\Gamma}_d^T\overline{P}\overline{\Gamma}_d+\overline{R}\right)^{-1}G^{T^\dagger}G^T\left(\overline{\Gamma}_d^T\overline{P}\overline{\Gamma}_d+\overline{R}\right)^{-1}\overline{K}_4\overline{C}\overline{C}^\dagger\right]\left(I_q-\overline{C}\overline{C}^\dagger\right)^\dagger \quad (76)$$

or, simply, by zeroing $\overline{K}_2$ and $\overline{K}_4$. On the other hand, (73) follows by replacing $\overline{K}_2$, Equation (72), in (68), and the obtained result, together with and (76), in (66). □

Remark 7. *Theorem 3(iii) implies that, in general, $\overline{G}$ is not unique in Theorem 3(ii). As a result, $\overline{P}$ is positive definite and unique in (64) once $\overline{G}$ has been fixed for each given symmetric positive matrix $\overline{R}$ if and only if the pair $\left(\overline{A}_d,\overline{\Gamma}_d\right)$ is controllable.*

It turns out that a general application of Theorem 3 might be very involved in the cases of a certain dimensionality, and generalized inverses not being coincident with the standard ones are involved in the computations. However, it can be useful for discussing in a closed form the existence of a stabilizing static linear output feedback controller for the extended discrete system.

5. Decentralized versus Centralized Control of the Extended Discrete System

It is now discussed if the stabilizing control gain can be sparse if not in its off-diagonal entries and how sparse it can be. As it is admitted to being more sparse in its off-diagonal part, more information could be deleted for each individual subsystem from the remaining ones while still keeping the stabilization property of the whole system. Note that the static controller gain is of the form:

$$\overline{K}=\overline{K}_d+\overline{K}_{od}=\begin{bmatrix}\overline{K}_1\\\overline{K}_2\end{bmatrix};\ \overline{K}_d:=\begin{bmatrix}\overline{K}_{1d}\\\overline{K}_{2d}\end{bmatrix};\ \overline{K}_{od}:=\begin{bmatrix}\overline{K}_{1od}\\\overline{K}_{2od}\end{bmatrix} \quad (77)$$

where the above six column matrix blocks are square q-matrices and $\overline{K}_{id}$ and $\overline{K}_{iod}$ are diagonal, respectively, and of diagonal zero entries, for $i=1,2$. Note that $\overline{K}$ has $2q$ diagonal entries and $2q(q-1)$ of non-diagonal ones. Assume that the whole family of such stabilizing controllers via linear output feedback of the modified extended discrete system is $\boldsymbol{K}$. Note that the above consideration is only of interest if $q>1$, i.e., if there are at least two coupled subsystems in the whole structure. The whole decentralization implies that each subsystem is controlled by a control input which has available information only on its own output. The two subsequent definitions rely on how strong the decentralization of the output information is to make possible the stabilization of the whole coupled system.

Definition 1. *The maximum decentralized degree of output linear feedback stabilization (MDdos) of the extended discrete system is the maximum number of non-diagonal zero entries* $i \in [0\ ,\ 2q(q-1)] \in \boldsymbol{Z}_{0+}$ *in* $\overline{K}_{od}$*, between all the gains* $\overline{K} \in \boldsymbol{K}$.

Definition 2. *The minimum centralized degree of output linear feedback stabilization (mCdos) of the extended discrete system is the minimum number of non-diagonal zero entries* $i \in [0\ ,\ 2q(q-1)] \in \boldsymbol{Z}_{0+}$ *in* $\overline{K}_{od}$*, between all the gains* $\overline{K} \in \boldsymbol{K}$.

It can be observed that Definitions 1 and 2 have only sense for $q \geq 2$ since, if $q = 1$, that is, the whole system consists of a single subsystem, then there is no distinction between centralized and decentralized control. Note that, trivially, $2q(q-1)$ = (mCdos) + (MDdos). Note also that if MDdos = $2q(q-1)$, then the linear output feedback stabilization of the extended discrete system may be performed with some fully decentralized control of gain $\overline{K} = \overline{K}_d \in \boldsymbol{K}$, that is, the whole closed-loop stabilization may be performed under individual controllers of each subsystem which only take information on the output of such a subsystem, that is, just of one of the components of the output vector which is the output of the involved subsystem. Furthermore, note that if MDdos = $2q(q-1)$, then the closed-loop stabilization can only be performed under fully centralized control, i.e., each subsystem has to acquire available information on the outputs of all the subsystems in the whole structure.

The subsequent result addresses the closed-loop fully decentralized stabilization of the modified extended discrete system via linear output feedback based on Theorem 2 and on Theorem 3.

Theorem 4. *Assume that* $D_c + D_d = 0$ *. Then, the following properties hold:*

(i) *Assume that there exists some convergent matrix* $\overline{A}_{cld}$ *such that* $\overline{\Gamma}_d \overline{K} \overline{C} = \overline{A}_{cld} - \overline{A}_d$ *is solvable with a solution:*

$$\overline{K} = \overline{K}_{diag} + \overline{K}_{odiag} = \overline{\Gamma}_d^{\dagger} (\overline{A}_{cld} - \overline{A}_d) \overline{C}^{\dagger} \in \boldsymbol{K} \tag{78}$$

or, equivalently,

$$vec(\overline{K}) = \left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger T} \right) vec\left(\overline{A}_{cld} - \overline{A}_d \right) \tag{79}$$

In addition, assume also that

$$rank\left[I_{2q^2} - \overline{\Gamma}_d^{\dagger} \overline{\Gamma}_d \otimes \overline{C}^{\dagger T} \overline{C}^{T T} \right] = rank\left[I_{2q^2} - \overline{\Gamma}_d^{\dagger} \overline{\Gamma}_d \otimes \overline{C}^{\dagger T} \overline{C}^{T}\ ,\ \left[\left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger T} \right) vec\left(\overline{A}_{cld} - \overline{A}_d \right) \right]_{od} \right] \tag{80}$$

Then, MDdos $= 2q(q-1)$ *so that the closed-loop modified extended discrete system can be stabilized with fully decentralized control which allocates the closed-loop modes of the modified extended system at the eigenvalues of* $\overline{A}_{cld}$.

