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Closed G_2 forms and special metrics

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Leioa, mayo de 2015

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Dedicado a
mis padres, mis hermanos y a mi sobrino Alejandro

Contents

Introduction

The main object of our study is *calibrated* G_2 *manifolds*. Such a manifold is a 7-dimensional Riemannian manifold admitting a G_2-structure whose fundamental 3-form, called the G_2 *form*, is closed. In particular, we construct new examples of those manifolds from *symplectic half-flat* manifolds of dimension six via mapping tori, and show the existence of *nilsoliton metrics* determined by closed G_2 forms. For each of these latter G_2 forms, we prove that the solution of its *Laplacian flow* has long time existence.

Let (M, g) be a Riemannian manifold. The existence of a certain differential form Ω on (M, g), such that Ω is parallel with respect to the Levi-Civita connection of the Riemannian metric g, gives a powerful restriction on the holonomy group of (M, g). This group acts in a natural way on the tangent space $T_p(M)$ of M in any point p. When (M, g) is complete, de Rham in [44] proved that unless this representation is irreducible, M has a finite covering, which is a product of Riemannian manifolds of smaller dimension. In 1955 Berger [15] gave the complete list of the possible holonomy groups of a simply connected, irreducible and non-symmetric Riemannian manifold (M, g) of dimension n:

i)	$\mathrm{SO}(n)$ acting on $\mathbb{R}^n$;	
ii)	$\mathrm{U}(m) \subset \mathrm{SO}(n)$ acting on $\mathbb{R}^{2m}$,	with $n = 2m$;
iii)	$\mathrm{SU}(m) \subset \mathrm{SO}(n)$ acting on $\mathbb{R}^{2m}$,	with $n = 2m$;
iv)	$\mathrm{Sp}(m) \subset \mathrm{SO}(n)$ acting on $\mathbb{R}^{4m}$,	with $n = 4m$;
v)	$\mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(n)$ acting on $\mathbb{R}^{4m}$,	with $n = 4m \geq 8$;
vi)	$\mathrm{G}_2 \subset \mathrm{SO}(7)$ acting on $\mathbb{R}^7$,	with $n = 7$;
vii)	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ acting on $\mathbb{R}^8$,	with $n = 8$.

It is remarkable that all but the two largest irreducible holonomy groups, $\mathrm{SO}(n)$ and $\mathrm{U}(m)$, force the metric to be Einstein and in some cases Ricci-flat. More precisely, Riemannian manifolds whose holonomy group is contained in $\mathrm{U}(m)$ are called *Kähler manifolds*; they have dimension $2m$, and the Kähler form is a non-degenerate 2-form which is parallel with respect to the Levi-Civita connection of the Kähler metric. Riemannian manifolds with holonomy $\mathrm{SU}(m)$ are Kähler manifolds such that the Kähler metric is Ricci-flat. On such a manifold, in addition to the Kähler form there is a complex volume form which is parallel, and if it is compact, then it is called *Calabi-Yau manifold*. A Riemannian manifold (M, g) of dimension $n = 4m$ is said to be *hyper-Kähler* if its holonomy group is contained in $\mathrm{Sp}(m)$. Since $\mathrm{Sp}(m) \subset \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, hyper-Kähler manifolds are Kähler and Ricci-flat, and they have three complex structures compatible with the hyper-Kähler metric. *Quaternionic Kähler manifolds* are those whose holonomy group is contained in $\mathrm{Sp}(m)\mathrm{Sp}(1)$. Such a manifold has a 4-form which is parallel. Alekseevsky in [2] proved that quaternionic Kähler manifolds of dimension ≥ 8 are Einstein. (Note that in dimension 4, an orientable Riemannian manifold is called quaternionic Kähler if it is Einstein and selfdual [117].) The groups G_2 and $\mathrm{Spin}(7)$ are the exceptional holonomy Lie groups. Riemannian manifolds with holonomy contained in G_2 or $\mathrm{Spin}(7)$ are Ricci-flat and have a 3-form or a 4-form, respectively, which is parallel with respect to the Levi-Civita connection [20].

The geometrical structures associated to the cases *ii*) to *vii*) of the list of Berger are known as *special geometrical structures* and the corresponding underlying Riemannian metric is called a *special metric*. For the exceptional holonomy groups, that is, G_2 and $\mathrm{Spin}(7)$ the corresponding geometrical structures are called *exceptional structures* and the corresponding metric is said to be an *exceptional metric*.

For many years after Berger's result, the theory of G_2 manifolds was a dormant subject. In fact, there were doubts whether the two exceptional entries (G_2 and $\mathrm{Spin}(7)$) in Berger's list can be realized as holonomy groups. Only in 80's, manifolds with holonomy G_2 were constructed. In 1987, Bryant in [23] constructed local examples, and then R. Bryant and S. Salamon in [26] produced complete manifolds with holonomy G_2. The first examples of compact manifolds with holonomy G_2 and $\mathrm{Spin}(7)$ were given by Joyce [86, 87]. Kovalev in [89] and, recently, Corti, Haskins, Nordström and Pacini in [43] produced new examples of compact manifolds with holonomy G_2. Since then, G_2 manifolds became a central subject of study not only in geometry and topology but also in mathematical physics, mainly in the context of string theory and supersymmetry [58, 59, 60, 67, 113].

Let us consider the space $\mathbb{O}$ of the Cayley numbers, which is a non-associative algebra over $\mathbb{R}$ of dimension 8. Thus, we can identify the 7-dimensional Euclidean space $\mathbb{R}^7$ with the subspace of $\mathbb{O}$ consisting of pure imaginary Cayley numbers. Then, the product on $\mathbb{O}$ defines on $\mathbb{R}^7$ the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245}, \tag{1}$$

where $\{e_1, \ldots, e_7\}$ is the standard basis of $\mathbb{R}^7$ and $\{e^1, \ldots, e^7\}$ is the dual basis. Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. The group G_2 is the stabilizer of the

3-form φ defined by (1) under the standard action of $\mathrm{GL}(7,\mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$, that is,

$$\mathrm{G}_2 = \{A \in \mathrm{GL}(7,\mathbb{R}) \mid A^* \varphi = \varphi\}.$$

The Lie group G_2 is compact, connected, simply connected and simple of dimension 14. It acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which $\{e_1, \ldots, e_7\}$ is an oriented orthonormal basis. Denote by $*_\varphi$ the Hodge star operator determined by the orientation and the metric. Then G_2 also preserves the 4-form

$$*_\varphi\varphi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} + e^{2467} + e^{3456}.$$

A G_2-*structure* on a 7-dimensional manifold M is a reduction of the structure group $\mathrm{GL}(7,\mathbb{R})$ of its frame bundle to the exceptional Lie group G_2. Therefore, a G_2-structure determines a Riemannian metric and an orientation on M. Manifolds admitting G_2-structure are called G_2 *manifolds*. The presence of such a structure is equivalent to the existence of a global 3-form φ, called G_2 *form*, which can be locally written as in (1) with respect to some (local) basis $\{e^1, \ldots, e^7\}$ of the (local) 1-forms on M (see subsection 1.1.2 for details). Such a G_2 form φ on a manifold M induces a Riemannian metric g_φ on M given by

$$g_\varphi(X,Y)\, vol_M = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for any vector fields X, Y on M, where vol_M is the volume form on M, and ι_X denotes the contraction by X. Let ∇ be the Levi-Civita connection of the Riemannian metric g_φ. The G_2-structure φ is called *torsion free* G_2-*structure* if $\nabla\varphi = 0$, so the holonomy group of (M, g_φ) is contained in G_2; and φ is said to be G_2-*structure with torsion* if $\nabla\varphi \neq 0$.

By [56], a manifold M with a G_2-structure comes equipped not only with a 3-form φ and a Riemannian metric g_φ determined by φ, but also with a *2-fold vector cross product* P, which satisfy the relation

$$\varphi(X,Y,Z) = g_\varphi(P(X,Y),Z),$$

where X, Y and Z are vector fields on M. Therefore, G_2 manifolds can be considered the G_2 analogues of almost Hermitian manifolds. Corresponding to the almost complex structure and the Kähler form, one has the vector cross product and the fundamental 3-form φ, respectively. It should be remarked that there is one fundamental difference between almost complex structures and 2-fold vector cross products. Almost complex structures are defined without reference to a metric (although if a metric exists, a compatibility condition is required). In contrast to this, a 2-fold vector cross product has a unique (positivie definite) metric associated with it (see subsection 1.1.2).

Fernández and Gray in [56] give a classification of G_2 manifolds. They prove that there are 16 classes according to how the covariant derivative of the fundamental 3-form behaves with respect to its decomposition into G_2 irreducible components. Within these classes one can find the G_2 analogues of some classes of almost Hermitian manifolds. For example, if φ is closed, then (M, φ) is a *calibrated* G_2 *manifold* in the sense of Harvey and Lawson [79], a G_2 analogue of an almost Kähler manifold; if $d\varphi$ is a multiple of $*_\varphi \varphi$, (M, φ) is a nearly parallel G_2 manifold, a G_2 analogue of a nearly Kähler manifold; if φ is coclosed ($d *_\varphi \varphi = 0$) then M is a *cocalibrated* G_2 *manifold*, a G_2 analogue of semi-Kähler manifolds. Moreover, if φ is closed and coclosed, then the holonomy group of M is a subgroup of G_2 [56], that is, φ is parallel with respect to the Levi-Civita connection of the metric g_φ, and M is a G_2 analogue of a Kähler manifold.

Any orientable hypersurface $M \subset \mathbb{R}^8$ has a G_2-structure induced by the vector product of $\mathbb{R}^8$ [50, 56, 132]. In [56] it is proved that such a G_2-structure is always coclosed; moreover, it is nearly parallel if and only if M is the sphere S^7 and it is parallel if and only if M is totally geodesic.

However, constructing examples of compact calibrated G_2 manifolds is not a straightforward task. For instance, Cleyton and Swann in [36] classify calibrated G_2 manifolds on which a simple group acts with cohomogeneity one, but no compact manifold occurs in this list. On the other hand, Fernández in [51] exhibited the first example of a compact calibrated G_2 manifold that does not have holonomy G_2. This example is given in terms of a nilpotent Lie algebra $\mathfrak{g}$ and an element of $\Lambda^3\mathfrak{g}^*$ that corresponds to a closed left invariant 3-form on the associated simply connected nilpotent Lie group. Since the structure constants are rational, there exists a uniform discrete subgroup [102] such that the quotient is a compact manifold, called *compact nilmanifold*, which has a calibrated G_2-structure. A classification of compact nilmanifolds carrying left invariant closed G_2 forms was given recently in [38]. In Chapter 4 we shall return to this classification.

In the first and second chapter we pursue this approach and we produce new examples of calibrated G_2 manifolds, with a G_2-structure with torsion, via mapping tori of diffeomorphisms of $\mathrm{SU}(3)$-*manifolds* carrying a *symplectic half-flat* structure which is preserved by the diffeomorphism.

In section 1.1 we recall some results on $\mathrm{SU}(3)$-structures and G_2 manifolds. An $\mathrm{SU}(3)$-*structure* on a manifold M of real dimension 6 consists of an almost Hermitian structure (g, J), with Riemannian metric g and almost complex structure J, such that (M, g, J) carries a complex volume form $\Psi = \psi_+ + i\,\psi_-$. In general, neither the 3-form Ψ nor the Kähler form ω of (g, J) are closed. Since G_2 is the stabilizer of the transitive action of G_2 on the six-sphere S^6, it follows that a G_2-structure on a manifold induces an SU(3)-structure on any oriented hypersurface. If the G_2 manifold has holonomy group contained in G_2, then Chiossi and Salamon in [33] prove that the SU(3)-structure is *half-flat*. (As we explain in the subsection 1.1.1, the name *half-flat* structure is due to the behavior of the intrinsic torsion of such a structure [33].) In terms of differential forms, an SU(3)-structure $(g, J, \Psi = \psi_+ + i\,\psi_-)$ on a 6-manifold is *half-flat* if

$$\omega \wedge d\,\omega = 0, \qquad d\,\psi_+ = 0,$$

where ω is the Kähler form of (g, J). This means that half-flatness is characterized by the closure of $\omega^2 = \omega \wedge \omega$ and the real part ψ_+ of the complex volume form Ψ.

Conversely, it follows from a result of Hitchin [83] that every compact, real-analytic half-flat 6-manifold can be realized as a hypersurface in a manifold with holonomy contained in G_2, though this is no longer true if the real-analytic hypothesis is dropped [25]. Moreover, the G_2-structure can be obtained from the half-flat structure by solving certain evolution equations (PDE which turns into an ODE in the homogeneous case), so that the construction of half-flat structures is indirectly a means of constructing local metrics with holonomy in G_2.

A half-flat structure $(g, J, \Psi = \psi_+ + i\,\psi_-)$ is called *symplectic half-flat* if the Kähler form ω of (g, J) is closed, and so symplectic. In this case, we denote the Kähler form by F instead of ω. Thus, if the almost complex structure J is integrable or, equivalently, Ψ is closed (see subsection 1.1.1) then (M, g, J) is a Kähler manifold and, when M is compact, (M, g, J, Ψ) is a Calabi-Yau manifold of complex dimension 3 assumed that the norm of Ψ is constant. Therefore, symplectic half-flat manifolds can be considered as an extension of Calabi-Yau manifolds to the non-integrable case.

Regarding G_2-structures, it happens that if M has a symplectic half-flat structure (g, J, Ψ), and F is the Kähler form of (g, J), then the 3-form φ on $M \times \mathbb{R}$ given by

$$\varphi = F \wedge dt + \psi_+ \tag{2}$$

is a closed G_2 form with

$$*_\varphi \varphi = \psi_- \wedge dt + \frac{1}{2} F^2,$$

where t is the coordinate of $\mathbb{R}$. Moreover, the pair (dt, F) is a *cosymplectic structure* on $M \times \mathbb{R}$ in the sense of Libermann [97], or a *co-symplectic structure* in the sense of Li [96], that is, dt and F are both closed and $dt \wedge F^3$ is a volume form on $M \times \mathbb{R}$. Hence $M \times \mathbb{R}$ has a closed G_2 form φ defined by (2) and a cosymplectic structure (dt, F). This fact and the result of Li [96] mentioned below are the reasons for which in section 1.3 we study *mapping tori of diffeomorphisms preserving a symplectic half-flat structure.*

Let M be a 6-manifold with a symplectic half-flat structure (g, J, Ψ) with Kähler form F (so F is a symplectic form) and let $\nu \colon M \longrightarrow M$ be a diffeomorphism. Independently of the symplectic half-flat structure that we have on M, we can consider the *mapping torus* M_ν of the diffeomorphism ν, that is, the manifold obtained from $M \times [0,1]$ by identifying the ends with ν,

$$M_\nu = \frac{M \times [0,1]}{(x,0) \sim (\nu(x),1)}.$$

Clearly, $M_\nu = M \times S^1$ if $\nu \colon M \longrightarrow M$ is the identity, and in general M_ν is the total space of a locally trivial fiber bundle $\pi \colon M_\nu \longrightarrow S^1$ with fiber M. If the diffeomorphism ν preserves the symplectic form F, then ν is called a *symplectomorphism*, and the manifold M_ν is said to be a *symplectic mapping torus* of $\nu \colon (M, F) \longrightarrow (M, F)$. In this case, F defines a closed 2-form $\widetilde{F}$ on M_ν, and the pair $(\alpha, \widetilde{F})$ is a cosymplectic structure on M_ν in the sense of Libermann [97] since $d\alpha = 0 = d\widetilde{F}$ and $\alpha \wedge \widetilde{F}^3$ is a volume form, where α is the pullback to M_ν of the volume form of S^1. In [96], Li proves the following nice structure theorem for compact cosymplectic manifolds:

A compact manifold N has a cosymplectic structure if and only if N is a symplectic mapping torus.

If M has a symplectic half-flat structure $(g, J, \Psi = \psi_+ + i\,\psi_-)$, we say that a diffeomorphism $\nu \colon M \longrightarrow M$ is an SU(3)-*diffeomorphism* if

$$\nu^* g = g, \qquad \nu^* F = F, \qquad \nu^* \psi_+ = \psi_+.$$

Then, ν also preserves the almost complex structure J and the complex volume form Ψ since $\psi_- = J\,\psi_+$. Moreover, M_ν has a cosymplectic structure. In Theorem 1.3.2 we prove that M_ν has also a closed G_2 form. However, we show that the converse is not true. In fact, in Proposition 1.3.3, we construct a compact calibrated G_2 manifold which does not admit cosymplectic structure, and so it cannot be the mapping torus of any SU(3)-diffeomorphism of a symplectic half-flat manifold.

As we said, compact Riemannian manifolds whose holonomy group is a subgroup of G_2 can be considered the G_2 analogous of Kähler manifolds. Compact Kähler manifolds satisfy a collection of topological obstructions: theory of Kähler groups, evenness of odd-degree Betti numbers, Lefschetz property or the formality [45, 126]. In symplectic geometry, *formality* allows to distinguish compact symplectic manifolds which admit Kähler structures from those which do not [41, 45, 64].

Intuitively, a simply connected manifold is formal if its rational homotopy type is determined by its rational cohomology algebra. Simply connected compact manifolds of dimension less than or equal to 6 are formal [63, 106]. We shall say that M is formal if its minimal model is formal or, equivalently, if the de Rham complex $(\Omega^* M, d)$ of differential forms on M and the algebra of the de Rham cohomology $(H^*(M), d = 0)$ have the same minimal model. In [63] the concept of formality is extended to a weaker notion called s-formality. There, it is proved that an orientable compact connected manifold, of dimension 2n or 2n-1, is formal if and only if it is (n-1)-formal.

For any compact manifold (M, g) with holonomy G_2, Joyce [87] proves that its fundamental group is finite, the first Betti number vanishes and the cup product by the cohomology class of the G_2 form is an isomorphism between the de Rham cohomology groups $H^2(M)$ and $H^5(M)$ (a type of Lefschetz property). But nothing is known on the formality of such manifolds.

The first example of a compact calibrated G_2 manifold was given by Fernández in [51]. This 7-manifold is a compact nilmanifold whose first Betti number b_1 is $b_1 = 5$. The classification of compact G_2 nilmanifolds with a left invariant calibrated G_2-structure was given recently in [38]. The first Betti number of these manifolds is such that $2 \leq b_1 \leq 5$, or $b_1 = 7$ for the 7-torus $\mathbb{T}^7$. Moreover, excepting $\mathbb{T}^7$, all of them are non-formal. Examples of compact formal calibrated G_2 solvmanifolds (non-nilmanifolds) were given in [52]; in all these cases $b_1 = 3$. In section 1.2 we recall the results of Bazzoni, Muñoz and Fernández proved in [8] on the formality of mapping tori. Then, in section 1.4, using Theorem 1.3.2 mentioned previously, we show new examples and, in particular, we construct a compact calibrated G_2 manifold with $b_1 = 1$ which is formal.

In order to construct more examples of 7-dimensional manifolds with closed G_2 forms, a natural place to look is left invariant symplectic half-flat structures on 6-dimensional solvable Lie groups, and then take the direct product of such a Lie group by $\mathbb{R}$. According to Magnin [100], and Bazzoni-Muñoz classification [11], there are 34 isomorphism classes of nilpotent Lie groups, of which exactly 3 (including the Abelian Lie group) admit symplectic half-flat structure [40].

In Chapter 2, we classify the 6-dimensional solvable Lie algebras admitting symplectic half-flat structure. To this end, we need the following results. Nilpotent Lie algebras with half-flat structures have been classified by Conti [37]. Schulte-Hengesbach has classified in [115] direct sums of two 3-dimensional Lie algebras admitting half-flat structure, and the complete classification of decomposable half-flat Lie algebras is achieved by Freibert and Schulte-Hengesbach in [65]. Moreover, in [66] they classify arbitrary indecomposable Lie algebras admitting half-flat structure, except for the solvable case with 4-dimensional nilradical, which we study in sections 2.3 and 2.4 using, on the one hand, the classification of Turkowski [124] of 6-dimensional solvable Lie algebras with 4-dimensional nilradical and, on the other hand, the classification given in [99] of 6-dimensional unimodular solvable Lie algebras admitting symplectic form. That list is given using as starting point the original classification due to Mubarakzyanov [104].

In section 2.2 we investigate the case when the solvable Lie algebra is decomposable (Proposition 2.2.1, Proposition 2.2.2 and Proposition 2.2.3). The decomposable case $4\oplus 2$ is of special interest, because such Lie algebras (solvable or not) cannot admit a symplectic half-flat structure (Proposition 2.2.2). The indecomposable case is considered in section 2.3 for unimodular Lie algebras (Proposition 2.3.1), and in section 2.4 for non-unimodular (Proposition 2.4.1 and Proposition 2.4.2). As a consequence of the results proved, we see that the decomposable solvable Lie algebras having symplectic half-flat structure are unimodular and, by [19, 61, 70, 123, 130], the corresponding simply connected solvable Lie groups have a co-compact discrete subgroup. Thus, they produce compact calibrated G_2 solvmanifolds. Moreover, in sections 2.2-2.4, we give an explicit symplectic half-flat structure for the 8 (non-nilpotent) solvable Lie algebras admitting it, as well as, for the 2 one-parameter families of (non-nilpotent) solvable Lie algebras that also admit it (Proposition 2.2.1, Proposition 2.2.3, Proposition 2.3.1, Proposition 2.4.1 and Proposition 2.4.2).

To prove that the remaining Lie algebras having either symplectic form or half-flat structure do not admit a symplectic half-flat structure we consider, in section 2.1, two obstructions to the existence of symplectic half-flat structure on a Lie algebra.

The first obstruction is given in [65] as follows. Suppose that $\mathfrak{g}$ is a 6-dimensional Lie algebra with volume form μ. For any closed 3-form ρ on $\mathfrak{g}$, we consider the endomorphism $\widetilde{J}_\rho^*\colon \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ defined by

$$\widetilde{J}_\rho^*\alpha(X)\mu = \alpha\wedge(\iota_X\rho)\wedge\rho,$$

for $\alpha\in\mathfrak{g}^*$ and $X\in\mathfrak{g}$. Then in [65] it is proved the following result:

If $\mathfrak{g}$ is a 6-dimensional Lie algebra with a volume form $\nu\in\Lambda^6\mathfrak{g}^*$, and there is a non-zero 1-form $\alpha\in\mathfrak{g}^*$ such that for any closed 3-form $\rho\in\Lambda^3\mathfrak{g}^*$ and any closed 4-form $\sigma\in\Lambda^4\mathfrak{g}^*$, the following condition

$$\alpha\wedge\tilde{J}_\rho^*\alpha\wedge\sigma = 0,$$

is satisfied, then $\mathfrak{g}$ does not admit half-flat $SU(3)$-structures.

The second obstruction is a consequence of the obstruction to the existence of closed G_2 forms on 7-dimensional Lie algebras given in [38]. We have: *If $\mathfrak{g}$ is a*

6-dimensional Lie algebra, and there exists $X \in \mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$ such that

$$(\iota_X \phi)^3 = 0,$$

for any closed 3-form ϕ on $\mathfrak{h}$, where ι_X denotes the contraction by X, then $\mathfrak{g}$ has no symplectic half-flat structure.

Compact calibrated G_2 manifolds have interesting curvature properties. As we mentioned at the beginning of this introduction, a Riemannian manifold whose holonomy group is contained in G_2 is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated G_2 manifold, both the Einstein condition [35] and scalar-flatness [24] are equivalent to the holonomy being contained in G_2. In fact, Bryant in [24] shows that the scalar curvature is always non-positive.

The result mentioned of Cleyton and Ivanov in [35] can be considered a G_2 analogue of the *Goldberg conjecture* for compact almost Kähler manifolds. This conjecture states that *the almost complex structure of a compact Einstein almost Kähler manifold is integrable* [68]. In [116], Sekigawa gives a proof of this conjecture under the assumption that the scalar curvature of the almost Kähler manifold is non-negative. On the negative side, a complete Einstein almost Kähler manifold, with negative scalar curvature, which is not Kähler was constructed in [5], and in [81] it was shown that this example is an almost-Kähler solvmanifold, that is, a simply connected solvable Lie group endowed with a left invariant almost-Kähler structure. Moreover, this Lie group is non-unimodular, since by [46] left invariant Einstein metrics on unimodular solvable Lie groups are flat. We point out that a left invariant Ricci-flat metric on a solvmanifold is necessarily flat [4], but solvmanifolds can admit incomplete metrics with holonomy contained in G_2 as it is shown in [67, 32].

Therefore, in relation to the G_2 analogue of the Goldberg conjecture, by [24, 35] we know that a closed G_2 form on a compact manifold cannot induce an Einstein metric, unless the induced metric has holonomy contained in G_2. However, it is still an open problem to see if the same property holds on non-compact manifolds.

The goal of Chapter 3 is to study the existence of G_2-structures inducing Einstein metrics on non-compact homogeneous Einstein manifolds and, more precisely, on non-compact homogeneous Einstein solvmanifolds, since all the known examples of non-compact homogeneous Einstein manifolds belong to the class of solvmanifolds, that is, they are simply connected solvable Lie groups endowed with a left invariant Einstein metric [92]. More yet, according to a long standing conjecture attributed to D. Alekseevskii (see [16, 7.57]), these might exhaust the class of non-compact homogeneous Einstein manifolds.

In section 3.1 we recall some results on Einstein solvmanifolds. A left invariant metric on a Lie group S will be always identified with the inner product $\langle \cdot, \cdot \rangle$ determined on the Lie algebra $\mathfrak{s}$ of S. Lauret in [93] showed that Einstein solvmanifolds are *standard*, that is, satisfy the following additional condition: the corresponding metric Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ has the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a}$ abelian. The dimension of $\mathfrak{a}$ is also called the rank of $\mathfrak{s}$. Standard Einstein solvmanifolds have been deeply studied by J. Heber, who has obtained many remarkable structural and uniqueness results, by assuming only the standard condition (see [80]). In particular, Heber showed that a standard Einstein metric is unique up to isometry and scaling among left invariant metrics and that the study of standard Einstein solvmanifolds can be reduced to the rank-one case. Using these results, in section 3.2 (Theorem 3.2.2) we show that, in dimension six, the example given in [5, 81] is the unique example of Einstein (non-Kähler) almost-Kähler solvmanifold. In Theorem 3.2.2 we also classify the 6-dimensional solvmanifolds admitting a left invariant (non-flat) Kähler-Einstein metric.

In subsection 3.3.1 (Theorem 3.3.6) by using the classification of 7-dimensional Einstein solvable Lie algebras and some obstructions to the existence of calibrated G_2-structures we prove that, in contrast to the almost-Kähler case, *a seven dimensional solvmanifold cannot admit any left invariant calibrated* G_2*-structure* φ *such that the induced metric* g_φ *is Einstein, unless* g_φ *is flat.*

The class of cocalibrated G_2 manifolds includes nearly parallel G_2 manifolds, which are always Einstein with non-negative scalar curvature. Since 7-dimensional solvmanifolds cannot admit a left invariant nearly parallel G_2-structure, it is a natural problem to study the existence of left invariant cocalibrated G_2 forms such that the induced metric g_φ is Einstein. In subsection 3.3.2 (Theorem 3.3.12) by using some obstructions to the existence of cocalibrated G_2-structures we show that *a* 7*-dimensional Einstein solvmanifold* (S, g) *cannot admit any left invariant cocalibrated* G_2*-structure* φ *such that the induced metric* $g_\varphi = g$*.*

Seven dimensional 3-Sasakian manifolds are always Einstein with Einstein constant 6 and scalar curvature 42. By the results in [1] they admit a canonical cocalibrated G_2-structure inducing the Einstein metric. In section 3.4, using warped products, we construct a new example of a (non-nearly parallel) coclosed G_2 form φ on a (non-compact) manifold such that φ determines an Einstein metric whose Einstein constant is equal to 4, and so such a G_2-structure is not 3-Sasakian.

According to the results mentioned in Chapter 3, we know that simply connected solvable Lie groups can have an Einstein metric but they do not admit any closed G_2 form inducing an Einstein metric, unless the induced metric is flat. Natural generalizations of Einstein metrics are given by *Ricci solitons*, which have been introduced by Hamilton in [78]. A complete Riemannian metric g on a manifold M is called *Ricci soliton* if its Ricci curvature tensor $Ric(g)$ satisfies the

following condition
$$Ric(g) = \lambda g + \mathcal{L}_X g,$$
where λ is a real constant, X is a complete vector field on M, and $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Moreover, such a metric g is said to be *homogeneous* if its isometry group acts transitively on M, and hence g has bounded curvature [94]; and g is called *trivial* if it is an Einstein metric or is the product of a homogeneous Einstein metric with the Euclidean metric.

All known examples of non-trivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups, whose Ricci operator satisfies the condition
$$Ric(g) = \lambda I + D,$$
for some $\lambda \in \mathbb{R}$ and some derivation D of the corresponding Lie algebra. The left invariant metrics satisfying the previous condition are called *nilsolitons* if the Lie group is nilpotent [90]. Thus, in the context of closed G_2-structures, a natural question arises:

Do there exist seven dimensional simply connected (non-Abelian) nilpotent Lie groups with nilsoliton metric determined by a closed G_2 *form?*

In Chapter 4, we give a positive answer to this question. In fact, we classify 7-dimensional simply connected non-Abelian nilpotent Lie groups with a closed G_2 form which determines a nilsoliton metric. Moreover, for each one of those closed G_2 forms we solve its Laplacian flow.

In section 4.1, we recall some results on nilsoliton metrics and its existence. Between them is to be noted that not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to isometry and scaling [90]. Moreover, the nilsolitons metrics are strictly related to left invariant Einstein metrics on solvable Lie groups. Indeed, Lauret in [92] proves that *a simply connected nilpotent Lie group N has a nilsoliton metric if and only if its Lie algebra $\mathfrak{n}$ is an Einstein nilradical*, which means that $\mathfrak{n}$ has an inner product $\langle\cdot,\cdot\rangle$ and there is a *metric solvable extension* $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ of $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ such that the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ is Einstein. Furthermore, by [80, 93] such an Einstein metric has to be of standard type and it is unique, up to isometry and scaling.

In section 4.2 we determine the nilpotent Lie algebras which admit both closed G_2 forms and nilsoliton metrics but the nilsoliton is not induced by any closed G_2 form; and in section 4.3 we classify the s-step nilpotent Lie algebras ($s = 2, 3$) carrying a nilsoliton metric determined by a closed G_2 form. For this, we use the classification given in [38] of 7-dimensional nilpotent Lie algebras admitting closed G_2 forms. There it is proved that there are 12 isomorphism classes, including the Abelian case which has a trivial nilsoliton because it is flat. On the other hand, we use the classification of 7-dimensional indecomposable nilpotent Lie algebras admitting a nilsoliton given in [49]. Nevertheless, in [38] there appear two decomposable (non-Abelian) nilpotent Lie algebras, namely $\mathfrak{n}_2$ and $\mathfrak{n}_3$ in Theorem 4.2.1. So, in the aforementioned paper [49], $\mathfrak{n}_2$ and $\mathfrak{n}_3$ are not studied. As we explain below, except in the case of the Lie algebra denoted by $\mathfrak{n}_9$ in Theorem 4.2.1, we write explicitly the nilsoliton for all the cases having closed G_2-structures.

Using the classifications in [38] and [47], we show that, up to isomorphism, *there is a unique nilpotent Lie algebra, namely* $\mathfrak{n}_9$ *in Theorem 4.2.1, with a closed* G_2 *form but not admitting nilsoliton metrics.* However, all the other ten nilpotent Lie algebras have a nilsoliton, and we can determine explicitly the nilsoliton except for the Lie algebra $\mathfrak{n}_{10}$ which is 4-step nilpotent (see also [47, 48, 49]). In Proposition 4.2.3 we prove that $\mathfrak{n}_i$ $(i = 3, 5, 7, 8, 11)$ *do not carry closed* G_2*-structures inducing the nilsoliton.* Moreover, as we mention before, the existence of a nilsoliton on the Lie algebra $\mathfrak{n}_{10}$ was shown in [47, Example 2] but we do not know the nilsoliton metric explicitly, and so we do not know whether or not there is a closed G_2 form inducing the nilsoliton. This is the reason why the result of Theorem 4.3.1 is restricted to s-step nilpotent Lie algebras, with $s = 2, 3$. In fact, in Theorem 4.3.1, we show that, *up to isomorphism, there are exactly four s-step nilpotent Lie algebras* $(s = 2, 3)$ *with a nilsoliton determined by a closed* G_2 *form.*

The Ricci flow became a very important issue in Riemannian geometry and has been deeply studied. The same techniques are also useful in the study of the flow involving other geometrical structures, like for example, the Kähler Ricci flow that was studied by Cao in [29].

For any closed G_2 form φ_0 on a manifold M, Bryant in [24] introduced a natural flow, the so-called *Laplacian flow*, given by

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t\, \varphi(t), \\ d\, \varphi(t) = 0, \\ \varphi(0) = \varphi_0, \end{cases}$$

where $\varphi(t)$ is a closed G_2 form on M, and Δ_t is the Hodge Laplacian operator of the metric determined by $\varphi(t)$. Since the initial 3-form φ_0 is closed, the de Rham cohomology class $[\varphi(t)]$ is constant in t. The short time existence and uniqueness of solution for the Laplacian flow of any closed G_2-structure, on a compact manifold M, has been proved by Bryant and Xu in [27]. Also, long time existence and convergence of the Laplacian flow starting near a torsion-free G_2-structure was proved in [129].

In section 4.4 (Theorem 4.4.2, Theorem 4.4.5, Theorem 4.4.8 and Theorem 4.4.10) we show long time existence of solution for the Laplacian flow on the four compact nilmanifolds admitting an invariant closed G_2-structure which determines the nilsoliton on the corresponding Lie algebra (see Theorem 4.3.1). To our knowledge, these are the first examples of compact manifolds having a closed G_2-structure such that the solution of its Laplacian flow has long time existence.

Since the Laplacian flow is invariant by diffeomorphisms and the initial G_2-form φ_0 is invariant, the solution $\varphi(t)$ of the Laplacian flow has to be also invariant. Therefore, we show that the Laplacian flow is equivalent to a (nonlinear) system of ordinary differential equations which admits a unique solution. We prove that the solution for the four manifolds is defined for any $t \in [0, +\infty)$. Moreover, considering the Laplacian flow on the associated nilpotent Lie algebra as a bracket flow on $\mathbb{R}^7$, in a similar way as Lauret did in [107] for the Ricci flow, we study the convergence of the underlying metrics $g(t)$ of the solution. We also show that, for any $t \in [0, +\infty)$, the metric $g(t)$ is a nilsoliton isometric to $g(0)$.

Chapter 1

G_2 manifolds and mapping tori

"You take the blue pill, the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill, you stay in Wonderland, and I show you how deep the rabbit hole goes."

Morpheus

The topological description of cosymplectic manifolds is due to Li [96]. There he proves that a compact manifold M has a cosymplectic structure if and only if M is the mapping torus of a symplectomorphism of a symplectic manifold. This result may be considered an extension to cosymplectic manifolds of Tischler's Theorem [120] that asserts that the existence of a non-vanishing closed 1-form on a compact manifold M is equivalent to the condition that M is a mapping torus.

In this Chapter we study mapping tori of diffeomorphisms of symplectic half-flat manifolds such that the symplectic half-flat structure is preserved by the diffeomorphism. By Li [96], such mapping tori produce cosymplectic manifolds. In Theorem 1.3.2 we prove that these mapping tori are also calibrated G_2 manifolds. But, in Proposition 1.3.3, we show that the converse is not true even if we assume compactness of the G_2 manifold. Moreover, using Theorem 1.3.2, we show new examples of compact calibrated G_2 manifolds and, in particular, the first example of such a manifold whose first Betti number is $b_1 = 1$.

1.1 Special structures on manifolds

In this section we recall some definitions and properties about the geometric structures that we consider throughout this work. If M is a differentiable manifold of dimension m, and G is a subgroup of the linear group $\mathrm{Gl}(m, \mathbb{R})$, a *$G$-structure on M consists in a reduction of the structure group of the frame bundle of M*

to the Lie group G. Manifolds endowed with a G-structure are usually called *G manifolds.* We will focus our attention on SU(3) and G_2 manifolds.

From now on, we denote by $\Omega^*(M)$ and $\mathcal{F}(M)$ the algebras of differential forms and differentiable functions on a differentiable manifold M, respectively, and by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M.

1.1.1 $\mathrm{SU}(n)$-structures

Definition 1.1.1. *Let M be a differentiable manifold of dimension $2n$. A* $\mathrm{U}(n)$-structure*, or an* almost Hermitian structure*, on M is a pair (g, J), where g is a Riemannian metric and J is an almost complex structure on M, such that g and J are compatible in the following sense:*

$$g(JX, JY) = g(X, Y),$$

for $X, Y \in \mathfrak{X}(M)$. A manifold M with such a structure is called an almost Hermitian manifold*.*

If (M, g, J) is an almost Hermitian manifold, the *Kähler form* of (g, J) is the differential 2-form ω on M defined by

$$\omega(X, Y) = g(JX, Y),$$

for $X, Y \in \mathfrak{X}(M)$.

Since g is non-degenerate, the Kähler form of (g, J) is also non-degenerate, that is, $\omega^n \neq 0$ at each point of M, where ω^n denotes $\omega \wedge \overset{(n}{\ldots} \wedge \omega$. Moreover, the compatibility condition of g and J implies that ω is *compatible with J*, which means that

$$\omega(X, Y) = \omega(JX, JY),$$

for $X, Y \in \mathfrak{X}(M)$.

Gray and Hervella in [74] prove that there are sixteen different classes of almost Hermitian manifolds depending on the behavior of the covariant derivative of the Kähler form. We recall here those that are needed in this thesis:

- (g, J) is *almost Kähler* iff $d\omega = 0$;
- (g, J) is *nearly Kähler* iff $\nabla_X(J)X = 0$ or, equivalently, $d\omega = 3\nabla\omega$;
- (g, J) is *complex* iff J is an integrable almost complex structure, that is, the Nijenhuis tensor N_J vanishes, $N_J = 0$;
- (g, J) is *Kähler* iff it is complex and almost Kähler or, equivalently, $\nabla J = 0$ which is equivalent to $\nabla \omega = 0$;

where ∇ denotes the Levi-Civita connection of g, and the Nijenhuis tensor N_J is given by

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY],$$

for $X, Y \in \mathfrak{X}(M)$.

We would like to note the following. A *symplectic* manifold (M, F) is a pair consisting of a $2n$–dimensional differentiable manifold M with a closed 2–form F which is non-degenerate (that is, F^n never vanishes). The form F is called *symplectic*. Any symplectic manifold admits an almost Hermitian structure (g, J) such that F is compatible with J, and the Riemannian metric g is given by $g(X,Y) = F(X, JY)$, for any X, Y vector fields on M. Thus, (M, g, J) is an almost Kähler manifold, with Kähler form F.

Examples of compact almost Hermitian manifolds in the mentioned classes have been constructed by different authors [72, 73, 74, 88, 119]. Let us recall some of these examples. The complex projective space $\mathbb{CP}^n$ has a natural Hermitian metric h which is defined as follows. Fix a basis on $\mathbb{C}^{n+1}$, and a Hermitian metric h_0 on the tangent space $T_{p_0}(\mathbb{CP}^n)$ at $p_0 = [1 : 0 : \ldots : 0]$. Consider the (unique) metric h on $\mathbb{CP}^n$ which is $\mathrm{U}(n+1)$-invariant, that is, h is obtained by moving h_0 by the matrices in $\mathrm{U}(n+1)$. Take homogeneous coordinates $[z_0 : z_1 : \ldots : z_n]$, that is, for each j such that $0 \leq j \leq n$, consider the open set U_j of $\mathbb{CP}^n$ defined by $z_j \neq 0$. Now, take $U = \{[z_0 : z_1 : \ldots : z_n] | z_0 \neq 0\} \subset \mathbb{CP}^n$. Then, $U \cong \mathbb{C}^n$, and the point $[1 : z_1 : \ldots : z_n] \in U$ has coordinates $z = (z_1, ..., z_n) \in \mathbb{C}^n$. The Hermitian metric h and the Kähler form ω_0 on $\mathbb{CP}^n$ are expressed as

$$h(z) = \frac{(1+|z|^2)\sum dz_i \cdot d\bar{z}_i - \sum_{i,j} z_j \bar{z}_i \, dz_i \cdot d\bar{z}_j}{(1+|z|^2)^2},$$

and

$$\omega_0 = \frac{i}{2} \frac{(1+|z|^2)\sum dz_i \wedge d\bar{z}_i - \sum_{i,j} z_j \bar{z}_i \, dz_i \wedge d\bar{z}_j}{(1+|z|^2)^2}.$$

Then, it is easy to check that

$$\omega_0 = \frac{i}{2} \partial\overline{\partial} \log(1+|z|^2),$$

and so $d\omega_0 = 0$. Moreover, $h = g + i\,\omega_0$, where $g = \mathrm{Re}\, h$ is the Kähler metric on $\mathbb{CP}^n$. Also, any compact complex submanifold S of $\mathbb{CP}^n$ is a Kähler manifold whose Kähler metric is the pullback to S of the Kähler metric g of $\mathbb{CP}^n$.

The most well known examples of compact nearly Kähler manifolds are the 6-sphere S^6 and $S^3 \times S^3$ [57, 72, 74]. They do not admit Kähler metrics since the second Betti number is zero. The products of odd dimensional spheres $S^{2l+1} \times S^{2k+1}$ $(k, l \geq 1)$ constitute the Calabi-Eckmann compact complex manifolds. Its complex structure is defined from the complex structures of $\mathbb{C}^{l+1}$ and $\mathbb{C}^{k+1}$, and considering $S^{2l+1} \subset \mathbb{C}^{l+1}$ and $S^{2k+1} \subset \mathbb{C}^{k+1}$. More yet, Vaisman in [125] proved that $S^1 \times S^{2k+1}$, where $k \geq 1$, is locally conformally Kähler, but the natural complex structure of $S^{2l+1} \times S^{2k+1}$, for $k, l > 1$, is not locally conformally Kähler [74]. Of course, $S^1 \times S^{2k+1}$ cannot be globally conformally Kähler because it does not have the cohomology of a Kähler manifold.

The first example of a compact manifold admitting complex and symplectic structures but no Kähler metric is the Kodaira-Thurston manifold KT [88, 119]; and the first example of a compact symplectic manifold M not admitting complex structures was constructed by Fernández, Gotay and Gray in [55]. These 4-manifolds are not simply connected; they are actually nilmanifolds. (As we recall below, a compact nilmanifold is the compact quotient of a simply connected and nilpotent Lie group by a discrete subgroup.) In fact, KT is the compact nilmanifold defined by the structure equations $de^1 = de^2 = de^4 = 0$, $de^3 = e^1 \wedge e^2$; and M is the compact nilmanifold defined by the equations $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$, $de^4 = e^1 \wedge e^3$.

The classification of complex and symplectic nilmanifolds of dimension 6 was given by Salamon in [114]. Generalizations to higher dimension $2n \geq 6$ of the Kodaira-Thurston manifold are the generalized Iwasawa manifolds considered in [41]. Such manifolds have complex and symplectic structures but carry no Kähler metrics since they are non-formal. Examples of simply connected compact symplectic non-Kählerian manifolds were given in [9, 64, 69, 76, 103]

Definition 1.1.2. *An* SU(n)-structure *on a differentiable manifold M, of dimension $2n$, is a triple (g, J, Ψ) such that (g, J) is an almost Hermitian structure on M, and $\Psi = \psi_+ + i\,\psi_-$ is a complex $(n, 0)$-form, which satisfies*

$$(-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Psi \wedge \overline{\Psi} = \frac{1}{n!}\omega^n,$$

where $\overline{\Psi}$ is the complex form obtained from Ψ by conjugation, and ω is the Kähler form of (g, J).

It is clear that $\omega \wedge \psi_+ = \omega \wedge \psi_- = 0$ and $\psi_- = J\psi_+$, for any SU(n)-structure (g, J, Ψ) on M with Kähler form ω. Moreover, if Ψ is closed, then the almost complex structure J is integrable. In fact, for any $(1, 0)$-form μ, we have that $d\mu \wedge \Psi = d(\mu \wedge \Psi) = 0$ since $\mu \wedge \Psi$ is an $(n + 1, 0)$-form and Ψ is closed. Thus, the component of type $(n, 2)$ of $d\mu \wedge \Psi$ vanishes. This implies that $d\mu$ has no component of type $(0, 2)$.

The different classes of SU(n)-structures are defined in terms of the forms ω, ψ_+ and ψ_- in a similar way to the Gray-Hervella classification of U(n)-structures. Some of these classes are the following [33]:

- (g, J, Ψ) is *half-flat* SU(n)-structure iff $d\omega^{n-1} = d\psi_+ = 0$;
- (g, J, Ψ) is *symplectic half-flat* SU(n)-structure iff $d\omega = d\psi_+ = 0$;
- (g, J, Ψ) is *integrable* SU(n)-structure iff $d\omega = d\psi_+ = d\psi_- = 0$ or, equivalently, $\nabla\,\omega = 0$, $\nabla\,\psi_+ = 0$ and $\nabla\,\psi_- = 0$

Moreover, for $n = 3$, a new class of SU(3)-structures is introduced in [57] to produce examples of nearly parallel G_2 manifolds which are defined in subsection 1.1.2:

- (g, J, Ψ) is *nearly half-flat* SU(3)-structure iff $d\psi_- = -2\omega \wedge \omega$.

We will consider nearly half-flat SU(3) manifolds in Proposition 3.4.3 of Chapter 3.

In this context on classes of SU(n)-structures, we would like to resalt the following result due to Lichnerowicz [98] (see also [Corollary 2.97 and Proposition 10.29 in [16]]).