(ii) *Assume that the hypotheses of Theorem 3 and (72) hold. Assume also that* $rank\,\Omega_2 = rank\left[\Omega_2\ ,\ vec\,\Omega_{1_{od}} \right]$ *, where*

$$\begin{aligned} \Omega_1 = \Big[& \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} \overline{\Gamma}_d^T \overline{P} \overline{A} - \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} G^{T\dagger} \left(\overline{P} - \overline{A}_d^T \overline{P} \overline{A}_d - \overline{C}^T \overline{C} + \left(\overline{A}_d^T \overline{P} \overline{\Gamma}_d \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} + \overline{G}^T \right) \overline{\Gamma}_d^T \overline{P} \overline{A}_d \right) \\ & - \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} \overline{\Gamma}_d^T \overline{P} \Big] \overline{C}^{\dagger} = \Omega_{1d} + \Omega_{1od} \end{aligned} \tag{81}$$

$$\Omega_2 = \left[I_q \otimes \left(I_q - \overline{C}\,\overline{C}^{\dagger} \right)^T \left(I_q - \overline{C}\overline{C}^{\dagger} \right)^T - \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} G^{T\dagger} G^T \left(\overline{\Gamma}_d^T \overline{P} \overline{\Gamma}_d + \overline{R} \right)^{-1} \otimes \left(I_q - \overline{C}\,\overline{C}^{\dagger} \right)^T \left(I_q - \overline{C}\overline{C}^{\dagger} \right)^T \left(\overline{C}\overline{C}^{\dagger} \right)^T \right] \tag{82}$$

Then, MDdos $= 2q(q-1)$ *so that the closed-loop modified extended discrete system can be stabilized with fully decentralized control which allocates the closed-loop modes at the eigenvalues of some existing convergent matrix.*

Proof. Property (i) follows from (53) and (54) by taking into account that $\overline{C}^{T^{\dagger}} = \overline{C}^{\dagger^{T}}$ since (78) is a particular solution with $X = 0$, then $vec(X) = 0$, for $\overline{\Gamma}_d\overline{KC} = \overline{A}_{cld} - \overline{A}_d$ which is solvable if and only if (80) holds, and then there is a real matrix X of order $2q^2 \times q$ given by

$$\left[I_{2q^2} - \overline{\Gamma}_d^{\dagger}\overline{\Gamma}_d \otimes \overline{C}^T\overline{C}^{\dagger T}\right] vec\,(X) = vec\left[\overline{\Gamma}_d\overline{\Gamma}_d^{\dagger}\left(\overline{A}_{cld} - \overline{A}_d\right)\overline{C}^{\dagger}\overline{C}\right]_{od} = \left[\left(\overline{\Gamma}_d\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^T\overline{C}^{\dagger^{T}}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right)\right]_{od} \tag{83}$$

See (58), such that there is some $\overline{K} = \overline{K}_d \in \boldsymbol{K}$ since $\overline{K}_{od} = \overline{K} - \overline{K}_d = 0$ if (80) holds, since one has that the general solution in $\boldsymbol{K}$ which includes as a particular case (79) is:

$$\begin{aligned} vec\left(\overline{K}\right) &= vec\left(\overline{K}_d\right) + vec\left(\overline{K}_{od}\right) = vec\left(\overline{K}_d\right) \\ &= \left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger^{T}}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right) + \left[I_{2q^2} - \left(\overline{\Gamma}_d^{\dagger}\overline{\Gamma}_d \otimes \overline{C}^{\dagger T}\overline{C}^T\right)\right] vec\,(X) \end{aligned} \tag{84}$$

$$\begin{aligned} &= \left[\left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger^{T}}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right)\right]_d \\ &+ \left(\left[\left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger^{T}}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right)\right]_{od} + \left[I_{2q^2} - \left(\overline{\Gamma}_d^{\dagger}\overline{\Gamma}_d \otimes \overline{C}^{\dagger T}\overline{C}^T\right)\right] vec\,(X)\right) \end{aligned} \tag{85}$$

$$= \left[\left(\overline{\Gamma}_d^{\dagger} \otimes \overline{C}^{\dagger^{T}}\right) vec\left(\overline{A}_{cld} - \overline{A}_d\right)\right]_d \tag{86}$$

In addition, (86) holds by zeroing the second additive term of the right-hand side of (85) by the choice of a solution $vecX$ which exists since (80) holds. Property (i) has been proved. To prove Property (ii), note that if the hypotheses of Theorem 3 and (72) hold, then a set of stabilizing controller gains satisfying (73) can be calculated which can be vectorized as follows:

$$vec\left(\overline{K}\right) = vec(\Omega_{1d}) + \left(\Omega_{1od} + \Omega_2 vec\left(\overline{K}_4\right)\right) \tag{87}$$

Note that $vec(\Omega_{1od}) + \Omega_2 vec\left(\overline{K}_4\right) = 0$, if $rank(\Omega_2) = rank\left[\Omega_2\,,\, vec\,\Omega_{1_{od}}\right]$, for $vec\left(\overline{K}_4\right) = -\left(\Omega_2 \otimes I_{2q^2}\right) vec\left(\Omega_{1_{od}}\right)$, then $vec\left(\overline{K}\right) = vec\left(\overline{K}_d\right) = vec(\Omega_{1d})$ so that a fully stabilizing controller of gain $\overline{K} = \overline{K}_d \in \boldsymbol{K}$ stabilizes the closed-loop system under linear output feedback fully decentralized stabilization. Property (ii) has been proved. □

Remark 8. *Note that Theorem 4 relies on the fully decentralized output feedback stabilization through a static controller of the modified extended discrete system. Its extension to a partial decentralized stabilization is direct under similar tools via alternative, more general decompositions* $vec\left(\overline{K}\right) = vec\left(\overline{K}_{qd}\right) + vec\left(\overline{K}_{qod}\right)$ *in Theorem 4(i) and* $\Omega = \Omega_{1qd} + \Omega_{1qod}$ *for Theorem 4(ii) in quasi-diagonal and off-quasi-diagonal column matrix blocks by including the tentative minimum number of the quasi-diagonal entries coming, deleting them from the off-quasi-diagonal blocks. In this case, the decentralized stabilization is not full and can have different degrees of decentralization depending on the off-diagonal entries transferred to the quasi-diagonal column matrix blocks.*

Example 2. *Consider the following hybrid delay-free system consisting of two subsystems given by:*

$$\begin{aligned} \dot{x}_{c1}(t) &= x_{c2}(t) \\ x_{d1}[k] &= -x_{c1}[k] - 2x_{d1}[k] + u[k] \end{aligned} \tag{88}$$

$$\dot{x}_{c2}(t) = -x_{c2}(t) - 0.8x_{c1}(t) - 2x_{d1}[k] + u(t) + 0.8u[k]$$

$$y(t) = y_1(t) = x_{d1}[k]; \forall k \in \mathbf{Z}_{0+}, \forall t \in [kT\ (k+1)\,T)$$

subject to any given finite initial conditions, where T is the sampling period. The discretization of (88) yields the following description of third order through extended discrete vector $\overline{x}[k] = (x_{c1}[k]\,,\, x_{c2}[k]\,,\, x_{d1}[k])^T$ *:*

$$\begin{pmatrix} x_{c1}[k+1] \\ x_{c2}[k+1] \\ x_{d1}[k+1] \end{pmatrix} = \begin{bmatrix} -0.8\left(T+e^{-T}\right)+0.2 & 1-e^{-T} & -2\left(T-e^{-T}\right)-1 \\ 0.8\left(1-e^{-T}\right) & e^{-T} & 2\left(1-e^{-T}\right) \\ -1 & 0 & -2 \end{bmatrix} \begin{pmatrix} x_{c1}[k] \\ x_{c2}[k] \\ x_{d1}[k] \end{pmatrix} + \begin{pmatrix} 0.8\left(T+e^{-T}-1\right) \\ 0.8\left(e^{-T}-1\right) \\ 1 \end{pmatrix} u[k]$$
$$+1.5\begin{pmatrix} \int_0^T \left(1-e^{-(T-\tau)}\right) u(kT+\tau)d\tau \\ \int_0^T e^{-(T-\tau)} u(kT+\tau)d\tau \\ 0 \end{pmatrix}; y[k] = (0\ ,\ 0\ ,\ 1)\begin{pmatrix} x_{c1}[k] \\ x_{c2}[k] \\ x_{d1}[k] \end{pmatrix} \quad (89)$$