Theorem 1.1.3 [16, 98]**.** *Let (M, g, J) be a Kähler manifold of (real) dimension $2n$. Then, the Kähler metric g is Ricci-flat if and only if there exists a closed complex volume $(n, 0)$-form which is parallel with respect to the Levi-Civita connection of the Kähler metric g or, equivalently, the holonomy group of (M, g) is a subgroup of* SU(n)*.*

Remark 1.1.4. *Note that if (M, g, J, Ψ) is an integrable* SU(n)*-structure, then not only is (M, g) a Kähler manifold but also Definition 1.1.2 implies that Ψ has constant norm, and so g is Ricci-flat by Theorem 1.1.3. However, if (M, g, J) is a Kähler manifold of (real) dimension 2n, and Ψ is a closed $(n, 0)$-form on M, in general Ψ is not parallel with respect to the Levi-Civita connection of the Kähler metric g. In fact, Ψ is parallel only if Ψ has constant norm. Nevertheless, if there exists a closed $(n, 0)$-form Ψ on a Kähler manifold (M, g, J), then Yau's Theorem [131] implies that there exists a Ricci-flat Kähler metric $\widetilde{g}$ on M. But, in general, $\widetilde{g}$ is not the metric g, and in some cases such a metric $\widetilde{g}$ is not known explicitly. This happens, for example, with Fermat quintic considered in section 1.4.*

Definition 1.1.5. *An* SU(n)*-structure (g, J, Ψ) is called* torsion free SU(n)-structure *if it is integrable; otherwise (g, J, Ψ) is said to be* SU(n)-structure with torsion*.*

Next, we will focus on SU(3)-structures. In this case, one can provide more details. First, we would like to note that, as we explain in subsection 1.1.2 (see identities (1.8)), half-flat SU(3)-structures were used firstly by Hitchin [83] to produce metrics in dimension 7 with holonomy contained in G_2. However, the name *half-flat* SU(3)*-structure* on a 6-manifold is due to Chiossi and Salamon [33]. This nomenclature is due to the behavior of the intrinsic torsion of such a structure. Since SU(3) is the stabiliser in SO(6) of ω and $\Psi = \psi_+ + i\,\psi_-$, the information about the intrinsic torsion of an SU(3)-structure is contained in $\nabla\omega$ and $\nabla\Psi$, where ∇ denotes the Levi-Civita connection. More precisely, the intrinsic torsion of an SU(3)-structure belongs to the 42-dimensional space $\mathcal{W} \cong T^*\otimes\mathfrak{su}(3)^\perp$, where T^* denotes the real space underlying the complex space of $(1,0)$-forms and $\mathfrak{su}(3)^\perp$ denotes the orthogonal complement of $\mathfrak{su}(3)$ in $\mathfrak{so}(6,\mathbb{R})$. The space $\mathcal{W}$ decomposes in SU(3)-modules as

$$\mathcal{W} = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

with $\mathcal{W}_1^\pm \cong \mathbb{R}$, $\mathcal{W}_2^\pm \cong \mathfrak{su}(3)$, $\mathcal{W}_4, \mathcal{W}_5 \cong \mathbb{R}^6$ and $\mathcal{W}_3$ isomorphic to to real space underlying the space of complex symmetric 2-tensors over $\mathbb{C}^3$. By [33], $d\omega, d\psi_+$ and $d\psi_-$ are sufficient to know the intrinsic torsion. The conditions $d\omega^2 = 0$ and $d\psi_+ = 0$ for a half-flat SU(3)-structure force the intrinsic torsion to belong to the 21-dimensional space $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$ and so half of the total 42 dimensions is eliminated.

Examples of half-flat SU(3)-structures are given by nearly Kähler structures in dimension 6. Indeed, if M is a 6-dimensional manifold with a nearly Kähler structure (g, J), then $(M, g, J, \Psi = \psi_+ + i\,\psi_-)$ is a half-flat SU(3) manifold satisfying

$$3\psi_+ = d\omega, \qquad d\psi_- = -2\omega\wedge\omega.$$

Symplectic half-flat SU(3)-structures were used in [56] to show the first example of a (non-compact) calibrated G_2 manifold (see subsection 1.1.2). Examples of symplectic half-flat 6-manifolds are given at the end of this subsection. Also in Chapter 2 we return to these manifolds.

As it was mentioned before, the existence of an SU(3)-structure on a manifold M implies the existence of a certain metric on M (see [82]), but actually this metric can be described in terms of the forms (ω, ψ_+) as

$$g(X,Y)\,\omega^3 = -3\,\iota_X\omega\wedge\iota_Y(\psi_+)\wedge\psi_+,$$

where X, Y are vector fields on M, and ι_X denotes the contraction by X. We can also recover, up to scaling, its compatible almost complex structure as in [42]

$$(J^*_{\psi_+}\alpha)(X)\,\omega^3 = \alpha\wedge\iota_X\psi_+\wedge\psi_+,$$

or, equivalently,

$$\alpha(JX) = -J^*\alpha(X),$$

for any 1-form α on M.

Also, if $(M, g, J, \Psi = \psi_+ + i\,\psi_-)$ is an SU(3) manifold, we may choose a local orthonormal frame $\{e_1, \dots, e_6\}$ such that $Je_1 = e_2$, $Je_3 = e_4$ and $Je_5 = e_6$. Then, if $\{e^1, \dots, e^6\}$ is the orthonormal local basis of the 1-forms on (M, g) dual to $\{e_1, \dots, e_6\}$, we have that $Je^1 = -e^2$, $Je^3 = -e^4$ and $Je^5 = -e^6$. So, the Kähler form ω of (g, J) and the complex volume form Ψ can be locally written as

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \Psi = (e^1 + i\,e^2) \wedge (e^3 + i\,e^4) \wedge (e^5 + i\,e^6), \tag{1.1}$$

where, for notational simplicity, we write e^{ij} for the wedge product $e^i \wedge e^j$, e^{ijk} for $e^i \wedge e^j \wedge e^k$, and so on. Thus,

$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \qquad \psi_- = -e^{246} + e^{235} + e^{145} + e^{136}.$$

Moreover, for any SU(3)-structure $(g, J, \Psi = \psi_+ + i\,\psi_-)$ we can write the forms $d\omega$ (where ω is the Kähler form of (g, J)), $d\psi_+$ and $d\psi_-$ in terms of the so called *torsion forms* (see below Proposition 1.1.6 and Definition 1.1.7). To this end, we proceed as follows. Let us consider the inner product on $\Omega^q(M)$ given by

$$\langle \alpha, \beta \rangle vol_M = \alpha \wedge *\beta,$$

for $\alpha, \beta \in \Omega^q(M)$. In [14], it is proved that $\Omega^q(M)$ splits orthogonally into components $\Omega^q_l(M)$ of dimension l, which are irreducible under the action of SU(3). The representation of SU(3) on $\Omega^1(M)$ is the six dimensional irreducible representation, and the representation of SU(3) on $\Omega^q(M)$ and $\Omega^{6-q}(M)$ are the same because the Hodge star operator $* : \Omega^q(M) \longrightarrow \Omega^{6-q}(M)$ is an isometry. Therefore, it suffices to describe the representations of SU(3) on $\Omega^2(M)$ and $\Omega^3(M)$. By [14]

$$\begin{aligned} \Omega^2(M) &= \Omega^2_1(M) \oplus \Omega^2_6(M) \oplus \Omega^2_8(M), \\ \Omega^3(M) &= \Omega^3_{Re}(M) \oplus \Omega^3_{Im}(M) \oplus \Omega^3_6(M) \oplus \Omega^3_{12}(M), \end{aligned} \tag{1.2}$$

where, using the notation of [14], the summands appearing in (1.2) are:

$$\begin{aligned} &\Omega^2_1(M) = \mathbb{R}\omega, \\ &\Omega^2_6(M) = \{\star(\alpha \wedge \psi_+) \,|\, \alpha \in \Omega^1(M)\}, \\ &\Omega^2_8(M) = \{\beta \in \Omega^2(M) \,|\, \beta \wedge \psi_+ = 0 \text{ and } \star\beta = -\beta \wedge \omega\}, \end{aligned}$$

and

$$\begin{aligned} \Omega^3_{Re}(M) = \mathbb{R}\psi_+, &\qquad \Omega^3_{Im}(M) = \mathbb{R}\psi_-, \\ \Omega^3_6(M) &= \{\alpha \wedge \omega \,|\, \alpha \in \Omega^1(M)\} = \{\gamma \in \Omega^3(M) \,|\, \star\gamma = \gamma\}, \\ \Omega^3_{12}(M) &= \{\gamma \in \Omega^3(M) \,|\, \gamma \wedge \omega = 0, \gamma \wedge \psi_+ = 0, \gamma \wedge \psi_- = 0\}. \end{aligned}$$

Using these decompositions of the space $\Omega^q(M)$, in [14] it is proved the following.

Proposition 1.1.6 [14]**.** *The differential of the forms* (ω, ψ_+, ψ_-) *of an* SU(3)-*structure on* M *can be written as:*

$$\begin{aligned} d\omega &= -\frac{3}{2}\sigma_0\,\psi_+ + \frac{3}{2}\pi_0\,\psi_- + \nu_1\wedge\omega + \nu_3, \\ d\psi_+ &= \pi_0\,\omega^2 + \pi_1\wedge\psi_+ - \pi_2\wedge\omega, \\ d\psi_- &= \sigma_0\,\omega^2 + J\pi_1\wedge\psi_+ - \sigma_2\wedge\omega, \end{aligned} \tag{1.3}$$

where $\sigma_0, \pi_0 \in \mathcal{F}(M)$*,* $\nu_1, \pi_1 \in \Omega^1(M)$*,* $\sigma_2, \pi_2 \in \Omega^2_8(M)$ *and* $\nu_3 \in \Omega^3_{12}(M)$*.*

Definition 1.1.7. *The forms* $\pi_0, \sigma_0, \pi_1, \nu_1, \sigma_2, \pi_2$ *and* ν_3*, that appear in* (1.3)*, are called the* torsion forms of the SU(3)-structure.

From Proposition 1.1.6, the special classes of SU(3)-structures can also be described by the behavior of the torsion forms as follows:

- (M, g, J, Ψ) is *half-flat* SU(3)-structure iff $\pi_0 = \nu_1 = \pi_1 = \pi_2 = 0$;
- (M, g, J, Ψ) is *nearly half-flat* SU(3)-structure iff $\nu_1 = \pi_1 = \sigma_2 = 0$;
- (M, g, J, Ψ) is *nearly Kähler* iff $\pi_0 = \nu_1 = \pi_1 = \sigma_2 = \pi_2 = \nu_3 = 0$;
- (M, g, J, Ψ) is *symplectic half-flat* SU(3)-structure iff $\pi_0 = \sigma_0 = \nu_1 = \pi_1 = \pi_2 = \nu_3 = 0$;
- (M, g, J, Ψ) is *integrable* SU(3)-structure $\pi_0 = \sigma_0 = \nu_1 = \pi_1 = \sigma_2 = \pi_2 = \nu_3 = 0$.

An effective technique to obtain compact examples of half-flat manifolds and, in general, examples of some special Riemannian manifolds, consists in considering left invariant structures on a rational nilpotent Lie group, that is, on a connected, simply connected and nilpotent Lie group such that its structure constants are rational numbers, for some basis of left invariant 1-forms. Six dimensional nilpotent Lie algebras are classified in [100], and recently in [11] was given the classification of nilmanifolds up to rational and real homotopy type. Moreover, according with Mal'cev Theorem [102], each associated Lie group has a uniform discrete subgroup, giving rise to a compact quotient, called *compact nilmanifold.* Thus, an SU(3)-structure on the Lie algebra determines a left invariant SU(3)-structure on the associated nilmanifold, and viceversa.

Before showing examples of half-flat SU(3)-structures, let us recall that, for compact nilmanifolds, Nomizu in [108] proves the following result:

Theorem 1.1.8 [108]**.** *Let* $N = \Gamma\backslash G$ *be a compact nilmanifold, and let* $\mathfrak{g}$ *be the Lie algebra of* G*. Denote by* $(\bigwedge\mathfrak{g}^*, d)$ *the Chevalley-Eilenberg complex of forms on*

$\mathfrak{g}$. *Then the natural inclusion* $(\bigwedge \mathfrak{g}^*, d) \subset (\Omega^*(N), d)$ *induces an isomorphism in cohomology,*

$$H^k(N) \cong H^k(\mathfrak{g}^*),$$

where $H^*(N)$ *denotes the de Rham cohomology group, of degree* k*, of the nilmanifold* $N = \Gamma\backslash G$.

The compact nilmanifolds admitting symplectic half-flat structure have been classified by Conti and Tomassini in [40]. There it is proved the following result.

Theorem 1.1.9 [40]**.** *The 6-dimensional nilpotent Lie algebras admitting symplectic half-flat structures are:*

$$(0,0,0,0,0,0), \quad (0,0,0,0,12,13), \quad (0,0,0,12,13,23).$$

A symplectic half-flat structure (ω_1, ψ_1^+), (ω_2, ψ_2^+) *and* (ω_3, ψ_3^+), *respectively, is given by*

$$\begin{aligned} \omega_1 &= e^{12} + e^{34} + e^{56}, \qquad & \psi_1^+ &= e^{135} - e^{146} - e^{236} - e^{245}; \\ \omega_2 &= e^{14} + e^{26} + e^{35}, \qquad & \psi_2^+ &= e^{123} + e^{156} - e^{245} - e^{346}; \end{aligned}$$

and

$$\omega_3 = e^{16} + 2e^{25} + e^{34}, \qquad \psi_3^+ = e^{123} + 2e^{145} + e^{246} - 2e^{356}.$$

In all these cases, the dual basis $\{e_1, \ldots, e_6\}$ *of* $\{e^1, \ldots, e^6\}$ *is an orthonormal basis of* $\mathfrak{g}$.

Here and in what follows, we use the following notation for Lie algebras. For instance,

$$\mathfrak{g} = (0,0,0,0,12,13),$$

means that there is a basis $\{e^1, \ldots, e^6\}$ of $\mathfrak{g}^*$ such that the Chevalley-Eilenberg differential d is given by

$$de^1 = 0, \quad de^2 = 0, \quad de^3 = 0, \quad de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{13},$$

and similarly for the other Lie algebras.

The 6-dimensional compact nilmanifolds admitting either symplectic or half-flat structures have been classified by Salamon in [114] and by Conti in [37], respectively. In the following table we summarize the existence of symplectic and half-flat structures on 6-dimensional nilpotent Lie algebras. There, for every Lie algebra $\mathfrak{g}$, we suppose that $\{e_1, \ldots, e_6\}$ is an orthonormal basis of $\mathfrak{g}$.

Table 1.1: SU(3)**-structures on 6-dimensional nilpotent Lie algebras.**

Structure equations	Symplectic	Half-flat
$(0,0,0,0,0,0)$	$e^{12}+e^{34}+e^{56}$	$e^{12}+e^{34}+e^{56}$ $e^{135}-e^{146}-e^{236}-e^{245}$
$(0,0,0,0,0,12)$	$e^{16}+e^{25}+e^{34}$	$e^{13}+e^{24}+e^{56}$ $e^{125}-e^{146}-e^{236}-e^{245}$
$(0,0,0,0,0,12+34)$	—	$e^{12}-e^{34}+e^{56}$ $e^{145}-e^{136}-e^{246}-e^{235}$
$(0,0,0,0,12,13)$	$e^{14}+e^{26}+e^{35}$	$e^{14}+e^{26}+e^{35}$ $e^{123}+e^{156}-e^{245}-e^{346}$
$(0,0,0,0,12,34)$	$e^{15}+e^{36}+e^{24}$	$-e^{13}+e^{24}-e^{56}$ $-e^{126}+e^{236}+e^{145}+e^{345}+e^{146}+e^{125}$
$(0,0,0,0,12,14+23)$	$e^{16}-e^{35}+e^{24}$	$e^{13}-e^{56}+e^{24}$ $e^{126}-e^{145}+e^{235}-e^{346}$
$(0,0,0,0,13-24,14+23)$	$e^{16}+e^{25}+e^{34}$	$e^{12}+e^{34}+e^{56}$ $e^{135}-e^{146}-e^{236}-e^{245}$
$(0,0,0,0,12,14+25)$	$e^{13}+e^{26}+e^{45}$	$e^{14}+e^{46}-e^{25}-e^{36}$ $e^{156}-e^{123}+e^{236}-e^{345}+e^{246}$
$(0,0,0,0,12,15)$	$e^{16}+e^{25}+e^{34}$	—
$(0,0,0,0,12,15+34)$	—	$e^{13}-e^{45}-e^{26}$ $e^{156}+e^{124}-e^{235}-e^{346}$
$(0,0,0,12,13,23)$	$e^{16}+2e^{25}+e^{34}$	$e^{16}+2e^{25}+e^{34}$ $e^{123}+2e^{145}+e^{246}-e^{356}$
$(0,0,0,12,13,14)$	$e^{16}+e^{25}+e^{34}$	—
$(0,0,0,12,13,24)$	$e^{14}+e^{26}+e^{35}$	$e^{16}+e^{23}+e^{45}$ $e^{124}-e^{135}-e^{256}-e^{346}$
$(0,0,0,12,13,14+23)$	$-e^{16}+e^{25}+2e^{34}$	$-e^{12}+e^{25}+e^{16}-e^{56}-e^{14}-e^{36}$ $e^{246}+e^{123}+e^{136}+e^{134}+e^{345}+e^{156}$
$(0,0,0,12,13+14,24)$	$e^{16}+e^{25}+e^{34}$	—
$(0,0,0,12,14,13-24)$	$e^{15}+e^{26}+e^{34}$	—
$(0,0,0,12,13-24,14+23)$	$e^{15}-e^{26}+e^{25}+e^{34}$	—
$(0,0,0,12,14,24)$	—	—
$(0,0,0,12,23,14+35)$	—	$e^{13}+e^{45}-e^{26}$ $e^{146}+e^{125}-e^{234}+e^{356}$
$(0,0,0,12,23,14-35)$	—	$e^{13}+e^{26}-e^{45}$ $e^{125}+e^{146}+e^{234}+e^{356}$
$(0,0,0,12,13,14+35)$	—	$e^{26}-e^{34}-e^{12}-e^{15}$ $-e^{123}+e^{245}+e^{236}-e^{356}+e^{146}$
$(0,0,0,12,14,15)$	$e^{26}+e^{35}+e^{14}$	$e^{13}+e^{25}+e^{46}$ $e^{124}-e^{156}+e^{236}+e^{345}$
$(0,0,0,12,14,15+24)$	$e^{13}+e^{26}-e^{45}$	$e^{13}+e^{24}-e^{36}-e^{56}$ $e^{123}+e^{125}+e^{146}-e^{236}-e^{345}$
$(0,0,0,12,14,15+24+23)$	$e^{16}+e^{25}-e^{34}$	$e^{123}+e^{125}+e^{146}-e^{126}$ $e^{236}-e^{345}-e^{235}$
$(0,0,0,12,14,15+23)$	$e^{26}+e^{13}-e^{45}$	$e^{26}-e^{56}-e^{24}-e^{13}$ $-e^{234}-e^{345}-e^{124}+e^{145}$ $+e^{146}-e^{235}-e^{236}$
$(0,0,0,12,14-23,15+34)$	$e^{16}+e^{35}+e^{24}$	$e^{24}-e^{13}-e^{56}$ $e^{236}+e^{125}+e^{345}+e^{146}$

Table 1.1: SU(3)**-structures on 6-dimensional nilpotent Lie algebras.** Continued.

Structure equations	Symplectic	Half-flat
$(0,0,12,13,23,14+25)$	$e^{16}+e^{35}+e^{24}+e^{15}$	$e^{14}-e^{24}-e^{25}-e^{36}$ $e^{156}-e^{256}-e^{123}-e^{345}+e^{246}$
$(0,0,12,13,23,14-25)$	$e^{16}-e^{35}+e^{24}+e^{15}$	$e^{36}+e^{24}+e^{15}$ $-2e^{345}-e^{134}-e^{123}-e^{235}$ $e^{146}-2e^{456}-e^{256}$
$(0,0,12,13,23,14)$	$e^{15}+e^{24}+e^{26}-e^{34}$	$e^{15}+e^{24}+e^{36}$ $e^{123}-e^{146}+e^{256}+e^{345}$
$(0,0,12,13,14,15+23)$	$e^{16}+e^{24}+e^{25}-e^{34}$	—
$(0,0,12,13,14+23,15+24)$	$e^{16}+2e^{34}-e^{25}$	$e^{25}-e^{14}+e^{36}-e^{56}$ $\sqrt{2}e^{345}-\frac{1}{\sqrt{2}}e^{156}-\frac{1}{\sqrt{2}}e^{246}$ $+\sqrt{2}e^{123}-\sqrt{2}e^{125}$
$(0,0,12,13,14,15)$	$e^{16}+e^{25}-e^{34}$	—
$(0,0,12,13,14,34-25)$	—	—
$(0,0,12,13,14+23,34-25)$	—	—

1.1.2 G_2-structures

Similarly to the well-known vector product (or Gibbs vector product) of $\mathbb{R}^3$, there exists also a vector product on $\mathbb{R}^7$ defined via the product of octonions, or Cayley numbers, as follows. Let us consider the 8-dimensional real vector space $\mathbb{O}$ of the octonions, which is a non-associative algebra over $\mathbb{R}$ with identity 1. Recall that the product on $\mathbb{O}$, that we denote by $\circ$, is given by

$$(p_1,p_2)\circ(q_1,q_2)=(p_1q_1-q_2\overline{p_2},q_2p_1+p_2\overline{q_1}), \tag{1.4}$$

where p_i,q_i $(i=1,2)$ are quaternions, and $\overline{p}$ denotes the conjugate of a quaternion p, that is, if p is the quaternion $p=a_1+a_2i+a_3j+a_4k$ ($a_k\in\mathbb{R}$ and i,j,k such that $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i, ki=-ik=j$) then $\overline{p}=a_1-a_2i-a_3j-a_4k$. We take an orthonormal basis $\{1,e_0,\dots,e_6\}$ of $\mathbb{O}$, such that the product of the octonions satisfies

$$\begin{aligned}&e_i^2=-1, \qquad e_ie_j=-e_je_i, \qquad (i\neq j),\\ &e_ie_{i+1}=e_{i+3}, \quad e_{i+3}e_i=e_{i+1}, \quad e_{i+1}e_{i+3}=e_i \text{ with } i\in\mathbb{Z}_7.\end{aligned}$$

Such a basis is called *Cayley basis.* Consider the basis of $\mathbb{O}$, described as the pairs of quaternions

$$\{(1,0), \quad (i,0), \quad (j,0), \quad (k,0), \quad (0,1), \quad (0,i), \quad (0,j), \quad (0,k)\}.$$

From the expression (1.4) of the product on $\mathbb{O}$, can be easily obtained that the basis $\{1, e_0, \ldots, e_6\}$ given, for example, by

$$\begin{array}{llll} 1 = (1,0), & e_0 = (i,0), & e_1 = (j,0), & e_2 = (0,k), \\ e_3 = (k,0), & e_4 = (0,i), & e_5 = (0,1), & e_6 = (0,j), \end{array}$$

is a Cayley basis.

We identify $\mathbb{R}^7$ with the 7-dimensional subspace of $\mathbb{O}$ consisting of the pure imaginary Cayley numbers, that is, those such that the real part of the first quaternion is zero. Then, the *vector product* (or *2-fold vector cross product*) on $\mathbb{R}^7$ is the bilinear map

$$P : \mathbb{R}^7 \times \mathbb{R}^7 \longrightarrow \mathbb{R}^7$$

given by

$$P(x,y) = x \circ y + (x \cdot y)1,$$

for $x, y \in \mathbb{R}^7$, and where $x \cdot y$ denotes the *dot product* of x and y. As the Gibbs vector product on $\mathbb{R}^3$, the vector product P on $\mathbb{R}^7$ satisfies

$$(P(x,y)) \cdot x = (P(x,y)) \cdot y = 0, \qquad ||P(x,y)||^2 = ||x||^2 \, ||y||^2 - (x \cdot y)^2,$$

for $x, y \in \mathbb{R}^7$. So, $P(x,y) = -P(y,x)$. Gray in [71] proves that there exists a 2-fold vector cross product on $\mathbb{R}^m$ if and only if $m = 3$ or $m = 7$.

It is known that the vector product of $\mathbb{R}^3$ determines a volume form on $\mathbb{R}^3$, and so a 3-form. Similarly, the pair $(\cdot, P)$ on $\mathbb{R}^7$ determines the 3-form φ defined by

$$\varphi(x,y,z) = (P(x,y)) \cdot z.$$

Thus, with respect to a Cayley basis, φ has the expression

$$\varphi = \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+1} \wedge e^{i+3}.$$

Many authors use different bases in order to obtain an expression adapted to their purposes. We consider an orthonormal basis $\{e^1, \ldots, e^7\}$ of $(\mathbb{R}^7)^*$ so that the 3-form φ has the following expression

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}. \tag{1.5}$$

The subgroup of $\mathrm{Gl}(7,\mathbb{R})$ that fixes φ is the 14-dimensional compact, connected, simple Lie group G_2, that is,

$$\mathrm{G}_2 = \{A \in \mathrm{Gl}(7,\mathbb{R}) \mid A^*\varphi = \varphi\}.$$

The group G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which the basis $\{e_1, \ldots, e_7\}$ dual to the basis $\{e^1, \ldots, e^7\}$ is an oriented orthonormal basis of $\mathbb{R}^7$.

Definition 1.1.10. *Let (M, g) be a Riemannian manifold of dimension 7, with Riemannian metric g. A* vector product *on (M, g) is a tensor field*

$$P : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

satisfying

$$g(P(X, Y), X) = g(P(X, Y), Y) = 0, ||P(X, Y)||^2 = ||X||^2 \, ||Y||^2 - g(X, Y)^2,$$

for $X, Y \in \mathfrak{X}(M)$. A 7-dimensional Riemannian manifold with a vector product is called G_2 manifold*.*

Note that, in opposite with almost complex structures, if P is a vector product on (M, g), the Riemannian metric g is determined by P. In fact, Fernández and Gray in [56] prove that

$$P(X, P(X, P(X, Y))) = -||X||^2 \, P(X, Y),$$

for $X, Y \in \mathfrak{X}(M)$.

To describe the geometry of a G_2 manifold (M, g, P) it is very useful to consider the differential 3-form φ on M given by

$$\varphi(X, Y, Z) = g(P(X, Y), Z), \tag{1.6}$$

for $X, Y, Z \in \mathfrak{X}(M)$. This form φ, originally defined by Bonan [20], is called G_2 *form*, or *fundamental 3-form*, of (M, g, P), and it may be locally written as in (1.5). Note that there is also the 4-form $*\varphi$ on M associated to (g, P), where $*$ denotes de Hodge star operator of (M, g). By (1.5), we have

$$*\varphi = e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

Bryant in [23] proves that if (M, g, P) is a G_2 manifold, the G_2 form φ of (g, P) determines the metric $g_\varphi = g$ on M as

$$g_\varphi(X, Y) \, vol_M = \frac{1}{6} \, \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi, \tag{1.7}$$

for any vector fields X, Y on M, where vol_M denotes the volume form on M. The vector product P on M is given by (1.6). This leads to the following definition

Definition 1.1.11. *A* G_2-structure *on a 7-manifold M is defined by a differential 3-form φ on M such that* (1.7) *defines a Riemannian metric g_φ, and φ can be locally written as in* (1.5) *with respect to some (local) orthonormal basis $\{e^1, \ldots, e^7\}$ of the (local)* 1*-forms on M.*

In [56] it is given a classification of G_2 manifolds. Some of these classes are the following:

- (g, P, φ) is *almost parallel* or *calibrated* G_2*-structure* iff $d\varphi = 0$;
- (g, P, φ) is *semiparallel* or *cocalibrated* G_2*-structure* iff $d * \varphi = 0$;
- (g, P, φ) is *nearly parallel* iff $d\varphi = c * \varphi$, where $c \in \mathbb{R}$;
- (g, P, φ) is *parallel* iff $\nabla \varphi = 0$ or, equivalently [56], $d\varphi = d * \varphi = 0$.

In order to show examples of G_2 manifolds in some of these classes, firstly we note that using the product of the octonions one can define a 3-fold vector cross product $\widetilde{P}$ on $\mathbb{R}^8$ [50], so that $\widetilde{P} : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ is a trilinear map. In [56] it is proved that if M is an oriented hypersurface of $\mathbb{R}^8$, with unit normal vector field U, then M has a natural G_2-structure whose vector product P is defined by

$$P(X, Y) = \widetilde{P}(U, X, Y),$$

for $X, Y \in \mathfrak{X}(M)$. This G_2 manifold is always cocalibrated, and it is nearly parallel if and only if $M = S^7$; moreover, the G_2-structure is parallel if and only if M is flat [56].

Regarding calibrated G_2 manifolds, the first example of a compact calibrated G_2 manifold was given by Fernández in [51]. This example is described as follows. Let K^7 be the 7-dimensional nilpotent Lie group

$$K^7 = \mathrm{H}(1, 2) \times \mathbb{R}^2,$$

where $\mathrm{H}(1, 2)$ is the connected nilpotent Heisenberg group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & 0 & x_1 & z_1 \\ 0 & 1 & x_2 & z_2 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, z_1, z_2, y \in \mathbb{R}$. Then a global system of coordinates x_i, y, z_i $(i = 1, 2)$ for $\mathrm{H}(1, 2)$ is given by $x_i(a) = x_i$, $y(a) = y$, $z_i(a) = z_i$. Let u_1 and u_2 be the natural coordinates on $\mathbb{R}^2$. A standard calculation shows that a basis $\{e^1, \ldots, e^7\}$ for the left invariant 1–forms on K^7 is given by

$$\begin{aligned} &e^1 = dx_1, \qquad e^2 = dx_2, \qquad e^3 = dy, \qquad e^4 = dz_1 - x_1 dy, \\ &e^5 = dz_2 - x_2 dy, \quad e^6 = du_1, \quad e^7 = du_2. \end{aligned}$$

Thus, the 3-form φ on K^7 given by

$$\varphi = e^{367} + e^{157} + e^{247} + e^{123} + e^{345} - e^{146} + e^{256}$$

is a left invariant and closed G_2 form on K^7. Therefore, it defines a closed G_2 form on the compact nilmanifold

$$V^7 = \Gamma \backslash K^7,$$

where $\Gamma \subset K^7$ is the discrete subgroup of K^7 consisting of the matrices of K^7 whose entries are integer numbers.

In Chapter 4, Theorem 4.2.1, is described the classification of Conti and Fernández of the compact nilmanifolds having left invariant closed G_2 form.

It is well-known that if (g, P, φ) is a parallel G_2-structure on M (that is, φ is a closed and coclosed form), then the holonomy group of (M, g) is a subgroup of G_2 [56]. The first examples of compact G_2 manifolds whose holonomy group is G_2 were given by Joyce in [86]. On the other hand, Hitchin in [83] proves that if M is a 6-manifold with a half-flat SU(3)-structure (ω, ψ_+, ψ_-) which belongs to a family $(\omega(t), \psi_+(t), \psi_-(t))$ of half-flat structures on M, for some real parameter t lying in some interval $I = (t_-, t_+)$, and satisfying the evolution equations

$$\begin{cases} \partial_t \psi_+(t) = \hat{d}\omega(t), \\ \omega(t) \wedge \partial_t(\omega(t)) = -\hat{d}\psi_-(t), \end{cases}$$

then $M \times I$ has a Riemannian metric whose holonomy is contained in G_2. In fact, it is easy to check that the 3-form φ and the 4-form $*\varphi$ given by

$$\varphi = \omega(t) \wedge dt + \psi_+(t), \qquad *\varphi = \psi_-(t) \wedge dt + \frac{1}{2}\omega(t)^2 \tag{1.8}$$

are closed. In general, if (ω, ψ_+, ψ_-) is an SU(3)-structure on M, then the 3-form φ defined by

$$\varphi = \omega \wedge dt + \psi_+ \tag{1.9}$$

is a G_2 form on $M \times \mathbb{R}$. Moreover, as we mentioned in the subsection 1.1.1, it is clear that if (ω, ψ_+, ψ_-) is a symplectic half-flat SU(3)-structure on M (so we will denote by F, instead of ω, its Kähler form), then

$$\varphi = F \wedge dt + \psi_+ \tag{1.10}$$

is a closed G_2 form on $M \times \mathbb{R}$. Similarly, half-flat manifolds produce cocalibrated G_2 manifolds. In fact, if (ω, ψ_+, ψ_-) is a half-flat SU(3)-structure on M, then

$$\varphi = \omega \wedge dt - \psi_- \tag{1.11}$$

is a coclosed G_2 form on $M \times \mathbb{R}$ since

$$*\varphi = \psi_+ \wedge dt + \frac{1}{2}\omega^2$$

is closed.

Definition 1.1.12. *A* G_2*-structure* (g, P) *is called* torsion free G_2-structure *if it is parallel, and otherwise* (g, P) *is said to be* G_2-structure with torsion.

As in subsection 1.1.1, we denote by $\Omega^q(M)$ the space of the differential q-forms on M, and let $\langle , \rangle$ be the inner product on $\Omega^q(M)$ given by

$$\langle \alpha, \beta \rangle vol_M = \alpha \wedge *\beta,$$

for $\alpha, \beta \in \Omega^q(M)$. In [56] it is proved that $\Omega^q(M)$ splits orthogonally into components $\Omega^q_l(M)$ of dimension l, which are irreducible under the action of G_2. The representation of G_2 on $\Omega^1(M)$ is the seven-dimensional irreducible representation, and the representation of G_2 on $\Omega^q(M)$ and $\Omega^{7-q}(M)$ are the same because the Hodge star operator $* : \Omega^q(M) \longrightarrow \Omega^{7-q}(M)$ is an isometry. Therefore, it suffices to describe the representations of G_2 on $\Omega^2(M)$ and $\Omega^3(M)$. In [23, 28, 56, 86, 113] it is proved that

$$\begin{aligned} \Omega^2(M) &= \Omega^2_7(M) \oplus \Omega^2_{14}(M), \\ \Omega^3(M) &= \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M), \end{aligned} \tag{1.12}$$

where the summands of (1.12) are characterized as

$$\begin{aligned} \Omega^2_7(M) &= \{ *(\alpha \wedge *\varphi) \mid \alpha \in \Omega^1(M) \}, \\ &= \{ \beta \in \Omega^2(M) \mid \beta \wedge \varphi = 2 * \beta \}, \\ \Omega^2_{14}(M) &= \{ \beta \in \Omega^2(M) \mid \beta \wedge \varphi = - * \beta \}, \end{aligned}$$

and

$$\begin{aligned} \Omega^3_1(M) &= \{ a\varphi \mid a \in \mathbb{R} \}, \\ \Omega^3_7(M) &= \{ *(\alpha \wedge \varphi) \mid \alpha \in \Omega^1(M) \}, \\ \Omega^3_{27}(M) &= \{ \gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0, \gamma \wedge *\varphi = 0 \}. \end{aligned}$$

With this G_2 decomposition, Bryant gave in [24] a description of $d\varphi$ and $d * \varphi$.

Proposition 1.1.13 (Proposition 1, [24])**.** *Let* (M, g, P) *be a* G_2 *manifold with* G_2 *form* φ*. Then, the forms* $d\varphi$ *and* $d * \varphi$ *are such that*

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \\ d * \varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi, \end{aligned}$$

where $\tau_0 \in \mathcal{F}(M), \tau_1 \in \Omega^1(M), \tau_2 \in \Omega^2_{14}(M)$ *and* $\tau_3 \in \Omega^3_{27}(M)$.

Definition 1.1.14. *The forms* τ_0, τ_1, τ_2 *and* τ_3 *are called the* torsion forms of the G_2-structure.

Remark 1.1.15. *We denote the index of the torsion forms according to its degree. However, several authors write the index of the torsion forms according to the dimension of the invariant subspace they belong to. Thus, the torsion forms* τ_0, τ_1, τ_2 *and* τ_3 *are sometimes denoted in the related literature as* $\tau_1, \tau_7, \tau_{14}$ *and* τ_{27}*, respectively.*

From Proposition 1.1.13 the classes of G_2 manifolds mentioned before, can also be described in terms of the torsion forms as follows:

- (g, P, φ) is *almost parallel* or *calibrated* G_2*-structure* iff $\tau_0 = \tau_1 = \tau_3 = 0$;
- (g, P, φ) is *semiparallel* or *cocalibrated* G_2*-structure* iff $\tau_1 = \tau_2 = 0$;
- (g, P, φ) is *nearly parallel* iff $\tau_1 = \tau_2 = \tau_3 = 0$;
- (g, P, φ) is *parallel* iff $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$.

Now, we can characterize parallel G_2-structures as follows.

Proposition 1.1.16 [24, 56]**.** *Let* (M, g) *be a 7-dimensional Riemannian manifold with a* G_2 *form* φ*. Denote by* ∇ *the Levi-Civita connection of* g*. Then, the following conditions are equivalent:*

1. $Hol(\nabla) \subseteq \mathrm{G}_2$;
2. $\nabla\varphi = 0$;
3. $d\varphi = d * \varphi = 0$;
4. $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$.

1.1.3 Almost contact metric structures

We first recall some properties of almost contact metric structures, and then we see the existence of such a structure on any G_2 manifold with a non-zero vector field [6].

Definition 1.1.17. *Let* M *be a* $(2n+1)$*-dimensional manifold. An* almost contact metric structure *on* M *is a quadruplet* (ϕ, ξ, η, g)*, where* $\phi\colon \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ *is a tensor field on* M*,* ξ *is a nowhere vanishing vector field,* η *is a differential* 1*-form, and* g *is a Riemannian metric on* M *satisfying the conditions*

$$\phi^2 = -Id + \xi \otimes \eta, \qquad \eta(\xi) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (1.13)$$

for $X, Y \in \mathfrak{X}(M)$*. The vector field* ξ *is called* Reeb vector field*, and* (M, ϕ, η, ξ, g) *is said to be* almost contact metric manifold*.*

Thus, if (M, ϕ, η, ξ, g) is an almost contact metric manifold, the kernel of η defines a codimension one distribution $\mathcal{H} = \ker \eta$, and there is the orthogonal decomposition of the tangent bundle TM of M

$$TM = \mathcal{H} \oplus \mathcal{L}\,,$$

where $\mathcal{L}$ is the trivial line subbundle generated by ξ. Note that conditions (1.13) imply

$$\phi(\xi) \,=\, 0\,, \quad \eta \circ \phi \,=\, 0\,.$$

If (ϕ, ξ, η, g) is an almost contact metric structure on M, the *fundamental 2-form* F of (ϕ, ξ, η, g) is the differential 2-form on M defined by

$$F(X,Y) \,=\, g(\phi X, Y)\,,$$

where $X, Y \in \mathfrak{X}(M)$. Hence,

$$F(\phi X, \phi Y) = F(X,Y),$$

that is, F is compatible with ϕ, and $\eta \wedge F^n \neq 0$ everywhere. Therefore, $\eta \wedge F^n$ is a volume form.

Conversely (see [17, 18]), as for symplectic manifolds, it happens that if M is a $(2n+1)$-dimensional manifold with a 2-form F and a 1-form η such that $\eta \wedge F^n$ is a volume form on M, then there exists an almost contact metric structure (ϕ, ξ, η, g) on M whose fundamental form is F. In fact, since $\eta \wedge F^n$ is a volume form, it defines an isomorphism between the $\mathcal{F}(M)$-module of the vector fields on M and the $\mathcal{F}(M)$-module of the $2n$-forms on M. Thus, corresponding to the $2n$-form F^n, there exists a unique vector field ξ on M such that

$$\iota_\xi(\eta \wedge F^n) = F^n,$$

Therefore,

$$\iota_\xi F^n = 0,$$

which implies that

$$\iota_\xi(\eta) = 1, \quad \iota_\xi F = 0.$$

Then, F is a symplectic form on the distribution $\mathcal{H} = \ker(\eta)$. Thus, proceeding as in the case of symplectic manifolds, there exists an endomorphism $\phi_{\mathcal{H}} \colon \mathcal{H} \longrightarrow \mathcal{H}$ and a metric $g_{\mathcal{H}}$ on $\mathcal{H}$ such that

$$\phi_{\mathcal{H}}^2 = -Id_{\mathcal{H}},$$

and

$$F(X,Y) = g_{\mathcal{H}}(\phi_{\mathcal{H}} X, Y),$$

on $\mathcal{H}$. Now we define the metric g on M and the endomorphism $\phi\colon TM \longrightarrow TM$ by

$$g(X,Y) = g_{\mathcal{H}}(X,Y), \quad g(X,\xi) = 0, \quad g(\xi,\xi) = 1,$$

and

$$\phi(X) = \phi_{\mathcal{H}}(X), \quad \phi(\xi) = 0.$$

It is easy to check that (g,ϕ,η,ξ) is an almost contact metric structure on M with fundamental form F, that is,

$$F(X,Y) = g(\phi X, Y),$$

for any vector fields X and Y on M.

An almost contact metric structure (ϕ,ξ,η,g) on M is said to be *contact metric* if

$$g(\phi X, Y) \ = \ d\eta(X,Y)\,.$$

In this case η is a *contact form*, meaning $\eta\wedge(d\eta)^n \neq 0$ at every point of M, and the fundamental form F of (ϕ,ξ,η,g) is $F = d\eta$.

Just as in the case of an almost Hermitian structure, there is the notion of *integrability of an almost contact metric structure*. More precisely, an almost contact metric structure (ϕ,ξ,η,g) is called *normal* if the Nijenhuis tensor N_ϕ associated to the tensor field ϕ, defined by

$$N_\phi(X,Y) \ = \ \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X, Y] - \phi[X,\phi Y]$$

satisfies the condition

$$N_\phi \ = \ -d\eta\otimes\xi\,.$$

This last condition is equivalent to the condition that the almost complex structure J on $M\times\mathbb{R}$ given by

$$J\left(X, f\frac{\partial}{\partial t}\right) \ = \ \left(\phi X - f\xi, \eta(X)\frac{\partial}{\partial t}\right)$$

is integrable, that is, $(M\times\mathbb{R}, J)$ is a complex manifold, where f is a differentiable function on $M\times\mathbb{R}$, and t is the coordinate on $\mathbb{R}$ [17, 18]. In other words, ϕ defines a complex structure on $\ker(\eta)$ compatible with $d\eta$.

A *Sasakian structure* is a normal contact metric structure, in other words, an almost contact metric structure (η,ξ,ϕ,g) such that

$$N_\Phi = -d\eta\otimes\xi, \quad d\eta = F.$$

If (η,ξ,ϕ,g) is a Sasakian structure on M, then (M,η,ξ,ϕ,g) is called a *Sasakian manifold*.

Boyer and Galicki in [22] prove that Riemannian manifolds with a Sasakian structure can also be characterized in terms of the Riemannian cone over the manifold. More precisely, a Riemannian manifold (M, g) admits a compatible Sasakian structure if and only if $M \times \mathbb{R}^+$ equipped with the cone metric

$$g^c = t^2 g + dt \otimes dt$$

is Kähler (but not necessarily Einstein). Furthermore, the Reeb vector field of the Sasakian structure is Killing and the covariant derivative of ϕ with respect to the Levi-Civita connection of g is given by

$$(\nabla_X \phi)(Y) \;=\; g(\xi, Y)X - g(X, Y)\xi\,,$$

for any pair of vector fields X and Y on M.

In Chapter 3, we need some properties of *3-Sasakian manifolds* [22]. Let (M, g) be a Riemannian manifold of dimension 4n+3.

- (M, g) is 3-Sasakian if $M \times \mathbb{R}^+$ with the cone metric $g^c = t^2 g + dt \otimes dt$ is hyperkähler or, equivalently, the holonomy group of $(M \times \mathbb{R}^+, g^c)$ is a subgroup of $\mathrm{Sp}(n+1)$. Thus, g^c is Ricci-flat.

Taking into account that cone metrics are a particular type of warped product metrics, we have

Proposition 1.1.18 [22]**.** *Any 3-Sasakian manifold* (M, g) *of dimension* $4n+3$ *is Einstein with Einstein constant* $\lambda = 4n+2$*. Moreover, if* (M, g) *is complete, it is compact with finite fundamental group.*

In terms of tensor fields, 3-Sasakian manifolds can be defined as follows [22]. A Riemannian manifold (M, g) is 3-Sasakian if and only if there exist 3 almost contact metric structures $(\phi_i, \xi_i, \eta_i, g)$ $(i = 1, 2, 3)$, with respect to same metric g, such that $(\phi_i, \xi_i, \eta_i, g)$ is Sasakian, for any $i = 1, 2, 3$, and

$$\begin{aligned} &\phi_i \xi_j = -\phi_j \xi_i = \xi_k, \quad \eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k, \\ &\phi_k = \phi_i \phi_j - \eta_j \otimes \xi_i = -\phi_j \phi_i + \eta_i \otimes \xi_j, \end{aligned}$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$.

By analogy with the terminology used in almost Hermitian geometry, we say that:

- (ϕ, ξ, η, g) is *almost cokähler* iff $dF = d\eta = 0$;

- (ϕ, ξ, η, g) is *cokähler* iff it is almost cokähler and normal, that is, $dF = d\eta = 0$ and $N_\phi = -d\eta \otimes \xi$,

where F is the fundamental 2-form of (ϕ, ξ, η, g).

If (N, g, J) is an almost Kähler manifold with Kähler form ω, then the Riemannian product $N \times \mathbb{R}$ has an almost cokähler structure (ϕ, ξ, η, h), where

$$\eta = dt, \quad \xi = \frac{\partial}{\partial t}, \quad \phi(X) = JX, \quad \phi(\xi) = 0,$$

and

$$h(X, Y) = g(X, Y), \quad h(X, \xi) = 0, \quad h(\xi, \xi) = 1,$$

for any vector fields X and Y on N, and where t is the coordinate on $\mathbb{R}$.

Conversely, if (M, ϕ, ξ, η, g) is an almost cokähler manifold, then the Riemannian product $M \times \mathbb{R}$ (or $M \times S^1$) is an almost Kähler manifold (in particular, Kähler if (ϕ, ξ, η, g) is a cokähler structure) with Kähler form

$$\omega = F + \eta \wedge dt,$$

and (M, η, F) is a *cosymplectic* manifold in the sense of Libermann [97] since $d\eta = dF = 0$ and $\eta \wedge F^n$ is a volume form of M. Cosymplectic manifolds are also called *almost cosymplectic* [18], and recently were called *co-symplectic* by Li [96].

The following result shows that cosymplectic manifolds constitute the odd dimensional analogue of symplectic manifolds.

Proposition 1.1.19 (Proposition 1, [96])**.** *A manifold M admits a cosymplectic structure if and only if the product $M \times S^1$ admits an S^1-invariant symplectic form.*

Before going to the topological characterization of compact cosymplectic manifolds given by Li in [96], we would like to note the following. Suppose that (M, φ) is a G_2 manifold with a nowhere vanishing vector field ξ on M. (This happens for example if M is compact [118].) Denote by g_φ the Riemannian metric induced by φ. By normalizing ξ using g_φ, we may assume that $||\xi|| = 1$. Then, in [6] it is proved that M has an almost contact metric structure (η, F, g_φ) defined by

$$\eta(X) = g_\varphi(X, \xi), \qquad F = \iota_\xi\, \varphi, \tag{1.14}$$

for $X \in \mathfrak{X}(M)$. In [6] it is proved that if M is compact and $d\varphi = 0$, then the almost contact metric structure given by (1.14) does not define a contact structure on M compatible with the closed G_2 form φ, that is, such that $d\eta = \iota_\xi \varphi$. Also in [121] it is proved the following result

Proposition 1.1.20 [121]**.** *Let (M, φ) be a G_2 manifold. Then the fundamental 2-form F of the almost contact metric structure given by* (1.14) *is closed if and only if ξ is Killing. In this case, $d\eta = 0$ and the almost contact metric structure is normal and hence a cokähler structure.*

Corollary 1.1.21 [121]. *If (M,φ) is a G_2 manifold with full G_2 holonomy, then $dF \neq 0$. In particular, the almost contact metric structure constructed above cannot be almost cokähler, cokähler nor Sasakian.*

In the last years, the geometry and topology of cokähler and almost cokähler manifolds and, in particular, of cosymplectic manifolds have been studied by several authors (see for example [8, 10, 12, 18, 22, 31, 30, 96] and the references therein). Some of those results are given using mapping tori.