To define an auxiliary input sequence $\{v[k]\}_{k=0}^{\infty}$ generate the continuous control input $u(kT+\tau) = L(T\ ,\tau)v[k]$ in the intersample intervals with $L(T,\tau) = \begin{pmatrix} \frac{1}{1-e^{-(T-\tau)}} & \frac{1}{e^{-(T-\tau)}} & 0 \end{pmatrix}^T$; $\forall k \in \boldsymbol{Z}_{0+}$, $\forall \tau \in (0\ ,\ T)$. For a sampling period of $T = 0.4$ s, the matrix of dynamics of the uncontrolled extended discrete system Equation (89) has as eigenvalues $z_1 = 0.21226$ and $z_{2,3} = -1.26942 \pm 3.12457\boldsymbol{i}$, the two complex conjugate ones being unstable. The extended discrete control 3×2 matrix associated with the two-dimensional extended control sequence $\{u[k]\ ,\ v[k]\}_{k=0}^{\infty}$ for $T = 0.4$ s becomes:

$$\overline{\Gamma}_d = \left[\ \overline{B}_d \quad , \overline{B}_c\ \right] = \begin{bmatrix} 0.05625 & 0.6 \\ -0.263744 & 0.6 \\ 1 & 0 \end{bmatrix} \quad (90)$$

The static controller gain of the extended discrete system is of the form $\overline{K} = \left(\overline{K}_1\ ,\ \overline{K}_2\right)^T \in \boldsymbol{R}^2$ leading to the following closed-loop matrix of dynamics of the modified extended discrete system

$$\overline{A}_{cld} = \begin{bmatrix} -0.8\left(T+e^{-T}\right)+0.2 & 1-e^{-T} & 2\left(e^{-T}-T\right)-1+0.1125+0.6\overline{K}_2 \\ 0.8\left(1-e^{-T}\right) & e^{-T} & 0.65936-0.526549+0.6\overline{K}_2 \\ -1 & 0 & \overline{K}_1-2 \end{bmatrix} \quad (91)$$

Since $T = 0.4$, if $\overline{K}_1 = 2$ and $\overline{K}_2 = 0.34686/ = 0.5781$, then a solution to Equation (42) is $vec\left(\overline{K}\right) = \left(\overline{K}_1\ ,\overline{K}_2\right)^T = (2,\ 0.5781)$, which is solvable according to (35), for a targeted matrix of closed-loop dynamics given by the ordered row-per-row vector defined by:

$$vec\left(\overline{A}_{cld}\right) = (-0.656256,\ 0.32968\ ,\ 0\ ,\ 0.263744\ ,\ 0.67032\ ,\ 0.479671\ ,\ -1,\ 0,\ 0) \quad (92)$$

The ordered row-per-row vector corresponding to the matrix $\overline{\Gamma}_d\overline{KC}^T$ is given by:

$$vec\left(\overline{\Gamma}_d\overline{KC}^T\right) = \left(0, 0, 0.05625\overline{K}_1 + 0.6\overline{K}_2, 0,\ 0, -0.2632744\overline{K}_1 + 0.6\overline{K}_2,\ 0,\ 0, \overline{K}_1\right) \quad (93)$$

corresponding to the matrix:

$$\overline{A}_{cld} = \begin{bmatrix} -0.656256 & 0.32968 & 0 \\ 0.263744 & 0.67032 & 0.479671 \\ -1 & 0 & 0 \end{bmatrix}$$

with characteristic polynomial $p(z) = z^3 - 0.0146z^2 - 0.526853z + 0.150995$ whose zeros are all stable with values $z_1 = -0.83432$; $z_{2,3} = 0.42419 \pm 0.03213\boldsymbol{i}$. As a result, as $k \to \infty$, $\{\overline{x}[k]\} \to 0$, $\{y[k]\} \to 0$, $\{u[k]\} \to 0$, $\{v[k]\} \to 0$, $x_{d1}[k] \to 0$, $x_{c1}[k] \to 0$, $x_{c2}[k] \to 0$ for any given finite initial conditions. Since $\{v[k]\} \to 0$ as $k \to \infty$, $u(t) \to 0$ as $t \to \infty$, $x_{c1}(t) \to 0$, $x_{c2}(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. The controller is of decentralized type since it only picks up information of the first subsystem through its output which is also the global output of the whole system.

6. Cases of Small Influences of the Delayed Discrete Dynamics and of the Couplings between Subsystems

Note that the closed-loop extended discrete system can be re-formulated with its state evolution by taking into account a separation of terms with associated sufficiently small norms in the relevant equations of (17) to (27) and (32) and (33) associated with the delayed dynamics:

$$\begin{aligned}\overline{x}[k+1] &= \hat{A}_d\overline{x}[k] + \hat{B}_d u[k] + \left(\overline{B}_{c\tau}\right)u[k] + \widetilde{\overline{A}}_d\overline{x}[k] \\ &= \left(\hat{A}_d + \overline{\Gamma}_d\overline{KC} + \widetilde{\overline{A}}_d\right)\overline{x}[k] = \overline{A}_{cld}\,\overline{x}[k]\end{aligned} \tag{94}$$

where

$$\hat{A}_d = \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) \quad 0_{n_c\times pn_c} & \Gamma_{cd0}(T)\; 0_{n_c\times pn_d} \\ I_{pn_c} & 0_{p\times(p+1)n_d} \\ A_{dc0}\; 0_{n_c\times pn_c} & A_{d0}\; 0_{n_c\times pn_d} \\ 0_{p\times(p+1)n_c} & I_{pn_d} \end{bmatrix};$$
$$\widetilde{\overline{A}}_d = \begin{bmatrix} 0_{n_c\times n_c}\; \widetilde{\overline{\Psi}}_{cc}(T) & 0_{n_c\times n_d}\; \widetilde{\overline{\Psi}}_{cd}(T) \\ 0_{pn_c\times pn_c} & 0_{p\times(p+1)n_d} \\ \widetilde{\overline{\Psi}}_{dc} & \widetilde{\overline{\Psi}}_{dd} \\ 0_{p\times(p+1)n_c} & 0_{pn_d\times pn_d} \end{bmatrix} \tag{95}$$

$$\hat{A}_d + \overline{\Gamma}_d\overline{KC}$$
$$= \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) + \Gamma_{cs}(T)\overline{K}_1 + B_c\overline{K}_2)(C_c + C_{cs}) \;\; 0_{n_c\times pn_c} & \Gamma_{cd0}(T) + (\Gamma_{cs}(T)\overline{K}_1 + B_c\overline{K}2)C_d \;\; 0_{n_c\times pn_d} \\ I_{pn_c} & 0_{p\times(p+1)n_d} \\ A_{dc0} + B_d\overline{K}_1(C_c + C_{cs}) \;\; 0_{n_c\times pn_c} & A_{d0} + B_d\overline{K}_1 C_d \;\; 0_{n_c\times pn_d} \\ 0_{p\times(p+1)_{n_c}} & I_{pn_d} \end{bmatrix} \tag{96}$$