A theorem by Tischler [120] asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form. This result was extended recently to cosymplectic manifolds by Li [96]. Let us recall first some definitions.

Let N be a differentiable manifold and let $\nu : N \to N$ be a diffeomorphism. Then, we can define the diffeomorphism

$$\begin{array}{rccc} \widetilde{\nu}: & N \times \mathbb{R} & \longrightarrow & N \times \mathbb{R} \\ & (x,t) & \longmapsto & \big(\nu(x), t+1\big). \end{array}$$

Denote by $\Gamma_{\widetilde{\nu}}$ the infinite cyclic group of diffeomorphisms of $N \times \mathbb{R}$ generated by $\widetilde{\nu}$. Then, $\Gamma_{\widetilde{\nu}}$ is a discrete group which acts freely and proper discontinuously on $N \times \mathbb{R}$. The *mapping torus N_ν of ν* is the quotient

$$N_\nu = (N \times \mathbb{R})/\Gamma_{\widetilde{\nu}}. \tag{1.15}$$

Thus, N_ν is a differentiable manifold of dimension $\dim N_\nu = (\dim N) + 1$. From the topological point of view, the mapping torus N_ν is just $N \times [0,1]$ with the ends identified by ν, that is,

$$N_\nu = \frac{N \times [0,1]}{(x,0) \sim (\nu(x),1)}.$$

The natural map

$$\pi \colon N_\nu \to S^1,$$

defined by $\pi(x,t) = e^{2\pi i t}$ is the projection of a locally trivial fiber bundle with fiber N.

Definition 1.1.22. *Let (N,ω) be a symplectic manifold, and let $\nu : N \longrightarrow N$ be a diffeomorphism. We say that the mapping torus N_ν of ν is a* symplectic mapping torus *if ν is a symplectomorphism, that is, $\nu^*\omega = \omega$.*

Remark 1.1.23. *We would like to note that if (N,ω) is a symplectic manifold (not necessarily compact), and $\nu : (N,\omega) \longrightarrow (N,\omega)$ is a symplectomorphism, we can equip N_ν with a cosymplectic structure as follows. First, we consider the pullback to $N \times \mathbb{R}$ of the symplectic form ω via the projection map $N \times \mathbb{R} \longrightarrow N$, and then we get a closed 2-form ω on $N \times \mathbb{R}$. Since ν is a symplectomorphism,* *we know that ω is ν-invariant, and so its pullback to $N \times \mathbb{R}$ is a closed 2-form ω, which is $\Gamma_{\widetilde{\nu}}$-invariant. Hence, it induces a closed 2-form $\widetilde{\omega}$ on N_ν. Similarly, for the canonical 1-form dt on $\mathbb{R}$ we consider its pullback to $N \times \mathbb{R}$, and it is also $\Gamma_{\widetilde{\nu}}$-invariant, so dt defines a closed 1-form η on N_ν. The pair (dt, ω) is a cosymplectic structure on $N \times \mathbb{R}$ and thus $(\eta, \widetilde{\omega})$ is a cosymplectic structure on N_ν. Note also that the Reeb vector field on $N \times \mathbb{R}$ is just $\frac{\partial}{\partial t}$, which is preserved by $\widetilde{\nu}$, and hence reduces to be the Reeb vector field on N_ν.*

For compact manifolds, Li in [96] proves the following result.

Theorem 1.1.24 (Theorem 1, [96])**.** *A compact manifold M admits a cosymplectic structure if and only if it is a symplectic mapping torus $M = N_\nu$.*

Recently, Bazzoni and Goertsches in [10] show conditions under which the Reeb vector field of a symplectic mapping torus is Killing.

Proposition 1.1.25 (Proposition 2.12, [10])**.** *Let (N, g, J) be an almost Kähler manifold with Kähler form ω, and let $\nu : (N, \omega) \longrightarrow (N, \omega)$ a symplectomorphism such that $\nu^* g = g$. Then, the Reeb vector field of the symplectic mapping torus N_ν is a Killing vector field.*

1.2 Massey products and formality of mapping tori

In this section some definitions and results about formal manifolds and, in particular, about formality of mapping tori are reviewed [8, 45].

From now on, we work with graded algebras over the field of real numbers $\mathbb{R}$, and we denote by $|a|$ the degree of an element.

A *differential graded commutative algebra* $(\mathcal{A}, d)$ over $\mathbb{R}$ (DGA for short) is a pair $(\mathcal{A}, d)$, where $\mathcal{A}$ is a graded commutative algebra $\mathcal{A} = \oplus_{i \geq 0} A^i$ over $\mathbb{R}$, and $d\colon A^* \to A^{*+1}$ is a derivation of degree 1, that is, d is a linear map such that $d^2 = 0$ and, for homogeneous elements a and b,

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db).$$

Given a differential graded commutative algebra $(\mathcal{A},\, d)$, we denote its cohomology by $H^*(\mathcal{A})$. The cohomology of a differential graded algebra $H^*(\mathcal{A})$ is naturally a DGA with the product inherited from that on $\mathcal{A}$ and with the differential being identically zero. The DGA $(\mathcal{A},\, d)$ is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$, and $(\mathcal{A},\, d)$ is 1-*connected* if, in addition, $H^1(\mathcal{A}) = 0$.

In our context, the main examples of DGAs are the de Rham complex $(\Omega^*(M),\, d)$ of a differentiable manifold M, where d is the exterior differential, and the de Rham cohomology algebra $(H^*(M), d = 0)$.

If $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ are DGAs, a map

$$\phi : (\mathcal{A}, d_{\mathcal{A}}) \longrightarrow (\mathcal{B}, d_{\mathcal{B}}),$$

is called *morphism of DGA's* if ϕ is a morphism of algebras such that preserves the degree and commutes with the differential, so $\phi \circ d_{\mathcal{A}} = d_{\mathcal{B}} \circ \phi$.

Definition 1.2.1. *A DGA $(\mathcal{A}, d)$ is said to be* minimal *if*

- *$\mathcal{A}$ is the free algebra $\mathcal{A} = \bigwedge V$ over a graded (real) vector space $V = \oplus_k V^k$; and*
- *there exists a basis $\{x_i, i \in I\}$ of V, for a well-ordered index set I, such that $|x_i| \leq |x_j|$ if $i < j$, and each dx_j is expressed in terms of the preceding x_i $(i < j)$.*

Note that the second condition implies that dx_i does not have a linear part.

Definition 1.2.2. *Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{A}, d)$ be two DGA's. We say that $(\mathcal{M}, d_{\mathcal{M}})$ is a* minimal model *of $(\mathcal{A}, d)$ if $(\mathcal{M}, d_{\mathcal{M}})$ is minimal, so $\mathcal{M} = \bigwedge V$, and there exists a morphism*

$$\rho : (\mathcal{M}, d_{\mathcal{M}}) \longrightarrow (\mathcal{A}, d),$$

of DGAs, such that it induces an isomorphism in cohomology

$$\rho^* : H^*(\mathcal{M}) \xrightarrow{\cong} H^*(\mathcal{A}).$$

In [77], Halperin proved that any connected differential graded algebra $(\mathcal{A},\, d)$ has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved earlier by Deligne, Griffiths, Morgan and Sullivan [45].

Definition 1.2.3. *A* minimal model *of a connected differentiable manifold M is a minimal model $(\bigwedge V,\, d)$ of the de Rham complex $(\Omega^*(M),\, d)$ of differential forms on M.*

Thus, if $(\bigwedge V,\, d)$ is a minimal model of a manifold M, then the cohomology of $(\bigwedge V,\, d)$ is, up to isomorphism, the cohomology of M. Moreover, if M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i. This relation also happens when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for $j > 1$ (see [45]).

Definition 1.2.4. *A minimal differential algebra* $(\bigwedge V,\, d)$ *is called* formal *if there exists a morphism of differential algebras*

$$\psi\colon \big(\bigwedge V,\, d\big) \longrightarrow \big(H^*(\bigwedge V), 0\big),$$

inducing the identity map on cohomology. Also a differentiable manifold M *is called* formal *if its minimal model is formal.*

Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, homogeneous spaces, flag manifolds, and compact Kähler manifolds.

The formality of a minimal algebra is characterized as follows.

Theorem 1.2.5 [45]**.** *A minimal algebra* $(\bigwedge V,\, d)$ *is formal if and only if the space* V *can be decomposed into a direct sum* $V = C \oplus N$ *with* $d(C) = 0$ *and* d *injective on* N*, such that every closed element in the ideal* $I(N)$ *in* $\bigwedge V$ *generated by* N *is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [63].

Definition 1.2.6. *A minimal algebra* $(\bigwedge V,\, d)$ *is* s*-formal* ($s > 0$) *if for each* $i \leq s$ *the space* V^i *of generators of degree* i *decomposes as a direct sum* $V^i = C^i \oplus N^i$*, where the spaces* C^i *and* N^i *satisfy the three following conditions:*

1. $d(C^i) = 0$,
2. *the differential map* $d\colon N^i \longrightarrow \bigwedge V$ *is injective, and*
3. *any closed element in the ideal* $I_s = I(\bigoplus_{i\leq s} N^i)$*, generated by the space* $\bigoplus_{i\leq s} N^i$ *in the free algebra* $\bigwedge(\bigoplus_{i\leq s} V^i)$*, is exact in* $\bigwedge V$*.*

A differentiable manifold M is *s-formal* if its minimal model is s-formal. Clearly, if M is formal then M is s-formal, for any $s > 0$. The main result of [63] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 1.2.7 [63]**.** *Let* M *be a connected and orientable compact differentiable manifold of dimension* $2n$ *or* $(2n-1)$*. Then* M *is formal if and only if it is* $(n-1)$*-formal.*

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 1.2.7 implies that any simply connected compact manifold of dimension ≤ 6 is formal.

In order to detect non-formality, one can use also Massey products, which are obstructions to formality. The simplest type of Massey products are the triple (also known as ordinary) Massey products, which are defined as follows.

Let $(\mathcal{A}, d)$ be a DGA (in particular, it can be the de Rham complex of differential forms on a differentiable manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(\mathcal{A})$, $p_i > 0$, $1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write

$$a_1 \cdot a_2 = dx, \qquad a_2 \cdot a_3 = dy. \tag{1.16}$$

The *(triple) Massey product* of the classes $[a_i]$ is defined as

$$\langle [a_1], [a_2], [a_3] \rangle = [a_1 \cdot y + (-1)^{p_1+1} x \cdot a_3] \in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}.$$

Note that $a_1 \cdot y + (-1)^{p_1+1} x \cdot a_3$ is always closed, because $d(a_1 \cdot y + (-1)^{p_1+1} x \cdot a_3) = (-1)^{p_1} a_1 \cdot a_2 \cdot a_3 + (-1)^{p_1+1} a_1 \cdot a_2 \cdot a_3 = 0$, but its cohomology class in $H^{p_1+p_2+p_3-1}(\mathcal{A})$ is not well-defined since it depends of the representatives x and y in (1.16).

Theorem 1.2.8 [45]. *A DGA which has a non-zero Massey product is non-formal.*

To finish this section we recall the results proved in [8] about the formality of the mapping torus of an orientation-preserving diffeomorphism ν of a manifold N. Those results show that the formality of N_ν depends on ν but not on the formality of N.

Next, by *multiplicity* of the eigenvalue λ of an endomorphism $A : V \to V$ we mean the multiplicity of λ as a root of the minimal polynomial of A.

First we notice that if $\nu : N \longrightarrow N$ is a diffeomorphism, the Mayer-Vietoris sequence implies that the de Rham cohomology group of the mapping torus N_ν of ν is, up to isomorphism,

$$H^p(N_\nu) \cong K^p(N) \oplus [dt] \wedge C^{p-1}(N), \tag{1.17}$$

where

$$K^p(N) = \ker\Big(\nu^* - Id : H^p(N) \to H^p(N)\Big),$$
$$C^{p-1}(N) = \mathrm{coker}\Big(\nu^* - Id : H^{p-1}(N) \to H^{p-1}(N)\Big).$$

As it is described in [8], $[\alpha] \in K^p(N)$ defines a cohomology class $[\widetilde{\alpha}] \in H^p(N_\nu)$. In fact, if $[\alpha] \in K^p$, then $\nu^*[\alpha] = [\alpha]$. Therefore, $\nu^*\alpha = \alpha + d\beta$ for some $(p-1)$-form β on N. Consider now a function $\rho(t)$, such that $\rho = 0$ near $t = 0$ and $\rho = 1$ near $t = 1$. Thus, the closed p-form $\tilde{\alpha}$ on $N \times [0, 1]$ defined by

$$\tilde{\alpha}(x, t) = \alpha(x) + d(\rho(t)\beta(x)),$$

where $x \in N$ and $t \in [0,1]$ is such that

$$\nu^*\tilde{\alpha}(x,0) = \nu^*\alpha(x) = \alpha(x) + d\beta(x) = \tilde{\alpha}(x,1),$$

and thus defines a closed p-form on the mapping torus N_ν.

Theorem 1.2.9 (Theorem 13, [8]). *Let N be an oriented compact differentiable manifold of dimension n, and let $\nu : N \to N$ be an orientation-preserving diffeomorphism. Let $M = N_\nu$ be the mapping torus of ν. Suppose that, for some $p > 0$, the homomorphism $\nu^* : H^p(N) \to H^p(N)$ has eigenvalue $\lambda = 1$ with multiplicity 2. Then, $M = N_\nu$ is non-formal since there exists a non-zero (triple) Massey product. More precisely, if $[\alpha] \in K^p \subset H^p(N)$ is such that*

$$[\alpha] \in \mathrm{Im}\Big(\nu^* - Id : H^p(N) \to H^p(N)\Big),$$

the Massey product $\langle[\eta],[\eta],[\tilde{\alpha}]\rangle$ does not vanish.

Also in [8], under certain conditions of ν, a partial computation of the minimal model of N_ν is given.

Theorem 1.2.10. *With $M = N_\nu$ as above, suppose that there is some $p \geq 2$ such that $\nu^* : H^k(N) \to H^k(N)$ does not have the eigenvalue $\lambda = 1$ for any $k \leq (p-1)$, and that $\nu^* : H^p(N) \to H^p(N)$ does have the eigenvalue $\lambda = 1$ with some multiplicity $r \geq 1$. Put*

$$K_j = \ker\Big((\nu^* - Id)^j : H^p(N) \to H^p(N)\Big),$$

where $j = 0, \ldots, r$. So $\{0\} = K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_r$.

Write $G_j = K_j/K_{j-1}$, for $j = 1, \ldots, r$. The map $F = \nu^* - Id$ induces maps $F : G_j \to G_{j-1}$, $j = 1, \ldots, r$ (here $G_0 = 0$). Then the minimal model $(\Lambda W, d)$ of $M = N_\nu$ is, up to degree p, given by the following generators:

$$\begin{aligned}
&W^1 = \langle a\rangle, \qquad da = 0,\\
&W^k = 0, \qquad k = 2, \ldots, p-1,\\
&W^p = G_1 \oplus G_2 \oplus \ldots \oplus G_r, \qquad dw = a \cdot F(w),\ w \in G_j.
\end{aligned}$$

Taking into account Definition 1.2.6, Theorem 1.2.10 implies that the spaces C^i and N^i $(i = 1, \ldots, p)$ of the minimal model $(\bigwedge W,\, d)$ of N_ν are

$$\begin{aligned}
&C^1 = \langle a\rangle, \qquad N^1 = 0, \qquad C^i = 0 = N^i \quad (i = 2, \ldots, p-1),\\
&C^p = G_1, \qquad N^p = G_2 \oplus \ldots \oplus G_r.
\end{aligned}$$

Also if $w \in G_r$, then $a \cdot w \in I(N)$ and $d\,(a \cdot w) = 0$, but $a \cdot w$ is not exact. Thus,

Corollary 1.2.11. *Under the conditions of Theorem 1.2.10, the mapping torus N_ν is always $(p-1)$-formal (in the sense of Definition 1.2.6), and if $r=1$, then N_ν is p-formal. Moreover, if $r \geq 2$, then N_ν is non-formal.*

Note that if N is a compact symplectic manifold, and $\nu : N \to N$ is a symplectomorphism, then $\nu^* : H^2(N) \to H^2(N)$ always has the eigenvalue $\lambda = 1$, since $\nu^* : H^2(N) \to H^2(N)$ fixes the symplectic form on N. Thus, as a consequence of Theorem 1.2.10, we have

Corollary 1.2.12. *Let N be a compact symplectic $2n$-manifold, and assume that $\nu : N \to N$ is a symplectomorphism such that the map induced on cohomology $\nu^* : H^1(N) \to H^1(N)$ does not have the eigenvalue $\lambda = 1$. Then, N_ν is 2-formal if and only if the eigenvalue $\lambda = 1$ of $\nu^* : H^2(N) \to H^2(N)$ has multiplicity $r = 1$.*

We will need also the following

Lemma 1.2.13. *Let N be a compact manifold, and let $\nu : N \longrightarrow N$ be a diffeomorphism of finite order, that is, there exists $p \in \mathbb{N}$ such that $(\nu)^p = Id$. If there exists $k \geq 1$, such that $\lambda = 1$ is an eigenvalue of*

$$\nu^* : H^k(N) \longrightarrow H^k(N),$$

then the multiplicity of $\lambda = 1$ is $r = 1$.

Proof. Suppose that, for some $k \geq 1$, the map $\nu^* : H^k(N) \longrightarrow H^k(N)$ has the eigenvalue $\lambda = 1$ with multiplicity $r > 1$. Then, there exists a non-zero cohomology class $a \in H^k(N)$ such that

$$a \in \ker\big(\nu^* - Id : H^k(N) \longrightarrow H^k(N)\big),$$

and

$$a = (\nu^* - Id)(b),$$

where $b \in \ker\big((\nu^*-Id)^2 : H^k(N) \longrightarrow H^k(N)\big) - \ker\big(\nu^* - Id : H^k(N) \longrightarrow H^k(N)\big)$.

Since the order of ν is finite, there exists $p \in \mathbb{N}$ such that $(\nu^*)^p = Id$. Now let us consider the map $T : H^k(N) \longrightarrow H^k(N)$ given by

$$T = Id + \nu^* + (\nu^*)^2 + \cdots + (\nu^*)^{p-1}.$$

Then,

$$T(a) = p\,a. \tag{1.18}$$

On the other hand, we obtain

$$\begin{aligned} T(a) = T\big((\nu^* - Id)(b)\big) &= \big(Id + \nu^* + (\nu^*)^2 + \cdots + (\nu^*)^{p-1}\big)(\nu^* - Id)(b) \\ &= \big((\nu^*)^p - Id\big)(b) = 0, \end{aligned}$$

which contradicts (1.18) since $p \geq 1$ and the cohomology class a is non-zero. Hence

$$\ker(\nu^* - Id) \cap \operatorname{Im}(\nu^* - Id) = 0.$$

□

1.3 Mapping torus of an SU(3)-diffeomorphism

As we mention in subsection 1.1.2, the Riemannian product of a symplectic half-flat SU(3) manifold with $\mathbb{R}$ has a closed G_2 form. In this section, we show conditions under which the mapping torus of a diffeomorphism of a symplectic half-flat SU(3) manifold has such a G_2 form. Moreover, we show that if a 6-dimensional symplectic half-flat Lie algebra is endowed with a particular type of derivation, then one can construct a Lie algebra with a closed G_2 form.

Definition 1.3.1. *Let* $(N, g, J, \Psi = \psi_+ + i\,\psi_-)$ *be an* SU(3) *manifold, and let* $\nu : N \longrightarrow N$ *be a diffeomorphism. We say that* $\nu : N \longrightarrow N$ *is an* SU(3)-diffeomorphism *if* ν *preserves the* SU(3)*-structure, that is*

$$\nu^* g = g, \qquad \nu_* \circ J = J \circ \nu_*, \qquad \nu^*(\psi_+) = \psi_+.$$

Thus, ν also preserves the Kähler form ω of (g, J) and the 3-form $\psi_- = J\,\psi_+$, that is,

$$\nu^*\,\omega = \omega, \qquad \nu^*\,\psi_- = \psi_-\,.$$

Hence, if ω is closed, N_ν is a 7-dimensional cosymplectic manifold since ν is a symplectomorphism (see Remark 1.1.23). Moreover, we have

Theorem 1.3.2. *Let* $(N, g, J, \Psi = \psi_+ + i\,\psi_-)$ *be a symplectic half-flat* SU(3) *manifold, and let* $\nu : N \longrightarrow N$ *be an* SU(3)*-diffeomorphism. Then, the symplectic half-flat* SU(3)*-structure* $(g, J, \Psi = \psi_+ + i\,\psi_-)$ *on* N *induces a closed* G_2 *form on the mapping torus* N_ν *of* ν.

Proof. Since $(N, g, J, \Psi = \psi_+ + i\,\psi_-)$ is a symplectic half-flat SU(3) manifold, we know that the 3-form φ on $N \times \mathbb{R}$ given by

$$\varphi = \psi_+ + F \wedge dt \tag{1.19}$$

is a closed G_2 form, where F is the Kähler form (now, symplectic form) of (g, J) and t is the coordinate of $\mathbb{R}$. The form φ determines the metric $h = g + (dt)^2$ on $N \times \mathbb{R}$. Moreover, φ and h are both $\Gamma_{\widetilde{\nu}}$-invariant. So, taking into account (1.15), they define a closed 3-form $\widetilde{\varphi}$ and a Riemannian metric $\widetilde{h}$ on N_ν, respectively, where $\widetilde{h}$ is the metric induced by $\widetilde{\varphi}$. The local expression of $\widetilde{\varphi}$ is as in (1.5) because locally φ has such an expression with respect to some local orthonormal basis of 1-forms on $N \times \mathbb{R}$. Hence $\widetilde{\varphi}$ is a closed G_2 form. □

We would like to note that *the converse of the previous Theorem is not true even if we assume that* N_ν *is compact. For this, it is sufficient to show an example of a compact calibrated* G_2 *manifold which does not admit cosymplectic structures.*

Such an example is given as follows. Let us consider the 7-dimensional nilpotent Lie algebra $\mathfrak{g}$ whose dual space is spanned by $\{e^1, \ldots, e^7\}$ such that

$$\begin{aligned} &de^1 = 0\,, \quad de^2 = 0\,, \quad de^3 = e^{12}\,, \quad de^4 = e^{13}\,, \quad de^5 = e^{23}\,, \\ &de^6 = e^{15} + e^{24}\,, \quad de^7 = e^{16} + e^{34}\,. \end{aligned} \tag{1.20}$$

Mal'cev Theorem [102] implies that the simply connected Lie group G associated with $\mathfrak{g}$ has a uniform discrete subgroup Γ, so that

$$M = \Gamma \backslash G$$

is a compact nilmanifold.

Proposition 1.3.3. *The compact manifold* $M = \Gamma\backslash G$ *given by* (1.20) *has a closed* G_2 *form but does not carry cosymplectic structures.*

Proof. Using Theorem of Nomizu (Theorem 1.1.8) we have that the real cohomology groups of M are

$$\begin{aligned}
H^0(M) &= \langle 1 \rangle, \\
H^1(M) &= \langle [e^1], [e^2] \rangle, \\
H^2(M) &= \langle [e^{14}], [e^{25}], [e^{26} - e^{35}] \rangle, \\
H^3(M) &= \langle [e^{456} - e^{357} + e^{267}], [-e^{345} + e^{246} - e^{237}], [e^{256}], [e^{147}], [e^{126}], [e^{135}], [e^{137}] \rangle, \\
H^4(M) &= \langle [e^{2356}], [e^{2357} - e^{2456}], [e^{1467}], [e^{1347}], [e^{1357}], [e^{1237}], [e^{1247}] \rangle, \\
H^5(M) &= \langle [e^{12467}], [e^{13467}], [e^{23567}] \rangle, \\
H^6(M) &= \langle [e^{134567}], [e^{234567}] \rangle, \\
H^7(M) &= \langle [e^{1234567}] \rangle.
\end{aligned}$$

By [38] we know that a closed G_2 form φ on $M = \Gamma\backslash G$ is given by

$$\varphi = e^{127} + e^{147} + e^{125} + e^{136} + e^{234} - e^{256} - e^{456} - e^{267} + e^{357}.$$

Now, let us suppose that M has a cosymplectic structure (η, F). Then, η is a differential 1-form and F a differential 2-form on M such that $d\,\eta = d\,F = 0$ and $\eta \wedge F^3$ is a volume form. Hence η, F and $\eta \wedge F^3$ define non-zero cohomology classes on M. But

$$[\eta] = \lambda\,[e^1] + \mu\,[e^2], \qquad [F] = a\,[e^{14}] + b\,[e^{25}] + c\,[e^{26} - e^{35}],$$

for some real numbers λ, μ, a, b and c. Therefore,

$$[F^3] = -6\,a\,c^2\,e^{123456},$$

which implies that $[\eta \wedge F^3] = 0$, for any $a, b, c \in \mathbb{R}$, since $e^{123456} = de^{23457}$. This is not possible for a cosymplectic structure on a compact manifold. $\square$

For cocalibrated G_2 manifolds we have:

Theorem 1.3.4. *Let* $(N, g, J, \Psi = \psi_+ + i\,\psi_-)$ *be a half-flat* SU(3) *manifold, and let* $\nu : N \longrightarrow N$ *be an* SU(3)*-diffeomorphism. Then, the mapping torus* N_ν *of* ν *has a coclosed* G_2 *form.*

Proof. Since $(N, g, J, \Psi = \psi_+ + i\,\psi_-)$ is a half-flat SU(3) manifold, we know that the 3-form φ on $N \times \mathbb{R}$ given by

$$\varphi = F \wedge dt - \psi_-,$$

is such that

$$*\varphi = \frac{1}{2} F \wedge F + \psi_+ \wedge dt,$$

and therefore is a coclosed G_2 form, where F is the Kähler form of (g, J) and t is the coordinate of $\mathbb{R}$. The form φ determines the metric $h = g + (dt)^2$ on $N \times \mathbb{R}$. Moreover, φ and h are both $\Gamma_{\widetilde{\nu}}$-invariant. So, taking into account (1.15), they define a coclosed 3-form $\widetilde{\varphi}$ and a Riemannian metric $\widetilde{h}$ on N_ν, respectively, where $\widetilde{h}$ is the metric induced by $\widetilde{\varphi}$. The local expression of $\widetilde{\varphi}$ is as in (1.5) because locally φ has such an expression with respect to some local orthonormal basis of 1-forms on $N \times \mathbb{R}$. Hence $\widetilde{\varphi}$ is a coclosed G_2 form. $\square$

Next, we show that if a symplectic half-flat Lie algebra, of dimension 6, is endowed with a particular type of derivation, then one can construct a Lie algebra of dimension 7 with a closed G_2 form. For this, we use that if $\mathfrak{h}$ is a 6-dimensional Lie algebra, and D a derivation of $\mathfrak{h}$, the vector space

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$$

is a Lie algebra with the Lie bracket given by

$$[U, V] = [U, V]|_{\mathfrak{h}}, \qquad [\xi, U] = D(U), \tag{1.21}$$

for any $U, V \in \mathfrak{h}$.

We recall that a closed G_2 form on a real Lie algebra $\mathfrak{g}$ of dimension 7 is a closed 3-form φ on $\mathfrak{g}$ such that φ can be written as in (1.5) with respect to some basis $\{e^1, \dots, e^7\}$ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

On the one hand, we know that (see for example [14]) there exists a real representation of the complex matrices via

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(6, \mathbb{R}).$$

In fact, if $A \in \mathfrak{gl}(3, \mathbb{C})$, $\rho(A)$ is the matrix $(B_{ij})_{i,j=1}^3$ with

$$B_{ij} = \begin{pmatrix} ReA_{ij} & ImA_{ij} \\ -ImA_{ij} & ReA_{ij} \end{pmatrix},$$

where A_{ij} is the (i,j) component of A.

Let $(g, J, \Psi = \psi_+ + i\,\psi_-)$ be an SU(3)-structure on a 6-dimensional Lie algebra $\mathfrak{h}$. We know that the SU(3)-structure on $\mathfrak{h}$ guarantees the existence of a basis $\{e^1, \ldots, e^6\}$ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, such that the Kähler form ω of (g, J) and the 3-forms ψ_+ and ψ_- have the canonical expression (1.1), that is,

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}. \end{aligned} \tag{1.22}$$

Definition 1.3.5. *Let* $(g, J, \Psi = \psi_+ + i\,\psi_-)$ *be an* SU(3)*-structure on a 6-dimensional Lie algebra* $\mathfrak{h}$*, and let* ω *be the Kähler form of* (g, J)*. We say that a basis* $\{e_1, \ldots, e_6\}$ *of* $\mathfrak{h}$ *is an* SU(3)-basis *if* ω*,* ψ_+ *and* ψ_- *are expressed as in* (1.22) *with respect to the dual basis* $\{e^1, \ldots, e^6\}$ *of* $\mathfrak{h}^*$*.*

Now, suppose that $(g, J, \Psi = \psi_+ + i\,\psi_-)$ is a symplectic half-flat structure on a 6-dimensional Lie algebra $\mathfrak{h}$, and let D be a derivation of $\mathfrak{h}$ such that $D = \rho(A)$, where $A \in \mathfrak{sl}(3, \mathbb{C})$. Then, the matrix representation of D with respect to an SU(3)-basis (in the sense of Definition 1.3.5) $\{e_1, \ldots, e_6\}$ of $\mathfrak{h}$ is the following

$$D = \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & a_{11} & -a_{14} & a_{13} & -a_{16} & a_{15} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ -a_{32} & a_{31} & -a_{34} & a_{33} & -a_{36} & a_{35} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & -a_{11} - a_{33} & -a_{12} - a_{34} \\ -a_{52} & a_{51} & -a_{54} & a_{53} & a_{12} + a_{34} & -a_{11} - a_{33} \end{array}\right), \tag{1.23}$$

where $a_{ij} \in \mathbb{R}$.

Proposition 1.3.6. *Let* $(\mathfrak{h}, g, J, \Psi = \psi_+ + i\,\psi_-)$ *be a symplectic half-flat Lie algebra of dimension* 6*, and let* $D = \rho(A)$ $(A \in \mathfrak{sl}(3, \mathbb{C}))$ *be a derivation of* $\mathfrak{h}$ *whose matrix representation, with respect to an* SU(3)*-basis (in the sense of Definition 1.3.5)* $\{e_1, \ldots, e_6\}$ *of* $\mathfrak{h}$*, is as in* (1.23)*. Then, the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi,$$

with the Lie bracket given by (1.21)*, has a closed* G_2 *form.*

Proof. We define the G_2 form φ on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ by

$$\varphi = \omega \wedge \eta + \psi_+, \tag{1.24}$$

where η is the 1-form on $\mathfrak{g}$ such that $\eta(X) = 0$, for all $X \in \mathfrak{h}$, and $\eta(\xi) = 1$. Then, for $U, V, W, T \in \mathfrak{h}$ we have

$$d\varphi(U, V, W, T) = d\psi_+(U, V, W, T) = 0,$$

since ψ_+ is closed.
Now,

$$d\varphi(U,V,W,\xi) = -\varphi([U,V],W,\xi) + \varphi([U,W],V,\xi) - \varphi([U,\xi],V,W) \\ - \varphi([V,W],U,\xi) + \varphi([V,\xi],U,W) - \varphi([W,\xi],U,V),$$

which by definition of φ is

$$- \omega([U,V],W) + \omega([U,W],V) - \omega([V,W],U) - \psi_+([U,\xi],V,W) \\ + \psi_+([V,\xi],U,W) - \psi_+([W,\xi],U,V) = d\omega(U,V,W) + \psi_+(D(U),V,W) \\ + \psi_+(U,D(V),W) + \psi_+(U,V,D(W)).$$

Therefore, since ω is closed, using (1.21) we obtain

$$d\varphi(U,V,W,\xi) = \psi_+(D(U),V,W) + \psi_+(U,D(V),W) + \psi_+(U,V,D(W)).$$

Taking into account the expressions of D and ψ_+ in terms of the SU(3)-basis $\{e_1,\ldots,e_6\}$, we see that

$$\psi_+(D(e_i),e_j,e_k) + \psi_+(e_i,D(e_j),e_k) + \psi_+(e_i,e_j,D(e_k)) = 0,$$

for every triple (e_i,e_j,e_k) of elements of the basis $\{e_1,\ldots,e_6\}$ of $\mathfrak{h}$. For example, let us see that $\psi_+(D(e_1),e_2,e_3) + \psi_+(e_1,D(e_2),e_3) + \psi_+(e_1,e_2,D(e_3)) = 0$. We have

$$\begin{aligned}\psi_+(D(e_1),e_2,e_3) &+ \psi_+(e_1,D(e_2),e_3) + \psi_+(e_1,e_2,D(e_3)) \\ &= \psi_+(a_{11}e_1 - a_{12}e_2 + a_{31}e_3 - a_{32}e_4 + a_{51}e_5 - a_{52}e_6, e_2, e_3) \\ &+ \psi_+(e_1, a_{12}e_1 + a_{11}e_2 + a_{32}e_3 + a_{31}e_4 + a_{52}e_5 + a_{51}e_6, e_3) \\ &+ \psi_+(e_1, e_2, a_{13}e_1 - a_{14}e_2 + a_{33}e_3 - a_{34}e_4 + a_{53}e_5 - a_{54}e_6) \\ &= a_{52} - a_{52} = 0.\end{aligned}$$

Thus, the 3-form φ defined by (1.24) is a closed G_2 form on $\mathfrak{g}$. □

As an application of the previous Proposition, we show a *new example of a compact solvmanifold with a closed* G_2 *form.* Let $\mathfrak{h}$ be the 6-dimensional nilpotent Lie algebra defined by the equations

$$\mathfrak{h} = (0, e^{35}, 0, 2e^{15}, 0, e^{13}).$$

The almost Hermitian structure (g,J) on $\mathfrak{h}$ given by

$$g = \sum_{i=1}^{6} e^i \otimes e^i, \qquad Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\,\psi_-$, where

$$\begin{aligned} \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246} \end{aligned}$$

define a symplectic half-flat structure on $\mathfrak{h}$, since ω and ψ_+ are closed.

Consider now the derivation D of $\mathfrak{h}$ that with respect to the SU(3)-basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{h}$ has the following representation

$$\left(\begin{array}{cc|cc|cc} & & 1 & & & \\ & & & 1 & & \\ \hline 2 & & & & & \\ & 2 & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right),$$

that is,

$$D(e_1) = 2e_3, \quad D(e_2) = 2e_4, \quad D(e_3) = e_1, \quad D(e_4) = e_2, \quad D(e_5) = D(e_6) = 0.$$

Now, consider the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

which, according to (1.21), is defined by the equations

$$\mathfrak{g} = (e^{37}, e^{35} + e^{47}, 2e^{17}, 2e^{27} + 2e^{15}, 0, e^{13}, 0).$$

Then, Proposition 1.3.6 implies that the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

is a closed G_2 form on $\mathfrak{g}$. Indeed, using the equations defining $\mathfrak{g}$, one can check that φ is closed because $0 = d(e^{127} + e^{347} + e^{567}) = d(e^{135}) = d(e^{146} + e^{236}) = d(e^{245})$.

Let G be the simply connected solvable Lie group with Lie algebra $\mathfrak{g}$, and let H be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$. Denote by $e \in H$ the identity element. Note that $G = \mathbb{R} \ltimes_\phi H$, where ϕ is the unique action $\phi : \mathbb{R} \longrightarrow Aut(H)$ such that, for any $t \in \mathbb{R}$, the morphism $(\phi_t)_*|_e : \mathfrak{h} \longrightarrow \mathfrak{h}$ is given by

$$(\phi_t)_*|_e = \exp(tD),$$

where D is the derivation previously defined of the Lie algebra $\mathfrak{h}$ of H, and exp denotes the map $\exp : \mathrm{Der}(\mathfrak{h}) \to \mathrm{Aut}(\mathfrak{h})$.

In order to show that there exists a discrete subgroup Γ of G such that the quotient space Γ/G is compact we proceed as follows. The SU(3)-basis $\{e_1, \dots, e_6\}$ of $\mathfrak{h}$ is a rational basis for $\mathfrak{h}$ and, with respect to this basis, we have

$$\exp(tD) = \left(\begin{array}{cc|cc|cc} \cosh(\sqrt{2}t) & & \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & & & \\ & \cosh(\sqrt{2}t) & & \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & & \\ \hline \sqrt{2}\sinh(\sqrt{2}t) & & \cosh(\sqrt{2}t) & & & \\ & \sqrt{2}\sinh(\sqrt{2}t) & & \cosh(\sqrt{2}t) & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right).$$

In particular, if we consider $t_0 = \frac{\sqrt{2}}{2} \operatorname{arc\,cosh}(3)$, then $\cosh(\sqrt{2}\, t_0) = 3$ and $\sinh(\sqrt{2}\, t_0) = 2\sqrt{2}$. Thus $\exp(t_0 D)$ is a matrix whose entries are integer numbers. Therefore, $\exp^H(\mathbb{Z}\langle e_1, \dots, e_6\rangle)$ is a co-compact subgroup of H preserved by ϕ_{to}. Consequently,

$$\Gamma = (t_0\, \mathbb{Z}) \ltimes_\phi exp^H(\mathbb{Z}\langle e_1, \dots, e_6\rangle)$$

is a co-compact subgroup of G (see [19, Proposition 7.2.5]). Hence, the compact quotient Γ/G is a compact solvmanifold endowed with an invariant closed G_2 form.

Remark 1.3.7. *Note that the Lie algebra $\mathfrak{h}$ of the previous example is the third nilpotent Lie algebra that appears in Theorem 1.1.9. In fact, consider the basis $\{f^1, f^2, f^3, f^4, f^5, f^6\}$ of $\mathfrak{h}$ given by*

$$f^1 = e^1, \quad f^2 = e^3, \quad f^3 = e^5, \quad f^4 = e^6, \quad f^5 = \frac{1}{2}e^6, \quad f^6 = e^2.$$

Then, the structure equations of $\mathfrak{h}$ with respect to the basis $\{f^i\}$ are $(0, 0, 0, f^{12}, f^{13}, f^{23})$ which are the equations defining $\mathfrak{h}$ in Theorem 1.1.9.

For cocalibrated Lie algebras we have the following result.

Proposition 1.3.8. *Let $(\mathfrak{h}, g, J, \Psi = \psi_+ + i\,\psi_-)$ be a half-flat Lie algebra, and let D be a derivation of $\mathfrak{h}$ whose matrix representation, with respect to an* SU(3)*-basis (in the sense of Definition 1.3.5) $\{e_1, \dots, e_6\}$ of $\mathfrak{h}$, lies in $\mathfrak{sp}(6, \mathbb{R})$. Then, the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi,$$

with the Lie bracket given by (1.21)*, has a coclosed* G_2 *form.*

Proof. Since $D \in \mathfrak{sp}(6,\mathbb{R})$ with respect to the SU(3)-basis $\{e_1,\dots,e_6\}$, we can write

$$D = \left(\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & -a_{11} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline -a_{24} & a_{14} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{23} & -a_{13} & a_{43} & -a_{33} & a_{45} & a_{46} \\ \hline -a_{26} & a_{16} & -a_{46} & a_{36} & a_{55} & a_{56} \\ a_{25} & -a_{15} & a_{45} & -a_{35} & a_{65} & -a_{55} \end{array}\right).$$

Consider the G_2 form on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ given by

$$\varphi = \omega \wedge \eta - \psi_- . \tag{1.25}$$

Thus

$$*\varphi = \frac{1}{2}\omega \wedge \omega + \psi_+ \wedge \eta,$$

where η is the 1-form satisfying that $\eta(X) = 0$, for all $X \in \mathfrak{h}$, and $\eta(\xi) = 1$.
For $U, V, W, T, R \in \mathfrak{h}$, we have

$$d * \varphi(U,V,W,T,R) = d\omega \wedge \omega(U,V,W,T,R) = 0,$$

since $\omega \wedge \omega$ is closed.
Now, we determine $d * \varphi(U,V,W,T,\xi)$,

$$\begin{aligned} d * \varphi(U,V,W,T,\xi) = & - * \varphi([U,V],W,T,\xi) + *\varphi([U,W],V,T,\xi) - *\varphi([U,T],V,W,\xi) \\ & + *\varphi([U,\xi],V,W,T) - *\varphi([V,W],U,T,\xi) + *\varphi([V,T],U,W,\xi) \\ & - *\varphi([V,\xi],U,W,T) - *\varphi([W,T],U,V,\xi) + *\varphi([W,\xi],U,V,T) \\ & - *\varphi([T,\xi],U,V,W), \end{aligned}$$

which by the definition of $*\varphi$ is exactly

$$\begin{aligned} & - \psi_+([U,V],W,T) + \psi_+([U,W],V,T) - \psi_+([U,T],V,W) - \psi_+([V,W],U,T) \\ & + \psi_+([V,T],U,W) - \psi_+([W,T],U,V) + \frac{1}{2}\omega \wedge \omega([U,\xi],V,W,T) \\ & - \frac{1}{2}\omega \wedge \omega([V,\xi],U,W,T) + \frac{1}{2}\omega \wedge \omega([W,\xi],U,V,T) - \frac{1}{2}\omega \wedge \omega([T,\xi],U,V,W) \\ & = d\psi_+(U,V,W,T) + \frac{1}{2}\omega \wedge \omega(D(U),V,W,T) + \frac{1}{2}\omega \wedge \omega(U,D(V),W,T) \\ & + \frac{1}{2}\omega \wedge \omega(U,V,D(W),T) + \frac{1}{2}\omega \wedge \omega(U,V,W,D(T)). \end{aligned}$$

Therefore, since ψ_+ is closed, we have

$$\begin{aligned} d * \varphi(U,V,W,T,\xi) = & + \frac{1}{2}\omega \wedge \omega(D(U),V,W,T) + \frac{1}{2}\omega \wedge \omega(U,D(V),W,T) \\ & + \frac{1}{2}\omega \wedge \omega(U,V,D(W),T) + \frac{1}{2}\omega \wedge \omega(U,V,W,D(T)). \end{aligned}$$

By using the expressions of D and ω with respect to the SU(3)-basis $\{e_1,\dots,e_6\}$, we obtain

$$\begin{aligned}&\omega\wedge\omega(D(e_i),e_j,e_k,e_l)+\omega\wedge\omega(e_i,D(e_j),e_k,e_l)\\&+\omega\wedge\omega(e_i,e_j,D(e_k),e_l)+\omega\wedge\omega(e_i,e_j,e_k,D(e_l))=0,\end{aligned}$$

for every quadruplet (e_i,e_j,e_k,e_l) of elements of the SU(3)-basis of $\mathfrak{h}$. Thus, the G_2 form φ defined by (1.25) is coclosed on $\mathfrak{g}$. □

Next, we show a *new example of a compact solvmanifold with a coclosed* G_2 *form.* Let $\mathfrak{h}$ be the 6-dimensional Abelian Lie algebra defined by the structure equations

$$\mathfrak{h}=(0,0,0,0,0,0).$$

The almost Hermitian structure (g,J) on $\mathfrak{h}$ given by

$$g=\sum_{i=1}^{6}e^i\otimes e^i,\qquad Je_1=e_2,\qquad Je_3=e_4,\qquad Je_5=e_6$$

is such that its Kähler form is

$$\omega=e^{12}+e^{34}+e^{56}.$$

Thus, (g,J) together with the complex volume form $\Psi=\psi_++i\,\psi_-$, where

$$\begin{aligned}\psi_+&=e^{135}-e^{146}-e^{236}-e^{245},\\ \psi_-&=e^{136}+e^{145}+e^{235}-e^{246},\end{aligned}$$

define an SU(3)-structure on $\mathfrak{h}$. Clearly, $d\,\omega^2=d\,\psi_+=0$. So $(g,J,\Psi=\psi_++i\,\psi_-)$ is a half-flat SU(3)-structure on $\mathfrak{h}$.
Consider now the derivation D of $\mathfrak{h}$ given by

$$\left(\begin{array}{cc|cc|cc}1&&&&&\\&-1&&&&\\\hline&&1&&&\\&&&-1&&\\\hline&&&&1&\\&&&&&-1\end{array}\right)\in\mathfrak{sp}(6,\mathbb{R}),$$

that is,

$$\begin{aligned}&D(e_1)=e_1,\qquad &&D(e_2)=-e_2,\qquad &&D(e_3)=e_3,\\&D(e_4)=-e_4,\qquad &&D(e_5)=e_5,\qquad &&D(e_6)=-e_6.\end{aligned}$$

Take the Lie algebra

$$\mathfrak{g}=\mathfrak{h}\oplus_D\mathbb{R}e_7.$$

Thus, according to (1.21), the structure equations of $\mathfrak{g}$ are

$$\mathfrak{g} = (e^{17}, -e^{27}, e^{37}, -e^{47}, e^{57}, -e^{67}, 0).$$

Then, Proposition 1.3.8 implies that the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246}$$

is a coclosed G$_2$ form on $\mathfrak{g}$. In this case, the 4-form $*\varphi$ is given by

$$*\varphi = e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457},$$

which is closed because each term that appear in that expression is closed.

Let G be the simply connected and completely solvable Lie group of dimension 7 consisting of matrices of the form.

$$a = \left(\begin{array}{cc|cc|cc|cc}
e^{x_7} & & & & & & & x_1 \\
 & e^{-x_7} & & & & & & x_2 \\ \hline
 & & e^{x_7} & & & & & x_3 \\
 & & & e^{-x_7} & & & & x_4 \\ \hline
 & & & & e^{x_7} & & & x_5 \\
 & & & & & e^{-x_7} & & x_6 \\ \hline
 & & & & & & 1 & x_7 \\
 & & & & & & & 1
\end{array}\right),$$

where $x_i \in \mathbb{R}$, for $1 \leq i \leq 7$. Then a global system of coordinates $\{x_i,\ 1 \leq i \leq 7\}$ for G is defined by $x_i(a) = x_i$, and a standard calculation shows that a basis for the left invariant 1-forms on G consists of

$$\begin{array}{llll}
e^1 = e^{-x_7}dx_1, & e^2 = e^{x_7}dx_2, & e^3 = e^{-x_7}dx_3, & e^4 = e^{x_7}dx_4, \\
e^5 = e^{-x_7}dx_5, & e^6 = e^{-x_7}dx_6, & e^7 = dx_7, &
\end{array}$$

which means that $\mathfrak{g}$ is the Lie algebra of G. Now we notice that the Lie group G may be described as a semi direct product $G = \mathbb{R} \ltimes_\phi \mathbb{R}^6$, where $\mathbb{R}$ acts on $\mathbb{R}^6$ via the linear transformation ϕ_t of $\mathbb{R}^6$ given by the matrix

$$\phi_t = \left(\begin{array}{cc|cc|cc}
e^{t} & & & & & \\
 & e^{-t} & & & & \\ \hline
 & & e^{t} & & & \\
 & & & e^{-t} & & \\ \hline
 & & & & e^{t} & \\
 & & & & & e^{-t}
\end{array}\right).$$

Thus the operation on the group G is given by

$$a \cdot b = (b_1 e^{a_7} + a_1, b_2 e^{-a_7} + a_2, b_3 e^{a_7} + a_3, b_4 e^{-a_7} + a_4, b_5 e^{a_7} + a_5, b_6 e^{-a_7} + a_6, b_7 + a_7),$$

where $a = (a_1, \ldots, a_7)$ and $b = (b_1, \ldots, b_7)$. Hence $G = \mathbb{R} \ltimes_\phi \mathbb{R}^6$, where $\mathbb{R}$ is a connected Abelian sugbroup, and $\mathbb{R}^6$ is the nilpotent commutator subgroup.

Now we show that there exists a discrete subgroup Γ of G such that the quotient space $\Gamma\backslash G$ is compact. To construct Γ it suffices to find some real number t_0 such that the matrix defining ϕ_{t_0} is conjugated to an element A of the special linear group $SL(6,\mathbb{Z})$ with distinct real eigenvalues λ and λ^{-1}. In these conditions we could find a lattice Γ_0 in $\mathbb{R}^6$ which is invariant under ϕ_{t_0}, and take

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0.$$

To this end, we consider the matrix $A \in SL(6,\mathbb{Z})$ given by

$$A = \left(\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 1 & 1 & & & & \\ \hline & & 2 & 1 & & \\ & & 1 & 1 & & \\ \hline & & & & 2 & 1 \\ & & & & 1 & 1 \end{array}\right),$$

with triple eigenvalues $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. Taking $t_0 = \log(\frac{3+\sqrt{5}}{2})$, we have that the matrices ϕ_{t_0} and A are conjugated. In fact, take

$$P = \left(\begin{array}{cc|cc|cc} 1 & \frac{-1+\sqrt{5}}{2} & & & & \\ 1 & \frac{-1-\sqrt{5}}{2} & & & & \\ \hline & & 1 & \frac{-1+\sqrt{5}}{2} & & \\ & & 1 & \frac{-1-\sqrt{5}}{2} & & \\ \hline & & & & 1 & \frac{-1+\sqrt{5}}{2} \\ & & & & 1 & \frac{-1-\sqrt{5}}{2} \end{array}\right).$$

Then, a direct calculation shows that

$$PA = \phi_{t_0} P = \left(\begin{array}{cc|cc|cc} \frac{3+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & & & & \\ \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & & & & \\ \hline & & \frac{3+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & & \\ & & \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & & \\ \hline & & & & \frac{3+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ & & & & \frac{3-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{array}\right).$$

So, the lattice Γ_0 in $\mathbb{R}^6$ defined by

$$\Gamma_0 = P(m_1, m_2, m_3, m_4, m_5, m_6)^t,$$

where $m_1, m_2, m_3, m_4, m_5, m_6 \in \mathbb{Z}$ and $(m_1, m_2, m_3, m_4, m_5, m_6)^t$ is the transpose of the vector $(m_1, m_2, m_3, m_4, m_5, m_6)$, is invariant under the group $t_0\mathbb{Z}$. Thus

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0$$

is a co-compact subgroup of G. Hence, the compact solvmanifold $\Gamma\backslash G$ has an invariant coclosed G_2 form.

1.4 Examples of compact calibrated G_2 manifolds

In this section, we apply the results of the previous sections to produce new examples of compact calibrated G_2 manifolds. Moreover, we prove that 7-dimensional Lie algebras with $b_1 = 6$ cannot admit closed G_2 forms.

Example 1. *A compact calibrated* G_2 *manifold with* $b_1 = 1$

According with (1.10), we know that the product of a compact symplectic half-flat manifold M with S^1 has a closed G_2 form. Moreover, this G_2 form is also coclosed if the symplectic half-flat structure on M is integrable (in the sense mentioned in subsection 1.1.1).

Next, using the results of the previous section, we show a closed and coclosed G_2 form on a non-trivial mapping torus of an SU(3)-diffeomorphism of a 6-dimensional Calabi-Yau manifold.

We consider the so-called Fermat quintic X, that is, the Calabi-Yau manifold of (complex) dimension 3 defined as the zero-locus of

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0, \tag{1.26}$$

where $[z_0 : z_1 : z_2 : z_3 : z_4]$ denotes the five homogeneous coordinates on the complex projective space $\mathbb{CP}^4$. Then, X is a complex submanifold of $\mathbb{CP}^4$ of complex dimension 3. In fact, as we explained in subsection 1.1.1, the standard holomorphic atlas of $\mathbb{CP}^4$ contains 5 charts. If

$$U = \{[z_0 : z_1 : z_2 : z_3 : z_4] \,|\, z_0 \neq 0\} \subset \mathbb{CP}^4,$$

then $U \cong \mathbb{C}^4$, and the point $[1 : z_1 : z_2 : z_3 : z_4] \in U \subset \mathbb{CP}^4$ has coordinates $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. Now, if $[1 : z_1 : z_2 : z_3 : z_4] \in X$, there is a constraint on z_1, z_2, z_3 and z_4, namely,

$$1 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0. \tag{1.27}$$

Note that the point $[1 : 0 : 0 : 0 : 0] \in \mathbb{CP}^4$ does not live in X. Suppose that $z_4 \neq 0$ or, equivalently,

$$1 + z_1^5 + z_2^5 + z_3^5 \neq 0.$$

Then, equation (1.27) means that we can express z_4 in terms of the coordinates z_1, z_2 and z_3 as

$$z_4 = z_4(z_1, z_2, z_3) = -\lambda^k \sqrt[5]{1 + z_1^5 + z_2^5 + z_3^5}\,,$$

where $\lambda = e^{2\pi i/5}$ and $k \in \{0, 1, 2, 3, 4\}$. Therefore, this generates 5 charts for X. Also note that these charts could be also described as subsets of the collection of points of the form $[z_4' : z_1' : z_2' : z_3' : 1]$, with $z_4' \neq 0$. In a similar way, we can also express, for example, z_1 in terms of z_2, z_3 and z_4. So, we have $50 = 5 \cdot 5 \cdot (4/2)$ charts which describe completely the Fermat quintic X.

Consider the inclusion $j \colon X \hookrightarrow \mathbb{CP}^4$. Then, the pullback of the Kähler form ω_0 (given in subsection 1.1.1) on $\mathbb{CP}^4$ defines on the Fermat quintic X the Kähler form

$$\omega = j^*(\omega_0),$$

which is compatible with the metric g induced on X by the Fubini-Study metric on $\mathbb{CP}^4$. Moreover, the Kähler manifold (X, g, ω) has a non-vanishing holomorphic 3-form Ψ, which is given as follows [13, 112]. Let $[1 : z_1 : z_2 : z_3 : z_4(z_1, z_2, z_3)]$ be a chart of X. Then,

$$\Psi = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4^4}.$$

This form was defined in [13], and one can see in [112] that Ψ is preserved by changes of coordinates. Thus, Ψ is a (globally) defined holomorphic 3-form on X. Clearly, if we examine the behavior of Ψ on any chart of X, we see that it is nowhere vanishing. But (g, ω, Ψ) does not define a Calabi-Yau structure on X since the norm of Ψ is non-constant.

Therefore, by Yau's Theorem [131], we know that there exists a Ricci-flat Kähler metric $\widetilde{g}$ on the complex manifold (X, J). (But, as we mentioned in Remark 1.1.4, the explicit expression of the Ricci-flat Kähler metric $\widetilde{g}$ on X is not known.) Then, Lichnerowicz's Theorem (see Theorem 1.1.3) implies that there exists a closed complex volume $(n, 0)$-form Φ which is parallel with respect to the Levi-Civita connection of the metric $\widetilde{g}$. From now on, we denote by $\widetilde{\omega}$ the Kähler form of the Ricci-flat Kähler structure $(\widetilde{g}, J)$. Thus, $(\widetilde{g}, J, \Phi)$ is an integrable SU(3)-structure on X, that is, $(X, \widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$ is a Calabi-Yau manifold.

To determine the cohomology of the quintic X we use that X satisfies *Lefschetz theorem on hyperplane sections* [75, p.156]. This means that $j\colon X \hookrightarrow \mathbb{CP}^4$ is 3–connected, that is, up to homotopy $\mathbb{CP}^4$ is constructed out of X by attaching cells of dimension 4 and higher. Thus [21, p.217], for $r < 3$, there is an isomorphism $j^*\colon H^r(\mathbb{CP}^4) \to H^r(Z)$ induced by j on cohomology and, for $r = 3$, there is a monomorphism $j^*\colon H^3(\mathbb{CP}^4) \hookrightarrow H^3(X)$. Hence, the Betti numbers $b_i(X)$ of X are such that

$$b_0(X) = 1, \quad b_1(X) = b_5(X) = 0, \quad b_2(X) = b_4(X) = 1, \quad b_3(X) = 204,$$

because the Euler characteristic $\chi(X)$ of X is $\chi(X) = -200$ (see for example [112]). Therefore, if we denote by $h^{i,j}$ the dimension of the Dolbeault cohomology group $H^{i,j}(X)$, we conclude that $h^{1,0} = h^{0,1} = 0$, $h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 1$. Since $h^{3,0} = h^{0,3} = 1$, we obtain $h^{2,1} = h^{1,2} = 101$. Hence, if $\widetilde{\omega}$ and $\Phi = \phi_+ + i\,\phi_-$ are the Kähler form and the complex volume form on the Ricci-flat Kähler manifold $(X, \widetilde{g}, J)$, respectively, considered before, then $H^{1,1}(X) = \langle [\widetilde{\omega}] \rangle$, $H^{3,0}(X) = \langle [\Phi] \rangle$ and $H^{0,3}(X) = \langle [\bar{\Phi}] \rangle$.

In order to define an SU(3)-diffeomorphism of $(X, \widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$ we proceed as follows. Let G be the group of the Kähler isometries of $(X, \widetilde{g}, J)$. We know that G is a finite group because $(X, \widetilde{g})$ is a simply connected compact Ricci-flat manifold (Corollary 6.2 of [7]). Consider $\mu \in G$. Let us see when μ is an SU(3)-diffeomorphism of $(X, \widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$. Since μ defines an action of G on $H^{3,0}(X)$, we have a morphism of groups

$$\mu : G \to \mathrm{GL}(H^{3,0}(X)).$$

Because the space $H^{3,0}(X)$ has (complex) dimension 1, $\mathrm{GL}(H^{3,0}(X))$ is isomorphic to $\mathbb{C}^*$ and so Abelian. Now, the first group isomorphism theorem (or fundamental homomorphism theorem) implies that $G/\ker \mu$ is Abelian since it is isomorphic to a subgroup of $\mathrm{GL}(H^{3,0}(X))$. Thus, the commutator subgroup $[G, G]$ of G is such that

$$[G, G] \subset \ker \mu.$$

This means that any element of $[G, G]$ preserves the complex volume form Φ, and so it defines an SU(3)-diffeomorphism of the Fermat quintic $(X, \widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$ as a Calabi-Yau manifold.

In order to construct a compact G_2 calibrated manifold, we need also the following result on the Kähler isometries of a simply connected compact Calabi-Yau manifold.

Proposition 1.4.1 [101]**.** *Let M be a simply connected compact Calabi-Yau manifold of complex dimension n, and let f be an isometry of M. Then, f fixes the Kähler cohomology class $[\omega]$ on M if and only if the order of f is finite.*

Now, we have the following

Theorem 1.4.2. *Let $(\widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$ be the Calabi-Yau structure on the Fermat quintic X defined by* (1.26)*, and let G be the group of the Kähler isometries of $(X, \widetilde{g}, J)$. Take $\nu \in [G, G]$. Then, the mapping torus X_ν of ν is a compact formal calibrated and cocalibrated* G_2 *manifold with first Betti number $b_1(X_\nu) = 1$.*

Proof. By Theorem 1.3.2 and Theorem 1.3.4, we know that the SU(3)-structure $(\widetilde{g}, J, \Phi = \phi_+ + i\,\phi_-)$ on X induces the closed and coclosed G_2 form φ on X_ν given by

$$\varphi = \psi_+ + \widetilde{\omega} \wedge \eta.$$

In fact, φ is also coclosed because $*\varphi = \psi_- \wedge \eta + \frac{1}{2}\,\widetilde{\omega}^2$ is closed.

Since X is simply connected, (1.17) implies that $b_1(X_\nu) = 1$. Moreover, using again (1.17), and taking into account Lemma 1.2.13 and Proposition 1.4.1, we have that the de Rham cohomology groups of X_ν are

$$\begin{aligned}
H^0(X_\nu) &= \langle 1 \rangle,\\
H^1(X_\nu) &= \langle [\eta] \rangle,\\
H^2(X_\nu) &= \langle [\widetilde{\omega}] \rangle,\\
H^3(X_\nu) &= \langle [\eta \wedge \widetilde{\omega}] \rangle \oplus \ker\Big(\nu^* - Id : H^3(X) \to H^3(X)\Big),\\
H^4(X_\nu) &= \langle [\widetilde{\omega}^2] \rangle \oplus [\eta] \wedge H^3(X),\\
H^5(X_\nu) &= \langle [\eta \wedge \widetilde{\omega}^2] \rangle,\\
H^6(X_\nu) &= \langle [\widetilde{\omega}^3] \rangle,\\
H^7(X_\nu) &= \langle [\eta \wedge \widetilde{\omega}^3] \rangle.
\end{aligned}$$

Note that ϕ_+ and ϕ_- define cohomology classes in $H^3(X_\nu)$ since ϕ_+ and ϕ_- are closed ν^*-invariant forms on X, that is, the cohomology classes $[\phi_+]$ and $[\phi_-]$ belong both to the space $\ker(\nu^* - Id : H^3(X) \to H^3(X))$. More yet, if

$$x \in \ker(\nu^* - Id : H^3(X) \to H^3(X)),$$

then $[\eta] \wedge x$ is a non-zero cohomology class in $H^4(X_\nu)$. Indeed,

$$x \notin \mathrm{Im}(\nu^* - Id : H^3(X) \to H^3(X)),$$

because, by Lemma 1.2.13, the eigenvalue $\lambda = 1$ of the map

$$\nu^* - Id : H^3(X) \to H^3(X)$$

has multiplicity $r = 1$.

By Theorem 1.2.10, we know that X_ν is 2-formal (in the sense of Definition 1.2.6). To prove that it is 3-formal, we see that the minimal model of X_ν is the differential graded algebra $(\Lambda W, d)$, where $W^i = C^i + N^i$ is such that, for $i \leq 3$,

$$\begin{gathered} C^1 = \langle a \rangle, \quad N^1 = 0, \\ C^2 = \langle b \rangle, \quad N^2 = 0, \\ C^3 \cong \ker\big(\nu^* - Id : H^3(X) \longrightarrow H^3(X)\big), \quad N^3 = 0. \end{gathered}$$

Since $N^i = 0$ $(i = 1, 2, 3)$, we conclude that X_ν is 3-formal and, by Theorem 1.2.7, X_ν is formal. $\square$

Example 2. *Compact calibrated non-flat* G_2 *manifolds with* $b_1 = 7$.

Tomassini and Vezzoni in [122] construct a family of symplectic half-flat structures (J_t, ω, Ψ_t) on the 6-dimensional torus $\mathbb{T}^6$, coinciding with the standard Calabi-Yau structure for $t = 0$, but which is not Calabi-Yau for $t \neq 0$.

Theorem 1.4.3 [122]**.** *There exists a family* (J_t, ω, Ψ_t) *of symplectic half-flat structures on the* 6*-dimensional torus* $\mathbb{T}^6$*, such that* (J_0, ω, Ψ_0) *is the standard Calabi-Yau structure on* $\mathbb{T}^6$*, but* (J_t, ω, Ψ_t) *is not integrable for* $t \neq 0$*. Such a structure* (J_t, ω, Ψ_t) *is defined as follows:*

$$\begin{cases} J_t(\partial_r) & = \; e^{-t\lambda_r}\partial_{3+r}, \\ J_t(\partial_{3+r}) & = \; -e^{t\lambda_r}\partial_r \, , \end{cases}$$

for $r = 1, 2, 3$*, where* $\partial_r = \frac{\partial}{\partial x_r}$*, and* $a = a(x_1), b = b(x_2), c = c(x_3)$ *are three smooth functions on* $\mathbb{R}^6$ *such that*

$$\lambda_1 = b(x_2) - c(x_3), \quad \lambda_2 = -a(x_1) + c(x_3), \quad \lambda_3 = a(x_1) - b(x_2)$$

are $\mathbb{Z}^6$*-periodic;*

$$\omega = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6, \tag{1.28}$$

and

$$\Psi_t = i\,(dx_1 + ie^{t\lambda_1}dx_4) \wedge (dx_2 + ie^{t\lambda_2}dx_5) \wedge (dx_3 + ie^{t\lambda_3}dx_6)\,. \tag{1.29}$$

In [122] it is proved that $(\mathbb{T}^6, J_t, \omega, \Psi_t)$ is a symplectic half-flat manifold, for any $t \in \mathbb{R}$, and J_t is non-integrable for $t \neq 0$; moreover, in this case $(t \neq 0)$, $\lambda_1, \lambda_2, \lambda_3$ are non-constant. Note that the corresponding metric g_t is given by

$$\begin{aligned} g_t = {} & e^{(\lambda_2+\lambda_3)t}dx_1 \otimes dx_1 + e^{(\lambda_1+\lambda_3)t}dx_2 \otimes dx_2 + e^{(\lambda_1+\lambda_2)t}dx_3 \otimes dx_3 \\ & + e^{(2\lambda_1+\lambda_2+\lambda_3)t}dx_4 \otimes dx_4 + e^{(\lambda_1+2\lambda_2+\lambda_3)t}dx_5 \otimes dx_5 + e^{(\lambda_1+\lambda_2+2\lambda_3)t}dx_6 \otimes dx_6. \end{aligned}$$

Now, taking into account subsection 1.1.2 and, in particular, the expression of the G_2 form given by (1.10), we have

Proposition 1.4.4. *The 7-torus* $\mathbb{T}^7$ *has a family of closed* G_2 *forms* φ_t*, inducing a non-flat metric for* $t \neq 0$,

$$\varphi_t = \psi_t^+ + \omega_t \wedge \eta,$$

where $\omega_t, \Psi_t = \psi_t^+ + i\,\psi_t^-$ *are given by* (1.28) *and* (1.29)*, respectively, and* η *is the volume form on* S^1.

In order to prove that 7-dimensional Lie algebras with $b_1 = 6$ do not carry closed G_2 forms, we need the following restriction to the existence of closed G_2-structures proved in [38].

Proposition 1.4.5 [38]**.** *Let* $\mathfrak{g}$ *be a 7-dimensional Lie algebra. If there is a non-zero* X *in* $\mathfrak{g}$ *such that* $(\iota_X\varphi)^3 = 0$ *for every closed 3-form on* $\mathfrak{g}$*, then* $\mathfrak{g}$ *has no closed* G_2*-structures.*

Proposition 1.4.6. *Let* $\mathfrak{g}$ *be a 7-dimensional Lie algebra with* $b_1 = 6$*. Then,* $\mathfrak{g}$ *cannot admit closed* G_2 *forms.*

Proof. Since $b_1 = 6$, there exists a basis $\{e^1, \dots, e^7\}$ of $\mathfrak{g}^*$ such that $\mathfrak{g}$ is defined by the equations

$$\begin{aligned} de^j &= 0, \quad j = 1, \dots, 6,\\ de^7 &= \beta + e^7 \wedge \alpha, \end{aligned} \tag{1.30}$$

where $\beta \in \Lambda^2(e^1, \dots, e^6)$ and $\alpha \in \Lambda^1(e^1, \dots, e^6)$. (Note that $\mathfrak{g}$ has to be solvable because $\dim[\mathfrak{g}, \mathfrak{g}] = 1$.) Suppose that $\mathfrak{g}$ has a closed G_2 form φ. Then we can write

$$\varphi = \gamma + e^7 \wedge \delta,$$

where $\gamma \in \Lambda^3(e^1, \dots, e^6)$ and $\delta \in \Lambda^2(e^1, \dots, e^6)$. Using (1.30), we see that $d\,\gamma = 0$, since $\gamma \in \Lambda^3(e^1, \dots, e^6)$. Also, $d\,\delta = 0$ because $\delta \in \Lambda^2(e^1, \dots, e^6)$. Then, the condition $d\varphi = 0$ is equivalent to $de^7 \wedge \delta = 0$, that is, to

$$(\beta + e^7 \wedge \alpha) \wedge \delta = 0.$$

This gives

$$\beta \wedge \delta = 0, \quad \alpha \wedge \delta = 0.$$

Since $\delta \in \Lambda^2(e^1, \dots, e^6)$, $\gamma \in \Lambda^3(e^1, \dots, e^6)$ and $\varphi = \gamma + e^7 \wedge \delta$, we have that $\delta = \iota_{e_7}\varphi$. Now, by Proposition 1.4.5, δ is non-degenerate, that is, $\delta^3 \neq 0$. Moreover, $\alpha \in \Lambda^1(e^1, \dots, e^6)$ is a 1-form, so the condition $\alpha \wedge \delta = 0$ implies that $\alpha = 0$ and hence $\mathfrak{g}$ has to be nilpotent. But, by the Conti-Fernández classification [38], there are no nilpotent Lie algebras with $b_1 = 6$ and admitting closed G_2-structure. $\square$

Chapter 2

Six dimensional symplectic half-flat solvable Lie algebras

"La géométrie est l'art de raisonner juste sur des figures fausses."
René Descartes

In order to construct examples of 7-dimensional manifolds with a closed G_2 form, a natural place to look is left invariant symplectic half-flat structures on 6-dimensional Lie groups. According to Magnin's classification [100], there are 34 isomorphism classes of nilpotent Lie algebras, of which exactly 3 (including the Abelian Lie algebra) admit symplectic half-flat structures [40].

In this Chapter, we give a complete classification of 6-dimensional solvable Lie algebras for which the corresponding simply connected solvable Lie group has left invariant symplectic half-flat structure.

Taking into account the classification of 6-dimensional *decomposable* Lie algebras with half-flat structures given in [115] and [65], we show that there are 21 (non-nilpotent) solvable Lie algebras as well as 4 one-parameter families of solvable Lie algebras having both half-flat structures and symplectic forms. Then, we prove that only 2 of those 21 Lie algebras and only 1 of the 4 one-parameter families carry symplectic half-flat structures (Propositions 2.2.1, 2.2.2 and 2.2.3).

For *indecomposable* Lie algebras we study separately unimodular and non-unimodular ones. For *unimodular* Lie algebras we consider the classification given in [99] of 6-dimensional unimodular Lie algebras with symplectic forms. We show that only 3 of those Lie algebras have symplectic half-flat structures (Proposition 2.3.1). *Non-unimodular* indecomposable Lie algebras are studied taking into account the dimension of their nilradical. For those with 5-dimensional nilradical we consider the classification given in [66] of 6-dimensional Lie algebras (with 5-dimensional nilradical) endowed with half-flat structures. On the other hand, we use the list of Turkowski [124] of the 6-dimensional solvable Lie algebras with 4-dimensional nilradical. For these latter Lie algebras, there is no classification of those with a half-flat structure. In Propositions 2.4.1 and 2.4.2, we determine a symplectic half-flat structure on 4 non-unimodular indecomposable (non-nilpotent) solvable Lie algebras and on 1 one-parameter family. To prove that the remaining Lie algebras do not admit such a structure we consider, in section 2.1, two obstructions to the existence of symplectic half-flat structures on a Lie algebra. All these results appear in [62].

2.1 Obstructions

In this section we give some obstructions to the existence of symplectic half-flat structures on a 6-dimensional Lie algebra, and use them in the following sections to classify the solvable Lie algebras with such a structure. To this end, we need the characterization of SU(3)-structures given in [82, 83] in terms of certain *stable* forms which satisfy some additional compatibility conditions as we explain in the following:

Definition 2.1.1 [82, 83]**.** *A 3-form ρ on a 6-dimensional oriented vector space (V,ν) is* stable *if its orbit under the action of the group $GL(V)$ of the automorphisms of V is open.*

If (V,ν) is a 6-dimensional oriented vector space, we have the isomorphism

$$\begin{array}{rccc} \kappa: & \Lambda^5 V^* & \longrightarrow & V\otimes\Lambda^6 V^* \\ & \eta & \longmapsto & X\otimes\nu, \end{array}$$

where $X\in V$ is such that $\eta=\iota_X\nu$. Then, for any 3-form ρ on (V,ν), we have the linear transformation $K_\rho: V\longrightarrow V\otimes\Lambda^6V^*$ given by

$$K_\rho(X)=\kappa(\iota_X\rho\wedge\rho).$$

Thus, we also have $\lambda(\rho)\in(\Lambda^6V^*)^2$ defined by

$$\lambda(\rho)=\frac{1}{6}trK_\rho^2.$$

Moreover, $\lambda(\rho)$ enables us to construct a volume form $\phi(\rho)$ on V as

$$\phi(\rho)=\sqrt{|\lambda(\rho)|}\in\Lambda^6V^*.$$

Here, we would like to note that for any one-dimensional vector space L, an element $u\in L\otimes L$ is defined to be positive ($u>0$) if $u=s\otimes s$ for some $s\in L$; and u

is negative if $-u > 0$. Therefore, we have well-defined square roots of positive elements in $L \otimes L$.

In [83] it is shown that the stability of ρ is characterized by the open condition $\lambda(\rho) \neq 0$ or, equivalently, by $\phi(\rho) \neq 0$.

It is well known that a 2-form $F \in \Lambda^2 V^*$ is stable if and only if it is non-degenerate, that is, if

$$\phi(F) = \frac{1}{6}F^3 \neq 0.$$

Definition 2.1.2 [82, 83]**.** *A pair of stable forms* $(F, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ *is called* compatible *if*

$$F \wedge \rho = 0,$$

and normalized *if*

$$\phi(\rho) = 2\phi(F).$$

Note that if $\lambda(\rho) < 0$, the endomorphism

$$J_\rho = \frac{1}{\phi(\rho)} K_\rho$$

defined by a stable 3-form ρ, gives rise to an almost complex structure. The action of J_ρ^* on 1-forms is given by the formula

$$J_\rho^* \alpha(X)\phi(\rho) = \alpha \wedge \iota_X \rho \wedge \rho. \tag{2.1}$$

Thus, such a pair (F, ρ) induces a pseudo Euclidean metric, $g(\cdot, \cdot) = F(J_\rho \cdot, \cdot)$ which satisfies on 1-forms the identity

$$\alpha \wedge J_\rho^* \beta \wedge F^2 = \frac{1}{2} g(\alpha, \beta) F^3, \tag{2.2}$$

for $\alpha, \beta \in V^*$.

For us the relevant result is the following characterization of SU(3)-structures.

Proposition 2.1.3 [82, 83]**.** *An* SU(3)-structure *on a 6-dimensional oriented vector space* (V, ν) *is determined by a pair of compatible and normalized stable forms* $(\omega, \psi_+) \in \Lambda^2 V^* \times \Lambda^3 V^*$ *inducing a positive-definite metric and with* $\lambda(\psi_+) < 0$.

A *symplectic half-flat structure* on an oriented Lie algebra $\mathfrak{g}$ is an SU(3)-structure (ω, ψ_+) such that $d\omega = 0$ and $d\psi_+ = 0$, where d denotes the Chevalley-Eilenberg differential on the dual $\mathfrak{g}^*$.

Notice that if (ω, ψ_+) defines an SU(3)-structure, ω is non-degenerate and $\Psi = \psi_+ + i\,\psi_-$ is a complex volume form.

Let $\mathfrak{g}$ be a 6-dimensional Lie algebra with an SU(3)-structure (ω, ψ_+). We denote by $Z^k(\mathfrak{g})$ the space of closed k-forms on $\mathfrak{g}$, by $\mathcal{S}(\mathfrak{g})$ the space of symplectic forms on $\mathfrak{g}$, and by $Ann(\psi_+)$ the annihilator of ψ_+ in the exterior algebra $\Lambda^*\mathfrak{g}^*$, that is

$$\begin{aligned} Z^k(\mathfrak{g}) &= \{\theta \in \Lambda^k\mathfrak{g}^* | d\theta = 0\}, \\ \mathcal{S}(\mathfrak{g}) &= \{\omega \in Z^2(\mathfrak{g}) \mid \omega \wedge \omega \wedge \omega \neq 0\}, \\ Ann(\psi_+) &= \{\theta \in \Lambda^*\mathfrak{g}^* | \theta \wedge \psi_+ = 0\}. \end{aligned}$$

In [65] it is proved the following obstruction to the existence of half-flat structures on Lie algebras.

Proposition 2.1.4 [65]**.** *Let $\mathfrak{g}$ be a 6-dimensional Lie algebra with a volume form $\nu \in \Lambda^6\mathfrak{g}^*$. Then, $\mathfrak{g}$ does not admit any half-flat* SU(3)*-structures if there is a non-zero 1-form $\alpha \in \mathfrak{g}^*$ satisfying*

$$\alpha \wedge \tilde{J}^*_\rho\alpha \wedge \sigma = 0,$$

for all closed 3-forms $\rho \in \Lambda^3\mathfrak{g}^*$ and all closed 4-forms $\sigma \in \Lambda^4\mathfrak{g}^*$, where $\tilde{J}^*_\rho\alpha$ is given by

$$\tilde{J}^*_\rho\alpha(X)\nu = \alpha \wedge (\iota_X\rho) \wedge \rho, \tag{2.3}$$

for each $X \in \mathfrak{g}$.

Proposition 2.1.5. *Let $\mathfrak{g}$ be a 6-dimensional Lie algebra. Then $\mathfrak{g}$ has no symplectic half-flat structures if one of the following conditions is satisfied:*

1. *There is a non-zero 1-form α on $\mathfrak{g}$ such that*

$$\alpha \wedge \tilde{J}^*_\rho\alpha \wedge F^2 = 0,$$

for each $F \in \mathcal{S}(\mathfrak{g}) \cap Ann(\rho)$, and for each $0 \neq \rho \in Z^3(\mathfrak{g})$.

2. *There are some $X, Y \in \mathfrak{g}$ such that*

$$F\big(\tilde{J}_\rho(X), X\big) \cdot F\big(\tilde{J}_\rho(Y), Y\big) \leq 0,$$

for any $F \in \mathcal{S}(\mathfrak{g}) \cap Ann(\rho)$, and for any $0 \neq \rho \in Z^3(\mathfrak{g})$.

Proof. Suppose that $\mathfrak{g}$ has a symplectic half-flat structure (ω, ψ_+). From (2.3) we have that for any 1-form α on $\mathfrak{g}$ and any X in $\mathfrak{g}$,

$$J^*_{\psi_+}\alpha(X)\omega^3 = 3\alpha \wedge (\iota_X\psi_+) \wedge \psi_+.$$

On the other hand, if ν denotes a volume form on $\mathfrak{g}^*$

$$\tilde{J}^*_{\psi_+}\alpha(X)\nu = \alpha \wedge (\iota_X\psi_+) \wedge \psi_+.$$

Therefore, $J^*_{\psi_+}$ and $\tilde{J}^*_{\psi_+}$ are proportional since

$$\tilde{J}^*_{\psi_+}\alpha(X)\nu = \frac{1}{3}J^*_{\psi_+}\alpha(X)\omega^3.$$

Now, from (2.2) it follows that if the condition of the part *(1.)* is satisfied, then the induced metric by (ω, ψ_+) is degenerate since there exists $\alpha \in \mathfrak{g}^*$ such that

$$g(\alpha, \alpha) = 0.$$

So there is no symplectic half-flat structure on $\mathfrak{g}$.

To prove part *(2.)* we use that the metric induced by an SU(3)-structure (ω, ψ_+) is determined by $g(\cdot,\cdot) = \omega(\cdot, J_{\psi_+}\cdot)$. Hence, condition *(2.)* implies that there exist X, Y on $\mathfrak{g}$ such that

$$g(X, X) \cdot g(Y, Y) \leq 0,$$

and thus g is not positive-definite. □

Moreover, from Proposition 1.4.5, we obtain the following obstruction to the existence of symplectic half-flat structures on Lie algebras.

Lemma 2.1.6. *Let $\mathfrak{g}$ be a 6-dimensional Lie algebra. If there exists a non-zero vector X of the Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$ such that for all closed 3-forms ϕ on $\mathfrak{h}$ the 2-form $\iota_X\phi$ is degenerate, that is*

$$(\iota_X\phi)^3 = 0,$$

then $\mathfrak{g}$ has no symplectic half-flat structures.

Proof. We know that a symplectic half-flat structure (ω, ψ_+) on a 6-dimensional Lie algebra $\mathfrak{g}$ induces the closed G_2 form $\varphi = \omega \wedge dt + \psi_+$ on the Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$, where t is the coordinate of $\mathbb{R}$. Suppose that for any closed 3-form ϕ on $\mathfrak{h}$, there exists $X \in \mathfrak{h}$ such that $(\iota_X\phi)^3 = 0$. Then, by Proposition 1.4.5, $\mathfrak{h}$ does not admit closed G_2 forms, and so $\mathfrak{g}$ has no symplectic half-flat structures. □

2.2 Decomposable symplectic half-flat Lie algebras

We determine the 6-dimensional *decomposable* and (non-nilpotent) solvable Lie algebras admitting symplectic half-flat structure. A Lie algebra $\mathfrak{g}$ is said to be *decomposable* if $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$.

From now on, we use

$$p \oplus q,$$

to denote a decomposable Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\dim(\mathfrak{g}_1) = p$ and $\dim(\mathfrak{g}_2) = q$.

2.2.1 $3 \oplus 3$ Lie algebras

We use the classification of the 6-dimensional solvable Lie algebras with half-flat structures given in [115]. Then we see that the $3\oplus 3$ decomposable (non-nilpotent) solvable Lie algebras which have both symplectic and half-flat structures are those appearing in the following Table 2.1, where F denotes a symplectic form and (ω, ψ_+) a half-flat structure.

Table 2.1: $3 \oplus 3$ **decomposable (non-nilpotent) solvable Lie algebras admitting both symplectic and half-flat structures.**

$\mathfrak{g}$	**str. equations**	**half-flat str.**	**simplectic str.**
$\mathfrak{e}(2) \oplus \mathfrak{e}(2)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, e^{45})$	$\omega = 2e^{14} + e^{25} + e^{36}$ $\psi_+ = e^{123} - e^{156} + e^{246} - e^{345}$ $+e^{126} - e^{135} + e^{234} - e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45})$	$\omega = e^{14} + e^{23} + 2e^{56}$ $\psi_+ = e^{125} - e^{126} - e^{135} - e^{136}$ $+e^{245} + e^{246} + e^{345} - e^{346}$	$F = e^{14} + e^{23} + 2e^{56}$
$\mathfrak{e}(2) \oplus \mathbb{R}^3$	$(0, -e^{13}, e^{12}, 0, 0, 0)$	$\omega = e^{14} + e^{25} + e^{36}$ $\psi_+ = e^{126} - e^{135} + e^{234} - e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(1,1) \oplus \mathbb{R}^3$	$(0, -e^{13}, -e^{12}, 0, 0, 0)$	$\omega = e^{14} + e^{25} + e^{36}$ $\psi_+ = e^{126} - e^{135} + e^{234} - e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45})$	$\omega = e^{14} + e^{25} + e^{36}$ $\psi_+ = -2e^{234} + e^{135} - e^{126}$ $-e^{246} - e^{345} + e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(2) \oplus \mathfrak{h}$	$(0, -e^{13}, e^{12}, 0, 0, e^{45})$	$\omega = e^{14} + e^{25} + e^{36}$ $\psi_+ = -e^{234} + \frac{5}{4}e^{135} - e^{126}$ $+e^{345} + e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(1,1) \oplus \mathfrak{h}$	$(0, -e^{13}, -e^{12}, 0, 0, e^{45})$	$\omega = e^{14} + e^{25} + e^{36}$ $\psi_+ = -e^{234} + \frac{5}{4}e^{135} - e^{126}$ $-e^{345} + e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$(0, -e^{13}, e^{12}, 0, -e^{45}, 0)$	$\omega = e^{12} + e^{34} - e^{56}$ $\psi_+ = e^{236} - e^{245} + e^{135} + e^{146}$	$F = e^{16} + e^{23} + e^{45}$
$\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$	$(0, -e^{13}, -e^{12}, 0, -e^{45}, 0)$	$\omega = e^{12} + e^{34} - e^{56}$ $\psi_+ = e^{236} - e^{245} + e^{135} + e^{146}$	$F = e^{16} + e^{23} + e^{45}$

Note that, in Table 2.1, the only $3 \oplus 3$ *non-unimodular* solvable Lie algebras are $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.

Proposition 2.2.1. *Let $\mathfrak{g}$ be a 6-dimensional decomposable and (non-nilpotent) solvable Lie algebra such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $dim(\mathfrak{g}_1) = dim(\mathfrak{g}_2) = 3$. Then, $\mathfrak{g}$ has symplectic half-flat structure if and only if $\mathfrak{g} = \mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$.*

Proof. First we observe that in [122] it is proved that the Lie algebra $\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1)$ defined by the structure equations

$$\mathfrak{e}(\mathbf{1},\mathbf{1}) \oplus \mathfrak{e}(\mathbf{1},\mathbf{1}) = (\mathbf{0}, -\mathbf{e^{13}}, -\mathbf{e^{12}}, \mathbf{0}, -\mathbf{e^{46}}, -\mathbf{e^{45}}),$$

has a symplectic half-flat structure. The differential forms ω and ψ_+ appearing in Table 2.1 are closed and determine an SU(3)-structure since with the change of basis given by

$$f^1 = e^1, \quad f^2 = e^4, \quad f^3 = e^2, \quad f^4 = e^3, \quad f^5 = e^5 - e^6, \quad f^6 = e^5 + e^6,$$

the forms ω and ψ_+ have the following expression

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_+ &= f^{135} - f^{146} - f^{236} - f^{245}. \end{aligned}$$

Next, we show that no other Lie algebras included in Table 2.1 have a symplectic half-flat structure. For the following Lie algebras we use Proposition 2.1.5 *(2.)* for suitable vectors X and Y of the Lie algebra.

- For the Lie algebra $\mathfrak{e}(2) \oplus \mathfrak{e}(2)$ defined by the structure equations

$$\mathfrak{e}(\mathbf{2}) \oplus \mathfrak{e}(\mathbf{2}) = (\mathbf{0}, -\mathbf{e^{13}}, \mathbf{e^{12}}, \mathbf{0}, -\mathbf{e^{46}}, \mathbf{e^{45}}),$$

the spaces $Z^k\big(\mathfrak{e}(2) \oplus \mathfrak{e}(2)\big)$ $(k = 2, 3)$ are:

$$Z^2\big(\mathfrak{e}(2) \oplus \mathfrak{e}(2)\big) = \langle e^{12}, e^{13}, e^{14}, e^{23}, e^{45}, e^{46}, e^{56} \rangle,$$

and

$$\begin{aligned} Z^3\big(\mathfrak{e}(2) \oplus \mathfrak{e}(2)\big) =& \langle e^{123}, e^{124}, e^{134}, e^{145}, e^{146}, e^{156}, e^{234}, e^{136} + e^{245}, \\ & - e^{135} + e^{246}, -e^{126} + e^{345}, e^{125} + e^{346}, e^{456} \rangle. \end{aligned}$$

Thus, the expression of any pair $(F, \rho) \in Z^2\big(\mathfrak{e}(2) \oplus \mathfrak{e}(2)\big) \times Z^3\big(\mathfrak{e}(2) \oplus \mathfrak{e}(2)\big)$ is

$$F = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{23} + b_5 e^{45} + b_6 e^{46} + b_7 e^{56},$$

and

$$\begin{aligned} \rho =& a_1 e^{123} + a_2 e^{124} + a_{11} e^{125} - a_{10} e^{126} + a_3 e^{134} - a_9 e^{135} + a_8 e^{136} + a_4 e^{145} \\ & + a_5 e^{146} + a_6 e^{156} + a_7 e^{234} + a_8 e^{245} + a_9 e^{246} + a_{10} e^{345} + a_{11} e^{346} + a_{12} e^{456}, \end{aligned}$$

where a_i and b_j are real numbers for $1 \le i \le 12$ and $1 \le j \le 7$. From these expressions of F and ρ, we have

$$g(e_5, e_5) \cdot g(e_6, e_6) = F(\tilde{J}_\rho e_5, e_5) \cdot F(\tilde{J}_\rho e_6, e_6) = -(a_3 a_4 + a_6 a_7)^2 b_7^2 \le 0.$$

Then, Proposition 2.1.5 *(2.)* applies for $X = e_5$ and $Y = e_6$. Thus, $\mathfrak{e}(2) \oplus \mathfrak{e}(2)$ has no symplectic half-flat structure.

- For

$$\mathfrak{e}(\mathbf{2}) \oplus \mathbb{R}^{\mathbf{3}} = (\mathbf{0}, -\mathbf{e^{13}}, \mathbf{e^{12}}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

the spaces of closed 2-forms and 3-forms are

$$Z^2\big(\mathfrak{e}(2) \oplus \mathbb{R}^3\big) = \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{16}, e^{23}, e^{45}, e^{46}, e^{56}\rangle,$$

and

$$\begin{aligned} Z^3\big(\mathfrak{e}(2) \oplus \mathbb{R}^3\big) =& \langle e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, \\ & e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{456}\rangle. \end{aligned}$$

Then, if $(F, \rho) \in Z^2\big(\mathfrak{e}(2) \oplus \mathbb{R}^3\big) \times Z^3\big(\mathfrak{e}(2) \oplus \mathbb{R}^3\big)$ we have

$$\begin{aligned} F &= b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{45} + b_8 e^{46} + b_9 e^{56}, \\ \rho &= a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{126} + a_5 e^{134} + a_6 e^{135} + a_7 e^{136} + a_8 e^{145} \\ &\quad + a_9 e^{146} + a_{10} e^{156} + a_{11} e^{234} + a_{12} e^{235} + a_{13} e^{236} + a_{14} e^{456}, \end{aligned}$$

where a_i and b_j are real numbers. These expressions of F and ρ imply that

$$F(\tilde{J}_\rho e_2, e_2) = 0.$$

Again, Proposition 2.1.5 (part *(2.)*) applies for $X = Y = e_2$.

- For $\mathfrak{e}(1,1) \oplus \mathbb{R}^3$ defined by

$$\mathfrak{e}(\mathbf{1}, \mathbf{1}) \oplus \mathbb{R}^{\mathbf{3}} = (\mathbf{0}, -\mathbf{e^{13}}, -\mathbf{e^{12}}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

we have

$$Z^2\big(\mathfrak{e}(1,1) \oplus \mathbb{R}^3\big) = \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{16}, e^{23}, e^{45}, e^{46}, e^{56}\rangle,$$

and

$$\begin{aligned} Z^3\big(\mathfrak{e}(1,1) \oplus \mathbb{R}^3\big) =& \langle e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, \\ & e^{156}, e^{234}, e^{235}, e^{236}, e^{456}\rangle. \end{aligned}$$

Then, any pair $(F, \rho) \in Z^2\big(\mathfrak{e}(1,1) \oplus \mathbb{R}^3\big) \times Z^3\big(\mathfrak{e}(1,1) \oplus \mathbb{R}^3\big)$ is of the form

$$F = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{45} + b_8 e^{46} + b_9 e^{56},$$

$$\begin{aligned} \rho &= a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{126} + a_5 e^{134} + a_6 e^{135} + a_7 e^{136} + a_8 e^{145} \\ &\quad + a_9 e^{146} + a_{10} e^{156} + a_{11} e^{234} + a_{12} e^{235} + a_{13} e^{236} + a_{14} e^{456}, \end{aligned}$$

for any real numbers a_i and b_j. Now we obtain that

$$F(\tilde{J}_\rho e_2, e_2) = 0.$$

Hence, it is sufficient to consider $X = Y = e_2$.

- Consider
$$\mathfrak{e}(\mathbf{2}) \oplus \mathfrak{e}(\mathbf{1},\mathbf{1}) = (\mathbf{0}, -\mathbf{e^{13}}, \mathbf{e^{12}}, \mathbf{0}, -\mathbf{e^{46}}, -\mathbf{e^{45}}).$$
The spaces $Z^k\big(\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)\big)$, for $k=2,3$, are
$$Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)\big) = \langle e^{12}, e^{13}, e^{14}, e^{23}, e^{45}, e^{46}, e^{56}\rangle,$$
and
$$\begin{aligned} Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)\big) =& \langle e^{123}, e^{124}, e^{134}, e^{145}, e^{146}, e^{156}, e^{234}, \\ & -e^{136}+e^{245}, -e^{135}+e^{246}, e^{126}+e^{345}, e^{125}+e^{346}, e^{456}\rangle. \end{aligned}$$
Then, any pair $(F,\rho) \in Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)\big) \times Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{e}(1,1)\big)$ has the following expression
$$F = b_1e^{12} + b_2e^{13} + b_3e^{14} + b_4e^{23} + b_5e^{45} + b_6e^{46} + b_7e^{56},$$
$$\begin{aligned} \rho =& a_1e^{123} + a_2e^{124} + a_3e^{134} + a_4e^{145} + a_5e^{146} + a_6e^{156} + a_7e^{234} \\ & + a_8(-e^{136}+e^{245}) + a_9(-e^{135}+e^{246}) + a_{10}(e^{126}+e^{345}) \\ & + a_{11}(e^{125}+e^{346}) + a_{12}e^{456}, \end{aligned}$$
and we obtain that
$$F(\tilde{J}_\rho e_2, e_2)\cdot F(\tilde{J}_\rho e_3, e_3) = -4(a_8a_{10} - a_9a_{11})^2 b_4^2 \leq 0.$$
Thus, we take $X = e_2$ and $Y = e_3$ in Proposition 2.1.5 *(2.)*.