Because of its structure, the eigenvalues of $\hat{A}_d$ are a zero eigenvalue of multiplicity $2pn$ plus the $n = n_c + n_d$ additional eigenvalues of

$$\hat{A}_{d0} = \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) & \Gamma_{cd0}(T) \\ A_{dc0} & A_{d0} \end{bmatrix} \tag{97}$$

By the same reason, the set of eigenvalues of $\hat{A}_d + \overline{\Gamma}_d\overline{KC}$ are a zero eigenvalue of multiplicity $2pn$ plus the $n = n_c + n_d$ extra eigenvalues of

$$\hat{A}_{d*} = \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) + \left(\Gamma_{cs}(T)\overline{K}_1 + B_c\overline{K}_2\right)(C_c + C_{cs}) & +\left(\Gamma_{cs}(T)\overline{K}_1 + B_c\overline{K}_2\right)C_d \\ A_{dc0} + B_d\overline{K}_1(C_c + C_{cs}) & A_{d0} + B_d\overline{K}_1 C_d \end{bmatrix} \tag{98}$$

Furthermore, note that

$$zI_{\overline{n}} - \overline{A}_{cld} = zI_{\overline{n}} - \left(\hat{A}_d + \overline{\Gamma}_d\overline{KC}\right) - \widetilde{\overline{A}}_d = \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)\left(I_{\overline{n}} - \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right);\ \forall z \in \mathbf{C}\backslash\{sp\ (\hat{A}_d)\} \tag{99}$$

Additionally, that

$$det\left(zI_{\overline{n}} - \overline{A}_{cld}\right) = det\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right) \times det\left(I_{\overline{n}} - \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right) \tag{100}$$

Since $det\left(zI_{\overline{n}} - \overline{A}_{cld}\right)$ and $det\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)$ are entire functions, they have the same number of zeros in the open unit circle of the complex plane centered at the origin $\{z \in \mathbf{C} : z < 1\}$ if

$$\begin{aligned}&\left|det\left(zI_{\overline{n}} - \overline{A}_{cld}\right) - det\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)\right| \\ &= \left|\,det\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)\right|\left|1 - det\left(I_{\overline{n}} - \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right)\right| \\ &< det\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)\end{aligned} \tag{101}$$

at the boundary $\{z \in \boldsymbol{C} : z = 1\}$ of such a circle (Rouché theorem, [47]) provided that $\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}$ exists, equivalently, if

$$\left|1 - det\left(I_{\overline{n}} - \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right)\right| < 1 \text{ for } |z| = 1 \tag{102}$$

which holds if

$$0 < \left|det\left(I_{\overline{n}} - \left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right)\right| < 1 \text{ for } |z| = 1 \tag{103}$$

That is, guaranteed if the H_∞-norm $\left\|\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right\|_{H_\infty}$ of $\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d$ is less than unity, that is, if

$$\sigma\left[\left(e^{i\theta}I_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right] = \max_{0\le\theta<2\pi}\left\|\left(e^{i\theta}I_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right\|_2 < 1$$

which holds for sufficiently small $\left\|\widetilde{\overline{A}}_d\right\|_2$. Thus, since $\hat{A}_d + \overline{\Gamma}_d\overline{KC}$ is convergent if $\hat{A}_{d*}$ is convergent, we have proved the following closed-loop global asymptotic stability result by taking into account also Remark 4:

Theorem 5. *Assume that $D_c + D_d = 0$. If $\hat{A}_{d*}$ is convergent and $\left\|\widetilde{\overline{A}}_d\right\|_2$ is sufficiently small, according to $\left\|\left(zI_{\overline{n}} - \hat{A}_d - \overline{\Gamma}_d\overline{KC}\right)^{-1}\widetilde{\overline{A}}_d\right\|_{H_\infty} < 1$, then $\overline{A}_{cld}$ is convergent. As a result, the resulting closed-loop modified extended discrete system is globally asymptotically stable in the sense that, for any given finite initial conditions, the sequences $\{\overline{u}[k]\}_{k=0}^{\infty}$ and $\{\overline{x}[k]\}_{k=0}^{\infty}$ are bounded, and $\{\overline{u}[k]\}_{k=0}^{\infty} \to 0$ and $\{\overline{x}[k]\}_{k=0}^{\infty} \to 0$. Moreover, $u(t) \to 0$, $x_c(t) \to 0$, $x_c(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$, so that the complete hybrid system is also globally asymptotically stable.*

In view of (9)–(11) and (20)–(22), the delay-free dynamics couplings of each subsystem with the remaining ones within the whole network are reflected in $\hat{A}_d + \overline{\Gamma}_d\overline{KC}$ defined in (96) by the off-diagonal matrix blocks of the matrices $\Phi_c(T) = \Phi_{cd}(T) + \Phi_{cod}(T) = e^{A_{cd}T} + \left(\int_0^T e^{A_{cd}(T-\tau)}d\tau\right)A_{cod}$,where A_{cd} and A_{0cd} are the diagonal (subscripted with "d") and off-diagonal (subscripted with "od") matrix blocks of A_c, $A_{cs0} = A_{csd} + A_{csdo}$ and $A_{dc0} = A_{dcd} + A_{dcdo}$. To evaluate when the closed-loop stabilization by fully decentralized control is possible under sufficiently weak couplings between the various subsystems and, at the same time, sufficiently weak delayed dynamics, we now further decompose the controller gain as $\overline{K} = \left[\overline{K}_1^T, \overline{K}_2^T\right]^T = \overline{K}_d + \overline{K}_{od}$ according to (77) to yield:

$$\overline{A}_{cld} = \hat{A}_{dd} + \overline{\Gamma}_d\overline{K}_d\overline{C} + \left(\hat{A}_{dod} + +\overline{\Gamma}_d\overline{K}_{od}\overline{C} + \widetilde{\overline{A}}_d\right) \tag{104}$$

where

$$\begin{aligned}\bar{\hat{A}}_{da} &= \hat{A}_{dd} + \bar{\Gamma}_d\bar{K}_d\bar{C}\\ &= \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) + (\Gamma_{cs}(T)\bar{K}_{1d} + B_c\bar{K}_{2d})(C_c + C_{cs}) \;\; 0_{n_c\times pn_c} & \Phi_c(T) + \Gamma_{cs0}(T) + (\Gamma_{cs}(T)\bar{K}_{1d} + B_c\bar{K}_{2d})C_d \;\; 0_{n_c\times pn_d} \\ I_{pn_c} & 0_{p\times(p+1)n_d} \\ A_{dc0} + B_d\bar{K}_{1d}(C_c + C_{cs}) \;\; 0_{n_c\times pn_c} & A_{d0} + B_d\bar{K}_{1d}C_d \;\; 0_{n_c\times pn_d} \\ 0_{p\times(p+1)n_c} & I_{pn_d}\end{bmatrix}\end{aligned} \tag{105}$$

$$\begin{aligned}\widetilde{\bar{A}}_{da} &= \hat{A}_{dod} + \bar{\Gamma}_d\bar{K}_{od}\bar{C} + \widetilde{\bar{A}}_d\\ &\begin{bmatrix} (\Gamma_{cs}(T)\bar{K}_{1od} + B_c\bar{K}_{2od})(C_c + C_{cs}) + \widetilde{\bar{\Psi}}_{cc}(T) \;\; 0_{pn_c\times pn_c} & (\Gamma_{cs}(T)\overline{K}_{1od} + B_c\overline{K}_{2od})C_d + \widetilde{\bar{\Psi}}_{cd}(T) \\ 0_{pn_c\times pn_c} & 0_{p\times(p+1)n_d} \\ \widetilde{\bar{\Psi}}_{dc}(T) + B_d\overline{K}_{1od}(C_c + C_{cs}) & \widetilde{\bar{\Psi}}_{dd}(T) + B_d\overline{K}_{1od}C_d \\ 0_{pn_c\times(p+1)n_c} & 0_{pnd_c\times pn_d}\end{bmatrix}\end{aligned} \tag{106}$$

$$\hat{A}_{da*} = \begin{bmatrix} \Phi_c(T) + \Gamma_{cs0}(T) + \left(\Gamma_{cs}(T)\overline{K}_{1d} + B_c\overline{K}_{2d}\right)(C_c + C_{cs}) & +\left(\Gamma_{cs}(T)\overline{K}_{1d} + B_c\overline{K}_{2d}\right)C_d \\ A_{dc0} + B_d\overline{K}_{1d}(C_c + C_{cs}) & A_{d0} + B_d\overline{K}_{1d}C_d\end{bmatrix} \tag{107}$$

In addition, $\hat{A}_{da*}$ has the same number of structural nonzero eigenvalues as $\hat{A}_{dd}$ in the same way as it has $\hat{A}_{d*}$ Equation (98) versus $\hat{A}_{d0}$ Equation (97). The appropriate modification of Theorem 5 by taking into account (104)–(107) under sufficiently small couplings of mutual dynamics between pairs of subsystems leads to the subsequent result:

Theorem 6. *Assume that* $D_c + D_d = 0$ *. If* $\hat{A}_{da*}$ *is convergent and* $\left\|\overline{\widetilde{A}}_{da}\right\|$ *is sufficiently small satisfying* $\left\|\left(zI_{\overline{n}} - \hat{A}_{dd} - \overline{\Gamma}_d\overline{K}_d\overline{C}_d\right)^{-1}\overline{\widetilde{A}}_{da}\right\|_{H_\infty} < 1$ *, then* $\overline{A}_{cld}$ *is convergent under fully decentralized control, that is, MDdos =* $2q(q-1)$ *and the closed-loop modified extended discrete system is globally asymptotically stable in the sense that, for any given finite initial conditions, the sequences* $\{\overline{u}[k]\}_{k=0}^{\infty}$ *and* $\{\overline{x}[k]\}_{k=0}^{\infty}$ *are bounded, and* $\{\overline{u}[k]\}_{k=0}^{\infty} \to 0$ *and* $\{\overline{x}[k]\}_{k=0}^{\infty} \to 0$ *. Moreover,* $u(t) \to 0$ *,* $x_c(t) \to 0$ *,* $x_c(t) \to 0$ *and* $y(t) \to 0$ *as* $t \to \infty$ *, so that the complete hybrid system is also globally asymptotically stable.*

Remark 9. *It turns out that, for the partial decentralized stabilization problem with the maximum degree of decentralization and, correspondingly, with the minimum degree of centralization, Theorem 6 can be directly re-addressed as a parallel result in the sense that* $\overline{\hat{A}}_{da}$ *,* $\overline{\widetilde{A}}_{da}$ *and* $\hat{A}_{da*}$ *can be replaced, respectively, by* $\overline{\hat{A}}_{da}(MDdos)$ *,* $\overline{\widetilde{A}}_{da}(MDdos)$ *and* $\hat{A}_{da*}(MDdos)$ *defined accordingly to an estimation of the maximum decentralization degree* $MDdos = \{max\, i \in [0\,,\, 2q(q-1)] \in \mathbf{Z}_{0+}\}$ *such that:*

(a) $\overline{K}_{dao}$ *= minimum number between* $\overline{K}_{1od}$ *and* $\overline{K}_{2od}$ *of off-diagonal entries to be used in the re-definition of* $\overline{\hat{A}}_{da}(MDdos)$ *, previously defined in (105), by replacing* $\overline{K}_d \to \overline{K}_d + \overline{K}_{dao}$ *in* $\hat{A}_d$ *being defined in (95);*
(b) $\overline{K}_{od} - \overline{K}_{dao}$ *to be used in the re-definition of* $\overline{\widetilde{A}}_{da}(MDdos)$ *, previously defined in (106), by replacing* $\overline{K}_{od} \to \overline{K}_{od} - \overline{K}_{dao}$ *in* $\overline{\widetilde{A}}_d$ *being defined in (85);*
(c) Reformulate Theorem 6 according to the two above replacements.

The above modification of Theorem 6 is based on an estimation of the maximum decentralization degree, rather than on such a degree itself, since Theorem 6 is rather a local robustness stability result for sufficiently weak delayed dynamics and sufficiently weak coupling dynamics between the various pairs linking the q subsystems. In fact, the result is based on the stability of a nominal closed-loop system without delayed dynamics and couplings between each pair of the various subsystems and a sufficient smallness of the remaining contributive terms to the whole dynamics.

It is also possible to rewrite, equivalently, (94) by decomposing the controller into two parts, one to be used to address the nominal closed-loop design while the other being used to partially compensate the effect of uncertainties in the closed-loop dynamics. The resulting version of (94) is:

$$\overline{x}[k+1] = \left(\hat{A}_d + \overline{\Gamma}_d\overline{K}_*\overline{C} + \overline{\Gamma}_d\left(\overline{K} - \overline{K}_*\right)\overline{C} + \overline{\widetilde{A}}_d\right)\overline{x}[k] = \overline{A}_{cld}\,\overline{x}[k]$$

The subsequent example visualizes the above ideas.

Example 3. *Consider the following hybrid delay-free system of sampling period* $T = 0.1$ *s which consists of two subsystems described by:*

$$\begin{gathered}\dot{x}_{1c1}(t) = x_{1c2}(t) - x_{1c2}[k] \\ x_{1d1}[k+1] = -x_{1d1}[k] + x_{1c1}[k] + \alpha_{0.1}x_{1c1}[k] + \textstyle\sum_{i=1}^{3}\alpha_{12i}x_{2di}[k] + u[k] \\ x_{2d1}[k] = x_{2d2}[k]; x_{2d2}[k] = x_{2d3}[k] \\ x_{2d3}[k+1] = 0.15x_{2d1}[k] - 0.1x_{2d2}[k-1] + 1.05x_{2d3}[k] + \textstyle\sum_{i=1}^{2}\alpha_{21i}x_{1di}[k] \\ y(t) = y_1(t) = x_{1c1}[k]; \forall k \in \mathbf{Z}_{0+}, \forall t \in [kT\ (k+1)\,T)\end{gathered} \tag{108}$$