- The Lie algebra $\mathfrak{e}(2)\oplus\mathfrak{h}$ defined by the structure equations
$$\mathfrak{e}(\mathbf{2}) \oplus \mathfrak{h} = (\mathbf{0}, -\mathbf{e^{13}}, \mathbf{e^{12}}, \mathbf{0}, \mathbf{0}, \mathbf{e^{45}}),$$
is such that
$$Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{h}\big) = \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{23}, e^{45}, e^{46}, e^{56}\rangle,$$
and
$$\begin{aligned} Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{h}\big) =& \langle e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{145}, e^{146}, e^{156}, \\ & e^{234}, e^{235}, e^{136}+e^{245}, -e^{126}+e^{345}, e^{456}\rangle. \end{aligned}$$
Then, any pair $(F,\rho) \in Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{h}\big) \times Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{h}\big)$ is given by
$$F = b_1e^{12} + b_2e^{13} + b_3e^{14} + b_4e^{15} + b_5e^{23} + b_6e^{45} + b_7e^{46} + b_8e^{56},$$

$$\rho = a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{134} + a_5 e^{135} + a_6 e^{145} + a_7 e^{146} + a_8 e^{156} + a_9 e^{234} + a_{10} e^{235} + a_{11}(e^{136} + e^{245}) + a_{12}(-e^{126} + e^{345}) + a_{13} e^{456}.$$

It is not necessary to impose the condition $F \wedge \rho = 0$ since

$$F(\tilde{J}_\rho e_6, e_6) = 0.$$

Hence, we take $X = Y = e_6$.

- Take the Lie algebra

$$\mathfrak{e}(\mathbf{1},\mathbf{1}) \oplus \mathfrak{h} = (\mathbf{0}, -\mathbf{e^{13}}, -\mathbf{e^{12}}, \mathbf{0}, \mathbf{0}, \mathbf{e^{45}}).$$

We have

$$Z^2\big(\mathfrak{e}(1,1) \oplus \mathfrak{h}\big) = \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{23}, e^{45}, e^{46}, e^{56} \rangle,$$

and

$$Z^3\big(\mathfrak{e}(1,1) \oplus \mathfrak{h}\big) = \langle e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{136} + e^{245}, e^{126} + e^{345}, e^{456} \rangle.$$

Consequently, any pair $(F, \rho) \in Z^2\big(\mathfrak{e}(1,1) \oplus \mathfrak{h}\big) \times Z^3\big(\mathfrak{e}(1,1) \oplus \mathfrak{h}\big)$ can be expressed as

$$F = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{23} + b_6 e^{45} + b_7 e^{46} + b_8 e^{56},$$

$$\rho = a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{134} + a_5 e^{135} + a_6 e^{145} + a_7 e^{146} + a_8 e^{156} + a_9 e^{234} + a_{10} e^{235} + a_{11}(e^{136} + e^{245}) + a_{12}(e^{126} + e^{345}) + a_{13} e^{456}.$$

Exactly as for $\mathfrak{e}(2) \oplus \mathfrak{h}$ we also obtain that

$$F(\tilde{J}_\rho e_6, e_6) = 0,$$

Therefore, we can consider $X = Y = e_6$.

For the two Lie algebras $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ and $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ we use Proposition 2.1.5 *(1.)* for an appropriate 1-form α on the Lie algebra.

- For $\mathfrak{e}(2)\oplus\mathfrak{r}_2\oplus\mathbb{R}$,

$$\mathfrak{e}(\mathbf{2})\oplus\mathfrak{r}_{\mathbf{2}}\oplus\mathbb{R}=(\mathbf{0},-\mathbf{e^{13}},\mathbf{e^{12}},\mathbf{0},-\mathbf{e^{45}},\mathbf{0}),$$

the spaces Z^k $(k=2,3)$ are

$$Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)=\langle e^{12},e^{13},e^{14},e^{16},e^{23},e^{45},e^{46}\rangle,$$

and

$$\begin{aligned}Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)=&\langle e^{123},e^{124},e^{126},e^{134},e^{136},e^{145},e^{146},e^{234},\\&e^{236},-e^{135}+e^{245},e^{125}+e^{345},e^{456}\rangle.\end{aligned}$$

Thus, any pair $(F,\rho)\in Z^2\big(\mathfrak{e}(2)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)\times Z^3\big(\mathfrak{e}(2)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)$ is given by

$$F=b_1e^{12}+b_2e^{13}+b_3e^{14}+b_4e^{16}+b_5e^{23}+b_6e^{45}+b_7e^{46},$$

and

$$\begin{aligned}\rho=&a_1e^{123}+a_2e^{124}+a_3e^{126}+a_4e^{134}+a_5e^{136}+a_6e^{145}+a_7e^{146}+a_8e^{234}\\&+a_9e^{236}+a_{10}(-e^{135}+e^{245})+a_{11}(e^{125}+e^{345})+a_{12}e^{456}.\end{aligned}$$

An easy computation shows that

$$\begin{aligned}\tilde{J}_\rho^*e^4=&\,(2a_3a_{10}+2a_5a_{11})\,e^1+2a_9a_{11}e^2-2a_9a_{10}e^3\\&+(a_6a_9-a_5a_{10}+a_3a_{11}-a_1a_{12})\,e^4,\end{aligned}$$

for arbitrary real numbers a_i and b_j. So,

$$e^4\wedge\tilde{J}_\rho^*e^4\wedge F^2=0.$$

Hence, we can apply Proposition 2.1.5 *(1.)* with $\alpha=e^4$.

- For $\mathfrak{e}(1,1)\oplus\mathfrak{r}_2\oplus\mathbb{R}$,

$$\mathfrak{e}(\mathbf{1},\mathbf{1})\oplus\mathfrak{r}_{\mathbf{2}}\oplus\mathbb{R}=(\mathbf{0},-\mathbf{e^{13}},-\mathbf{e^{12}},\mathbf{0},-\mathbf{e^{45}},\mathbf{0}),$$

the spaces Z^k $(k=2,3)$ are

$$Z^2\big(\mathfrak{e}(1,1)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)=\langle e^{12},e^{13},e^{14},e^{16},e^{23},e^{45},e^{46}\rangle,$$

and

$$\begin{aligned}Z^3\big(\mathfrak{e}(1,1)\oplus\mathfrak{r}_2\oplus\mathbb{R}\big)=&\langle e^{123},e^{124},e^{126},e^{134},e^{136},e^{145},e^{146},e^{234},\\&e^{236},-e^{135}+e^{245},-e^{125}+e^{345},e^{456}\rangle.\end{aligned}$$

Then, any pair $(F, \rho) \in Z^2\big(\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}\big) \times Z^3\big(\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}\big)$ can be expressed as follows

$$F = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{16} + b_5 e^{23} + b_6 e^{45} + b_7 e^{46},$$

and

$$\begin{aligned}\rho = & a_1 e^{123} + a_2 e^{124} + a_3 e^{126} + a_4 e^{134} + a_5 e^{136} + a_6 e^{145} + a_7 e^{146} + a_8 e^{234} \\ & + a_9 e^{236} + a_{10}(-e^{135} + e^{245}) + a_{11}(-e^{125} + e^{345}) + a_{12} e^{456}.\end{aligned}$$

Now we have

$$\begin{aligned}\tilde{J}_\rho^* e^4 = & \,(2a_3 a_{10} + 2a_5 a_{11})\, e^1 - 2a_9 a_{11} e^2 - 2a_9 a_{10} e^3 \\ & + (a_6 a_9 - a_5 a_{10} + a_3 a_{11} - a_1 a_{12})\, e^4,\end{aligned}$$

for arbitrary real numbers a_i and b_j. So,

$$e^4 \wedge \tilde{J}_\rho^* e^4 \wedge F^2 = 0.$$

Hence, we can apply Proposition 2.1.5 *(1.)* with $\alpha = e^4$.

□

2.2.2 $4 \oplus 2$ Lie algebras

According to [65], there are 6 (non-nilpotent) solvable Lie algebras as well as a one-parameter family of type $4 \oplus 2$ with half-flat structures. Below, in Table 2.2, we consider only those also admitting symplectic forms and such that cannot be decomposed as $3 \oplus 3$. As usual, we denote by F a symplectic form and by (ω, ψ_+) a half-flat structure.

Table 2.2: $4 \oplus 2$ **decomposable (non-nilpotent) solvable Lie algebras admitting both symplectic and half-flat structures.**

$\mathfrak{g}$	**str. equations**	**half-flat str.**	**symplectic str.**
$A_{4,1} \oplus \mathfrak{r}_2$	$(e^{24}, e^{34}, 0, 0, 0, e^{56})$	$\omega = -e^{16} + e^{25} - e^{34}$ $\psi_+ = e^{123} - e^{145} + e^{156}$ $-e^{246} + e^{345} - 2e^{356}$	$F = e^{14} + e^{23} + e^{56}$
$A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$	$(\frac{1}{2}e^{14} + e^{23}, e^{24}, -\frac{1}{2}e^{34}, 0, 0, e^{56})$	$\omega = e^{16} - 3e^{24} + 2e^{25} + e^{35}$ $\psi_+ = \sqrt{3}(e^{124} + 2e^{134} - e^{135}$ $+e^{146} - 2e^{156} + 2e^{236}$ $+4e^{245} - e^{345} + \frac{29}{2}e^{456})$	$F = e^{13} + e^{24} + e^{56}$
$A_{4,12} \oplus \mathfrak{r}_2$	$(e^{13} + e^{24}, -e^{14} + e^{23}, 0, 0, 0, e^{56})$	$\omega = e^{16} - 2e^{23} + e^{25} + e^{34} - e^{36}$ $\psi_+ = e^{123} + 2e^{134} - e^{136} + e^{145}$ $+e^{156} - e^{235} - e^{246} + 2e^{356}$	$F = e^{13} + e^{24} + e^{56}$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$	$(0, -e^{12}, 0, -e^{34}, 0, -e^{56})$	$\omega = e^{12} - e^{23} - e^{25} - e^{35} + e^{46}$ $\psi_+ = e^{124} - e^{126} + 2e^{134} + 3e^{156}$ $-e^{234} + e^{256} + e^{345} + 2e^{356}$	$F = e^{12} + e^{34} + e^{56}$

We should remark that all the Lie algebras appearing in Table 2.2 are non-unimodular.

Proposition 2.2.2. *Let $\mathfrak{g}$ be a 6-dimensional decomposable and (non-nilpotent) solvable Lie algebra such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $dim(\mathfrak{g}_1) = 4$, $dim(\mathfrak{g}_2) = 2$. Then, $\mathfrak{g}$ has no symplectic half-flat structure.*

Proof. We use Proposition 2.1.5 *(1.)* for an appropriate 1-form α on the Lie algebra.

- For the Lie algebra

$$\mathbf{A_{4,1}} \oplus \mathfrak{r}_\mathbf{2} = (\mathbf{e^{24}}, \mathbf{e^{34}}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e^{56}}),$$

we have

$$Z^2\big(A_{4,1} \oplus \mathfrak{r}_2\big) = \langle e^{14}, e^{15}, e^{24}, e^{34}, e^{45}, e^{56}\rangle,$$

and

$$\begin{aligned} Z^3\big(A_{4,1} \oplus \mathfrak{r}_2\big) =&\langle e^{124}, e^{134}, e^{145}, e^{156}, e^{234}, e^{245}, -e^{246} + e^{256}, \\ &e^{345}, -e^{346} + e^{356}, e^{456}\rangle. \end{aligned}$$

Then, any pair $(F, \rho) \in Z^2\big(A_{4,1} \oplus \mathfrak{r}_2\big) \times Z^3\big(A_{4,1} \oplus \mathfrak{r}_2\big)$ is of the form

$$\begin{aligned} F =& b_1e^{14} + b_2e^{15} + b_3e^{24} + b_4e^{34} + b_5e^{45} + b_6e^{56}, \\ \rho =& a_1e^{124} + a_2e^{134} + a_3e^{145} + a_4e^{156} + a_5e^{234} + a_6e^{245} \\ &- a_7e^{246} + a_7e^{256} + a_8e^{345} - a_9e^{346} + a_9e^{356} + a_{10}e^{456}, \end{aligned}$$

for any real numbers a_i and b_j. So,

$$e^4 \wedge \tilde{J}^*_\rho e^4 \wedge F^2 = 0.$$

Hence, we can apply Proposition 2.1.5 *(1.)* with $\alpha = e^4$.

- The Lie algebra $A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$ with structure equations

$$\mathbf{A_{4,9}^{-\frac{1}{2}}} \oplus \mathfrak{r}_\mathbf{2} = \Big(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{14}} + \mathbf{e^{23}}, \mathbf{e^{24}}, -\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{34}}, \mathbf{0}, \mathbf{0}, \mathbf{e^{56}}\Big),$$

is such that

$$Z^2\big(A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2\big) = \langle e^{13}, \frac{1}{2}e^{14} + e^{23}, e^{24}, e^{34}, e^{45}, e^{56}\rangle,$$

and

$$Z^3\big(A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2\big) = \langle e^{124}, e^{134}, e^{135}, e^{234}, \frac{1}{2}e^{145} + e^{235}, \frac{1}{2}e^{146} - e^{156} + e^{236}, \\ e^{245}, -e^{246} + e^{256}, e^{345}, \frac{1}{2}e^{346} + e^{356}, e^{456}\rangle.$$

Then, any pair $(F,\rho) \in Z^2\big(A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2\big) \times Z^3\big(A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2\big)$ is given by

$$\begin{aligned} F =& b_1e^{13} + \frac{b_2}{2}e^{14} + b_2e^{23} + b_3e^{24} + b_4e^{34} + b_5e^{45} + b_6e^{56}, \\ \rho =& a_1e^{124} + a_2e^{134} + a_3e^{135} + \frac{a_5}{2}e^{145} + \frac{a_6}{2}e^{146} - a_6e^{156} + a_4e^{234} + a_5e^{235} \\ &+ a_6e^{236} + a_7e^{245} - a_8e^{246} + a_8e^{256} + a_9e^{345} + \frac{a_{10}}{2}e^{346} + a_{10}e^{356} + a_{12}e^{456}. \end{aligned}$$

Therefore, the expression of $\tilde{J}_\rho^* e^4$ is

$$\tilde{J}_\rho^* e^4 = 2a_3a_6e^3 + (a_4a_6 + a_2a_8 - a_3a_8 - a_1a_{10})\, e^4 + (2a_5a_6 + 2a_3a_8)\, e^5 + 2a_6^2e^6,$$

so that

$$e^4 \wedge \tilde{J}_\rho^* e^4 \wedge F^2 = 0.$$

Thus, we apply Proposition 2.1.5 *(1.)* with $\alpha = e^4$.

- For $A_{4,12} \oplus \mathfrak{r}_2$, that is,

$$\mathbf{A_{4,12}} \oplus \mathfrak{r}_\mathbf{2} = (\mathbf{e^{13}} + \mathbf{e^{24}}, -\mathbf{e^{14}} + \mathbf{e^{23}}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{e^{56}}),$$

the spaces $Z^k\big(A_{4,12} \oplus \mathfrak{r}_2\big)$, with $k = 2, 3$, are

$$Z^2\big(A_{4,12} \oplus \mathfrak{r}_2\big) = \langle e^{13} + e^{24}, -e^{14} + e^{23}, e^{34}, e^{35}, e^{45}, e^{56}\rangle,$$

and

$$Z^3\big(A_{4,12} \oplus \mathfrak{r}_2\big) = \langle e^{123}, e^{134}, e^{234}, -e^{145} + e^{235}, e^{135} + e^{245}, e^{136} - e^{156} + e^{246}, \\ e^{146} - e^{236} + e^{256}, e^{345}, e^{356}, e^{456}\rangle.$$

Thus, any pair $(F,\rho) \in Z^2\big(A_{4,12} \oplus \mathfrak{r}_2\big) \times Z^3\big(A_{4,12} \oplus \mathfrak{r}_2\big)$ has the following expression

$$\begin{aligned} F =& b_2e^{13} - b_1e^{14} + b_1e^{23} + b_2e^{24} + b_3e^{34} + b_4e^{35} + b_5e^{45} + b_6e^{56}, \\ \rho =& a_1e^{123} + a_2e^{134} + a_5e^{135} + a_6e^{136} - a_4e^{145} + a_7e^{146} - a_6e^{156} + a_3e^{234} \\ &+ a_4e^{235} - a_7e^{236} + a_5e^{245} + a_6e^{246} + a_7e^{256} + a_8e^{345} + a_9e^{356} + a_{10}e^{456}. \end{aligned}$$

So,

$$e^3 \wedge \tilde{J}_\rho^* e^3 \wedge F^2 = 0,$$

that is, we can apply Proposition 2.1.5 *(1.)* with $\alpha = e^3$.

- The Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$ defined by

$$\boldsymbol{\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2 = (0, -e^{12}, 0, -e^{34}, 0, -e^{56})}$$

is such that

$$Z^2\big(\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2\big) = \langle e^{12}, e^{13}, e^{15}, e^{34}, e^{35}, e^{56}\rangle,$$

and

$$\begin{aligned} Z^3\big(\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2\big) =& \langle e^{123}, e^{125}, e^{134}, e^{135}, e^{156}, -e^{124} + e^{234}, \\ & - e^{126} + e^{256}, e^{345}, e^{356}, -e^{346} + e^{456}\rangle. \end{aligned}$$

Then, any pair $(F, \rho) \in Z^2(\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2) \times Z^3(\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2)$ is given by

$$\begin{aligned} F =& b_1 e^{12} + b_2 e^{13} + b_3 e^{15} + b_4 e^{34} + b_5 e^{35} + b_6 e^{56}, \\ \rho =& a_1 e^{123} - a_6 e^{124} + a_2 e^{125} - a_7 e^{126} + a_3 e^{134} + a_4 e^{135} + a_5 e^{156} \\ & + a_6 e^{234} + a_7 e^{256} + a_8 e^{345} - a_{10} e^{346} + a_9 e^{356} + a_{10} e^{456}. \end{aligned}$$

Hence

$$(e^1 + e^3) \wedge \tilde{J}_\rho^* (e^1 + e^3) \wedge F^2 = 0,$$

so, we can apply Proposition 2.1.5 *(1.)* with $\alpha = e^1 + e^3$.

□

2.2.3 $5 \oplus 1$ Lie algebras

According to Theorem 1.1.9, the unique (non-Abelian) nilpotent Lie algebra which is decomposable and has symplectic half-flat structure is the Lie algebra defined by $(0, 0, 0, 0, 12, 13)$. Thus, we use the classification given in [65] of $5 \oplus 1$ decomposable (non-nilpotent) solvable Lie algebras which have half-flat structure. There it is proved that there are 13 (non-nilpotent) solvable Lie algebras and 4 one-parameter families with such a structure. In the following table we describe which of them also admit a symplectic form.

Table 2.3: $5 \oplus 1$ **decomposable (non-nilpotent) solvable Lie algebras admitting both symplectic and half-flat structures.**

$\mathfrak{g}$	**str. equations**	**half-flat str.**	**symplectic str.**
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ $0 < \beta < 1$	$(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\psi_+ = e^{126} + e^{145} + e^{235} + e^{346}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\psi_+ = e^{126} + e^{145} + e^{235} + e^{346}$	$F = -e^{13} + e^{24} + e^{56}$
$A_{5,8}^{-1} \oplus \mathbb{R}$	$(e^{25}, 0, e^{35}, -e^{45}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\psi_+ = e^{126} + e^{145} + e^{235} + e^{346}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ $\gamma > 0$	$(e^{15}, -e^{25}, \gamma e^{45}, -\gamma e^{35}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\rho = e^{126} + e^{145} + e^{235} + e^{346}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,14}^{0} \oplus \mathbb{R}$	$(e^{25}, 0, e^{45}, -e^{35}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\psi_+ = e^{126} + e^{145} + e^{235} + e^{346}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,15}^{-1} \oplus \mathbb{R}$	$(e^{15} + e^{25}, e^{25}, -e^{35} + e^{45},$ $-e^{45}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\psi_+ = e^{125} + e^{146} - e^{236} - e^{345}$	$F = -e^{14} + e^{23} + e^{56}$
$A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$ $0 < \gamma < 1$	$(e^{25}, -e^{15}, \gamma e^{45}, -\gamma e^{35}, 0, 0)$	$\omega = -e^{13} + e^{24} + e^{56}$ $\rho = e^{126} + e^{145} + e^{235} + e^{346}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ $\alpha \geq 0$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25},$ $-\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0)$	$\omega = e^{13} + e^{24} + e^{56}$ $\rho = e^{125} - e^{146} + e^{236} - e^{345}$	$F = e^{13} + e^{24} + e^{56}$
$A_{5,18}^{0} \oplus \mathbb{R}$	$(e^{25} + e^{35}, -e^{15} + e^{45}, e^{45},$ $-e^{35}, 0, 0)$	$\omega = e^{12} - e^{34} - e^{56}$ $\psi_+ = e^{136} + e^{145} - e^{235} + e^{246}$	$F = e^{13} + e^{24} + e^{56}$
$A_{5,19}^{-1,2} \oplus \mathbb{R}$	$(-e^{15} + e^{23}, e^{25}, -2e^{35},$ $2e^{45}, 0, 0)$	$\omega = e^{13} + e^{24} - 2e^{25} - e^{56}$ $\psi_+ = -e^{126} + e^{145} - e^{234}$ $e^{346} - e^{356}$	$F = e^{12} + e^{34} + e^{56}$
$A_{5,36} \oplus \mathbb{R}$	$(e^{14} + e^{23}, e^{24} - e^{25},$ $e^{35}, 0, 0, 0)$	$\omega = \frac{1}{12} e^{12} + e^{13} + e^{16}$ $-\frac{1}{4} e^{24} + e^{46} + e^{56}$ $\psi_+ = -\frac{1}{6} e^{124} + \frac{1}{2} e^{125} - e^{134} - e^{135}$ $+4e^{146} + 4e^{236} + 3e^{345} + 3e^{456}$	$F = e^{14} + e^{23} + e^{56}$
$A_{5,37} \oplus \mathbb{R}$	$(2e^{14} + e^{23}, e^{24} + e^{35},$ $-e^{25} + e^{34}, 0, 0, 0)$	$\omega = -\frac{1}{3} e^{16} + 3e^{24} + e^{35}$ $\psi_+ = -e^{125} + 3e^{134} + 2e^{146}$ $e^{236} + 6e^{345} - \frac{13}{3} e^{456}$	$F = 2e^{14} + e^{23} + e^{56}$

Note that, in Table 2.3, the only $5 \oplus 1$ *non-unimodular* solvable Lie algebras are $A_{5,36} \oplus \mathbb{R}$ and $A_{5,37} \oplus \mathbb{R}$.

Proposition 2.2.3. *Let $\mathfrak{g}$ be a 6-dimensional decomposable and (non-nilpotent) solvable Lie algebra such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $dim(\mathfrak{g}_1) = 5$ and $dim(\mathfrak{g}_2) = 1$. Then, $\mathfrak{g}$ has symplectic half-flat structure if and only if $\mathfrak{g}$ is either $\mathfrak{g} = A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ or $\mathfrak{g} = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$, with α a non-negative real number.*

Proof. We show first that $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ have a symplectic half-flat structure.

- For $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ defined by

$$\mathbf{A_{5,7}^{-1,-1,1}} \oplus \mathbb{R} = (\mathbf{e^{15}}, -\mathbf{e^{25}}, -\mathbf{e^{35}}, \mathbf{e^{45}}, \mathbf{0}, \mathbf{0}),$$

the differential forms ω and ψ_+ appearing in Table 2.3 are closed and determine an SU(3)-structure. In fact, let us consider the change of basis given by

$$f^1 = -e^1, \quad f^2 = e^3, \quad f^3 = e^2, \quad f^4 = e^4, \quad f^5 = -e^6, \quad f^6 = e^5.$$

Then, in this new basis, ω and ψ_+ are

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_+ &= f^{135} - f^{146} - f^{236} - f^{245}. \end{aligned}$$

- For the one-parameter family of Lie algebras

$$\mathbf{A}_{\mathbf{5,17}}^{\alpha,-\alpha,\mathbf{1}} \oplus \mathbb{R} = (\alpha\mathbf{e^{15}} + \mathbf{e^{25}}, -\mathbf{e^{15}} + \alpha\mathbf{e^{25}}, -\alpha\mathbf{e^{35}} + \mathbf{e^{45}}, -\mathbf{e^{35}} - \alpha\mathbf{e^{45}}, \mathbf{0}, \mathbf{0}),$$

we have that any Lie algebra corresponding to a non-negative real number α has a symplectic half-flat structure. Actually, the differential forms ω and ψ_+ appearing in Table 2.3 are closed and determine an SU(3)-structure. This can be checked from the fact that with the change of basis

$$f^1 = e^1, \quad f^2 = e^3, \quad f^3 = e^2, \quad f^4 = e^4, \quad f^5 = e^5, \quad f^6 = e^6,$$

the forms ω and ψ_+ are described canonically.

Notice that $A_{5,17}^{\alpha,\beta,\gamma} \cong A_{5,17}^{-\alpha,-\beta,\gamma}$, thus we can restrict the study to $\alpha \geq 0$.

Now, we prove that there are no more Lie algebras from Table 2.3 having a symplectic half-flat structure. For the next ones we use Proposition 2.1.5 *(2.)* with appropriate vectors X and Y.

- The Lie algebra $A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ (with $0 < \beta < 1$) has the following structure equations

$$\mathbf{A}_{\mathbf{5,7}}^{-\mathbf{1},\beta,-\beta} \oplus \mathbb{R} = (\mathbf{e^{15}}, -\mathbf{e^{25}}, \beta\mathbf{e^{35}}, -\beta\mathbf{e^{45}}, \mathbf{0}, \mathbf{0}).$$

Thus, any pair $(F, \rho) \in Z^2(A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}) \times Z^3(A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R})$ is given by

$$F = b_1e^{12} + b_2e^{15} + b_3e^{25} + b_4e^{34} + b_5e^{35} + b_6e^{45} + b_7e^{56},$$

and

$$\begin{aligned} \rho = {} & a_1e^{125} + a_2e^{126} + a_3e^{135} + a_4e^{145} + a_5e^{156} + a_6e^{235} + a_7e^{245} \\ & + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}, \end{aligned}$$

where a_i and b_j are real numbers. These expressions of F and ρ imply that

$$F(\tilde{J}_\rho e_1, e_1) = 0,$$

that is, Proposition 2.1.5 *(2.)* can be applied for $X = Y = e_1$.

- For

$$\mathbf{A_{5,8}^{-1}} \oplus \mathbb{R} = (\mathbf{e^{25}}, \mathbf{0}, \mathbf{e^{35}}, -\mathbf{e^{45}}, \mathbf{0}, \mathbf{0}),$$

any pair $(F, \rho) \in Z^2(A_{5,8}^{-1} \oplus \mathbb{R}) \times Z^3(A_{5,8}^{-1} \oplus \mathbb{R})$ is of the form

$$F = b_1 e^{12} + b_2 e^{15} + b_3 e^{25} + b_4 e^{26} + b_5 e^{34} + b_6 e^{35} + b_7 e^{45} + b_8 e^{56},$$

and

$$\begin{aligned}\rho = {} & a_1 e^{125} + a_2 e^{126} + a_3 e^{135} + a_4 e^{145} + a_5 e^{156} + a_6 e^{234} + a_7 e^{235} \\ & + a_8 e^{245} + a_9 e^{256} + a_{10} e^{345} + a_{11} e^{346} + a_{12} e^{356} + a_{13} e^{456}.\end{aligned}$$

Now, we obtain that

$$F(\tilde{J}_\rho e_3, e_3) = 0.$$

Hence, it is sufficient to consider $X = Y = e_3$.

- The Lie algebra $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ (with $0 < \gamma$) with structure equations

$$\mathbf{A_{5,13}^{-1,0,\gamma}} \oplus \mathbb{R} = (\mathbf{e^{15}}, -\mathbf{e^{25}}, \gamma\mathbf{e^{45}}, -\gamma\mathbf{e^{35}}, \mathbf{0}, \mathbf{0})$$

is such that the pair $(F, \rho) \in Z^2(A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}) \times Z^3(A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R})$ is

$$F = b_1 e^{12} + b_2 e^{15} + b_3 e^{25} + b_4 e^{34} + b_5 e^{35} + b_6 e^{45} + b_7 e^{56},$$

and

$$\begin{aligned}\rho = {} & a_1 e^{125} + a_2 e^{126} + a_3 e^{135} + a_4 e^{145} + a_5 e^{156} + a_6 e^{235} + a_7 e^{245} \\ & + a_8 e^{256} + a_9 e^{345} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}.\end{aligned}$$

These expressions of F and ρ imply that

$$F(\tilde{J}_\rho e_1, e_1) = 0.$$

Then, Proposition 2.1.5 *(2.)* applies with $X = Y = e_1$.

- For
$$\mathbf{A^0_{5,14}} \oplus \mathbb{R} = (\mathbf{e^{25}}, \mathbf{0}, \mathbf{e^{45}}, -\mathbf{e^{35}}, \mathbf{0}, \mathbf{0}),$$
the pairs $(F, \rho) \in Z^2(A^0_{5,14} \oplus \mathbb{R}) \times Z^3(A^0_{5,14} \oplus \mathbb{R})$ are given by
$$F = b_1 e^{12} + b_2 e^{15} + b_3 e^{25} + b_4 e^{26} + b_5 e^{34} + b_6 e^{35} + b_7 e^{45} + b_8 e^{56},$$
and
$$\begin{aligned}\rho = &a_1 e^{125} + a_2 e^{126} + a_3 e^{135} + a_4 e^{145} + a_5 e^{156} + a_6 e^{234} + a_7 e^{235} \\ &+ a_8 e^{245} + a_9 e^{256} + a_{10} e^{345} + a_{11} e^{346} + a_{12} e^{356} + a_{13} e^{456}.\end{aligned}$$
We obtain again that
$$F(\tilde{J}_\rho e_1, e_1) = 0,$$
and hence, it is sufficient to consider Proposition 2.1.5 *(2.)* with $X = Y = e_1$.

- The Lie algebra $A^{-1}_{5,15} \oplus \mathbb{R}$ is defined by
$$\mathbf{A^{-1}_{5,15}} \oplus \mathbb{R} = (\mathbf{e^{15}} + \mathbf{e^{25}}, \mathbf{e^{25}}, -\mathbf{e^{35}} + \mathbf{e^{45}}, -\mathbf{e^{45}}, \mathbf{0}, \mathbf{0}).$$
Thus, any pair $(F, \rho) \in Z^2(A^{-1}_{5,15} \oplus \mathbb{R}) \times Z^3(A^{-1}_{5,15} \oplus \mathbb{R})$ is given by
$$F = -b_2 e^{14} + b_1 e^{15} + b_2 e^{23} + b_3 e^{24} + b_4 e^{25} + b_5 e^{35} + b_6 e^{45} + b_7 e^{56},$$
and
$$\begin{aligned}\rho = &a_1 e^{125} + a_2 e^{135} + a_3 e^{145} - a_6 e^{146} + a_4 e^{156} + a_5 e^{235} + a_6 e^{236} + a_7 e^{245} \\ &+ a_8 e^{246} + a_9 e^{256} + a_{10} e^{345} + a_{11} e^{356} + a_{12} e^{456}.\end{aligned}$$
The expressions of F and ρ imply that
$$F(\tilde{J}_\rho e_1, e_1) = 0.$$
Thus, Proposition 2.1.5 *(2.)* applies for $X = Y = e_1$.

- For
$$\mathbf{A^{0,0,\gamma}_{5,17}} \oplus \mathbb{R} = (\mathbf{e^{25}}, -\mathbf{e^{15}}, \gamma\mathbf{e^{45}}, -\gamma\mathbf{e^{35}}, \mathbf{0}, \mathbf{0}),$$
(with $0 < \gamma < 1$), the pairs $(F, \rho) \in Z^2(A^{0,0,\gamma}_{5,17} \oplus \mathbb{R}) \times Z^3(A^{0,0,\gamma}_{5,17} \oplus \mathbb{R})$ are of the form
$$F = b_1 e^{12} + b_2 e^{15} + b_3 e^{25} + b_4 e^{34} + b_5 e^{35} + b_6 e^{45} + b_7 e^{56},$$

and

$$\begin{aligned}\rho = & a_1e^{125} + a_2e^{126} + a_3e^{135} + a_4e^{145} + a_5e^{156} + a_6e^{235} \\ & + a_7e^{245} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

We obtain that

$$F(\tilde{J}_\rho e_2, e_2) = 0,$$

and then, it is sufficient to consider $X = Y = e_2$.

- The Lie algebra $A^0_{5,18} \oplus \mathbb{R}$, given by the structure equations

$$\mathbf{A^0_{5,18}} \oplus \mathbb{R} = (\mathbf{e^{25}} + \mathbf{e^{35}}, -\mathbf{e^{15}} + \mathbf{e^{45}}, \mathbf{e^{45}}, -\mathbf{e^{35}}, \mathbf{0}, \mathbf{0}),$$

is such that the pair $(F, \rho) \in Z^2(A^0_{5,18} \oplus \mathbb{R}) \times Z^3(A^0_{5,18} \oplus \mathbb{R})$ is of the form

$$\begin{aligned}F = & b_2e^{13} + b_1e^{15} + b_2e^{24} + b_3e^{25} + b_4e^{34} + b_5e^{35} + b_6e^{45} + b_7e^{56}, \\ \rho = & a_1e^{125} + a_2e^{135} + a_7e^{136} + a_3e^{145} + a_4e^{156} + a_5e^{235} + a_6e^{245} \\ & + a_7e^{246} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

These expressions of F and ρ imply that

$$F(\tilde{J}_\rho e_2, e_2) = 0.$$

Then, Proposition 2.1.5 *(2.)* applies for $X = Y = e_2$.

- For

$$\mathbf{A^{-1,2}_{5,19}} \oplus \mathbb{R} = (-\mathbf{e^{15}} + \mathbf{e^{23}}, \mathbf{e^{25}}, -\mathbf{2e^{35}}, \mathbf{2e^{45}}, \mathbf{0}, \mathbf{0}),$$

the pairs $(F, \rho) \in Z^2(A^{-1,2}_{5,19} \oplus \mathbb{R}) \times Z^3(A^{-1,2}_{5,19} \oplus \mathbb{R})$ are given by

$$F = b_1e^{12} - b_2e^{15} + b_2e^{23} + b_3e^{25} + b_4e^{34} + b_5e^{35} + b_6e^{45} + b_7e^{56},$$

and

$$\begin{aligned}\rho = & a_1e^{125} + a_2e^{126} + a_3e^{135} - a_4e^{145} - a_6e^{156} + a_4e^{234} + a_5e^{235} + a_6e^{236} \\ & + a_7e^{245} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

We obtain that

$$F(\tilde{J}_\rho e_4, e_4) = 0,$$

thus, it is sufficient to consider Proposition 2.1.5 *(2.)* for $X = e_4$ and $Y = e_4$.

For the two Lie algebras left, that is, $A_{5,36} \oplus \mathbb{R}$ and $A_{5,37} \oplus \mathbb{R}$ we use Proposition 2.1.5 *(1.)* with a convenient 1-form α on the Lie algebra.

- For the Lie algebra
$$\mathbf{A_{5,36}} \oplus \mathbb{R} = (\mathbf{e^{14}} + \mathbf{e^{23}}, \mathbf{e^{24}} - \mathbf{e^{25}}, \mathbf{e^{35}}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$
any pair $(F, \rho) \in Z^2(A_{5,36} \oplus \mathbb{R}) \times Z^3(A_{5,36} \oplus \mathbb{R})$ is described by
$$F = b_1 e^{14} + b_1 e^{23} - b_2 e^{24} + b_2 e^{25} + b_3 e^{35} + b_4 e^{45} + b_5 e^{46} + b_6 e^{56},$$
and
$$\begin{aligned}\rho = & -2a_1 e^{124} + a_1 e^{125} + a_2 e^{134} + a_2 e^{135} + a_4 e^{145} + a_5 e^{146} + a_3 e^{234} + a_4 e^{235} \\ & + a_5 e^{236} + a_6 e^{245} - a_7 e^{246} + a_7 e^{256} + a_8 e^{345} + a_9 e^{356} + a_{10} e^{456}.\end{aligned}$$
Computing the expression of $\tilde{J}^*_\rho$ we obtain that
$$\tilde{J}^*_\rho e^4 = 2a_1 a_5 e^2 + 2a_2 a_5 e^3 + 2a_1 a_9 e^4 + (2a_2 a_7 - 2a_1 a_9)\, e^5,$$
for arbitrary real numbers a_i and b_j. So,
$$e^4 \wedge \tilde{J}^*_\rho e^4 \wedge F^2 = 0.$$
Hence, we can apply Proposition 2.1.5 *(1.)* with $\alpha = e^4$.

- The Lie algebra $A_{5,37} \oplus \mathbb{R}$ is defined by the structure equations
$$\mathbf{A_{5,37}} \oplus \mathbb{R} = (\mathbf{2e^{14}} + \mathbf{e^{23}}, \mathbf{e^{24}} + \mathbf{e^{35}}, -\mathbf{e^{25}} + \mathbf{e^{34}}, \mathbf{0}, \mathbf{0}, \mathbf{0}).$$
Thus, the pairs $(F, \rho) \in Z^2(A_{5,37} \oplus \mathbb{R}) \times Z^3(A_{5,37} \oplus \mathbb{R})$ are of the form
$$F = 2b_1 e^{14} + b_1 e^{23} + b_3 e^{24} - b_2 e^{25} + b_2 e^{34} + b_3 e^{35} + b_4 e^{45} + b_5 e^{46} + b_6 e^{56},$$
and
$$\begin{aligned}\rho = & \, 3a_2 e^{124} + a_1 e^{125} - 3a_1 e^{134} + a_2 e^{135} + 2a_4 e^{145} + 2a_5 e^{146} + a_3 e^{234} + a_4 e^{235} \\ & + a_5 e^{236} + a_6 e^{245} + a_9 e^{246} - a_8 e^{256} + a_7 e^{345} + a_8 e^{346} + a_9 e^{356} + a_{10} e^{456}.\end{aligned}$$
An easy computation shows that
$$\tilde{J}^*_\rho e^4 = 2a_1 a_5 e^2 + 2a_2 a_5 e^3 + (2a_1 a_8 - 2a_2 a_9)\, e^4 + (-2a_2 a_8 - 2a_1 a_9)\, e^5,$$
for real numbers a_i and b_j. Therefore,
$$e^4 \wedge \tilde{J}^*_\rho e^4 \wedge F^2 = 0,$$
Hence, we can apply again Proposition 2.1.5 *(1.)* with $\alpha = e^4$. □

Remark 2.2.4. The Lie algebra $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ is unimodular and the corresponding simply connected solvable Lie group admits a compact quotient as it is shown in [123]. For the simply connected solvable Lie groups corresponding to $A_{5,7}^{-1,-1,1}$ and $A_{5,17}^{\alpha,-\alpha,1}$ with $\alpha \geq 0$, conditions for the existence of lattice are given in [19, Propositions 7.2.1 and 7.2.14]. In particular, there is a lattice for the cases $A_{5,7}^{-1,-1,1}$ and $A_{5,17}^{\alpha,-\alpha,1}$ for $\alpha = 0$ and for some positive α.

2.3 Unimodular indecomposable symplectic half-flat Lie algebras

This section is dedicated to studying symplectic half-flat structures on unimodular indecomposable solvable Lie algebras of dimension six. The classification of these Lie algebras is given in Proposition 2.3.1.

By Theorem 1.1.9, we know that $\mathfrak{g}_{6,N3} = (0,0,0,12,13,23)$ is the only indecomposable nilpotent Lie algebra admitting symplectic half-flat structures. Therefore, in the following we focus our attention on (non-nilpotent) solvable Lie algebras.

By [99] there exist 15 unimodular indecomposable 6-dimensional (non-nilpotent) solvable Lie algebras with symplectic forms. We show that only 3 of them have symplectic half-flat structures.

Table 2.4: **Unimodular indecomposable (non-nilpotent) solvable Lie algebras admitting symplectic forms** [99].

$\mathfrak{g}$	**str. equations**	**symplectic str.**
$\mathfrak{g}_{6,3}^{0,-1}$	$(e^{26}, e^{36}, 0, e^{46}, -e^{56}, 0)$	$F = e^{16} + e^{23} + e^{45}$
$\mathfrak{g}_{6,10}^{0,0}$	$(e^{26}, e^{36}, 0, e^{56}, -e^{46}, 0)$	$F = e^{16} + e^{23} + e^{45}$
$\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$	$(-\frac{1}{2}e^{16} + e^{23}, -e^{26}, \frac{1}{2}e^{36}, e^{46}, 0, 0)$	$F = e^{13} + e^{24} + e^{56}$
$\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$	$(-\frac{1}{2}e^{16} + e^{23}, \frac{1}{2}e^{26}, -e^{36}, e^{46}, 0, 0)$	$F = e^{12} + e^{34} + e^{56}$
$\mathfrak{g}_{6,15}^{-1}$	$(e^{23}, e^{26}, -e^{36}, e^{26} + e^{46}, e^{36} - e^{56}, 0)$	$F = e^{16} + \frac{1}{2}e^{25} + e^{34}$
$\mathfrak{g}_{6,18}^{-1,-1}$	$(e^{23}, -e^{26}, e^{36}, e^{36} + e^{46}, -e^{56}, 0)$	$F = e^{16} + e^{24} + e^{35}$
$\mathfrak{g}_{6,21}^{0}$	$(e^{23}, 0, e^{26}, e^{46}, -e^{56}, 0)$	$F = e^{12} + e^{36} + e^{45}$
$\mathfrak{g}_{6,36}^{0,0}$	$(e^{23}, 0, e^{26}, -e^{56}, e^{46}, 0)$	$F = 2e^{16} - e^{34} + e^{25}$
$\mathfrak{g}_{6,38}^{0}$	$(e^{23}, -e^{36}, e^{26}, e^{26} - e^{56}, e^{36} + e^{46}, 0)$	$F = -2e^{16} + e^{34} - e^{25}$
$\mathfrak{g}_{6,54}^{0,-1}$	$(e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0)$	$F = e^{14} + e^{23} + e^{56}$
$\mathfrak{g}_{6,70}^{0,0}$	$(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0)$	$F = e^{13} + e^{24} + e^{56}$
$\mathfrak{g}_{6,78}$	$(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36} + e^{46}, e^{46}, -e^{56}, 0)$	$F = e^{14} + e^{35} - e^{26}$
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$(-e^{16} + e^{25}, -e^{15} - e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0)$	$F = e^{14} + e^{23} - e^{56}$
$\mathfrak{n}_{6,84}^{\pm 1}$	$(-e^{45}, -e^{15} - e^{36}, -e^{14} + e^{26} \mp e^{56}, e^{56}, -e^{46}, 0)$	$F = \mp e^{16} + e^{25} + e^{34}$

Proposition 2.3.1. *Let $\mathfrak{g}$ be a 6-dimensional unimodular indecomposable and (non-nilpotent) solvable Lie algebra. Then, $\mathfrak{g}$ has symplectic half-flat structures if and only if $\mathfrak{g} = \mathfrak{g}_{6,38}^{0}$, $\mathfrak{g} = \mathfrak{g}_{6,54}^{0,-1}$ or $\mathfrak{g} = \mathfrak{g}_{6,118}^{0,-1,-1}$.*

Proof. First we give a symplectic half-flat structure on $\mathfrak{g}_{6,38}^{0}$, $\mathfrak{g}_{6,54}^{0,-1}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$.

- The Lie algebra $\mathfrak{g}^0_{6,38}$ is defined by the structure equations

$$\mathfrak{g}^0_{6,38} = (\mathbf{e^{23}}, -\mathbf{e^{36}}, \mathbf{e^{26}}, \mathbf{e^{26}} - \mathbf{e^{56}}, \mathbf{e^{36}} + \mathbf{e^{46}}, \mathbf{0}).$$

Therefore, the forms

$$\begin{aligned} \omega &= -2e^{12} + e^{34} - e^{25}, \\ \psi_+ &= -2e^{135} - 2e^{124} + e^{236} - e^{456} \end{aligned}$$

are closed, and they determine an SU(3)-structure since with the change of basis given by

$$\{f^1 = -2e^1, \quad f^2 = e^6, \quad f^3 = e^3, \quad f^4 = e^4, \quad f^5 = e^5, \quad f^6 = e^2\},$$

the forms ω and ψ_+ have the following expression

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_+ &= f^{135} - f^{146} - f^{236} - f^{245}. \end{aligned}$$

- For

$$\mathfrak{g}^{0,-1}_{6,54} = (\mathbf{e^{16}} + \mathbf{e^{35}}, -\mathbf{e^{26}} + \mathbf{e^{45}}, \mathbf{e^{36}}, -\mathbf{e^{46}}, \mathbf{0}, \mathbf{0}),$$

the forms

$$\begin{aligned} \omega &= e^{14} + e^{23} + e^{56}, \\ \psi_+ &= e^{125} - e^{136} + e^{246} + e^{345} \end{aligned}$$

are closed, and they define an SU(3)-structure. In fact, with respect to the new basis

$$\{f^1 = e^1, \quad f^2 = e^4, \quad f^3 = e^2, \quad f^4 = e^3, \quad f^5 = e^5, \quad f^6 = e^2\},$$

the pair (ω, ψ_+) is given by the canonical expression.

- The Lie algebra $\mathfrak{g}^{0,-1,-1}_{6,118}$ is defined by the structure equations

$$\mathfrak{g}^{0,-1,-1}_{6,118} = (-\mathbf{e^{16}} + \mathbf{e^{25}}, -\mathbf{e^{15}} - \mathbf{e^{26}}, \mathbf{e^{36}} - \mathbf{e^{45}}, \mathbf{e^{35}} + \mathbf{e^{46}}, \mathbf{0}, \mathbf{0}).$$

Thus, the forms

$$\begin{aligned} \omega &= e^{14} + e^{23} - e^{56}, \\ \psi_+ &= e^{126} - e^{135} + e^{245} + e^{346} \end{aligned}$$

are closed, and they determine an SU(3)-structure, because with the change of basis given by

$$\{f^1 = e^1, \quad f^2 = e^4, \quad f^3 = e^2, \quad f^4 = e^3, \quad f^5 = e^6, \quad f^6 = e^5\},$$

the forms ω and ψ_+ are expressed as

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_+ &= f^{135} - f^{146} - f^{236} - f^{245}. \end{aligned}$$

For the remaining Lie algebras of Table 2.4 we show the details of how Proposition 2.1.5 rejects the existence of symplectic half-flat structures.