The $\alpha_{(.)}$ *takes account for small dynamic coupling uncertainties of not very precise knowledge. The whole extended discrete system of state* $\overline{x}[k] = (x_{1c1}[k],\ x_{1d1}[k],\ x_{2d1}[k],\ x_{2d2}[k],\ x_{2d3}[k])^T$*, with the continuous part discretized for the period* $T = 0.1$*, is described by the following equations:*

$$\begin{gathered} x_{1c1}[(k+1)T] = x_{1c2}[kT] \\ x_{1d1}[k+1] = -x_{1d1}[k] + x_{1c1}[k] + \alpha_{111}x_{1c1}[k] + \textstyle\sum_{i=1}^{3} \alpha_{12i}x_{2di}[k] + u[k] \\ x_{2d1}[k] = x_{2d2}[k]; x_{2d2}[k] = x_{2d3}[k] \\ x_{2d3}[k+1] = 0.15x_{2d1}[k] - 0.1x_{2d2}[k-1] + 1.05x_{2d3}[k] + \textstyle\sum_{i=1}^{2} \alpha_{21i}x_{1di}[k] + u[k] \\ y(t) = y_1(t) = x_{1c1}[k]; \forall k \in \mathbf{Z}_{0+}, \forall t \in [kT\ (k+1)\,T) \end{gathered} \tag{109}$$

which can be rewritten in a compact form, which is also in companion controllability form [27,48], for each of the subsystems as follows:

$$\overline{A}_d = \begin{bmatrix} \overline{A}_{d11} & \overline{A}_{d12} \\ \overline{A}_{d21} & \overline{A}_{d22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.15 & -0.1 & 1.05 \end{bmatrix} \tag{110}$$

$$\overline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \overline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{111}$$

$$\tilde{\overline{A}}_d = \begin{bmatrix} \tilde{\overline{A}}_{d11} & \tilde{\overline{A}}_{d12} \\ \tilde{\overline{A}}_{d21} & \tilde{\overline{A}}_{d22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{121} & \alpha_{122} & \alpha_{123} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_{211} & \alpha_{212} & 0 & 0 & 0 \end{bmatrix} \tag{112}$$

$$vec\left(\tilde{\overline{A}}_d\right) = (0,0,0,0,0,0,0,\alpha_{121},\alpha_{122},\alpha_{123},0,0,0,0,0,0,0,0,0,0,\alpha_{211},\alpha_{212},0,0,0)^{\mathrm{T}}$$

where $\tilde{\overline{A}}_d$ *is the matrix dynamics of the uncertainties. The matrix* $\overline{A}_d$ *is not convergent since it has two unstable eigenvalues* $z = 1.08521$ *and* $z = -\frac{1+\sqrt{5}}{2}$*. The controller is proposed to have the structure:*

$$\overline{K} = \begin{bmatrix} \overline{K}_1 \\ \overline{K}_2 \end{bmatrix} = \begin{bmatrix} \overline{K}_{111} & \overline{K}_{211} \\ \overline{K}_{112} & \overline{K}_{212} \\ \overline{K}_{121} & \overline{K}_{221} \\ \overline{K}_{122} & \overline{K}_{222} \end{bmatrix} \tag{113}$$

Leading to a closed-loop dynamics of the whole extended discrete system given by the matrix:

$$\overline{A}_{cld} = \overline{A}_d + \overline{BKC} + \tilde{\overline{A}}_d = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1+\overline{K}_{111} & -1 & \overline{K}_{112}+\alpha_{121} & \alpha_{122} & \alpha_{123} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \overline{K}_{221}+\alpha_{211} & \alpha_{212} & 0.15+\overline{K}_{222} & -0.1 & 1.05 \end{bmatrix} \tag{114}$$

which can be equivalently decomposed also as $\overline{A}_{cld} = \overline{A}_{cld*} + \widetilde{\overline{A}}_{cld}$ *in terms of a closed-loop coupling nominal and uncertain dynamics between both subsystems being given by the matrices:*

$$\overline{A}_{cld*} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1+\overline{K}_{111} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.15+\overline{K}_{222} & -0.1 & 1.05 \end{bmatrix} \quad (115)$$

$$\widetilde{\overline{A}}_{cld} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{K}_{112}+\alpha_{121} & \alpha_{122} & \alpha_{123} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \overline{K}_{221}+\alpha_{211} & \alpha_{212} & 0 & 0 & 0 \end{bmatrix} \quad (116)$$

Then,

$$vec\left(\overline{K}\right) = \left(\overline{K}_{111}, \overline{K}_{112}, \overline{K}_{121}, \overline{K}_{122}, \overline{K}_{211}, \overline{K}_{212}, \overline{K}_{221}, \overline{K}_{222}\right)^T$$

Because of the sparse structure of the matrix of dynamics, the whole number of controller entries is simplified by zeroing directly $\overline{K}_{121}$ *,* $\overline{K}_{122}$*,* $\overline{K}_{211}$ *,* $\overline{K}_{212}$*. Moreover,* $\overline{K}_{112}$ *and* $\overline{K}_{221}$ *are used to address the achievement of the sufficient norm smallness of the uncertainties vector, so they are also zeroed in the unknowns vector* $vec\left(\overline{K}\right)$ *and transferred to* $vec\left(\widetilde{\overline{A}}_d\right)$ *so that the nominal linear algebraic Equation (42) is solved in the unknown vector:*

$$vec\left(\overline{K}\right) = \left(\overline{K}_{111}, 0, 0, 0, 0, 0, 0, \overline{K}_{222}\right)^T$$

with

$$vec\left(\overline{A}_d\right) = (0,1,0,0,0,1,-1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0.15,-0.1,1.05)^T$$

and

$$vec\left(\widetilde{\overline{A}}_d\right) = \left(0,0,0,0,0,0,0,\overline{K}_{112}+\alpha_{121},\alpha_{122},\alpha_{123},0,0,0,0,0,0,0,0,0,0,\overline{K}_{221}+\alpha_{211},\alpha_{212},0,0,0\right)^T$$

One checks the static controller synthesis solvability for three intended matrices of the nominal closed-loop dynamics (that is, excluding the contribution of the uncertainties, which are incorporated to the matrix $\widetilde{\overline{A}}_d$*, in this first synthesis step) which are, respectively, defined depending on the unknown* $\overline{K}_{111}$ *by:*

$$vec\left(\overline{A}_{cld*1}\right) = \left(0,1,0,0,0,1+\overline{K}_{111},-1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0.20,-0.1,1.05\right)^T$$
$$vec\left(\overline{A}_{cld*2}\right) = \left(0,1,0,0,0,1+\overline{K}_{111},-1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0.15,0.30,1.05\right)^T$$
$$vec\left(\overline{A}_{cld^*3}\right) = \left(0,1,0,0,0,1+\overline{K}_{111},-1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0.15,0.40,1.05\right)^T$$

Now, note that the closed-loop characteristic polynomials which define the respective closed-loop self-dynamics of both subsystems in the extended discretized system, after compensation via static linear output feedback, are:

$$p_1(z) = z^2 + z - \left(1+\overline{K}_{111}\right); p_2(z) = z^3 - 1.05z^2 + 0.1z - \left(0.15+\overline{K}_{222}\right) \quad (117)$$

The first one depends on the still undetermined $\overline{K}_{111}$*. The eigenvalues of* $\overline{A}_{cld*}$ *are trivially the zeros of the product of both characteristic polynomials* $p_1(z)p_2(z)$ *since the matrices of targeted closed-loop dynamics are in companion forms in the self-dynamics of both subsystems integrated in the extended discrete one. Note that* $p_1(z)$ *is stable for* $\overline{K}_{111} \in \left(-\frac{5}{4}, -\frac{1}{2}\right)$ *while* $p_2(z)$ *is stable for* $\overline{K}_{222} = -0.35$ *with zeros* -0.34454 *and* $0.69727 \pm 0.30707\boldsymbol{i}$*, for* $\overline{K}_{222} = -0.45$ *with zeros* -0.41913 *and* $0.73457 \pm 0.41973\boldsymbol{i}$ *or for* $\overline{K}_{222} = -0.55$ *with zeros* -0.47971 *and* $0.7649 \pm 0.49881\boldsymbol{i}$*. Those zeros are in* $\overline{A}_{cld*i}$*, respectively, for i = 1, 2, 3. How-*

ever, in the absence of closed-loop compensation through the choice $\overline{K}_{222} = 0$, the polynomial $p_2(z) = z^3 - 1.05z^2 + 0.1z - 0.15$ is not stable having a zero $z = 1.08521$. In summary, the above system in the absence of coupling dynamics is unstable in the absence of control, that is, the open-loop system is unstable. However, the closed-loop one can be stabilized with linear static output feedback control just with two nonzero scalar gains, that is, with two nonzero entries in the control gain matrix (113). With both self-dynamics being stable under the conditions given for the choices of $\overline{K}_{111}$ and $\overline{K}_{222}$, one concludes that $\overline{A}_{cld*}$ is convergent.

It turns out that any norm of $\widetilde{\overline{A}}_{cld}$ is arbitrary small for $\alpha = max(|\alpha_{122}|, |\alpha_{123}|, |\alpha_{212}|, |\overline{K}_{112} + \alpha_{121}|, |\overline{K}_{221} + \alpha_{211}|)$ being arbitrary small. Under the given conditions which guarantee that $p_1(z)$ and $p_2(z)$ are stable, so that $\overline{A}_{cld*}$ is convergent, it follows that $\overline{A}_{cld}$ is also convergent if α is sufficiently small related to $1/\left\|(zI_5 - A_{cld*})^{-1}\right\|_{H_\infty}$ so since, for any complex number z which is not an eigenvalue of A_{cld*}, one has that

$$zI_5 - A_{cld} = (zI_5 - A_{cld*})\left(I_5 - (zI_5 - A_{cld*})^{-1}\widetilde{\overline{A}}_{cld*}\right) \tag{118}$$

so that the eigenvalues of A_{cld} are not in $\mathbf{C}_1$. In particular, note the following features:

(a) *Assume that α_{121} and α_{211} are known precisely. Then, the additional choices of the previously unspecified gains $\overline{K}_{112} = -\alpha_{121}$ and $\overline{K}_{221} = -\alpha_{211}$ as entries of the controller gain guarantee that $\overline{A}_{cld}$ is convergent, so that the extended closed-loop system is stable if $\alpha_0 = min(|\alpha_{122}|, |\alpha_{123}|, |\alpha_{212}|)$ is sufficiently small satisfying $\alpha_0 < 1/\left(\sqrt{5}\sup\limits_{0\le\theta<2\pi}\left\|(e^{i\theta}I_5 - A_{cld*})^{-1}\right\|_2\right)$ after using the norm inequality $\left\|\widetilde{\overline{A}}_{cld}\right\|_2 \le \sqrt{5}min\left(\left\|\widetilde{\overline{A}}_{cld}\right\|_\infty, \left\|\widetilde{\overline{A}}_{cld}\right\|_1\right)$ [49], for the matrix $\widetilde{\overline{A}}_{cld}$ of order 5.*

(b) *Assume that α_{121} and α_{211} are not known precisely but they are known to belong to known respective real subsets $\left[\underline{\alpha}_{121}, \overline{\alpha}_{121}\right]$ and $\left[\underline{\alpha}_{211}, \overline{\alpha}_{211}\right]$, which is a reasonable assumption in practice. Then, choose the previously unspecified gains $\overline{K}_{112} = -\alpha_{121*} = -\frac{\underline{\alpha}_{121}+\overline{\alpha}_{121}}{2}$ and $\overline{K}_{221} = -\alpha_{221*} = -\frac{\underline{\alpha}_{221}+\overline{\alpha}_{221}}{2}$ as entries of the controller gain guarantee that $\overline{A}_{cld}$ is convergent, so that the extended closed-loop system is stable if $\alpha = max(|\alpha_{122}|, |\alpha_{123}|, |\alpha_{212}|, \left|\underline{\alpha}_{121}\right|, |\overline{\alpha}_{121},|, \frac{1}{2}\left|\overline{\alpha}_{121} - \underline{\alpha}_{121}\right|, \frac{1}{2}\left|\overline{\alpha}_{221} - \underline{\alpha}_{221}\right|)$ is sufficiently small so that $\alpha < 1/\left(\sqrt{5}\sup\limits_{0\le\theta<2\pi}\left\|(e^{i\theta}I_5 - A_{cld*})^{-1}\right\|_2\right)$.*

As a result, as $k \to \infty$, $\{\overline{x}[k]\} \to 0$, $\{y[k]\} \to 0$, $\{u[k]\} \to 0$, $x_{2dj}[k] \to 0$ $(j = 1,2,3)$, $x_{1ci}[k] \to 0$ $(i = 1,2)$ for any given finite initial conditions. Moreover, $u(t) \to 0$ as $t \to \infty$, $x_{1ci}(t) \to 0$ $(i = 1,2)$ and $y(t) \to 0$ as $t \to \infty$. The controller is of decentralized type since it only picks up information of the first subsystem through its output which is also the global output of the whole system.

Note that for the sparse control and output matrices defined in (111), four of the eight control gains which are entries of the control matrix (113) do not play a role in the closed-loop matrix of dynamics and can be zeroed.

Example 4. *Assume a fifth order system as that of Example 4 but with, in general, a less sparse parameterization of the control and output matrices. In this case, the stability of the self-dynamics of both subsystems and the influence of the coupling dynamics to keep the achieved closed-loop stability might be more difficult to deal with. The general idea of stabilizing the uncoupled dynamics under a sufficiently small influence of the coupling one will be addressed as follows. Assume that the matrix of the closed-loop dynamics is partitioned into four block matrices as:*

$$\overline{A}_{cld} = \begin{bmatrix} \overline{A}_{cld11} & \overline{A}_{cld12} \\ A_{cld21} & \overline{A}_{cld22} \end{bmatrix} \tag{119}$$