- For the Lie algebra $\mathfrak{g}_{6,3}^{0,-1}$, whose structure equations are

$$\mathbf{\mathfrak{g}_{6,3}^{0,-1}} = (\mathbf{e^{26}}, \mathbf{e^{36}}, \mathbf{0}, \mathbf{e^{46}}, -\mathbf{e^{56}}, \mathbf{0}),$$

any pair (F, ρ) with $F \in Z^2(\mathfrak{g}_{6,3}^{0,-1})$ and $\rho \in Z^3(\mathfrak{g}_{6,3}^{0,-1})$ is given by

$$F = b_1 e^{16} + b_2 e^{23} + b_3 e^{26} + b_4 e^{36} + b_5 e^{45} + b_6 e^{46} + b_7 e^{56},$$

and

$$\begin{aligned} \rho = {} & a_1 e^{123} + a_2 e^{126} + a_3 e^{136} + a_4 e^{146} + a_5 e^{156} + a_6 e^{236} + a_7 e^{246} \\ & + a_8 e^{256} + a_9 e^{345} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}. \end{aligned}$$

Now $\tilde{J}_\rho^*$ is such that

$$\tilde{J}_\rho^* e^6 = (-a_2 a_9 - a_1 a_{12})\, e^6 - 2 a_1 a_9 e^3,$$

for arbitrary real numbers a_i and b_j. Therefore,

$$e^6 \wedge \tilde{J}_\rho^* e^6 \wedge F^2 = 0.$$

Thus, we apply Proposition 2.1.5 *(1.)* for the 1-form $\alpha = e^6$ and consequently $\mathfrak{g}_{6,3}^{0,-1}$ does not admit symplectic half-flat structures.

- The Lie algebra $\mathfrak{g}_{6,10}^{0,0}$ has structure equations

$$\mathbf{\mathfrak{g}_{6,10}^{0,0}} = (\mathbf{e^{26}}, \mathbf{e^{36}}, \mathbf{0}, \mathbf{e^{56}}, -\mathbf{e^{46}}, \mathbf{0}).$$

Then, any pair (F, ρ) with $F \in Z^2(\mathfrak{g}_{6,10}^{0,0})$ and $\rho \in Z^3(\mathfrak{g}_{6,10}^{0,0})$ is given by

$$F = b_1 e^{16} + b_2 e^{23} + b_3 e^{26} + b_4 e^{36} + b_5 e^{45} + b_6 e^{46} + b_7 e^{56},$$

and

$$\rho = a_1e^{123} + a_2e^{126} + a_3e^{136} + a_4e^{146} + a_5e^{156} + a_6e^{236} + a_7e^{246} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.$$

Now $\tilde{J}_\rho^*$ is such that

$$\tilde{J}_\rho^* e^6 = -2a_1a_9e^3 + (-a_2a_9 - a_1a_{12})\, e^6.$$

Thus,

$$e^6 \wedge \tilde{J}_\rho^* e^6 \wedge F^2 = 0.$$

Hence, we can apply Proposition 2.1.5 *(1.)* for the 1-form $\alpha = e^6$.

- For the Lie algebra $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$, whose structure equations are

$$\boldsymbol{\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0} = \Big(-\frac{1}{2}e^{16} + e^{23}, -e^{26}, \frac{1}{2}e^{36}, e^{46}, 0, 0\Big)},$$

any pair $(F,\rho) \in Z^2\big(\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}\big) \times Z^3\big(\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}\big)$ is given by

$$F = b_1e^{13} + b_2\Big(-\frac{1}{2}e^{16} + e^{23}\Big) + b_3e^{24} + b_4e^{26} + b_5e^{36} + b_6e^{46} + b_7e^{56},$$
$$\rho = a_1e^{126} + a_2e^{135} + a_3e^{136} + a_4\Big(\frac{1}{2}e^{146} - e^{234}\Big) + a_5\Big(\frac{1}{2}e^{156} + e^{235}\Big) + a_6e^{236} + a_7e^{245} + a_8e^{246} + a_9e^{256} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.$$

We have that $\tilde{J}_\rho$ is such that

$$\tilde{J}_\rho e_1 = (a_4a_5 - a_3a_7 + a_2a_8)e_1 - a_2a_4e_2 + 2a_1a_2e_4,$$

which implies

$$F(\tilde{J}_\rho e_1, e_1) = 0.$$

Therefore, we apply Proposition 2.1.5 *(2.)* with $X = Y = e_1$.

- The Lie algebra $\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}$ is described by

$$\boldsymbol{\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0} = \Big(-\frac{1}{2}e^{16} + e^{23}, \frac{1}{2}e^{26}, -e^{36}, e^{46}, 0, 0\Big)}.$$

Thus, any pair $(F,\rho) \in Z^2\big(\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}\big) \times Z^3\big(\mathfrak{g}_{6,13}^{\frac{1}{2},-1,0}\big)$ is of the form

$$F = b_1e^{12} - \frac{b_2e^{16}}{2} + b_2e^{23} + b_3e^{26} + b_4e^{34} + b_5e^{36} + b_6e^{46} + b_7e^{56},$$

and

$$\begin{aligned}\rho = {} & a_1e^{125} + a_2e^{126} + a_3e^{136} - \frac{a_4e^{146}}{2} + \frac{a_5e^{156}}{2} + a_4e^{234} + a_5e^{235} + a_6e^{236} \\ & + a_7e^{246} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

A direct calculation shows that

$$\tilde{J}_\rho e_1 = (-a_4a_5 + a_2a_9 - a_1a_{10})\, e_1 - a_1a_4e_3 - 2a_1a_3e_4,$$

for arbitrary real numbers a_i and b_j. Then

$$F(\tilde{J}_\rho e_1, e_1) = 0.$$

Now, we apply Proposition 2.1.5 *(2.)* with $X = e_1$ and $Y = e_1$.

- For the Lie algebra $\mathfrak{g}_{6,15}^{-1}$ defined by the structure equations

$$\boldsymbol{\mathfrak{g}_{6,15}^{-1}} = (\mathbf{e^{23}}, \mathbf{e^{26}}, -\mathbf{e^{36}}, \mathbf{e^{26}} + \mathbf{e^{46}}, \mathbf{e^{36}} - \mathbf{e^{56}}, \mathbf{0}),$$

any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,15}^{-1}) \times Z^3(\mathfrak{g}_{6,15}^{-1})$ is given by

$$\begin{aligned}F = {} & b_2e^{16} - b_4e^{16} + b_1e^{23} + b_2e^{25} + b_3e^{26} + b_4e^{34} + b_5e^{36} + b_6e^{46} + b_7e^{56}, \\ \rho = {} & a_1e^{123} + a_3e^{125} + a_2e^{126} + a_3e^{134} + a_4e^{136} - a_5e^{146} + a_6e^{156} + a_5e^{234} \\ & + a_6e^{235} + a_7e^{236} + a_8e^{246} + a_9e^{256} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

Thus,

$$\tilde{J}_\rho e_4 = (-2a_5a_{12})\, e_1 + (2a_3a_{12})e_2 + (-a_3a_9 + a_3a_{10} + a_1a_{12})\, e_4 + \left(2a_5^2 + 2a_3a_8\right) e_5,$$

which implies that

$$F(\tilde{J}_\rho e_4, e_4) = 0.$$

Therefore, we can apply Proposition 2.1.5 *(2.)* with $X = Y = e_4$.

- For the Lie algebra $\mathfrak{g}_{6,18}^{-1,-1}$ whose structure equations are

$$\boldsymbol{\mathfrak{g}_{6,18}^{-1,-1}} = (\mathbf{e^{23}}, -\mathbf{e^{26}}, \mathbf{e^{36}}, \mathbf{e^{36}} + \mathbf{e^{46}}, -\mathbf{e^{56}}, \mathbf{0}),$$

any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,18}^{-1,-1}) \times Z^3(\mathfrak{g}_{6,18}^{-1,-1})$ is of the form

$$F = b_2e^{16} + b_1e^{23} + b_2e^{24} + b_3e^{26} + b_4e^{35} + b_5e^{36} + b_6e^{46} + b_7e^{56},$$

and

$$\begin{aligned}\rho = {} & a_1e^{123} + a_2e^{126} + a_3e^{135} + a_4e^{136} - a_5e^{146} + a_6e^{156} + a_5e^{234} + a_6e^{235} \\ & + a_7e^{236} + a_8e^{246} + a_9e^{256} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}.\end{aligned}$$

Now $\tilde{J}_\rho^*$ is such that

$$\tilde{J}_\rho^* e^3 = (-2a_5a_6 - a_3a_8 + a_1a_{12})\, e^3 + (2a_6a_8 + 2a_5a_9 + 2a_2a_{12})\, e^6,$$

therefore

$$e^3 \wedge \tilde{J}_\rho^* e^3 \wedge F^2 = 0.$$

Thus, we consider Proposition 2.1.5 *(1.)* for the 1-form $\alpha = e^3$.

- The Lie algebra $\mathfrak{g}_{6,21}^0$ is defined by

$$\boldsymbol{\mathfrak{g}_{6,21}^0} = (\mathbf{e^{23}}, \mathbf{0}, \mathbf{e^{26}}, \mathbf{e^{46}}, -\mathbf{e^{56}}, \mathbf{0}).$$

We have that any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,21}^0) \times Z^3(\mathfrak{g}_{6,21}^0)$ is of the form

$$\begin{aligned} F =& b_1e^{12} + b_2e^{23} + b_3e^{26} + b_4e^{36} + b_5e^{45} + b_6e^{46} + b_7e^{56}, \\ \rho =& a_1e^{123} + a_2e^{126} + a_3e^{136} - a_4e^{146} + a_5e^{156} + a_4e^{234} + a_5e^{235} \\ &+ a_6e^{236} + a_7e^{245} + a_8e^{246} + a_9e^{256} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}. \end{aligned}$$

Then, $\tilde{J}_\rho^*$ satisfies that

$$\tilde{J}_\rho^* e^2 = (-2a_4a_5 - a_3a_7 + a_1a_{12})\, e^2 + (-2a_5a_{10} - 2a_4a_{11} - 2a_3a_{12})\, e^6.$$

So,

$$e^2 \wedge \tilde{J}_\rho^* e^2 \wedge F^2 = 0,$$

and we apply Proposition 2.1.5 *(1.)* with the 1-form $\alpha = e^2$.

- Consider the Lie algebra $\mathfrak{g}_{6,36}^{0,0}$ whose structure equations are

$$\boldsymbol{\mathfrak{g}_{6,36}^{0,0}} = (\mathbf{e^{23}}, \mathbf{0}, \mathbf{e^{26}}, -\mathbf{e^{56}}, \mathbf{e^{46}}, \mathbf{0}).$$

Any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,36}^{0,0}) \times Z^3(\mathfrak{g}_{6,36}^{0,0})$ is given by

$$\begin{aligned} F =& b_1e^{12} + b_2e^{23} + b_3e^{26} + b_4e^{36} + b_5e^{45} + b_6e^{46} + b_7e^{56}, \\ \rho =& a_1e^{123} + a_2e^{126} + a_3e^{136} + a_4e^{146} + a_5e^{156} + a_5e^{234} - a_4e^{235} + a_6e^{236} \\ &+ a_7e^{245} + a_8e^{246} + a_9e^{256} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}. \end{aligned}$$

Then, $\tilde{J}_\rho^*$ is such that

$$\tilde{J}_\rho^* e^2 = \left(-a_4^2 - a_5^2 - a_3a_7 + a_1a_{12}\right) e^2 + (-2a_5a_{10} + 2a_4a_{11} - 2a_3a_{12})\, e^6,$$

therefore

$$e^2 \wedge \tilde{J}_\rho^* e^2 \wedge F^2 = 0.$$

Thus, we take $\alpha = e^2$ in Proposition 2.1.5 *(1.)*.

- For the Lie algebra $\mathfrak{g}_{6,70}^{0,0}$, defined by the equations

$$\mathfrak{g}_{\mathbf{6,70}}^{\mathbf{0,0}} = (-\mathbf{e^{26}} + \mathbf{e^{35}}, \mathbf{e^{16}} + \mathbf{e^{45}}, -\mathbf{e^{46}}, \mathbf{e^{36}}, \mathbf{0}, \mathbf{0}),$$

any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,70}^{0,0}) \times Z^3(\mathfrak{g}_{6,70}^{0,0})$ is given by

$$\begin{aligned} F &= b_1e^{13} + b_1e^{24} + b_2e^{34} - b_3e^{26} + b_3e^{35} + b_4e^{36} + b_5e^{16} + b_5e^{45} + b_6e^{46} + b_7e^{56}, \\ \rho &= a_1e^{125} + a_2e^{136} + a_3e^{156} - a_4e^{145} + a_4e^{235} + a_5e^{146} + a_5e^{236} + a_6e^{135} \\ &\quad + a_6e^{245} + a_7e^{246} + a_8e^{256} + a_9e^{345} + a_{10}e^{346} + a_{11}e^{356} + a_{12}e^{456}. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} \tilde{J}_\rho e_1 = {} & (2a_4a_5 - a_2a_6 + a_6a_7 - a_1a_{10})\, e_1 + (-2a_2a_4 - 2a_5a_6)\, e_2 + (2a_1a_5)e_3 \\ & - (2a_1a_2)e_4, \end{aligned}$$

and

$$\begin{aligned} \tilde{J}_\rho e_2 = {} & (2a_4a_7 - 2a_5a_6)\, e_1 + (-2a_4a_5 + a_2a_6 - a_6a_7 - a_1a_{10})\, e_2 + (2a_1a_7)e_3 \\ & - (2a_1a_5)e_4, \end{aligned}$$

which implies

$$F(\tilde{J}_\rho e_1, e_1) = -2b_1a_1a_5, \quad \text{and} \quad F(\tilde{J}_\rho e_2, e_2) = 2b_1a_1a_5.$$

Thus, we can consider Proposition 2.1.5 *(2.)* with the pair $X = e_1$ and $Y = e_2$.

- Take the Lie algebra $\mathfrak{g}_{6,78}$ defined by

$$\mathfrak{g}_{\mathbf{6,78}} = (-\mathbf{e^{16}} + \mathbf{e^{25}}, \mathbf{e^{45}}, \mathbf{e^{24}} + \mathbf{e^{36}} + \mathbf{e^{46}}, \mathbf{e^{46}}, -\mathbf{e^{56}}, \mathbf{0}).$$

Any pair $(F, \rho) \in Z^2(\mathfrak{g}_{6,78}) \times Z^3(\mathfrak{g}_{6,78})$ is of the form

$$\begin{aligned} F &= b_1e^{14} + b_2e^{16} + b_3e^{24} - b_2e^{25} + b_1e^{26} + b_1e^{35} + b_3e^{36} + b_4e^{45} + b_5e^{46} + b_6e^{56}, \\ \rho &= a_1e^{124} + a_2e^{126} - a_2e^{145} + a_3e^{146} + a_4e^{156} + a_5e^{235} + a_6e^{236} + a_7e^{245} \\ &\quad + a_8e^{246} + a_9e^{256} - a_6e^{345} + a_{10}e^{346} + a_3e^{356} - a_5e^{356} + a_{11}e^{456}. \end{aligned}$$

We obtain that

$$\tilde{J}_\rho e_1 = (a_1a_3 + a_5a_3 - a_1a_5)\, e_1 + \left(2a_2^2 - 2a_1a_4\right) e_3,$$

which implies that

$$F(\tilde{J}_\rho e_1, e_1) = 0.$$

Hence, we apply Proposition 2.1.5 *(2.)* for $X = Y = e_1$.

- For the Lie algebras $\mathfrak{g} = \mathfrak{n}_{6,84}^{\pm}$, whose structure equations are

$$\mathfrak{n}_{\mathbf{6,84}}^{\pm} = (-\mathbf{e^{45}}, -\mathbf{e^{15}} - \mathbf{e^{36}}, -\mathbf{e^{14}} + \mathbf{e^{26}} \mp \mathbf{e^{56}}, \mathbf{e^{56}}, -\mathbf{e^{46}}, \mathbf{0}),$$

any pair $(F, \rho) \in Z^2(\mathfrak{n}_{6,84}^{\pm}) \times Z^3(\mathfrak{n}_{6,84}^{\pm})$ is given by

$$\begin{aligned} F = & b_1 e^{14} - b_1 e^{26} + b_2 e^{15} + b_2 e^{36} + b_3 e^{16} \mp b_3 e^{25} \mp b_3 e^{34} + b_4 e^{45} + b_5 e^{46} + b_6 e^{56}, \\ \rho = & a_1 e^{124} - a_1 e^{135} + a_2 e^{125} + a_2 e^{134} \pm a_2 e^{246} + a_3 e^{126} + a_3 e^{345} + a_4 e^{136} \\ & - a_4 e^{245} + a_5 e^{145} + a_6 e^{146} + a_7 e^{156} + a_8 e^{125} \pm a_8 e^{356} + a_8 e^{134} + a_9 e^{256} \\ & + a_{10} e^{346} + a_{11} e^{456}. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{J}_\rho e_2 = & \left(-a_1 a_2 + a_3^2 - a_4^2 \pm a_1 a_8 + a_2 a_9 + a_8 a_9 - a_2 a_{10} - a_8 a_{10}\right) e_2 \\ & + \left(\mp 2 a_2^2 + 2 a_3 a_4 \mp 2 a_2 a_8 - 2 a_1 a_9\right) e_3, \end{aligned}$$

and so,

$$F(\tilde{J}_\rho e_2, e_2) = 0.$$

Now, we apply Proposition 2.1.5 *(2.)* for $X = e_2$ and $Y = e_2$. □

Remark 2.3.2. The simply connected solvable Lie group whose Lie algebra is $\mathfrak{g}_{6,38}^{0}$ admits a lattice by [19, Proposition 8.3.3]. For $\mathfrak{g}_{6,54}^{0,-1}$, it is shown in [61] that the corresponding simply connected Lie group has also a compact quotient by a lattice. Finally, the simply connected solvable Lie group corresponding to $\mathfrak{g}_{6,118}^{0,-1,-1}$ admits a lattice by [130].

2.4 Non-unimodular indecomposable symplectic half-flat Lie algebras

In this section we complete the classification of 6-dimensional solvable Lie algebras carrying symplectic half-flat structures.

A 6-dimensional solvable Lie algebra with nilradical of dimension lower than 4 is decomposable or nilpotent [104]. So, we are left to study Lie algebras with 4 and 5-dimensional nilradical. We study which of those admit also symplectic forms, and describe them in Table 2.5, where F denotes a symplectic form and (ω, ψ_+) a half-flat structure.

Table 2.5: **Non-unimodular indecomposable solvable Lie algebras with 5-dimensional nilradical admitting both symplectic and half-flat structures.**

$\mathfrak{g}$	**str. equations**	**half-flat str.**	**symplectic str.**
$A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$	$(-\frac{1}{3}e^{16}+e^{23},-\frac{2}{3}e^{26},\frac{1}{3}e^{36},$ $e^{46},-e^{56},0)$	$\omega=-2e^{16}+e^{34}+e^{52}$ $\psi_+=-2e^{135}-2e^{124}-e^{356}+e^{246}$	$F=-2e^{16}+e^{34}-e^{25}$
$A_{6,39}^{\frac{3}{2},-\frac{3}{2}}$	$(-\frac{1}{2}e^{16}+e^{45},e^{15}+\frac{1}{2}e^{26},$ $\frac{3}{2}e^{36},-\frac{3}{2}e^{46},e^{56},0)$	$\omega=-e^{12}-e^{34}+2e^{56}$ $\psi_+=e^{135}+2e^{146}+2e^{236}-e^{245}$	$F=-2e^{15}+e^{34}+e^{26}$
$A_{6,39}^{1,-1}$	$(e^{45},e^{15}+e^{26},e^{36},-e^{46},e^{56},0)$	$\omega=2e^{16}+e^{23}-e^{34}+e^{45}$ $\psi_+=-e^{124}+e^{135}+2e^{236}$ $+2e^{256}+2e^{346}$	$F=e^{15}+e^{26}+e^{34}$
$A_{6,42}^{-1}$	$(e^{45},e^{15}+e^{26},e^{36}+e^{56},$ $-e^{46},e^{56},0)$	$\omega=2e^{16}+e^{23}-e^{34}+e^{45}$ $\psi_+=-e^{124}+e^{135}+2e^{236}$ $+2e^{256}+2e^{346}$	$F=e^{15}+e^{26}+e^{34}-e^{16}$
$A_{6,51}^{\pm 1}$	$(e^{45},e^{15}\pm e^{46},e^{36},0,0,0)$	$\omega=e^{16}+e^{23}-e^{34}+e^{45}$ $\psi_+=-e^{124}+e^{135}+e^{236}$ $+e^{256}+e^{346}$	$F=e^{14}+e^{16}\pm e^{25}+e^{36}$
$A_{6,54}^{-1,-2}$	$(e^{16}+e^{35},-2e^{26}+e^{45},2e^{36},$ $-e^{46},-e^{56},0)$	$\omega=-e^{13}-e^{24}+e^{56}$ $\psi_+=e^{126}-e^{145}+e^{235}-e^{346}$	$F=e^{14}+e^{23}+e^{56}$
$A_{6,54}^{\alpha,\alpha-1}$ $0<\alpha<2$	$(e^{16}+e^{35},(\alpha-1)e^{26}+e^{45},$ $(1-\alpha)e^{36},-e^{46},\alpha e^{56},0)$	$\omega=-e^{13}-e^{24}-\alpha e^{56}$ $\psi_+=-\alpha e^{126}-e^{145}+e^{235}+\alpha e^{346}$	$F=e^{14}+e^{23}+e^{56}$
$A_{6,54}^{2,1}$	$(e^{16}+e^{35},e^{26}+e^{45},-e^{36},$ $-e^{46},2e^{56},0)$	$\omega=e^{31}+e^{42}+2e^{65}$ $\psi_+=2e^{346}+e^{235}-e^{145}-2e^{126}$	$F=-e^{13}-e^{24}-2e^{56}$
$A_{6,56}^{1}$	$(e^{16}+e^{35},e^{36}+e^{45},0,$ $-e^{46},e^{56},0)$	$\omega=-e^{13}-e^{24}-e^{56}$ $\psi_+=-e^{126}-e^{145}+e^{235}+e^{346}$	$F=e^{14}+e^{23}+e^{56}$
$A_{6,65}^{1,2}$	$(e^{16}+e^{35},e^{16}+e^{26}+e^{45},$ $-e^{36},e^{36}-e^{46},2e^{56},0)$	$\omega=-e^{13}-e^{24}-2e^{56}$ $\psi_+=-2e^{126}-e^{145}+e^{235}+2e^{346}$	$F=e^{13}+e^{16}+e^{26}+e^{45}$
$A_{6,70}^{\alpha,\frac{\alpha}{2}}$ $\alpha\neq 0$	$(\frac{\alpha}{2}e^{16}-e^{26}+e^{35},$ $e^{16}+\frac{\alpha}{2}e^{26}+e^{45},-\frac{\alpha}{2}e^{36}-e^{46},$ $e^{36}-\frac{\alpha}{2}e^{46},\alpha e^{56},0)$	$\omega=e^{13}+e^{24}-\alpha e^{65}$ $\psi_+=-\alpha e^{126}-e^{145}+e^{235}+\alpha e^{346}$	$F=e^{13}+e^{24}-\alpha e^{65}$
$A_{6,71}^{-\frac{3}{2}}$	$(\frac{3}{2}e^{16}+e^{25},\frac{1}{2}e^{26}+e^{35},$ $-\frac{1}{2}e^{36}+e^{45},-\frac{3}{2}e^{46},e^{56},0)$	$\omega=e^{41}+e^{23}+2e^{56}$ $\psi_+=-e^{245}+2e^{346}-2e^{126}-e^{135}$	$F=-e^{14}+e^{23}+2e^{56}$
$A_{6,76}^{-3}$	$(-5e^{16}+e^{25},-2e^{26}+e^{45},$ $e^{24}-e^{36},e^{46},-3e^{56},0)$	$\omega=e^{13}-8e^{26}-\frac{3}{4}e^{34}+e^{45}$ $\psi_+=e^{124}-6e^{136}-8e^{156}$ $+e^{235}-8e^{346}$	$F=-5e^{16}+e^{25}+e^{34}$
$A_{6,82}^{1,5,9}$	$(2e^{16}+e^{24}+e^{35},6e^{26},$ $10e^{36},-4e^{46},-8e^{56},0)$	$\omega=e^{14}-3e^{24}-12e^{26}-e^{35}$ $\psi_+=e^{125}-12e^{136}-e^{234}$ $36e^{236}-12e^{456}$	$F=2e^{16}+e^{24}+e^{35}$
$A_{6,94}^{-3}$	$(-e^{16}+e^{25}+e^{34},-2e^{26}+e^{35},$ $-3e^{36},2e^{46},e^{56},0)$	$\omega=-e^{14}-e^{25}-\frac{3}{2}e^{34}-3e^{36}$ $\psi_+=3e^{126}-e^{135}+e^{234}$ $-\frac{9}{2}e^{236}-3e^{456}$	$F=-e^{16}+e^{25}+e^{34}$
$A_{6,94}^{-\frac{5}{3}}$	$(\frac{1}{3}e^{16}+e^{25}+e^{34},-\frac{2}{3}e^{26}+e^{35},$ $-\frac{5}{3}e^{36},2e^{46},e^{56},0)$	$\omega=e^{12}-\frac{1}{7}e^{23}+\frac{7}{2}e^{25}+$ $e^{34}-\frac{7}{3}e^{36}+\frac{7}{3}e^{56}$ $\psi_+=\frac{1}{3}e^{126}-e^{135}+\frac{7}{3}e^{146}+e^{234}$ $+\frac{7}{3}e^{236}+e^{245}+\frac{7}{6}e^{256}+\frac{49}{6}e^{456}$	$F=\frac{1}{3}e^{16}+e^{25}+e^{34}$
$A_{6,94}^{-1}$	$(e^{16}+e^{25}+e^{34},e^{35},$ $-e^{36},2e^{46},e^{56},0)$	$\omega=e^{14}+e^{15}-3e^{16}$ $-3e^{26}+e^{34}$ $\psi_+=e^{123}-3e^{146}+3e^{156}$ $-e^{245}+3e^{246}+3e^{356}$	$F=e^{16}+e^{25}+e^{34}$

Proposition 2.4.1. *Let $\mathfrak{g}$ be a 6-dimensional non-unimodular indecomposable and (non-nilpotent) solvable Lie algebra with 5-dimensional nilradical. Then, $\mathfrak{g}$ admits symplectic half-flat structure if and only if $\mathfrak{g} = A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$, $\mathfrak{g} = A_{6,54}^{2,1}$, $\mathfrak{g} = A_{6,70}^{\alpha,\frac{\alpha}{2}}$ $(\alpha \neq 0)$, or $\mathfrak{g} = A_{6,71}^{-\frac{3}{2}}$.*

Proof. The half-flat structure (ω, ψ_+) given in Table 2.5 for the Lie algebras $\mathfrak{g} = A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1}$, $\mathfrak{g} = A_{6,54}^{2,1}$, $\mathfrak{g} = A_{6,70}^{\alpha,\frac{\alpha}{2}}$ $(\alpha \neq 0)$ and $\mathfrak{g} = A_{6,71}^{-\frac{3}{2}}$ is symplectic half-flat.

In the following, we show that there are no more Lie algebras of Table 2.5 having a symplectic half-flat structure. To this end, we use Lemma 2.1.6 for appropriated vector X of the corresponding Lie algebra.

- For the Lie algebra $A_{6,39}^{\frac{3}{2},-\frac{3}{2}}$, defined by the structure equations

$$\mathbf{A_{6,39}^{\frac{3}{2},-\frac{3}{2}}} = \Big(-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{16}} + \mathbf{e^{45}}, \mathbf{e^{15}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{26}}, \frac{\mathbf{3}}{\mathbf{2}}\mathbf{e^{36}}, -\frac{\mathbf{3}}{\mathbf{2}}\mathbf{e^{46}}, \mathbf{e^{56}}, \mathbf{0} \Big),$$

  we consider the 7-dimensional Lie algebra

$$A_{6,39}^{\frac{3}{2},-\frac{3}{2}} \oplus \mathbb{R}.$$

  Then, any 3-form $\phi = Z^3\big(A_{6,39}^{\frac{3}{2},-\frac{3}{2}} \oplus \mathbb{R}\big)$ is of the form

$$\begin{aligned}\phi = &a_1 e^{146} + a_2 e^{156} + a_3(e^{135} + 2e^{236}) + a_4 e^{245} - a_5(e^{145} - e^{246}) + a_6 e^{256} \\ &+ a_7(2e^{157} + e^{267}) - a_8(e^{136} - e^{345}) + a_9 e^{346} + a_{10} e^{347} + a_{11} e^{356} \\ &+ a_{12} e^{367} + e_{13} e^{456} - a_{14}(e^{167} - 2e^{457}) + a_{15} e^{467} + a_{16} e^{567},\end{aligned}$$

  where a_i are arbitrary real numbers. Thus, the 2-form $\mu = \iota_{e_1}\phi$ has the expression

$$\mu = a_1 e^{46} + a_2 e^{56} + a_3 e^{35} - a_5 e^{45} + 2a_7 e^{57} - a_8 e^{36} - a_{14} e^{67}.$$

  Therefore, μ is degenerate and Lemma 2.1.6 applies for $X = e_1$.

- The Lie algebra $A_{6,39}^{1,-1}$ is defined by the structure equations

$$\mathbf{A_{6,39}^{1,-1}} = (\mathbf{e^{45}}, \mathbf{e^{15}} + \mathbf{e^{26}}, \mathbf{e^{36}}, -\mathbf{e^{46}}, \mathbf{e^{56}}, \mathbf{0}).$$

  Then, $\phi \in Z^3\big(A_{6,39}^{1,-1} \oplus \mathbb{R}\big)$ is given by

$$\begin{aligned}\phi = &a_1 e^{135} + a_2 e^{136} + a_3 e^{145} + a_4 e^{146} + a_5 e^{156} + a_6 e^{157} + a_7 e^{167} + 2a_1 e^{236} \\ &+ a_8 e^{245} - a_3 e^{246} + a_9 e^{256} + \frac{a_6}{2} e^{267} - a_2 e^{345} + a_{10} e^{346} + a_{11} e^{347} \\ &+ a_{12} e^{356} + a_{13} e^{367} + a_{14} e^{456} - 2a_7 e^{457} + a_{15} e^{467} + a_{16} e^{567}.\end{aligned}$$

  Thus, Lemma 2.1.6 applies for $X = e_2$.

- For the Lie algebra $A_{6,42}^{-1}$, given by

$$\mathbf{A_{6,42}^{-1}} = (\mathbf{e^{45}}, \mathbf{e^{15}} + \mathbf{e^{26}}, \mathbf{e^{36}} + \mathbf{e^{56}}, -\mathbf{e^{46}}, \mathbf{e^{56}}, \mathbf{0}),$$

we consider a closed 3-form $\phi \in Z^3\big(A_{6,42}^{-1} \oplus \mathbb{R}\big)$. So,

$$\begin{aligned}\phi =\,& a_1e^{124} + a_2e^{126} + a_3e^{134} + a_4e^{135} + a_5e^{136} + a_6e^{145} + a_7e^{146} + a_8e^{156}\\ &+ a_9e^{157} + a_{10}e^{167} + 2a_4e^{236} - a_2e^{245} - a_3e^{246} + a_{11}e^{256} + a_9e^{267}\\ &- a_5e^{345} + a_{12}e^{346} - a_{10}e^{347} + a_{13}e^{356} + a_{14}e^{367} + a_{15}e^{456} + a_{16}e^{457}.\end{aligned}$$

Then, Lemma 2.1.6 can be applied with $X = e_2$.

- The Lie algebras $A_{6,51}^{\pm 1}$ are defined by

$$\mathbf{A_{6,51}^{\pm 1}} = (\mathbf{e^{45}}, \mathbf{e^{15}} \pm \mathbf{e^{46}}, \mathbf{e^{36}}, \mathbf{0}, \mathbf{0}, \mathbf{0}).$$

Any closed 3-form $\phi \in Z^3\big(A_{6,51}^{\pm 1} \oplus \mathbb{R}\big)$ has the following expression

$$\begin{aligned}\phi =\,& a_1e^{124} + a_2e^{125} + a_3e^{135} + a_4e^{136} + a_5e^{145} + a_6e^{146} + a_7e^{147} + a_8e^{156}\\ &+ a_9e^{157} + a_{10}e^{167} + a_3e^{236} + a_{11}e^{245} \pm a_2e^{246} + a_{12}e^{256} \pm a_{10}e^{257} - a_4e^{345}\\ &+ a_{13}e^{346} + a_{14}e^{356} + a_{15}e^{367} + a_{16}e^{457} + a_{17}e^{467} + a_{18}e^{567}.\end{aligned}$$

Therefore, Lemma 2.1.6 fulfils for $X = e_3$.

- For the Lie algebra $A_{6,54}^{-1,-2}$, with structure equations

$$\mathbf{A_{6,54}^{-1,-2}} = (\mathbf{e^{16}} + \mathbf{e^{35}}, -\mathbf{2e^{26}} + \mathbf{e^{45}}, \mathbf{2e^{36}}, -\mathbf{e^{46}}, -\mathbf{e^{56}}, \mathbf{0}),$$

a closed 3-form on $A_{6,54}^{-1,-2} \oplus \mathbb{R}$ is

$$\begin{aligned}\phi =\,& a_1e^{126} + a_2e^{136} - a_1e^{145} + a_3e^{146} + a_4e^{147} + a_5e^{156} + a_6e^{157} + a_7e^{167}\\ &+ a_1e^{235} + a_3e^{236} + a_4e^{237} + a_8e^{246} + a_9e^{256} + a_{10}e^{267} + a_{11}e^{345} + a_{12}e^{346}\\ &+ a_{13}e^{356} + a_7e^{357} + a_{14}e^{367} + a_{15}e^{456} - \frac{a_{10}}{2}e^{457} + a_{16}e^{467} + a_{17}e^{567}.\end{aligned}$$

Thus, we apply Lemma 2.1.6 for $X = e_1$.

- Take the family of Lie algebras $A_{6,54}^{\alpha,\alpha-1}$ with $0 < \alpha < 2$. It is given by

$$\mathbf{A_{6,54}^{\alpha,\alpha-1}} = (\mathbf{e^{16}} + \mathbf{e^{35}}, (\alpha - \mathbf{1})\mathbf{e^{26}} + \mathbf{e^{45}}, (\mathbf{1} - \alpha)\mathbf{e^{36}}, -\mathbf{e^{46}}, \alpha\mathbf{e^{56}}, \mathbf{0}).$$

The expression of any $\phi \in Z^3\big(A_{6,54}^{\alpha,\alpha-1} \oplus \mathbb{R}\big)$ is

$$\begin{aligned}\phi =\,& \alpha a_2e^{126} + a_1e^{136} + a_2e^{145} + a_3e^{146} + a_4e^{147} + a_5e^{156} + a_6e^{167} - a_2e^{235}\\ &+ a_3e^{236} + a_4e^{237} + a_7e^{246} + a_8e^{256} + (\alpha - 1)a_{14}e^{267} + a_9e^{345} + a_{10}e^{346}\\ &+ a_{11}e^{356} + a_6e^{357} + a_{12}e^{367} + a_{13}e^{456} + a_{14}e^{457} + a_{15}e^{467} + a_{16}e^{567}.\end{aligned}$$

Now, we can apply Lemma 2.1.6 with $X = e_1$.

- For the Lie algebra $\mathfrak{g} = A^1_{6,56}$, given by the structure equations

$$\mathbf{A^1_{6,56}} = (\mathbf{e^{16}} + \mathbf{e^{35}}, \mathbf{e^{36}} + \mathbf{e^{45}}, \mathbf{0}, -\mathbf{e^{46}}, \mathbf{e^{56}}, \mathbf{0}),$$

  any $\phi \in Z^3\big(A^1_{6,56} \oplus \mathbb{R}\big)$ is of the form

$$\begin{aligned}\phi = {} & a_1e^{126} + a_2e^{134} + a_3e^{136} + a_1e^{145} + a_4e^{146} + a_5e^{147} + a_6e^{156} + a_7e^{167} - a_1e^{235} \\ & + a_8e^{236} + a_5e^{237} + a_4e^{245} - a_8e^{245} + a_9e^{246} + a_{10}e^{256} + a_{11}e^{345} + a_{12}e^{346} \\ & + a_{13}e^{356} + a_7e^{357} + a_{14}e^{367} + a_{15}e^{456} + a_{16}e^{457} + a_{17}e^{467} + a_{18}e^{567},\end{aligned}$$

  thus, we use Lemma 2.1.6 with $X = e_1$. Hence $A^1_{6,56}$ does not admit symplectic half-flat structure.

- The Lie algebra $\mathfrak{g} = A^{1,2}_{6,65}$ is defined by

$$\mathbf{A^{1,2}_{6,65}} = (\mathbf{e^{16}} + \mathbf{e^{35}}, \mathbf{e^{16}} + \mathbf{e^{26}} + \mathbf{e^{45}}, -\mathbf{e^{36}}, \mathbf{e^{36}} - \mathbf{e^{46}}, \mathbf{2e^{56}}, \mathbf{0}).$$

  A closed 3-form ϕ on $A^{1,2}_{6,65} \oplus \mathbb{R}$ is expressed as

$$\begin{aligned}\phi = {} & a_1e^{126} + a_2e^{136} + a_3e^{137} + \frac{a_1}{2}e^{145} + a_4e^{146} + a_5e^{156} + a_6e^{167} - \frac{a_1}{2}e^{235} \\ & + a_4e^{236} + a_7e^{246} + a_8e^{256} + a_9e^{267} + a_{10}e^{345} + a_{11}e^{346} + a_{12}e^{356} + a_6e^{357} \\ & - a_9e^{357} + a_{13}e^{367} + a_{14}e^{456} + a_9e^{457} + a_{15}e^{467} + a_{16}e^{567}.\end{aligned}$$

  Thus, we apply Lemma 2.1.6 for $X = e_2$.

  Following the same procedure we obtain that for the remaining Lie algebras of Table 2.5, that is, $A^{-3}_{6,76}$, $A^{1,5,9}_{6,82}$, $A^{-3}_{6,94}$, $A^{\frac{5}{3}}_{6,94}$ and $A^{-1}_{6,94}$ we can apply Lemma 2.1.6 with $X = e_1$.

□

It remains to study solvable Lie algebras with 4-dimensional nilradical. To this end, we use the list of [124] that contains 12 Lie algebras and 31 families depending at least of one parameter. Indeed, there are 14 one-parameter families, 10 two-parameter families, 4 three-parameter families and 3 four-parameter families. We prove that from all them only 1 Lie algebra has symplectic half-flat structures. First we study which of those Lie algebras admit symplectic forms and we describe them in the following table.

Table 2.6: **Non-unimodular indecomposable symplectic solvable Lie algebras with 4-dimensional nilradical.**

$\mathfrak{g}$	str. equations	symplectic str.
$N_{6,1}^{\alpha,\beta,-\alpha,-\beta}$ $\alpha\beta\neq 0$	$(\alpha e^{15}+\beta e^{16},-\alpha e^{25}-\beta e^{26},e^{36},e^{45},0,0)$	$F=e^{12}+e^{36}+e^{45}$
$N_{6,1}^{\alpha,\beta,0,-1}$ $\alpha\beta\neq 0$	$(\alpha e^{15}+\beta e^{16},-e^{26},e^{36},e^{45},0,0)$	$F=e^{23}+e^{45}+\alpha e^{15}+\beta e^{16}$
$N_{6,1}^{\alpha,\beta,-1,0}$ $\alpha\beta\neq 0$	$(\alpha e^{15}+\beta e^{16},-e^{25},e^{36},e^{45},0,0)$	$F=e^{24}+e^{36}+\alpha e^{15}+\beta e^{16}$
$N_{6,2}^{-1,\beta,-\beta}$	$(-e^{15}+\beta e^{16},e^{25}-\beta e^{26},e^{36},e^{35}+e^{46},0,0)$	$F=e^{12}+e^{35}+e^{46}$
$N_{6,2}^{0,-1,\gamma}$	$(-e^{16},e^{25}+\gamma e^{26},e^{36},e^{35}+e^{46},0,0)$	$F=\gamma e^{26}+e^{25}+e^{13}+e^{35}+e^{46}$
$N_{6,7}^{0,\beta,0}$ $\beta\neq 0$	$(-e^{26},e^{16},e^{35},e^{35}+\beta e^{36}+e^{45},0,0)$	$F=e^{12}+e^{35}+\beta e^{36}+e^{45}$
$N_{6,13}^{\alpha,\beta,-\alpha,-\beta}$ $\alpha^2+\beta^2\neq 0$ $(\alpha,\beta)\neq(0,\pm 2)$	$(\alpha e^{15}+\beta e^{16},-\alpha e^{25}-\beta e^{26},e^{36}-e^{45},$ $e^{35}+e^{46},0,0)$	$F=e^{12}+e^{36}-e^{45}$
$N_{6,13}^{0,-2,0,2}$	$(-2e^{16},2e^{26},e^{36}-e^{45},e^{35}+e^{46},0,0)$	$F=e^{12}+e^{35}+e^{46}$
$N_{6,14}^{\alpha,\beta,0}$ $\alpha\beta\neq 0$	$(\alpha e^{15}+\beta e^{16},e^{26},-e^{45},e^{35},0,0)$	$F=\alpha e^{15}+\beta e^{16}+e^{34}+e^{26}$
$N_{6,15}^{0,\beta,\gamma,0}$ $\beta\neq 0$	$(e^{15}+\gamma e^{16}-e^{26},e^{16}+e^{25}+\gamma e^{26},$ $-\beta e^{45},\beta e^{35},0,0)$	$F=e^{15}+\gamma e^{16}+e^{34}-e^{26}$
$N_{6,16}^{0,0}$	$(e^{16},e^{15}+e^{26},-e^{45},e^{35},0,0)$	$F=e^{15}+e^{26}+e^{34}$
$N_{6,17}^{0}$	$(0,e^{15},e^{36}-e^{45},e^{35}+e^{46},0,0)$	$F=e^{12}+e^{35}+e^{46}$
$N_{6,18}^{0,\beta,0}$ $\beta\neq 0$	$(e^{16}-e^{25},e^{15}+e^{26},-\beta e^{45},\beta e^{35},0,0)$	$F=-e^{15}+e^{26}+e^{34}$
$N_{6,20}^{0,-1}$	$(-e^{56},-e^{26},e^{36},e^{45},0,0)$	$F=e^{23}+e^{45}+e^{16}$
$N_{6,22}^{\alpha,0}$ $\alpha\neq 0$	$(e^{15}+\alpha e^{16},e^{26},0,e^{35},0,0)$	$F=e^{15}+\alpha e^{16}+e^{34}+e^{26}$
$N_{6,23}^{\alpha,0}$	$(e^{15}-e^{26},e^{16}+e^{25},0,e^{35}+\alpha e^{36},0,0)$	$F=e^{16}+e^{25}+e^{34}$
$N_{6,26}^{0}$	$(-e^{56},e^{26},-e^{45},e^{35},0,0)$	$F=e^{15}+e^{26}+e^{34}$
$N_{6,28}$	$(-e^{24}+e^{15},-e^{34}+e^{26},-e^{35}+2e^{36},$ $e^{45}-e^{46},0,0)$	$F=e^{15}-e^{24}-e^{35}+2e^{26}$
$N_{6,29}^{\alpha,\beta}$ $\alpha^2+\beta^2\neq 0$	$(-e^{23}+e^{15}+e^{16},e^{25},e^{36},\alpha e^{45}+\beta e^{46},0,0)$	$F=-e^{23}+e^{15}+e^{16}+\alpha e^{45}+\beta e^{46}$
$N_{6,30}^{\alpha}$	$(-e^{23}+2e^{15},e^{25},e^{26}+e^{35},\alpha e^{45}+e^{46},0,0)$	$F=e^{15}-e^{23}+e^{46}+\alpha e^{45}$
$N_{6,32}^{\alpha}$	$(-e^{23}+e^{45}+e^{16},e^{25}+\alpha e^{26},$ $(1-\alpha)e^{36}-e^{35},e^{46},0,0)$	$F=-e^{23}+e^{45}+e^{16}$
$N_{6,33}$	$(-e^{23}+e^{15}+e^{16},e^{25},e^{36},e^{36}+e^{46},0,0)$	$F=-e^{23}+e^{15}+e^{46}+e^{16}$
$N_{6,34}^{\alpha}$	$(-e^{23}+e^{15}+(1+\alpha)e^{16},e^{25}+\alpha e^{26},$ $e^{36},e^{35}+e^{46},0,0)$	$F=-e^{23}+e^{15}+(1+\alpha)e^{16}+e^{35}+e^{46}$
$N_{6,35}^{\alpha,\beta}$ $\alpha\neq 0$	$(-e^{23}+2e^{16},-e^{35}+e^{26},e^{36}+e^{25},$ $\alpha e^{45}+\beta e^{46},0,0)$	$F=-e^{23}+2e^{16}+\alpha e^{45}+\beta e^{46}$
$N_{6,37}^{\alpha}$	$(-e^{23}+e^{45}+2e^{16},e^{26}-e^{35}-\alpha e^{36},$ $e^{25}+\alpha e^{26}+e^{36},2e^{46},0,0)$	$F=-e^{23}+e^{45}+2e^{16}$
$N_{6,38}$	$(-e^{23}+e^{15}+e^{16},e^{25},e^{36},-e^{56},0,0)$	$F=-e^{23}+e^{15}+e^{16}+e^{46}$
$N_{6,39}$	$(-e^{23}+2e^{16},-e^{35}+e^{26},e^{25}+e^{36},-e^{56},0,0)$	$F=-e^{23}+e^{45}+2e^{16}$

Proposition 2.4.2. *Let $\mathfrak{g}$ be a 6-dimensional non-unimodular indecomposable and (non-nilpotent) solvable Lie algebra with 4-dimensional nilradical. Then, $\mathfrak{g}$ has symplectic half-flat structure if and only if $\mathfrak{g}=N_{6,13}^{0,-2,0,2}$.*

Proof. The Lie algebra $N_{6,13}^{0,-2,0,2}$, defined by the structure equations

$$\mathbf{N_{6,13}^{0,-2,0,2}=(-2e^{16},2e^{26},e^{36}-e^{45},e^{35}+e^{46},0,0)},$$

has a symplectic half-flat structure. In fact, the differential forms

$$\omega = e^{12} + e^{35} + e^{46},$$

and

$$\psi_+ = e^{134} - e^{156} - e^{236} + e^{245},$$

are closed and determine an SU(3)-structure since, with the change of basis given by

$$\{f^1 = e^1, \quad f^2 = e^2, \quad f^3 = e^3, \quad f^4 = e^5, \quad f^5 = e^4, \quad f^6 = e^6\},$$

the forms ω and ψ_+ have the canonical expression, that is,

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_+ &= f^{135} - f^{146} - f^{236} - f^{245}. \end{aligned}$$

Next, using Proposition 2.1.5 and Lemma 2.1.6, we show that no more Lie algebras appearing in Table 2.6 have symplectic half-flat structure. For the Lie algebras

$$\mathbf{N}_{\mathbf{6,1}}^{\alpha,\beta,-\alpha,-\beta}(\alpha\beta \neq \mathbf{0}), \quad \mathbf{N}_{\mathbf{6,2}}^{-\mathbf{1},\beta,-\beta}, \quad \mathbf{N}_{\mathbf{6,2}}^{\mathbf{0},-\mathbf{1},\gamma}, \quad \mathbf{N}_{\mathbf{6,7}}^{\mathbf{0},\beta,\mathbf{0}},$$
$$\mathbf{N}_{\mathbf{6,13}}^{\alpha,\beta,\alpha,\beta} \text{ with } \alpha^{\mathbf{2}} + \beta^{\mathbf{2}} \neq \mathbf{0}, \big((\alpha,\beta) \neq (\mathbf{0}, \pm\mathbf{2})\big) \text{ and } \mathbf{N}_{\mathbf{6,17}}^{\mathbf{0}},$$

we discard the existence of a symplectic half-flat structure by applying Lemma 2.1.6 with $X = e_4$.

For the Lie algebras

$$\mathbf{N}_{\mathbf{6,1}}^{\alpha,\beta,\mathbf{0},-\mathbf{1}}(\alpha\beta \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,1}}^{\alpha,\beta,-\mathbf{1},\mathbf{0}}(\alpha\beta \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,14}}^{\alpha,\beta,\mathbf{0}}(\alpha\beta \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,15}}^{\mathbf{0},\beta,\gamma,\mathbf{0}}, \mathbf{N}_{\mathbf{6,16}}^{\mathbf{0},\mathbf{0}}, \mathbf{N}_{\mathbf{6,18}}^{\mathbf{0},\beta,\mathbf{0}}(\beta \neq \mathbf{0}),$$
$$\mathbf{N}_{\mathbf{6,20}}^{\mathbf{0},-\mathbf{1}}, \mathbf{N}_{\mathbf{6,22}}^{\alpha,\mathbf{0}}(\alpha \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,23}}^{\alpha,\mathbf{0}}, \mathbf{N}_{\mathbf{6,26}}^{\mathbf{0}}, \mathbf{N}_{\mathbf{6,29}}^{\alpha,\beta}(\alpha^{\mathbf{2}} + \beta^{\mathbf{2}} \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,30}}^{\alpha}, \mathbf{N}_{\mathbf{6,32}}^{\alpha}, \mathbf{N}_{\mathbf{6,33}}, \mathbf{N}_{\mathbf{6,34}}^{\alpha},$$
$$\mathbf{N}_{\mathbf{6,35}}^{\alpha,\beta}(\alpha \neq \mathbf{0}), \mathbf{N}_{\mathbf{6,37}}^{\alpha}, \mathbf{N}_{\mathbf{6,38}}, \mathbf{N}_{\mathbf{6,39}},$$

we apply Lemma 2.1.6 with $X = e_1$.

Finally, on $\mathrm{N}_{6,28}$ we can apply Proposition 2.1.5 *(2.)* for $X = e_5$ and $Y = e_6$, which completes the proof. □

Therefore, by Theorem 1.1.9, Proposition 2.2.1, Proposition 2.2.2, Proposition 2.2.3 and Proposition 2.3.1 we have

Theorem 2.4.3. *A unimodular (non-Abelian) solvable Lie algebra $\mathfrak{g}$ has symplectic half-flat structure if and only if it is isomorphic to one in the following list:*

$$
\begin{aligned}
&\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1) = (0,-e^{13},-e^{12},0,-e^{46},-e^{45}),\\
&\qquad \omega = e^{14}+e^{23}+2e^{56}, \quad \psi_+ = (e^{12}+e^{34})\wedge(e^5-e^6)+(-e^{13}+e^{24})\wedge(e^5+e^6);\\
&\mathfrak{g}_{5,1}\oplus\mathbb{R} = (0,0,0,0,e^{12},e^{13}),\\
&\qquad \omega = e^{14}+e^{26}+e^{35}, \quad \psi_+ = e^{123}+e^{156}+e^{245}-e^{346};\\
&A_{5,7}^{-1,-1,1}\oplus\mathbb{R} = (e^{15},-e^{25},-e^{35},e^{45},0,0),\\
&\qquad \omega = -e^{13}+e^{24}+e^{56}, \quad \psi_+ = -e^{126}-e^{145}-e^{235}-e^{346};\\
&A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R} = (\alpha e^{15}+e^{25},-e^{15}+\alpha e^{25},-\alpha e^{35}+e^{45},-e^{35}-\alpha e^{45},0,0),\ \alpha\geq 0,\\
&\qquad \omega = e^{13}+e^{24}+e^{56}, \quad \psi_+ = e^{125}-e^{146}+e^{236}-e^{345};\\
&\mathfrak{g}_{6,N3} = (0,0,0,e^{12},e^{13},e^{23}),\\
&\qquad \omega = e^{16}+2e^{25}+e^{34}, \quad \psi_+ = e^{123}+2e^{145}+e^{246}-2e^{356};\\
&\mathfrak{g}_{6,38}^{0} = (e^{23},-e^{36},e^{26},e^{26}-e^{56},e^{36}+e^{46},0),\\
&\qquad \omega = -2e^{16}+e^{34}-e^{25}, \quad \psi_+ = e^{123}-2e^{124}+e^{236}-e^{456};\\
&\mathfrak{g}_{6,54}^{0,-1} = (e^{16}+e^{35},-e^{26}+e^{45},e^{36},-e^{46},0,0),\\
&\qquad \omega = e^{14}+e^{23}+e^{56}, \quad \psi_+ = e^{125}-e^{136}+e^{246}+e^{345};\\
&\mathfrak{g}_{6,118}^{0,-1,-1} = (-e^{16}+e^{25},-e^{15}-e^{26},e^{36}-e^{45},e^{35}+e^{46},0,0),\\
&\qquad \omega = e^{14}+e^{23}-e^{56}, \quad \psi_+ = e^{126}-e^{135}+e^{245}+e^{346}.
\end{aligned}
$$

Remark 2.4.4. *Note that in the previous theorem, only the first 4 Lie algebras are decomposable while the remaining 4 are indecomposable; and the Lie algebras $\mathfrak{g}_{5,1}\oplus\mathbb{R}$ and $\mathfrak{g}_{6,N3}$ are the unique (non-Abelian) nilpotent Lie algebras admitting symplectic half-flat structure. (These Lie algebras $\mathfrak{g}_{5,1}\oplus\mathbb{R}$ and $\mathfrak{g}_{6,N3}$ are, in Theorem 1.1.9, the second and third algebras, respectively.) Moreover, in Proposition 2.2.1 was defined a symplectic half-flat structure on $\mathfrak{e}(1,1)\oplus\mathfrak{e}(1,1)$; in Proposition 2.2.3 was defined such a structure on $A_{5,7}^{-1,-1,1}\oplus\mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1}\oplus\mathbb{R}$, where $\alpha\geq 0$, and a symplectic half-flat structure on $\mathfrak{g}_{6,38}^{0}$, $\mathfrak{g}_{6,54}^{0,-1}$ and $\mathfrak{g}_{6,118}^{0,-1,-1}$ was defined in Proposition 2.3.1.*

Regarding non-unimodular solvable Lie algebras, from Proposition 2.4.1 and Proposition 2.4.2, we have:

Theorem 2.4.5. *A non-unimodular solvable Lie algebra $\mathfrak{g}$ has symplectic half-flat* *structure if and only if it is isomorphic to one in the following list:*

$$A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1} = \Big(-\frac{1}{3}e^{16} + e^{23}, -\frac{2}{3}e^{26}, \frac{1}{3}e^{36}, e^{46}, -e^{56}, 0 \Big),$$
$$\omega = -2e^{16} + e^{34} + e^{52}, \qquad \psi_+ = -2e^{135} - 2e^{124} - e^{356} + e^{246};$$
$$A_{6,54}^{2,1} = (e^{16} + e^{35}, e^{26} + e^{45}, -e^{36}, -e^{46}, 2e^{56}, 0),$$
$$\omega = -e^{13} - e^{24} - 2e^{56}, \qquad \psi_+ = e^{346} + e^{235} - e^{145} - 2e^{126};$$
$$A_{6,70}^{\alpha,\frac{1}{2}\alpha} = \Big(\frac{\alpha}{2}e^{16} - e^{26} + e^{35}, e^{16} + \frac{\alpha}{2}e^{26} + e^{45}, -\frac{\alpha}{2}e^{36} - e^{46}, e^{36} - \frac{\alpha}{2}e^{46}, \alpha e^{56}, 0 \Big),$$

with $\alpha \in \mathbb{R} - \{0\}$,

$$\omega = e^{13} + e^{24} + \alpha e^{56}, \qquad \psi_+ = \alpha e^{126} - e^{145} + e^{235} + \alpha e^{346};$$
$$A_{6,71}^{-\frac{3}{2}} = \Big(\frac{3}{2}e^{16} + e^{25}, \frac{1}{2}e^{26} + e^{35}, -\frac{1}{2}e^{36} + e^{45}, -\frac{3}{2}e^{46}, e^{56}, 0 \Big),$$
$$\omega = -e^{14} + e^{23} + 2e^{56}, \qquad \psi_+ = -e^{245} + 2e^{346} - 2e^{126} - e^{135};$$
$$N_{6,13}^{0,-2,0,2} = (-2e^{16}, 2e^{26}, e^{36} - e^{45}, e^{35} + e^{46}, 0, 0),$$
$$\omega = e^{12} + e^{35} + e^{46}, \qquad \psi_+ = e^{134} - e^{156} - e^{236} + e^{245}.$$

Therefore, in all these cases, $\mathfrak{g}$ *is indecomposable.*

Chapter 3

G_2-structures on Einstein solvmanifolds

"Dios, concédeme Serenidad para aceptar las cosas que no puedo cambiar, Valor para cambiar aquellas que puedo, y Sabiduría para reconocer la diferencia."

Plegaria de la Serenidad

The G_2 analogue of the Goldberg conjecture for compact Einstein almost Kähler manifolds was studied in [35]. There, Cleyton and Ivanov proved that if φ is a closed G_2 form inducing an Einstein metric g_φ on a compact manifold, then φ is parallel with respect to the Levi-Civita connection of the metric g_φ. But nothing is known for closed G_2 forms inducing Einstein metrics on non-compact manifolds.

In this Chapter, we consider (non-compact) solvmanifolds and, for those manifolds, we study a G_2 analogue of the Goldberg conjecture. In Theorem 3.3.5 we prove that 7-dimensional solvable Lie groups do not carry any left invariant closed G_2 form defining an Einstein metric, unless the metric is flat. Moreover, in section 3.3.2 we study a G_2 analogue for coclosed G_2 forms of the Goldberg conjecture. In Theorem 3.3.11 we show that 7-dimensional solvable Lie groups do not admit any left invariant coclosed G_2 form defining an Einstein metric, unless the metric is flat. On the other hand, in section 3.4, using warped products we construct a new example of a (non-nearly parallel) coclosed G_2 form φ on a non-compact manifold such that φ induces an Einstein metric. The results appearing in this chapter can be found in [53].

3.1 Standard solvable Lie algebras and Einstein metrics

In this section, we recall some results on solvable Lie groups, of arbitrary dimension, with a left invariant metric g which is Einstein in the Riemannian sense, that is, the Ricci curvature tensor $Ric(g)$ satisfies

$$Ric(g) = \lambda g,$$

where λ is a constant.

Note that all the known examples of non-compact homogeneous Einstein manifolds are simply connected solvable Lie groups S endowed with a left invariant metric (see for instance the survey [92]). Moreover, according to a long standing conjecture attributed to D. Alekseevskii (see [16, 7.57]), these might exhaust the class of non-compact homogeneous Einstein manifolds. In [46] the following result concerning unimodular solvable Lie groups admitting a left invariant Einstein metric is proved.

Theorem 3.1.1 [46]**.** *Left invariant Einstein metrics on unimodular solvable Lie groups are flat.*

Therefore, we will study non-unimodular solvable Lie groups. A left invariant metric on a Lie group S will always be identified with the inner product $\langle \cdot, \cdot \rangle$ determined on the Lie algebra $\mathfrak{s}$ of S. The pair $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ is said to be a *metric Lie algebra.*

Lauret in [93] characterizes the Einstein metric solvable Lie algebras as the metric solvable Lie algebras which are *standard* in the following sense.

Definition 3.1.2. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric nilpotent Lie algebra. A* metric solvable extension of *$(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is a metric solvable Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ such that $\mathfrak{s}$ has the orthogonal decomposition $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{n}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{s}}|_{\mathfrak{n} \times \mathfrak{n}} = \langle \cdot, \cdot \rangle$. The metric solvable Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ is* standard, or has standard type, *if $\mathfrak{a}$ is an Abelian subalgebra of $\mathfrak{s}$; in this case, the dimension of $\mathfrak{a}$ is called the rank of the metric solvable extension.*

Remark 3.1.3. *Note that an standard metric solvable extension $(\mathfrak{s}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ of a metric nilpotent Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle_{\mathfrak{n}})$ is not unique even if we fix the dimension of $\mathfrak{s}$. For example, consider the metric nilpotent Lie algebra $(\mathfrak{h}_3, \langle \cdot, \cdot \rangle)$, where $\mathfrak{h}_3$ denotes the 3-dimensional Heisenberg Lie algebra defined by*

$$\mathfrak{h}_3 = (0, 0, e^{12}),$$

and where the inner product $\langle\cdot,\cdot\rangle$ is given by $\langle\cdot,\cdot\rangle = (e^1)^2 + (e^2)^2 + (e^3)^2$. Then, it admits different standard metric solvable extensions like

$$\mathfrak{s}_1 = \big(\frac{1}{2}e^{14}, \frac{1}{2}e^{24}, e^{12} + e^{34}, 0\big),$$

and

$$\mathfrak{s}_2 = (e^{14}, e^{24}, e^{12} + 2e^{34}, 0).$$

In these examples, the first one is Einstein, while the second one is not.

Theorem 3.1.4 [93]**.** *Any Einstein metric solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle_{\mathfrak{s}})$ has to be of standard type (in the sense of Definition 3.1.2).*

Standard Einstein, simply connected, solvable Lie groups constitute a distinguished class that has been deeply studied by J. Heber, who has obtained many remarkable structural and uniqueness results, by assuming only the standard condition (see [80]).

Theorem 3.1.5 [80]**.** *(Uniqueness) A simply connected solvable Lie group admits at most one standard Einstein left invariant metric up to isometry and scaling.*

In [94], it is proved that any nilpotent Lie algebra of dimension lower than or equal to 5 admits an Einstein solvable extension. Moreover, in [127] it is shown that the same is true for any of the 34 nilpotent Lie algebras of dimension six. Using these results, we obtain a classification of all 7-dimensional rank-one Einstein solvable Lie algebras (see Table 3.1, at the end of the subsection 3.3.1). Also, a classification of 6 and 7-dimensional Einstein solvable Lie algebras of higher rank was given in [128].

Furthermore, the study of standard Einstein simply connected solvable Lie groups can be reduced to the rank-one case, that is, $\dim\mathfrak{a} = 1$, where $\mathfrak{a}$ is the Abelian part in the decomposition given in Definition 3.1.2 (see [80]). More precisely, in [80, Sections 4.5,4.6] Heber shows the following theorem.

Theorem 3.1.6 [80]**.** *Let $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ be a non-unimodular solvable Lie algebra, of standard type, endowed with an Einstein inner product $\langle\cdot,\cdot\rangle$, with Einstein constant λ. Then $\lambda < 0$ and, up to isometry, it can be assumed that ad_A is symmetric for any $A \in \mathfrak{a}$. In that case, the following conditions hold.*

1. *There exists $H \in \mathfrak{a}$ such that the eigenvalues of $ad_H|_{\mathfrak{n}}$ are all positive integers without common divisors.*
2. *The restriction of $\langle\cdot,\cdot\rangle$ to the solvable Lie algebra $\mathbb{R}H \oplus \mathfrak{n}$ is also Einstein.*
3. *$\mathfrak{a}$ is an Abelian Lie algebra of symmetric derivations of $\mathfrak{n}$ and the inner product on $\mathfrak{a}$ must be given by $\langle X, Y\rangle = -\frac{1}{\lambda}tr\,(ad_X ad_Y)$, for all $X, Y \in \mathfrak{a}$.*

Standard Einstein solvable Lie algebras are closely related to solvable metric Lie algebras of Iwasawa type. Such a Lie algebra is defined as follows

Definition 3.1.7. *A solvable metric Lie algebra* $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ *is of Iwasawa type if it satisfies the conditions:*

1. $\mathfrak{s}$ *is the orthogonal decomposition* $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, *where* $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ *and* $\mathfrak{a}$ *is Abelian;*

2. *the operator* ad_H *is symmetric, for* $H \in \mathfrak{a}$*;*

3. *there exists* $H_0 \in \mathfrak{a}$ *such that* $ad_{H_0} : \mathfrak{n} \longrightarrow \mathfrak{n}$ *has positive eigenvalues.*

One defines $H_0 \in \mathfrak{a}$ such that $\langle H_0, X\rangle = tr(ad_X)$ holds for all $X \in \mathfrak{s}$. Then, for some positive multiple $H = kH_0$, the normal operator $ad_H|_{\mathfrak{n}}$ has eigenvalues whose real parts $\mu_1, \dots, \mu_m$ are positive integers with no common divisors. Thus, Theorem 3.1.6 implies the following

Corollary 3.1.8. *Any standard Einstein simply connected solvable Lie group is isometric to a solvable Lie group whose underlying metric Lie algebra is of Iwasawa type.*

3.2 Almost Kähler manifolds

As in section 1.1.1, an almost Hermitian manifold (M, g, J), with Riemannian metric g and almost complex structure J, is said to be *almost Kähler* if the corresponding Kähler 2-form ω defined by $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is closed.

In this section we study the existence of almost Kähler structures on 6-dimensional solvable Lie groups whose underlying metric is Einstein. We obtain that the unique 6-dimensional solvable Lie group carrying left invariant Einstein (non-Kähler) almost Kähler structure is the example given by Apostolov, Draghici and Moroianu in [5]. Moreover, we determine the 6-dimensional solvable Lie groups admitting left invariant Kähler-Einstein metric.

For convenience, we will use the following notation for the coefficients appearing on the Einstein extensions of the Lie algebras. We will always denote by a the coefficient that appears on the rank-one Einstein extension. However, the coefficients of the higher rank Einstein extensions will be denoted by b_i.

Proposition 3.2.1 [91]**.** *If* $\mathfrak{k}$ *is a 6-dimensional Einstein solvable Lie algebra of rank one, then* $\mathfrak{k}$ *is one of the following eight Lie algebras:*

$$\begin{aligned}
\mathfrak{k}_1 =&\Big(\frac{2}{13}ae^{16}, \frac{9}{13}ae^{26}, \frac{10}{13}\sqrt{3}ae^{12} + \frac{11}{13}ae^{36}, \frac{20}{13}ae^{13} + ae^{46}, \frac{10}{13}\sqrt{3}ae^{14} + \frac{15}{13}ae^{56}, 0\Big),\\
\mathfrak{k}_2 =&\Big(\frac{1}{4}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{4}\sqrt{30}ae^{12} + \frac{3}{4}ae^{36}, \frac{1}{4}\sqrt{30}ae^{13} + ae^{46},\\
&\qquad -\frac{1}{2}\sqrt{5}ae^{14} - \frac{1}{2}\sqrt{5}ae^{23} + \frac{5}{4}ae^{56}, 0\Big),\\
\mathfrak{k}_3 =&\Big(\frac{3}{10}ae^{16}, \frac{2}{5}ae^{26}, \frac{3}{5}ae^{36}, \frac{1}{5}\sqrt{30}ae^{12} + \frac{7}{10}ae^{46}, \frac{1}{5}\sqrt{15}ae^{23} + \frac{1}{5}\sqrt{30}ae^{14} + ae^{56}, 0\Big),\\
\mathfrak{k}_4 =&\Big(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}ae^{36}, \frac{1}{2}ae^{46}, ae^{12} + ae^{34} + ae^{56}, 0\Big),\\
\mathfrak{k}_5 =&\Big(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, 2ae^{12} + ae^{36}, \sqrt{3}ae^{13} + \frac{3}{2}ae^{46}, \sqrt{3}ae^{23} + \frac{3}{2}ae^{56}, 0\Big),\\
\mathfrak{k}_6 =&\Big(\frac{1}{3}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}ae^{36}, ae^{12} + \frac{5}{6}ae^{46}, ae^{13} + \frac{5}{6}ae^{56}, 0\Big),\\
\mathfrak{k}_7 =&\Big(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}\sqrt{7}ae^{12} + ae^{36}, \frac{3}{4}ae^{46}, \frac{3}{4}ae^{56}, 0\Big),\\
\mathfrak{k}_8 =&\Big(\frac{1}{4}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{4}\sqrt{26}ae^{12} + \frac{3}{4}ae^{36}, \frac{1}{4}\sqrt{26}ae^{13} + ae^{46}, \frac{3}{4}ae^{56}, 0\Big),
\end{aligned}$$

where $\{e_1, \dots, e_6\}$ is an orthonormal basis of $\mathfrak{k}$ with respect to the Einstein metric.

Theorem 3.2.2. *Let (S, g) be a 6-dimensional solvable Lie group, and let $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ be the metric Lie algebra of (S, g). Then,*

1. *(S, g) admits a left invariant Einstein (non-Kähler) almost Kähler structure if and only if $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isometric to the rank-two Einstein solvable Lie algebra*

$$\mathfrak{s} = \Big(\frac{a}{4}e^{15}+\frac{3a}{4}e^{16}, \frac{a}{2}e^{25}-ae^{26}, \frac{\sqrt{5}a}{2}e^{12}+\frac{3a}{4}e^{35}-\frac{a}{4}e^{36}, \frac{1}{2}\sqrt{5}ae^{13}+ae^{45}+\frac{a}{2}e^{46}, 0, 0\Big).$$

2. *(S, g) admits a left invariant Kähler-Einstein structure if and only if $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isometric either to the rank-one solvable Lie algebra*

$$\mathfrak{s} = \Big(\frac{a}{2}e^{16}, \frac{a}{2}e^{26}, \frac{a}{2}e^{36}, \frac{a}{2}e^{46}, ae^{12} + ae^{34} + ae^{56}, 0\Big),$$

or to the rank-two Einstein solvable Lie algebra

$$\begin{aligned}
\mathfrak{s} =&\Big(\frac{a}{2}e^{15} + b_1e^{16} + b_2e^{26}, \frac{a}{2}e^{25} + b_2e^{16} + b_{10}e^{26}, \frac{\sqrt{22}}{4}ae^{12} + ae^{35} + (b_1 + b_{10})e^{36},\\
&\frac{3}{4}ae^{45} - 2(b_1 + b_{10})e^{46}, 0, 0\Big),
\end{aligned}$$

where $a = \frac{4\sqrt{66}}{33}\sqrt{3b_1^2 + 5b_{10}b_1 + b_2^2 + 3b_{10}^2}$, *or to the rank-three Einstein solvable Lie algebra*

$$\mathfrak{s} = \Big(ae^{14} - \frac{\sqrt{6}}{2}ae^{15} + \frac{\sqrt{2}}{2}e^{16}, ae^{24} + \frac{\sqrt{6}}{2}ae^{25} + \frac{\sqrt{2}}{2}e^{26}, ae^{34} - \sqrt{2}ae^{36}, 0, 0, 0\Big).$$

Moreover, in all the Lie algebras that appear in 1. *and* 2.*, the Einstein inner product is such that the basis* $\{e_1, \dots, e_6\}$ *of the corresponding Lie algebra is orthonormal.*

Proof. A 6-dimensional Einstein solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is necessarily standard, so one has the orthogonal decomposition (with respect to $\langle\cdot,\cdot\rangle$)

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},$$

with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ nilpotent and $\mathfrak{a}$ Abelian.

We consider separately the different possibilities according to the rank of $\mathfrak{s}$, that is, to the dimension of $\mathfrak{a}$.

Rank one

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is Abelian, then it is known by [80, Proposition 6.12] that the structure equations of $\mathfrak{s}$ are

$$(ae^{16}, ae^{26}, ae^{36}, ae^{46}, ae^{56}, 0),$$

where a is a non-zero real number. For this Lie algebra $\mathfrak{s}$ we obtain that any closed 2-form ω is degenerate, that is, satisfies $\omega^3 = 0$ and so $\mathfrak{s}$ does not admit symplectic forms.

If $\dim \mathfrak{a} = 1$, but $\mathfrak{n}$ is (non-Abelian) nilpotent, then $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isometric to one of the solvable Lie algebras $\mathfrak{k}_i$ $(i = 1, \dots, 8)$ defined in Proposition 3.2.1 endowed with the inner product $\langle\cdot,\cdot\rangle$ such that the basis $\{e_1, \dots, e_6\}$ is orthonormal.

For $\mathfrak{k}_1$, $\mathfrak{k}_j$, $5 \leq j \leq 8$, we have again that any closed 2-form ω is degenerate, so they do not carry symplectic form.

The Lie algebras $\mathfrak{k}_2$ and $\mathfrak{k}_3$ admit symplectic forms. However, one can check that any almost complex structure J on $\mathfrak{k}_i$ $(i = 2, 3)$ is such that $\langle\cdot,\cdot\rangle \neq \omega(\cdot, J\cdot)$.

For $\mathfrak{k}_4$ we have that a symplectic form is

$$\omega = \mu_{1,2}e^{12} + \mu_{1,6}e^{16} + \mu_{2,6}e^{26} + \mu_{1,2}e^{34} + \mu_{3,6}e^{36} + \mu_{4,6}e^{46} + \mu_{1,2}e^{56},$$

where $\mu_{i,j}$ are real numbers with $\mu_{1,2} \neq 0$. In fact, such a form ω is closed and $\omega^3 = 2\mu_{1,2}^3 e^{123456} \neq 0$. Thus, if J is the almost complex structure on $(\mathfrak{k}_4, \langle\cdot,\cdot\rangle)$ satisfying $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$, then J is given as follows

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6,$$

where $\{e_1, \ldots, e_6\}$ is the dual basis to $\{e^1, \ldots, e^6\}$. Such a J is integrable. Therefore, (g, J, ω) are Kähler-Einstein structures on $\mathfrak{k}_4$.

Rank two

In order to determine all the 6-dimensional rank-two Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions

$$\mathfrak{s}_5 = \mathfrak{n}_4 \oplus \mathbb{R}\langle e_5 \rangle$$

of the 4-dimensional nilpotent Lie algebras $\mathfrak{n}_4$, and then we consider the standard solvable Lie algebra

$$\mathfrak{s}_6 = \mathfrak{s}_5 \oplus \mathbb{R}\langle e_6 \rangle,$$

or, equivalently,

$$\mathfrak{s}_6 = \mathfrak{n}_4 \oplus \mathfrak{a},$$

where $\mathfrak{a} = \mathbb{R}\langle e_5, e_6 \rangle$ is Abelian, and such that the basis $\{e_1, \ldots, e_6\}$ is orthonormal.

If $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is Abelian, the rank-one Einstein extension $\mathfrak{s}_5$ is defined by

$$(ae^{15}, ae^{25}, ae^{35}, ae^{45}, 0).$$

To find the rank-two Einstein solvable extension $\mathfrak{s}_6$ we have to consider the structure equations

$$\begin{cases} de^1 = ae^{15} + b_1 e^{16} + b_2 e^{26} + b_3 e^{36} + b_4 e^{46}, \\ de^2 = ae^{25} + b_5 e^{16} + b_6 e^{26} + b_7 e^{36} + b_8 e^{46}, \\ de^3 = ae^{35} + b_9 e^{16} + b_{10} e^{26} + b_{11} e^{36} + b_{12} e^{46}, \\ de^4 = ae^{45} + b_{13} e^{16} + b_{14} e^{26} + b_{15} e^{36} + b_{16} e^{46}, \\ de^5 = de^6 = 0. \end{cases}$$

Now, we impose that the inner product on $\mathfrak{s}_6$, making the basis $\{e_1, \ldots, e_6\}$ orthonormal, has to be Einstein and that $d^2 e^j = 0$, for $j \in \{1, \ldots, 6\}$. Solving these conditions, we find that the structure equations of $\mathfrak{s}_6$ are

$$\begin{cases} de^1 = ae^{15} + b_1 e^{16}, \\ de^2 = ae^{25} + (-b_1 - b_3 - b_4)\, e^{26}, \\ de^3 = ae^{35} + b_3 e^{36}, \\ de^4 = ae^{45} + b_4 e^{46}, \\ de^5 = de^6 = 0, \end{cases}$$

where $a = \frac{1}{2}\sqrt{2(b_1^2 + b_3^2 + b_4^2 + b_1 b_3 + b_1 b_4 + b_3 b_4)}$. But this Lie algebra does not admit symplectic forms.

The other possibilities for $\mathfrak{s}_5$ are obtained beginning with the 4-dimensional (non-Abelian) nilpotent Lie algebras, that is, the nilpotent Lie algebras defined by the structure equations $(0,0,e^{12},0)$ and $(0,0,e^{12},e^{13})$. Their rank-one solvable extensions are

$$\Big(\frac{1}{2}ae^{15},\frac{1}{2}ae^{25},\frac{1}{4}\sqrt{22}ae^{12}+ae^{35},\frac{3}{4}ae^{45},0\Big),$$

and

$$\Big(\frac{1}{4}ae^{15},\frac{1}{2}ae^{25},\frac{1}{2}\sqrt{5}ae^{12}+\frac{3}{4}ae^{35},\frac{1}{2}\sqrt{5}ae^{13}+ae^{45},0\Big),$$

respectively. To obtain the rank-two solvable Einstein extension of $\mathfrak{n}=(0,0,e^{12},0)$ we should consider the Lie algebra

$$\begin{cases} de^1=\frac{1}{2}ae^{15}+b_1e^{16}+b_2e^{26}+b_3e^{36}+b_4e^{46},\\ de^2=\frac{1}{2}ae^{25}+b_5e^{16}+b_6e^{26}+b_7e^{36}+b_8e^{46},\\ de^3=\frac{1}{4}\sqrt{22}ae^{12}+ae^{35}+b_9e^{16}+b_{10}e^{26}+b_{11}e^{36}+b_{12}e^{46},\\ de^4=\frac{3}{4}ae^{45}+b_{13}e^{16}+b_{14}e^{26}+b_{15}e^{36}+b_{16}e^{46},\\ de^5=de^6=0.\end{cases}$$

Then, we impose the Jacoby identity and the condition that the inner product has to be Einstein. We obtain the rank-two Einstein extension $\mathfrak{s}_6$ defined by the equations

$$\begin{cases} de^1=\frac{1}{2}ae^{15}+b_1e^{16}+b_2e^{26},\\ de^2=\frac{1}{2}ae^{25}+b_2e^{16}+b_{10}e^{26},\\ de^3=\frac{1}{4}\sqrt{22}ae^{12}+ae^{35}+(b_1+b_{10})e^{36},\\ de^4=\frac{3}{4}ae^{45}-2(b_1+b_{10})e^{46},\\ de^5=de^6=0,\end{cases}$$

where $a=\frac{4\sqrt{66}}{33}\sqrt{3b_1^2+5b_{10}b_1+b_2^2+3b_{10}^2}$, which admits the Kähler-Einstein structures given by

$$\begin{aligned}\omega=\mu_{1,2}\big(ae^{12}+2\sqrt{\tfrac{2}{11}}ae^{35}+2\sqrt{\tfrac{2}{11}}(b_1+b_{10})e^{36}\big)+\mu_{1,5}(ae^{15}+2b_1e^{16}+2b_2e^{26})\\ +\mu_{2,5}(2b_2e^{16}+ae^{25}+2b_{10}e^{26})+\mu_{4,5}\big(3ae^{45}-8(b_1+b_{10}e^{46})\big)+\mu_{5,6}e^{56},\end{aligned}$$

$$Je_1=e_2,\quad Je_2=-e_1,\quad Je_3=2\sqrt{\tfrac{2}{11}}e_5+\sqrt{\tfrac{3}{11}}e_6,\quad Je_4=\sqrt{\tfrac{3}{11}}e_5-2\sqrt{\tfrac{2}{11}}e_6,$$

$$Je_5=-2\sqrt{\tfrac{2}{11}}e_3-\sqrt{\tfrac{3}{11}}e_4,\qquad Je_6=-\sqrt{\tfrac{3}{11}}e_3+2\sqrt{\tfrac{2}{11}}e_4,$$

where μ_i are real parameters satisfying $(b_1+b_{10})\mu_{1,2}^2\mu_{4,5}\neq 0$. The almost complex structure J is indeed integrable since the Nijenhuis tensor of J vanishes.

From the rank-one Einstein solvable extension of $\mathfrak{n} = (0, 0, e^{12}, e^{13})$, we obtain the rank-two Einstein solvable extension. For this, we have to consider the Lie algebra

$$\begin{cases} de^1 = \frac{a}{4}e^{15} + b_1 e^{16} + b_2 e^{26} + b_3 e^{36} + b_4 e^{46}, \\ de^2 = \frac{a}{2}e^{25} + b_5 e^{16} + b_6 e^{26} + b_7 e^{36} + b_8 e^{46}, \\ de^3 = \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35} + b_9 e^{16} + b_{10} e^{26} + b_{11} e^{36} + b_{12} e^{46}, \\ de^4 = \frac{1}{2}\sqrt{5}ae^{13} + ae^{45} + b_{13} e^{16} + b_{14} e^{26} + b_{15} e^{36} + b_{16} e^{46}, \\ de^5 = de^6 = 0. \end{cases}$$

Therefore, imposing the condition $d^2e^j = 0$ $(j = 1, \ldots, 6)$ and the inner product to be Einstein, the corresponding structure equations of $\mathfrak{s}_6$ become

$$\begin{cases} de^1 = \frac{a}{4}e^{15} + \frac{3}{4}ae^{16}, \\ de^2 = \frac{a}{2}e^{25} - ae^{26}, \\ de^3 = \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35} - \frac{a}{4}e^{36}, \\ de^4 = \frac{1}{2}\sqrt{5}ae^{13} + ae^{45} + \frac{a}{2}e^{46}, \\ de^5 = de^6 = 0. \end{cases} \tag{3.1}$$

Moreover, $\mathfrak{s}_6$ has the Einstein (non-Kähler) almost Kähler structure given by

$$\omega = e^{13} + \tfrac{2}{\sqrt{5}}e^{45} + \tfrac{1}{\sqrt{5}}e^{46} - \tfrac{1}{\sqrt{5}}e^{25} + \tfrac{2}{\sqrt{5}}e^{26}\,;$$

$$Je_1 = e_3, \quad Je_3 = -e_1, \quad Je_2 = -\tfrac{1}{\sqrt{5}}e_5 + \tfrac{2}{\sqrt{5}}e_6, \quad Je_4 = \tfrac{2}{\sqrt{5}}e_5 + \tfrac{1}{\sqrt{5}}e_6,$$

$$Je_5 = \tfrac{1}{\sqrt{5}}e_2 - \tfrac{2}{\sqrt{5}}e_4, \qquad Je_6 = -\tfrac{1}{\sqrt{5}}e_2 - \tfrac{2}{\sqrt{5}}e_4,$$

which defines the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}_6$ given by

$$\langle\cdot,\cdot\rangle = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2.$$

Now, let (S, g) be the simply connected Lie group whose Lie algebra is $\mathfrak{s}_6$. Then, the Einstein almost Kähler structure on $\mathfrak{s}_6$ defines a left invariant almost Kähler structure (g, J) on S whose Kähler form is ω. Clearly the metric g is given by

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2,$$

and its Ricci curvature tensor is such that

$$Ric(g) = -\frac{15}{4}a^2 g.$$

This means that g is Einstein with non-positive scalar curvature. The almost complex structure J is not integrable since its Nijenhuis tensor N_J does not vanish. In fact,

$$N_J(e_1, e_2) = -\sqrt{5}ae_3 \neq 0, \quad N_J(e_1, e_5) = ae_1 \neq 0, \quad N_J(e_1, e_6) = -2ae_1 \neq 0.$$

Rank three

For the rank-three Einstein extensions we proceed as for the previous extensions. If $\dim \mathfrak{a} = 3$ and $\mathfrak{n}$ is Abelian, we already know that the rank-one Einstein extension of $\mathfrak{n}$ is exactly described by

$$(ae^{14}, ae^{24}, ae^{34}, 0).$$

In order to find the rank-three Einstein solvable extension we have to consider the structure equations

$$\begin{cases} de^1 = ae^{14} + b_1e^{15} + b_2e^{25} + b_3e^{35} + b_4e^{16} + b_5e^{26} + b_6e^{36}, \\ de^2 = ae^{24} + b_7e^{15} + b_8e^{25} + b_9e^{35} + b_{10}e^{16} + b_{11}e^{26} + b_{12}e^{36}, \\ de^3 = ae^{34} + b_{13}e^{15} + b_{14}e^{25} + b_{15}e^{35} + b_{16}e^{16} + b_{17}e^{26} + b_{18}e^{36}, \\ de^4 = de^5 = de^6 = 0. \end{cases}$$

After imposing the Einstein condition on the inner product and that $d^2e^j = 0, j = 1, \dots, 6$ we obtain that the structure equations of the corresponding Lie algebra are

$$\begin{cases} de^1 = ae^{14} - \frac{\sqrt{6}}{2}ae^{15} + \frac{\sqrt{2}}{2}ae^{16}, \\ de^2 = ae^{24} + \frac{\sqrt{6}}{2}ae^{25} + \frac{\sqrt{2}}{2}ae^{26}, \\ de^3 = ae^{34} - \sqrt{2}ae^{36}, \\ de^4 = de^5 = de^6 = 0, \end{cases}$$

which admits the almost Kähler structure given by

$$\begin{aligned} \omega = {} & \mu_{1,4}(\sqrt{2}e^{14} - \sqrt{3}e^{15} + e^{16}) + \mu_{2,4}(\sqrt{2}e^{24} + \sqrt{3}e^{25} + e^{26}) + \mu_{3,4}(-e^{34} + \sqrt{2}e^{36}) \\ & + \mu_{4,5}e^{45} + \mu_{4,6}e^{46} + \mu_{5,6}e^{56}; \end{aligned}$$

$$Je_1 = \tfrac{1}{\sqrt{3}}e_4 - \tfrac{1}{\sqrt{2}}e_5 + \tfrac{1}{\sqrt{6}}e_6, \quad Je_2 = \tfrac{1}{\sqrt{3}}e_4 + \tfrac{1}{\sqrt{2}}e_5 + \tfrac{1}{\sqrt{6}}e_6, \quad Je_3 = -\tfrac{1}{\sqrt{3}}e_4 + \sqrt{\tfrac{2}{3}}e_6,$$

$$Je_4 = -\tfrac{1}{\sqrt{3}}e_1 - \tfrac{1}{\sqrt{3}}e_2 + \tfrac{1}{\sqrt{3}}e_3, \quad Je_5 = \tfrac{1}{\sqrt{2}}e_1 - \tfrac{1}{\sqrt{2}}e_2, \quad Je_6 = -\tfrac{1}{\sqrt{6}}e_1 - \tfrac{1}{\sqrt{6}}e_2 - \sqrt{\tfrac{2}{3}}e_3,$$

where $\mu_{1,4}\mu_{2,4}\mu_{3,4} \neq 0$ and actually the almost complex structure J is integrable since $N_J = 0$. Thus, the metric

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2,$$

is Einstein because its Ricci curvature tensor is

$$Ric(g) = -3a^2g.$$

If dim $\mathfrak{a} = 3$ and $\mathfrak{n}$ is (non-Abelian) nilpotent, then $\mathfrak{n}$ is exactly $\mathfrak{h}_3$ (the 3-dimensional real Heisenberg Lie algebra) defined by the structure equations

$$(0, 0, e^{12}).$$

We find the following rank-one Einstein solvable extension

$$(\tfrac{a}{2}e^{14}, \tfrac{a}{2}e^{24}, ae^{12} + ae^{34}, 0).$$

Proceeding similarly as in the previous examples we find that $\mathfrak{h}_3$ does not admit a rank-three Einstein solvable extension unless it is flat. In fact, the rank-three Einstein solvable extension of $\mathfrak{h}_3$ has to be of the form

$$\begin{cases} de^1 = \frac{a}{2}e^{14} + b_1e^{15} + b_2e^{25} + b_3e^{35} + b_4e^{16} + b_5e^{26} + b_6e^{36}, \\ de^2 = \frac{a}{2}e^{24} + b_7e^{15} + b_8e^{25} + b_9e^{35} + b_{10}e^{16} + b_{11}e^{26} + b_{12}e^{36}, \\ de^3 = ae^{12} + ae^{34} + b_{13}e^{15} + b_{14}e^{25} + b_{15}e^{35} + b_{16}e^{16} + b_{17}e^{26} + b_{18}e^{36}, \\ de^4 = de^5 = de^6 = 0. \end{cases}$$

After imposing that $de^j = 0$, for any $j \in \{1, \ldots, 6\}$, and solving the equations corresponding to the Einstein condition on the inner product we obtain that the structure equations have to be

$$\begin{cases} de^1 = \frac{a}{2}e^{14} + b_1e^{15} + b_4e^{16}, \\ de^2 = \frac{a}{2}e^{24} - b_1e^{25} - b_4e^{26}, \\ de^3 = ae^{12} + ae^{34}, \\ de^4 = de^5 = de^6 = 0. \end{cases}$$

Thus, the metric

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2$$

is such that its Ricci curvature tensor is exactly

$$\begin{aligned} Ric(g) = &- \frac{3}{2}a^2(e^1)^2 - \frac{3}{2}a^2(e^2)^2 - \frac{3}{2}a^2(e^3)^2 - \frac{3}{2}a^2(e^4)^2 - 2b_1^2(e^5)^2 - 2b_4^2(e^6)^2 \\ &- 4b_1b_4(e^5 \otimes e^6). \end{aligned}$$

Therefore, it cannot be Einstein unless it is flat.

Higher rank Einstein solvable extensions are always Ricci-flat, and therefore because of Theorem 3.1.1 they are flat. □

Remark 3.2.3. *We would like to note that Lie algebras with structure equations* (3.1) *are all isomorphic, for the different values of the parameter $a \neq 0$, since by Theorem 3.1.5 the rank-one Einstein metric solvable extensions are unique. The solvable Lie group corresponding to the Lie algebra defined by* (3.1) *is the example given by Apostolov, Draghici and Moroianu in [5]. That example is a non-compact, Einstein and non-Kähler almost Kähler manifold.*

3.3 Ricci curvature of G_2 manifolds

Bryant in [24] gave a formula for the Ricci curvature of a G_2 manifold in terms of the torsion forms and its exterior derivatives. To describe this formula we define the linear map

$$j_\varphi : \Lambda^3(\mathbb{R}^7) \longrightarrow S^2(\mathbb{R}^7),$$

given by

$$j_\varphi(\gamma)(v,w) = *_\varphi\big((\iota_v\varphi)\wedge(\iota_w\varphi)\wedge\gamma\big),$$

for $\gamma \in \Lambda^3(\mathbb{R}^7)$ and $v, w \in \mathbb{R}^7$. Note that j_φ satisfies $j_\varphi(\varphi) = 6g_\varphi$.

Using the map j_φ, Bryant in [24] describes the Ricci curvature tensor of a G_2 manifold in terms of the torsion forms τ_0, τ_1, τ_2 and τ_3 as

$$\begin{aligned} Ric(g_\varphi) = &-\Big(\frac{3}{2}\delta\tau_1 - \frac{3}{8}\tau_0^2 + 15|\tau_1|^2 - \frac{1}{4}|\tau_2|^2 + \frac{1}{2}|\tau_3|^2\Big)g_\varphi \\ &+ j_\varphi\Big(-\frac{5}{4}d\big(*_\varphi(\tau_1\wedge *_\varphi\varphi)\big) - \frac{1}{4}d\tau_2 + \frac{1}{4}*_\varphi d\tau_3 \\ &+ \frac{5}{2}\tau_1\wedge *_\varphi(\tau_1\wedge *_\varphi\varphi) - \frac{1}{8}\tau_0\tau_3 + \frac{1}{4}\tau_1\wedge\tau_2 \\ &+ \frac{3}{4}*_\varphi(\tau_1\wedge\tau_3) + \frac{1}{8}*_\varphi(\tau_2\wedge\tau_2) + \frac{1}{64}Q(\tau_3,\tau_3)\Big), \end{aligned} \tag{3.2}$$

where δ is the codifferential operator associated to g_φ, $|\cdot|$ denotes the norm of differential forms and the map

$$Q : \Lambda^3(\mathbb{R}^7)\times\Lambda^3(\mathbb{R}^7) \longrightarrow \Lambda^3(\mathbb{R}^7),$$

is defined as follows: if $\{e_i\}_{i=1}^7$ is a local orthonormal frame, for $\alpha, \beta \in \Omega^3(M)$

$$Q(\alpha,\beta) = *_\varphi[\epsilon_{ijkl}\big(\iota_{(e_i\wedge e_j)} *_\varphi \alpha\big)\wedge\big(\iota_{(e_i\wedge e_j)} *_\varphi \beta\big)],$$

with ϵ_{ijkl}, the value of $*_\varphi\varphi(e_i, e_j, e_k, e_l)$.

In general, for a n-dimensional Riemannian manifold (N^n, g) the codifferential operator δ is the map

$$\delta : \Omega^p(N^n) \longrightarrow \Omega^{p-1}(N^n),$$

given by

$$\delta = (-1)^{pn+n+1} * d*,$$

so, in particular, for a G_2 manifold M

$$\delta : \Omega^p(M) \longrightarrow \Omega^{p-1}(M),$$

the codifferential operator δ of g_φ is exactly

$$\delta = (-1)^p *_\varphi d *_\varphi .$$

It is also assumed that the norm $|\cdot|$ is the natural norm of differential forms, in explanation, if $F = f_{ij}e^i \wedge e^j$ with $\{e^k\}$ an orthonormal coframe, then $|F| = f_{ij} \cdot f_{ij}$.

The scalar curvature of a G_2 manifold can be described in terms of the torsion forms as

$$Scal(g_\varphi) = 12\delta\tau_1 + \frac{21}{8}\tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2. \tag{3.3}$$

In general, the formulas for the Ricci curvature tensor and the scalar curvature are complicated, but when we restrict them to special classes of G_2-structures they simplify.