Assume that its diagonal part, which contains the coupling-free self-dynamics of both subsystems

$$\overline{A}_{cld} = \begin{bmatrix} \overline{A}_{cld11} & \overline{A}_{cld12} \\ A_{cld21} & \overline{A}_{cld22} \end{bmatrix} \tag{120}$$

is convergent. Then, the characteristic polynomial of the whole system is given by

$$\begin{aligned} det\left(zI_5 - \overline{A}_{cld}\right) &= det\begin{bmatrix} zI_2 - \overline{A}_{cld11} & -\overline{A}_{cld12} \\ -A_{cld21} & zI_3 - \overline{A}_{cld22} \end{bmatrix} \\ = det\ \left(zI_2 - \overline{A}_{cld11}\right) & det\left(zI_3 - \overline{A}_{cld22}\right)\ det\left(I - \left(zI_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\left(zI_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right) \end{aligned} \tag{121}$$

which has no zeros on $z \in \boldsymbol{C}_1$ if the system is stabilizable by some linear output feedback static controller, so that the matrix $\overline{A}_{cld}$ is convergent, then all its poles are in $z(< 1) \in \boldsymbol{C}$ if the following constraint holds (Banach's Perturbation Lemma [49]; see also existence and calculation of the inverses of partitioned non-polynomial and polynomial matrices [[50],[51],[52],[53],[54]):

$$\begin{aligned} &\left\|\left(zI_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\left(zI_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right\|_\infty = \underset{0\le\omega<2\pi}{sup}\ \sigma\left[\left(e^{i\theta} I_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\left(e^{i\theta}\omega I_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right] < \sqrt{5}min \\ &\times\left[\underset{0\le\theta<2\pi}{max}\left\|\left(e^{i\theta}\omega I_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\left(e^{i\theta}\omega I_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right\|_2, \underset{0\le\theta<2\pi}{max}\left\|\left(e^{i\theta}\omega I_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\left(e^{i\theta}\omega I_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right\|_1\right] \\ &\le \left\|\left(zI_3 - \overline{A}_{cld22}\right)^{-1} A_{cld21}\right\|_{H_\infty}\left\|\left(zI_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right\|_{H_\infty} \\ &\le \left\|\left(zI_3 - \overline{A}_{cld22}\right)^{-1}\right\|_{H_\infty}\left\|\left(zI_2 - \overline{A}_{cld11}\right)^{-1}\overline{A}_{cld12}\right\|_{H_\infty}\|A_{cld21}\|_2\|\overline{A}_{cld12}\|_2 \end{aligned} \tag{122}$$

for any $z \in \boldsymbol{C}\backslash\left(sp\overline{A}_{cld11} \cup sp\overline{A}_{cld22}\right)$, since $\overline{A}_{cld22}$ and $\overline{A}_{cld11}$ are convergent. This constraint holds if

$$\begin{aligned} \|\overline{A}_{cld12}\|_2\|\overline{A}_{cld21}\|_2 &< 1/\left(\left\|\left(zI - \overline{A}_{cld11}\right)^{-1}\right\|_{H_\infty}\left\|\left(zI - \overline{A}_{cld22}\right)^{-1}\right\|_{H_\infty}\right) \\ &= 1/\left(\left\|\left(e^{i\theta} I_3 - \overline{A}_{cld22}\right)^{-1}\right\|_2\left\|\left(e^{i\theta} I_2 - \overline{A}_{cld11}\right)^{-1}\right\|_2\right) \end{aligned} \tag{123}$$

which is also guaranteed if, for some $\varepsilon \in \boldsymbol{R}_+$,

$$\begin{aligned} max\left(\|\overline{A}_{cld12}\|_2, \|\overline{A}_{cld21}\|_2\right) \le \varepsilon < \overline{\varepsilon} &= 1/\left(max\left(\left\|\left(zI - \overline{A}_{cld11}\right)^{-1}\right\|_{H_\infty}, \left\|\left(zI - \overline{A}_{cld22}\right)^{-1}\right\|_{H_\infty}\right)\right) \\ &= 1/\left(\left\|\left(e^{i\theta} I_3 - \overline{A}_{cld22}\right)^{-1}\right\|_2\left\|\left(e^{i\theta} I_2 - \overline{A}_{cld11}\right)^{-1}\right\|_2\right) \end{aligned} \tag{124}$$

Then, $\overline{A}_{cld}$ is convergent, provided that $\overline{A}_{cldd}$is convergent, implying that det $\left(zI - \overline{A}_{cld11}\right)$ $det\left(zI - \overline{A}_{cld22}\right) \neq 0$ for $|z| \ge 1$ under (123) or under (124). Thus, the extended discrete closed-loop system has been stabilized for small off-diagonal dynamics which has not been considered by the designed stabilizing controller. The specific solution is found by following generically the basic ideas of Example 3.

The invoked discretization tools on the continuous substate are based on the use of a zero-order-hold on the continuous time-input to obtain its sampled value at sampling instants which is kept constant along the current intersample time period. A potential extension for the use of fractional order holds can be performed by using first-order and rate correction sampling and hold discretization. See, for instance [55].

7. Conclusions

This paper has studied a hybrid dynamic system which consists of a set of single-input single-output continuous-time systems with mutual dynamic couplings on a set of digital subsystems. Each one of the combined continuous-time/discrete subsystems is assumed of single-input single-output (SISO) type. The dynamics of the whole system can also be eventually affected by discrete-time delayed dynamics for a finite number of point delays and it is driven, in general, by a combined action of the continuous-time input along the intersample time interval and its values at the sampling instants. An extended discretized system, built with the discretization of the continuous parts of the whole hybrid system being eventually coupled with the digital ones, is formulated and an ad hoc particular version of it is also given where the continuous-time input in the intersample period is generated for an auxiliary discrete control sequence. Both discrete sequences, the discretized version of the primary control and the auxiliary discrete sequence, are used as a double control channel to stabilize through static linear output feedback control the extended discrete dynamic system associated with the original hybrid one. The stabilization objective is the first and main intended step for the stabilization of the whole hybrid system. Later on, one deals with the stabilization through linear static output feedback of the modified extended discrete system with zero input–output direct interconnection gains, which implies basically that the relative degree, or pole-zero excess in the transfer function, is greater than one. The mechanism for designing the controller gain is of algebraic type and it is based on converting the set of equations to be solved into a linear algebraic system of equations with an equivalent vector form presentation of the controller gain matrix. The initial algebraic linear system is derived from the synthesis problem initial statement of the needed equations in terms of ad hoc Kronecker products of matrices and vectors of the unknowns (that is, the entries of the controller gain matrix) and the data (that is, the entries of the targeted close-loop matrix of dynamics). In general, the algebraic problem to be solved can be either non-compatible, so that it has no solution for a pre-defined suited stable closed-loop dynamics of the extended discrete system being defined by a convergent matrix of closed-loop dynamics, or it can be algebraically compatible with either one or infinitely many solutions for the controller to be synthesized. Then, the existence of static linear output feedback stabilization of the modified extended discrete system is investigated through special matrix Riccati algebraic equalities. The final part of the manuscript is devoted to the characterization of keeping the stabilization under a total of partial degree of decentralized control. Roughly speaking, such a decentralization is related to the achievement of the closed-loop stabilization under a total or partial lack of information of couplings of dynamics between subsystems being transmitted to the overall controller so that controllers with just local information about its own subsystem with eventually minimum information taken about the mutual dynamic couplings between the various subsystems are able to achieve the closed-loop stabilization. Section 6 addresses the cases of small influences of the delayed discrete dynamics and that of the couplings between subsystems in the whole dynamics of the hybrid system.

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