If the G_2 form is **closed**, then (as we explain in Chapter 1) the torsion forms satisfy $\tau_0 = \tau_1 = \tau_3 = 0$. From (3.2) and (3.3), we obtain that the expression of the Ricci curvature tensor and the scalar curvature of a manifold endowed with a closed G_2-structure are given respectively by

$$Ric(g_\varphi) = \frac{1}{4}|\tau_2|^2 g_\varphi + j_\varphi\Big(-\frac{1}{4}d\tau_2 + \frac{1}{8} *_\varphi (\tau_2 \wedge \tau_2)\Big),$$

and

$$Scal(g_\varphi) = -\frac{1}{2}|\tau_2|^2.$$

Thus, the scalar curvature of a closed G_2-structure is always non-positive. If a G_2 manifold (M, φ) is such that φ is closed and the induced metric g_φ is Ricci-flat, then it is parallel and therefore g_φ has holonomy contained in G_2 [56]. Regarding Einstein metrics, if we take into account that $j_\varphi(\varphi) = 6g_\varphi$ one can check [24, Corollary 2] that the Einstein condition is equivalent to the torsion form τ_2 satisfying the equation

$$d\tau_2 = \frac{3}{14}|\tau_2|^2\varphi + \frac{1}{2} *_\varphi (\tau_2 \wedge \tau_2). \tag{3.4}$$

Remark 3.3.1. *Note that from the* G_2*-type decomposition, if* φ *is closed, then* τ_2 *is exactly* $\delta\varphi$*. Thus, the Einstein condition* (3.4) *implies that the Hodge Laplacian operator* $\Delta = \delta \circ d + d \circ \delta$ *of* φ *is exactly*

$$\Delta\varphi = \frac{3}{14}|\tau_2|^2\varphi + \frac{1}{2} *_\varphi (\tau_2 \wedge \tau_2).$$

From the G_2-type decomposition of exterior forms we already know that $\tau_2 \in \Omega^2_{14}(M)$. It is also known that, if a 2-form β belongs to $\Omega^2_{14}(M)$, then $\beta \wedge \beta \in \Omega^4_1(M) \oplus \Omega^4_{27}(M)$, that is, it does not have part in $\Omega^4_7(M)$. Even more, by [35, Lemma 5.8] the $\Omega^4_{27}(M)$ part of $\beta \wedge \beta$ vanishes if and only if β vanishes itself.

With all these considerations we have the following:

Proposition 3.3.2 [35]. *(Cleyton-Ivanov) No compact manifold M can support a closed G_2-structure φ whose underlying metric g_φ is Einstein unless φ is parallel (with respect to the Levi-Civita connection ∇ of g_φ).*

Proof. (Bryant technique). From the Einstein condition (3.4), and the fact that $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_2 \wedge \tau_2 \in \Omega^4_1(M) \oplus \Omega^4_{27}(M)$, we obtain that

$$d\Big(\frac{1}{3}\tau_2^3\Big) = \frac{2}{7}|\tau_2|^4 *_\varphi 1,$$

where $*_\varphi 1$ denotes the volume form. By Stokes' theorem we get

$$0 = \int_M d\Big(\frac{1}{3}\tau_2^3\Big) = \int_M \frac{2}{7}|\tau_2|^4 *_\varphi 1,$$

which implies that $\tau_2 = 0$ and therefore $\nabla\varphi = 0$. $\square$

For **nearly parallel** G_2**-structures** we know that the torsion forms are such that $\tau_1 = \tau_2 = \tau_3 = 0$. Therefore, from (3.2) and (3.3), the expressions for the Ricci curvature tensor and the scalar curvature of a nearly parallel G_2 manifold (M, φ) are respectively given by

$$Ric(g_\varphi) = \frac{3}{8}|\tau_0|^2 g_\varphi,$$

and

$$Scal(g_\varphi) = \frac{21}{8}|\tau_0|^2.$$

Thus, nearly parallel G_2-structures are always Einstein with non-negative scalar curvature. It is also clear that a Ricci-flat nearly parallel manifold is parallel.

Finally, **coclosed** G_2**-structures** are determined by the vanishing of the torsion forms τ_1 and τ_2. In general, for a G_2 manifold (M, φ), with φ coclosed, from (3.2) and (3.3), we have that the Ricci curvature tensor and the scalar curvature are respectively given by

$$Ric(g_\varphi) = \Big(\frac{3}{8}\tau_0^2 - \frac{1}{2}|\tau_3|^2\Big) g_\varphi + j_\varphi\Big(\frac{1}{4} *_\varphi d\tau_3 - \frac{1}{8}\tau_0\tau_3 + \frac{1}{64}Q(\tau_3, \tau_3)\Big),$$

and

$$Scal(g_\varphi) = \frac{21}{8}\tau_0^2 - \frac{1}{2}|\tau_3|^2.$$

So, in general, nothing is known about the sign of the scalar curvature of the Riemannian metric induced by a coclosed G_2-structure. In contrast with the closed and nearly parallel ones, manifolds with a coclosed G_2-structure which are Ricci-flat do not need to be parallel.

Remark 3.3.3. *We would like to note that for cocalibrated* G_2 *manifolds an analogue of Goldberg conjecture is no longer true. Indeed, there exist compact* G_2 *manifolds* (M, φ)*, with* φ *coclosed and such that the induced metric* g_φ *is Einstein, but* φ *is non-parallel. For example, nearly parallel manifolds. But not all coclosed* G_2 *forms inducing Einstein metric are nearly parallel; 3-Sasakian manifolds constitute a counterexample. In section 3.4 we describe an example of Einstein non-nearly parallel manifold with a coclosed* G_2 *form such that the induced metric is not 3-Sasakian. Such an example is given using warped products.*

3.3.1 Closed G_2-structures

In this section we study the existence of closed G_2 forms φ on 7-dimensional solvable Lie algebras whose underlying Riemannian metric g_φ is Einstein. For this, we use Proposition 1.4.5 (an obstruction to the existence of closed G_2-structures), and the following results.

Lemma 3.3.4. *Let* $\mathfrak{g}$ *be a* 7*-dimensional Lie algebra and* φ *a* G_2*-structure on* $\mathfrak{g}$*. Then the bilinear form* $g_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *defined by*

$$g_\varphi(X, Y)\, vol = \frac{1}{6}\,(\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi),$$

has to be a Riemannian metric.

Proof. It is a direct consequence of (1.7). □

By [115, Proposition 4.5], if a 3-form φ defines a G_2-structure on a 7-dimensional Lie algebra and we choose a non-zero vector $X \in \mathfrak{g}$ of length one, with respect to g_φ, then on the orthogonal complement of the span of X one has the SU(3)-structure defined by the pair (α, β), where α and β are the 2-form and the 3-form, respectively, given by

$$\alpha = \iota_X\varphi, \quad \beta = \varphi - \alpha \wedge \eta, \tag{3.5}$$

where $\eta = \iota_X(g_\varphi)$. So, α and β satisfy

$$\alpha^3 = \alpha \wedge \alpha \wedge \alpha \neq 0, \quad \alpha \wedge \beta = 0. \tag{3.6}$$

In contrast with the almost Kähler case, we have the following theorem.

Theorem 3.3.5. *A* 7*-dimensional solvable Lie group does not admit any left invariant closed* G_2 *form* φ *such that* g_φ *is Einstein, unless* g_φ *is flat.*

Proof. An Einstein solvable Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is necessarily standard, so one has the orthogonal decomposition (with respect to $\langle\cdot,\cdot\rangle$), $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ nilpotent and $\mathfrak{a}$ Abelian. We will consider separately the different cases according to the rank of $\mathfrak{s}$, that is, to the dimension of $\mathfrak{a}$.

Rank one

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is Abelian, then by [80, Proposition 6.12] the structure equations of the Einstein solvable extension $\mathfrak{s}$ are

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where a is a non-zero real number. Calculating the general expression of a closed 3-form on $\mathfrak{s}$, it is easy to check that $\mathfrak{s}$ cannot admit closed G_2 forms since Proposition 1.4.5 is satisfied for $X = e_1, \dots, e_6$.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is (non-Abelian) nilpotent, then Will in [127] proves that $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i$, with $i \in \{1, \dots, 33\}$, that appear in Table 3.1 which is included at the end of this subsection, endowed with the inner product such that the basis $\{e_1, \dots, e_7\}$ is orthonormal. We may apply Proposition 1.4.5 with $X = e_6$ to all the Lie algebras $\mathfrak{g}_i$ $(i = 1, \dots, 33)$ with the exception of the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_4$, $\mathfrak{g}_9$, $\mathfrak{g}_{18}$ and $\mathfrak{g}_{28}$, showing in this way that they do not admit any closed G_2-structure. For the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}$ and $\mathfrak{g}_{28}$ we first determine a generic closed 3-form φ and then, we apply [115, Proposition 4.5], for each $X = e_1, \dots, e_7$ and $\eta = \iota_X(g_\varphi)$. Thus, for each X, taking α and β as in (3.5), we know by (3.6) that $\alpha \wedge \alpha \wedge \alpha \neq 0$ and $\alpha \wedge \beta = 0$.

Moreover, we have that the closed 3-form φ defines a G_2-structure if and only the matrix associated to the symmetric bilinear form g_φ, with respect to the orthonormal basis $\{e_1, \dots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a closed G_2-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the basis $\{e_1, \dots, e_7\}$, is a multiple of the identity matrix. Then by a direct calculation we have that the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}$ and $\mathfrak{g}_{28}$ admit a closed G_2-structure (see Table 3.2) but they do not admit any closed G_2-structure inducing an Einstein (non-flat) metric. Next, we show the details of this result for the Lie algebra $\mathfrak{g}_{28}$. To this end, we see that any closed 3-form φ on $\mathfrak{g}_{28}$ has the following expression:

$$\begin{aligned}\varphi =& \rho_{1,2,7}e^{127} - \frac{1}{2}\rho_{5,6,7}e^{136} + \rho_{2,4,7}e^{137} + \frac{1}{2}\rho_{5,6,7}e^{145} - \rho_{2,3,7}e^{147} - \rho_{2,6,7}e^{157} \\ &+ \rho_{2,5,7}e^{167} + \frac{1}{2}\rho_{5,6,7}e^{235} + \rho_{2,3,7}e^{237} + \frac{1}{2}\rho_{5,6,7}e^{246} + \rho_{2,4,7}e^{247} + \rho_{2,5,7}e^{257} \\ &+ \rho_{2,6,7}e^{267} + \rho_{3,4,7}e^{347} + \rho_{3,5,7}e^{357} + \rho_{3,6,7}e^{367} + \rho_{3,6,7}e^{457} - \rho_{3,5,7}e^{467} + \rho_{5,6,7}e^{567},\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary real numbers denoting the coefficients of e^{ijk}.

Thus, the induced metric $g = g_\varphi$ is given by the matrix G with elements $G_{i,j} = g(e_i, e_j)$:

$$
\begin{array}{lll}
g(e_1,e_1) = -\frac{1}{4}\rho_{1,2,7}\rho_{5,6,7}^2, & g(e_1,e_2) = 0, & g(e_1,e_3) = \frac{1}{4}\rho_{2,3,7}\rho_{5,6,7}^2, \\
g(e_1,e_4) = \frac{1}{4}\rho_{2,4,7}\rho_{5,6,7}^2, & g(e_1,e_5) = \frac{1}{4}\rho_{2,5,6}\rho_{5,6,7}^2, & g(e_1,e_6) = \frac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2, \\
g(e_1,e_7) = 0, & g(e_2,e_2) = -\frac{1}{4}\rho_{1,2,7}\rho_{5,6,7}^2, & g(e_2,e_3) = -\frac{1}{4}\rho_{2,4,7}\rho_{5,6,7}^2, \\
g(e_2,e_4) = \frac{1}{4}\rho_{2,3,7}\rho_{5,6,7}^2, & g(e_2,e_5) = \frac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2, & g(e_2,e_6) = -\frac{1}{4}\rho_{2,5,7}\rho_{5,6,7}^2, \\
g(e_2,e_7) = 0, & g(e_3,e_3) = -\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2, & g(e_3,e_4) = 0, \\
g(e_3,e_5) = \frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2, & g(e_3,e_6) = -\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, & g(e_3,e_7) = 0, \\
g(e_4,e_4) = -\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2, & g(e_4,e_5) = -\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, & g(e_4,e_6) = -\frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2, \\
g(e_4,e_7) = 0, & g(e_5,e_5) = \frac{\rho_{5,6,7}^3}{4}, & g(e_5,e_6) = 0, \\
g(e_5,e_7) = 0, & g(e_6,e_6) = \frac{\rho_{5,6,7}^3}{4}, & g(e_6,e_7) = 0,
\end{array}
$$

$$
\begin{aligned}
g(e_7,e_7) = {} & -\rho_{5,6,7}\rho_{2,3,7}^2 + \rho_{1,2,7}\rho_{3,5,7}^2 + \rho_{1,2,7}\rho_{3,6,7}^2 + \rho_{2,5,7}^2\rho_{3,4,7} + \rho_{2,6,7}^2\rho_{3,4,7} \\
& +\rho_{2,5,7}\left(2\rho_{2,3,7}\rho_{3,6,7} - 2\rho_{2,4,7}\rho_{3,5,7}\right) - 2\rho_{2,6,7}\left(\rho_{2,3,7}\rho_{3,5,7} + \rho_{2,4,7}\rho_{3,6,7}\right)
\end{aligned}
$$

Then, the system $G = kI_7$, with k a non-vanishing real number and I_7 the identity matrix, does not have solution. Equivalently, the Lie algebra $\mathfrak{g}_{28}$ does not admit any closed G_2-structure inducing an Einstein inner product.

Rank two

In order to determine all the 7-dimensional rank-two Einstein solvable Lie algebras, we need first to consider the rank-one Einstein solvable extensions $\mathfrak{s}_6 = \mathfrak{n}_5 \oplus \mathbb{R}\langle e_6\rangle$ of any of the eight 5-dimensional nilpotent Lie algebras $\mathfrak{n}_5$ (see Proposition 3.2.1). Then we obtain the standard solvable Lie algebra $\mathfrak{s}_7 = \mathfrak{s}_6 \oplus \mathbb{R}\langle e_7\rangle = \mathfrak{n}_5 \oplus \mathfrak{a}$, with $\mathfrak{a} = \mathbb{R}\langle e_6, e_7\rangle$ Abelian and such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal.

For the rank-two Einstein extension of the nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, e^{12}, e^{13}, e^{14}),$$

we take into account the expression of its rank-one Einstein extension $\mathfrak{k}_1$ which is described in Proposition 3.2.1. Thus the structure equations of the rank-two Einstein solvable extension are

$$
\begin{cases}
de^1 = \frac{2}{13}ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\
de^2 = \frac{9}{13}ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\
de^3 = \frac{10}{13}\sqrt{3}ae^{12} + \frac{11}{13}ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\
de^4 = \frac{20}{13}ae^{13} + ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\
de^5 = \frac{10}{13}\sqrt{3}ae^{14} + \frac{15}{13}ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\
de^6 = de^7 = 0.
\end{cases}
$$

Imposing that the inner product $\sum_i (e^i)^2$ is Einstein and the condition $d^2e^j = 0$, where $j \in \{1,\dots,7\}$, we have that the structure equations of the corresponding 7-dimensional rank-two Einstein solvable Lie algebra are

$$\begin{cases} de^1 = \frac{1}{3}\sqrt{6}ae^{16} + 4ae^{17}, \\ de^2 = \frac{3}{2}\sqrt{6}ae^{26} - 7ae^{27}, \\ de^3 = \frac{5}{3}\sqrt{18}ae^{12} + \frac{11}{6}\sqrt{6}ae^{36} - 3ae^{37}, \\ de^4 = \frac{10}{3}\sqrt{6}ae^{13} + \frac{13}{6}\sqrt{6}ae^{46} + ae^{47}, \\ de^5 = \frac{5}{3}\sqrt{18}ae^{14} + \frac{5}{2}\sqrt{6}ae^{56} + 5ae^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

Now, calculating the general expression of a closed 3-form φ, and using Lemma 3.3.4 and [115, Proposition 4.5], for $\eta = e^i$ $(i \in \{1,\dots,7\})$, we have that the matrix associated to g_φ, with respect to the basis $\{e_1,\dots,e_7\}$, cannot be a multiple of the identity matrix. Thus, the Lie algebra previously defined does not admit a closed G_2-structure inducing the Einstein metric.

For the nilpotent Lie algebra

$$\mathfrak{n} = (0,0,e^{12},e^{13},e^{14}+e^{23}),$$

whose rank one Einstein extension is $\mathfrak{k}_2$, is not obtained any 7-dimensional Einstein Lie algebra of rank two. Concretely, a rank two Einstein extension of $\mathfrak{n}$ would have to be of the form

$$\begin{cases} de^1 = \frac{1}{4}ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = \frac{1}{2}ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = \frac{1}{4}\sqrt{30}ae^{12} + \frac{3}{4}ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = \frac{1}{4}\sqrt{30}ae^{13} + ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = -\frac{1}{2}\sqrt{5}ae^{14} - \frac{1}{2}\sqrt{5}ae^{23} + \frac{5}{4} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

However, after imposing the conditions $d^2e^j = 0$ $(j = 1,\dots,7)$ and the inner product to be Einstein, we obtain that there are no values on the parameters b_i satisfying these conditions.

The nilpotent Lie algebra

$$\mathfrak{n} = (0,0,0,e^{12},e^{14}+e^{23})$$

has a rank-one Einstein extension given by $\mathfrak{k}_3$. Therefore the structure equations of the rank-two Einstein solvable extension are

$$\begin{cases} de^1 = \frac{3}{10}ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = \frac{2}{5}ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = \frac{3}{5}ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = \frac{1}{5}\sqrt{30}ae^{12} + \frac{7}{10}ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = \frac{1}{5}\sqrt{15}ae^{23} + \frac{1}{5}\sqrt{30}ae^{14} + ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

After solving the equations corresponding to the condition $d^2e^j = 0$ $(j = 1, \dots, 7)$ and the inner product to be Einstein, the structure equations of the rank-two Einstein extension are

$$\begin{cases} de^1 = \frac{1}{7}\sqrt{21}ae^{16} - ae^{17}, \\ de^2 = \frac{4}{21}\sqrt{21}ae^{26} + 2ae^{27}, \\ de^3 = \frac{2}{7}\sqrt{21}ae^{36} - 2ae^{37}, \\ de^4 = \frac{2}{21}\sqrt{30}\sqrt{21}ae^{12} + \frac{1}{3}\sqrt{21}ae^{46} + ae^{47}, \\ de^5 = \frac{2}{21}\sqrt{30}\sqrt{21}ae^{14} + \frac{2}{21}\sqrt{15}\sqrt{21}ae^{23} + \frac{10}{21}\sqrt{21}ae^{56}, \\ de^6 = de^7 = 0. \end{cases}$$

Calculating the expression of a generic closed 3-form φ and using Lemma 3.3.4 and [115, Proposition 4.5] with $\eta = e^1, \dots, e^7$ or equivalently, with $X = e_1, \dots, e_7$, we get that the matrix associated to g_φ, with respect to the basis $\{e_1, \dots, e_7\}$, cannot be a multiple of the identity matrix. Therefore, the metric induced by a closed 3-form cannot be Einstein.

We already know that the rank-one Einstein extension of the nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, 0, 0, e^{12} + e^{34})$$

is described by $\mathfrak{k}_4$ (see Proposition 3.2.1). Thus, the rank-two Einstein extension of $\mathfrak{n}$ is of the form

$$\begin{cases} de^1 = ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = 2ae^{12} + 2ae^{34} + 2ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

Therefore, we solve the corresponding conditions and obtain that the structure equations of the 7-dimensional rank-two Einstein extension are

$$\begin{cases} de^1 = ae^{16} - b_7e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47}, \\ de^2 = ae^{26} + b_2e^{17} + b_7e^{27} + b_4e^{37} - b_3e^{47}, \\ de^3 = ae^{36} + b_3e^{17} + b_4e^{27} - b_{19}e^{37} + b_{14}e^{47}, \\ de^4 = ae^{46} + b_4e^{17} - b_3e^{27} + b_{14}e^{37} + b_{19}e^{47}, \\ de^5 = 2a(e^{12} + e^{34} + e^{56}), \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \frac{1}{2}\sqrt{b_7^2 + b_2^2 + 2b_3^2 + 2b_4^2 + b_{14}^2 + b_{19}^2}$. Then, if we compute the expression of a generic closed 3-form on this Lie algebra, we can apply Proposition 1.4.5 with $X = e_5$. Thus the rank-two Einstein extension does not admit closed G_2 forms.

Now, we consider the nilpotent Lie algebra defined by the structure equations

$$\mathfrak{n} = (0, 0, e^{12}, e^{13}, e^{23}).$$

It admits the rank-one Einstein extension given by $\mathfrak{k}_5$. Then, the rank-two Einstein extension of $\mathfrak{n}$ is of the form

$$\begin{cases} de^1 = ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = 4ae^{12} + 2ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = 2\sqrt{3}ae^{13} + 3ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = 2\sqrt{3}ae^{23} + 3ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

We solve the equations corresponding to the inner product to be Einstein and $d^2e^j = 0, j = 1, \dots, 7$. Thus, the structure equations of the rank-two Einstein extension of $\mathfrak{n}$ are

$$\begin{cases} de^1 = ae^{16} + b_{19}e^{17} + b_{20}e^{27}, \\ de^2 = ae^{26} + b_{20}e^{17} - b_{19}e^{27}, \\ de^3 = 4ae^{12} + 2ae^{36}, \\ de^4 = 2\sqrt{3}ae^{13} + 3ae^{46} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = 2\sqrt{3}ae^{23} + 3ae^{56} + b_{20}e^{47} - b_{19}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

We take the expression of a closed 3-form, namely φ. After calculating its corresponding metric g_φ and its associated matrix G with respect to $\{e_1, \dots, e_7\}$, using [115, Proposition 4.5] for $\eta = e^1, \dots, e^7$ we have that the system $G = kI_7$ (where k is a non-zero real number) has no solution.

For the nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, 0, e^{12}, e^{13}),$$

the rank-one Einstein extension is given by $\mathfrak{k}_6$. Thus, the structure equations of the rank-two Einstein extension associated to $\mathfrak{n}$ are of the form

$$\begin{cases} de^1 = \frac{1}{3}ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = \frac{1}{2}ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = \frac{1}{2}ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = ae^{12} + \frac{5}{6}ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = ae^{13} + \frac{5}{6}ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

Therefore, imposing the Einstein condition and the differential operator to vanish when applied twice we obtain the two families of 7-dimensional rank-two Einstein Lie algebras

$$1) \begin{cases} de^1 = 2ae^{16} + 2(b_{19} + b_{25})e^{17}, \\ de^2 = 3ae^{26} - (b_{19} + 2b_{25})e^{27} + b_{12}e^{37}, \\ de^3 = 3ae^{36} + b_{12}e^{27} - (b_{19} + 2b_{25})e^{37}, \\ de^4 = 6ae^{12} + 5ae^{46} + b_{19}e^{47} + b_{12}e^{57}, \\ de^5 = 6ae^{13} + 5ae^{56} + b_{12}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

and

$$2) \begin{cases} de^1 = \sqrt{2}b_{25}e^{16} + 4b_{25}e^{17}, \\ de^2 = \frac{3}{2}\sqrt{2}b_{25}e^{26} - 3b_{25}e^{27} - b_{12}e^{37}, \\ de^3 = \frac{3}{2}\sqrt{2}b_{25}e^{36} + b_{12}e^{27} - 3b_{25}e^{37}, \\ de^4 = 3\sqrt{2}b_{25}e^{12} + \frac{5}{2}\sqrt{2}b_{25}e^{46} + b_{25}e^{47} - b_{12}e^{57}, \\ de^5 = 3\sqrt{2}b_{25}e^{13} + \frac{5}{2}\sqrt{2}b_{25}e^{56} + b_{12}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

For 1) we calculate first a generic closed 3-forms φ and then, using [115, Proposition 4.5] for $X = e_7$, equivalently $\eta = e^7$, we impose the condition $\alpha \wedge \beta = 0$. By this condition we have, in particular, that

$$\rho_{1,2,3}\rho_{1,3,4}(b_{19} + b_{25}) = 0,$$

where by $\rho_{i,j,k}$ we denote the coefficient of e^{ijk} in φ. One can immediately exclude the case $\rho_{1,3,4} = 0$, since otherwise the element $G_{4,4}$ of the matrix associated to the metric g_φ has to be zero. Then we study separately the cases $\rho_{1,2,3} = 0$ and $b_{19} = -b_{25}$. In both cases we do not find any solution for the system $S = G - kI_7$.

For 2) we consider separately the cases $b_{12}b_{25} \neq 0$, $b_{12} = 0$ and $b_{25} = 0$. In the case $b_{12}b_{25} \neq 0$ we compute first a generic closed 3-form φ and then, using [115, Proposition 4.5] for $X = e_7$, we impose the condition $\alpha \wedge \alpha \wedge \alpha \neq 0$, getting the condition $\rho_{1,2,5} \neq 0$. Thus, we take the system $S = G - kI_7$ and get the values of $\rho_{2,5,6}, \rho_{3,4,6}, \rho_{3,5,6}$ and $\rho_{2,3,6}$ from $S_{5,5}, S_{3,5}, S_{4,4}$ and $S_{3,4}$, respectively. Now $S_{3,3} = -k$, and the system does not admit any solution.

For $b_{12} = 0$ we first compute the expression of a general closed 3-form φ and then we use [115, Proposition 4.5] with $X = e_7$, obtaining that $\rho_{2,3,6}, \rho_{2,4,6}, \rho_{3,5,6}$ and $\rho_{4,5,6}$ are all different from zero. Thus, the system $S = G - kI_7$ does not have solution. In the case $b_{25} = 0$ we first compute a generic closed 3-form φ and then we apply Proposition 1.4.5 with $X = e_1, \ldots, e_5$.

The nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, e^{12}, 0, 0)$$

admits a rank-one Einstein extension described by $\mathfrak{k}_7$. Therefore, the rank-two Einstein extension of $\mathfrak{n}$ is given by

$$\begin{cases} de^1 = ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = \frac{1}{2}\sqrt{7}ae^{12} + ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = \frac{3}{4}ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = \frac{3}{4}ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

We solve the equations corresponding to the inner product to be Einstein and $d^2e^j = 0, j = 1, \ldots, 7$. Then, we obtain the four families of 7-dimensional rank-two Einstein Lie algebras, with structure equations

$$1) \begin{cases} de^1 = ae^{16} + (-b_7 + b_{13})e^{17} + b_6e^{27}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27}, \\ de^3 = a(\sqrt{7}e^{12} + 2e^{36}) + b_{13}e^{37}, \\ de^4 = \frac{3}{2}ae^{46} - (2b_{13} + b_{25})e^{47} + b_{24}e^{57}, \\ de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \frac{2}{21}\sqrt{21b_7^2 - 21b_7f_{13} + 63b_{13}^2 + 21b_6^2 + 42b_{25}b_{13} + 21b_{25}^2 + 21b_{24}^2}$;

$$2)\begin{cases} de^1 = ae^{16} + (-b_7 + b_{13})e^{17} + b_6e^{27}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27}, \\ de^3 = a(\sqrt{7}e^{12} + 2e^{36}) + b_{13}e^{37}, \\ de^4 = \frac{3}{2}ae^{46} - b_{13}e^{47} - b_{24}e^{57}, \\ de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} - b_{13}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

with $a = \frac{2}{21}\sqrt{21b_7^2 - 21b_7b_{13} + 42b_{13}^2 + 21b_6^2}$;

$$3)\begin{cases} de^1 = ae^{16} + \frac{1}{2}b_{13}e^{17} - b_6e^{27}, \\ de^2 = ae^{26} + b_6e^{17} + \frac{1}{2}b_{13}e^{27}, \\ de^3 = a(\sqrt{7}e^{12} + 2e^{36}) + b_{13}e^{37}, \\ de^4 = \frac{3}{2}ae^{46} - (2b_{13} + b_{25})e^{47} + b_{24}e^{57}, \\ de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \frac{1}{21}\sqrt{231b_{13}^2 + 168b_{13}b_{25} + 84b_{25}^2 + 84b_{24}^2}$; and

$$4)\begin{cases} de^1 = \frac{1}{3}\sqrt{3}b_{25}e^{16} - \frac{1}{2}b_{25}e^{17} - b_6e^{27}, \\ de^2 = \frac{1}{3}\sqrt{3}b_{25}e^{26} + b_6e^{17} - \frac{1}{2}b_{25}e^{27}, \\ de^3 = \frac{1}{3}\sqrt{3}b_{25}(\sqrt{7}e^{12} + 2e^{36}) - b_{25}e^{37}, \\ de^4 = \frac{1}{2}\sqrt{3}b_{25}e^{46} + b_{25}e^{47} - b_{24}e^{57}, \\ de^5 = \frac{1}{2}\sqrt{3}b_{25}e^{46} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

For all of them, after calculating a generic closed 3-form we can apply Proposition 1.4.5 with $X = e_3$. Thus, the rank-two Einstein extension of $\mathfrak{n}$ does not admit closed G_2-structures.

The nilpotent Lie algebra given by

$$\mathfrak{n} = (0, 0, e^{12}, e^{13}, 0)$$

has the rank-one Einstein extension described by $\mathfrak{k}_8$. Thus, the structure equations of the rank-two Einstein extension of $\mathfrak{n}$ are

$$\begin{cases} de^1 = ae^{16} + b_1e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47} + b_5e^{57}, \\ de^2 = 2ae^{26} + b_6e^{17} + b_7e^{27} + b_8e^{37} + b_9e^{47} + b_{10}e^{57}, \\ de^3 = \sqrt{26}ae^{12} + 3ae^{36} + b_{11}e^{17} + b_{12}e^{27} + b_{13}e^{37} + b_{14}e^{47} + b_{15}e^{57}, \\ de^4 = \sqrt{26}ae^{13} + 4ae^{46} + b_{16}e^{17} + b_{17}e^{27} + b_{18}e^{37} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = 3ae^{56} + b_{21}e^{17} + b_{22}e^{27} + b_{23}e^{37} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

Therefore, after solving the equations corresponding to the Einstein condition, and the vanishing of the differential operator when it is applied twice, we obtain that the structure equations of the rank-two Einstein solvable extension of $\mathfrak{n}$ are

$$\begin{cases} de^1 = ae^{16} + (-b_{13} + b_{19})e^{17}, \\ de^2 = ae^{26} + (2b_{13} - b_{19})e^{27}, \\ de^3 = a(\sqrt{26}e^{12} + 3e^{36}) + b_{13}e^{37}, \\ de^4 = a(\sqrt{26}e^{13} + 4e^{46}) + b_{19}e^{47}, \\ de^5 = 3ae^{56} - (2b_{13} + b_{19})e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \frac{1}{39}\sqrt{390b_{13}^2 - 78b_{13}b_{19} + 156b_{19}^2}$.

We study separately the cases $b_{13}b_{19} \neq 0$, $b_{13} = 0$ and $b_{19} = 0$.

For $b_{13}b_{19} \neq 0$, using [115, Proposition 4.5] (that is, $\alpha\wedge\alpha\wedge\alpha \neq 0$, with $X = e_7$), we may suppose $\rho_{1,2,4}\rho_{1,3,5} \neq 0$ for a generic closed 3-form φ. Now, we consider the system

$$S = G - kI_7,$$

where G is the matrix associated to g_φ and k a non-zero real number. From the equation corresponding to $S_{4,5}, S_{3,4}$ and $S_{2,2}$ we obtain the values of the parameters $\rho_{1,2,5}, \rho_{1,2,3}$ and a. Substituting these values in the remaining equations and considering the equations corresponding to $S_{2,4}$ and $S_{2,5}$ we conclude that the system $S = 0$ does not have solution. Indeed, the two equations corresponding to $S_{2,4}$ and $S_{2,5}$ imply that $b_{13} = -\frac{19}{5}b_{19}$ and $b_{13} = \frac{7}{31}b_{19}$, which is a contradiction since $b_{13}b_{19} \neq 0$.

In the case $b_{13} = 0$, using [115, Proposition 4.5] with $\eta = e^7$, we may suppose $\rho_{1,3,5}\rho_{3,4,7} \neq 0$ for a generic closed 3-form. Then, we get the expression of $\rho_{2,5,7}, k$ and $\rho_{2,3,7}$ from $S_{5,5}, S_{5,3}$ and $S_{2,3}$, respectively. After substituting these values on S we have that

$$S_{4,4} = \frac{7\rho_{1,3,5}\left(49\rho_{1,2,5}^2 + 152\rho_{3,4,7}^2\right)}{76\sqrt{78}},$$

and thus, $S_{4,4} \neq 0$, since $\rho_{1,3,5}$ and $\rho_{3,4,7}$ cannot vanish.

For $b_{19} = 0$, using [115, Proposition 4.5] with $\eta = e^7$, we can consider $\rho_{1,3,5}\rho_{3,4,7} \neq 0$ for any closed 3-form. Then we obtain $\rho_{1,2,3}, \rho_{2,5,6}, \rho_{2,3,7}$ and $\rho_{1,3,5}$ from $S_{3,4}, S_{2,3}, S_{2,4}$ and $S_{3,3}$, respectively. Substituting these values of $\rho_{1,2,3}, \rho_{2,5,6}, \rho_{2,3,7}$ and $\rho_{1,3,5}$ in the remaining equations we have that

$$S_{4,4} = -\frac{959322\rho_{1,2,5}^2\rho_{3,4,7}^4 + 59711k^2}{59711k},$$

which implies again $S_{4,4} \neq 0$. Then, the rank-two Einstein extension of $\mathfrak{n}$ does not admit any closed G_2-structure inducing the Einstein metric.

If $\mathfrak{n}$ is the 5-dimensional Abelian Lie algebra, in order to obtain the rank-two Einstein extension of $\mathfrak{n}$, namely $\mathfrak{s}$, we have to consider for $\mathfrak{s}$ the structure equations

$$\begin{cases} de^1 = ae^{16} + b_1e^{17}, \\ de^2 = ae^{26} + b_2e^{27}, \\ de^3 = ae^{36} + b_3e^{37}, \\ de^4 = ae^{46} + b_4e^{47} \\ de^5 = ae^{56} + b_5e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

By imposing that $\mathfrak{s}$ is an Einstein Lie algebra (the inner product is the one for which $\{e_1, \ldots, e_7\}$ is orthonormal), we get that the family of Lie algebras with structure equations

$$\begin{cases} de^1 = ae^{16} + (-b_2 - b_3 - b_4)e^{17}, \\ de^2 = ae^{26} + b_2e^{27}, \\ de^3 = ae^{36} + b_3e^{37}, \\ de^4 = ae^{46} + b_4e^{47} \\ de^5 = ae^{56}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \sqrt{10b_7^2 + 10b_7b_{13} + 10b_7b_{19} + 10b_{13}^2 + 10b_{13}b_{19} + 10b_{19}^2}$. For these Lie algebras we first calculate a generic closed 3-form φ, and then we may apply Proposition 1.4.5 with $X = e_1, \ldots, e_5$.

Rank three

In order to determine all the 7-dimensional rank-three Einstein solvable Lie algebras, we need to find first the rank-one Einstein solvable extensions

$$\mathfrak{s}_5 = \mathfrak{n}_4 \oplus \mathbb{R}\langle e_5 \rangle$$

of the two 4-dimensional (non-Abelian) nilpotent Lie algebras $\mathfrak{n}_4$ as well as of the Abelian one. Then we consider the standard solvable Lie algebra $\mathfrak{s}_7 = \mathfrak{s}_5 \oplus \mathbb{R}\langle e_6, e_7 \rangle = \mathfrak{n}_4 \oplus \mathfrak{a}$, with $\mathfrak{a} = \mathbb{R}\langle e_5, e_6, e_7 \rangle$ Abelian and such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal. We begin with the 4-dimensional nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, e^{12}, e^{13}),$$

which has the rank-one Einstein extension defined by

$$(\frac{1}{4}ae^{15}, \frac{1}{2}ae^{25}, \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35}, \frac{1}{2}\sqrt{5}ae^{13} + ae^{45}, 0).$$

Thus, the structure equations of the rank-three Einstein solvable extensions are

$$\begin{cases} de^1 &= \frac{1}{4}ae^{15} + b_1e^{16} + b_2e^{26} + b_3e^{36} + b_4e^{46} + b_5e^{17} + b_6e^{27} + b_7e^{37} + b_8e^{47}, \\ de^2 &= \frac{1}{2}ae^{25} + b_9e^{16} + b_{10}e^{26} + b_{11}e^{36} + b_{12}e^{46} + b_{13}e^{17} + b_{14}e^{27} + b_{15}e^{37} \\ & \quad +b_{16}e^{47}, \\ de^3 &= \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35} + b_{17}e^{16} + b_{18}e^{26} + b_{19}e^{36} + b_{20}e^{46} + b_{21}e^{17} + b_{22}e^{27} \\ & \quad +b_{23}e^{37} + b_{24}e^{47}, \\ de^4 &= \frac{1}{2}\sqrt{5}ae^{13} + ae^{45} + b_{25}e^{16} + b_{26}e^{26} + b_{27}e^{36} + b_{28}e^{46} + b_{29}e^{17} + b_{30}e^{27} \\ & \quad +b_{31}e^{37} + b_{32}e^{47}, \\ de^5 &= de^6 = de^7 = 0. \end{cases} \tag{3.7}$$

After considering the equations corresponding to the inner product to be Einstein and the condition of the differential operator to vanish when applied twice, we have that the family of Lie algebras $\mathfrak{n}$ does not admit a rank-three Einstein extension.

Consider now the nilpotent Lie algebra

$$\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R} = (0, 0, e^{12}, 0),$$

where $\mathfrak{h}_3$ is the Lie algebra of the 3-dimensional nilpotent Heisenberg group, that is, $\mathfrak{h}_3 = (0, 0, e^{12})$. Then, the rank-one Einstein extension of $\mathfrak{n}$ is

$$(\frac{1}{2}ae^{15}, \frac{1}{2}ae^{25}, \frac{1}{4}\sqrt{22}ae^{12} + ae^{35}, \frac{3}{4}ae^{45}, 0).$$

Therefore, the structure equations of the corresponding rank-three Einstein solvable extensions are

$$\begin{cases} de^1 &= \frac{1}{2}ae^{15} + b_1e^{16} + b_2e^{26} + b_3e^{36} + b_4e^{46} + b_5e^{17} + b_6e^{27} + b_7e^{37} + b_8e^{47}, \\ de^2 &= \frac{1}{2}ae^{25} + b_9e^{16} + b_{10}e^{26} + b_{11}e^{36} + b_{12}e^{46} + b_{13}e^{17} + b_{14}e^{27} + b_{15}e^{37} + b_{16}e^{47}, \\ de^3 &= \frac{1}{4}\sqrt{22}ae^{12} + \frac{3}{4}ae^{35} + b_{17}e^{16} + b_{18}e^{26} + b_{19}e^{36} + b_{20}e^{46} + b_{21}e^{17} + b_{22}e^{27} \\ & \quad +b_{23}e^{37} + b_{24}e^{47}, \\ de^4 &= \frac{3}{4}ae^{45} + b_{25}e^{16} + b_{26}e^{26} + b_{27}e^{36} + b_{28}e^{46} + b_{29}e^{17} + b_{30}e^{27} + b_{31}e^{37} + b_{32}e^{47}, \\ de^5 &= de^6 = de^7 = 0. \end{cases}$$

Then, we obtain that the 7-dimensional rank-three Einstein extension of $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R}$ has the structure equations

$$\begin{cases} de^1 = \frac{1}{2}ae^{15} - (b_{10} + \frac{1}{2}b_{28})e^{16} + b_2e^{26} + (-b_{14} + b_{23})e^{17} + b_6e^{27}, \\ de^2 = \frac{1}{2}ae^{25} + b_9e^{16} + b_{10}e^{26} + b_{13}e^{17} + b_{14}e^{27}, \\ de^3 = \frac{1}{4}\sqrt{22}ae^{12} + ae^{35} - \frac{1}{2}b_{28}e^{36} + b_{23}e^{37}, \\ de^4 = \frac{3}{4}ae^{45} + b_{28}e^{46} - 2b_{23}e^{47}, \\ de^5 = de^6 = de^7 = 0, \end{cases} \tag{3.8}$$

satisfying the conditions $d^2e^i = 0$ $(i = 1, \dots, 4)$ and one of the two following:

$$
\begin{aligned}
1)\quad & a = \sqrt{\frac{32b_{14}^2 - 32b_{14}b_{23} + 96b_{23}^2 + 32b_{13}^2}{33}}, \quad b_6 = b_{13},\\
& b_2 = b_9 = \pm\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}\, b_{13}b_{23},\\
& b_{10} = \mp\frac{1}{11}\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}\,(-13b_{14}b_{23} + 6b_{23}^2 + 2b_{14}^2 + 2b_{13}^2),\\
& b_{28} = \pm\frac{2}{11}\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}\,(4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2);\\
2)\quad & a = 2\sqrt{\frac{2}{3}}b_{23}, \quad b_2 = \pm\frac{1}{2}\sqrt{11b_{23}^2 - 4b_{10}^2}, \quad b_6 = b_{13} = b_{28} = 0, \quad b_{14} = \frac{1}{2}b_{23}.
\end{aligned}
$$

For 1) we determine the general expression of a generic closed 3-form φ. Using [115, Proposition 4.5] with $X = e_7$, and the fact that $a \neq 0$, we consider the condition $\alpha \wedge \beta = 0$, where

$$
\begin{aligned}
\alpha =& \rho_{1,2,7}e^{12} + \left(\frac{2\left(b_{14} - b_{23}\right)\rho_{1,5,6} - \left(2b_{10} + b_{28}\right)\rho_{1,5,7} - 2b_{13}\rho_{2,5,6} + 2b_9\rho_{2,5,7}}{a}\right)e^{16}\\
&+ \rho_{1,5,7}e^{15} + \rho_{2,5,7}e^{25} + \left(-\frac{2b_6\rho_{1,5,6} + 2b_2\rho_{1,5,7} - 2b_{14}\rho_{2,5,6} + 2b_{10}\rho_{2,5,7}}{a}\right)e^{26}\\
&+ \frac{2\sqrt{\frac{2}{11}}b_{23}\rho_{1,2,4}}{a}e^{34} + \left(2\sqrt{\frac{2}{11}}\rho_{1,2,7} - \frac{2\sqrt{\frac{2}{11}}b_{23}\rho_{1,2,5}}{a}\right)e^{35} + \rho_{4,5,7}e^{45}\\
&+ \left(-\frac{2\sqrt{\frac{2}{11}}b_{23}\rho_{1,2,6}\sqrt{\frac{2}{11}}b_{28}\rho_{1,2,7}}{a}\right)e^{36} + \left(\frac{8b_{23}\rho_{4,5,6} + 4b_{28}\rho_{4,5,7}}{3a}\right)e^{46} + \rho_{5,6,7}e^{56}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta =& \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,5,6}e^{156} + \rho_{2,5,6}e^{256} - \frac{7\rho_{1,2,4}}{\sqrt{22}}e^{345}\\
&- \frac{\sqrt{\frac{2}{11}}b_{28}\rho_{1,2,4}}{a}e^{346} + \left(\frac{\sqrt{\frac{2}{11}}b_{28}\rho_{1,2,5}}{a} + 2\sqrt{\frac{2}{11}}\rho_{1,2,6}\right)e^{356} + \rho_{4,5,6}e^{456}.
\end{aligned}
$$

Take the metric induced by φ, namely g_φ, whose corresponding matrix in terms of the basis $\{e_1, \dots, e_7\}$ is denoted by G. Then, the Einstein condition is equivalent to the vanishing of the matrix system

$$S = G - kI_7,$$

where k is a non-zero real number. Considering the equations corresponding to $S_{3,4}$ and $S_{3,3}$, we have that $S_{2,2} \neq 0$. Therefore, the family of Lie algebras satisfying condition 1) does not admit closed G_2 form inducing the Einstein metric.

For 2) we start considering the conditions

$$a = \frac{2}{3}\sqrt{6}b_{23}, \qquad b_6 = b_{13} = b_{28} = 0, \qquad b_{14} = \frac{1}{2}b_{23},$$

where $b_{23} \neq 0$ since $a \neq 0$.

We calculate the expression of a generic closed 3-form φ. Then we use the condition $\alpha \wedge \beta \neq 0$ in [115, Proposition 4.5] for $X = e_7$. Studying separately the solutions of the equations corresponding to $S_{3,3}, S_{3,4}, S_{2,3}$ and $S_{1,3}$, can be checked that this Lie algebra does not carry any closed G_2 form inducing the Einstein metric.

If $\mathfrak{n}$ is the 4-dimensional Abelian Lie algebra, to get the rank-three Einstein extension of $\mathfrak{n}$, namely $\mathfrak{s}$, we should consider the structure equations

$$\begin{cases} de^1 = ae^{15} + b_1e^{16} + b_2e^{17}, \\ de^2 = ae^{25} + b_3e^{26} + b_4e^{27}, \\ de^3 = ae^{35} + b_5e^{36} + b_6e^{37}, \\ de^4 = ae^{45} + b_7e^{46} + b_8e^{47}, \\ de^5 = de^6 = de^7 = 0. \end{cases} \tag{3.9}$$

The conditions of the inner product to be Einstein and $d^2(e^i) = 0$ imply

$$\begin{aligned} &b_7 = -b_1 - b_3 - b_5, \qquad b_8 = -b_2 - b_4 - b_6, \\ &2b_1b_2 + 2b_3b_4 + 2b_5b_6 + b_2b_3 + b_2b_5 + b_1b_4 + b_4b_5 + b_1b_6 + b_3b_6 = 0, \\ &2b_1^2 + 2b_3^2 + 2b_5^2 + 2b_1b_3 + 2b_1b_5 + 2b_3b_5 = 4a^2. \end{aligned}$$

We take the expression of a generic closed 3-form φ and use [115, Proposition 4.5] with $X = e_7$. Then, we consider the equations $S_{1,2}, S_{1,3}, S_{1,4}$ and $S_{2,3}$ obtaining that

$$\rho_{1,2,5}\rho_{1,3,5}\rho_{1,4,5} = 0,$$

where $\rho_{i,j,k}$ denotes the coefficient of e^{ijk} in φ. But, from $S_{1,1} = 0$ we obtain that $\rho_{1,2,5}\rho_{1,3,5}\rho_{1,4,5} \neq 0$. So the system $S = 0$ does not have solution, that is, the rank-three Einstein solvable extension of the 4-dimensional Abelian Lie algebra does not admit any closed G_2 form inducing the Einstein metric. □

Remark 3.3.6. *Note that in the proof of Theorem 3.3.5, we study only the existence of closed* G_2 *forms inducing Einstein metrics on the Einstein extensions of rank two and three, which is sufficient to prove Theorem 3.3.5. But, we do not care about closed* G_2 *forms whose corresponding metric is not Einstein.*

Remark 3.3.7. *Note that for the Lie algebra* $\mathfrak{g}_{28}$ *with structure equations*

$$\mathfrak{g}_{28} = (e^{17}, e^{27}, e^{37}, e^{47}, 2e^{13} - 2e^{24} + 2e^{57}, 2e^{14} + 2e^{23} + 2e^{67}, 0),$$

we are able to solve 48 of the 49 equations of the system $G = kI_7$*. The 3-form* φ *on* $\mathfrak{g}_{28}$ *given by*

$$\varphi = -2e^{127} - 2e^{347} - e^{136} + e^{145} + e^{235} + e^{246} + 2e^{567},$$

is a closed G_2 *form, which induces the metric*

$$g_\varphi = 2(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6 + 4e^7 \otimes e^7).$$

We finish this subsection with Table 3.1 and Table 3.2, which were used in the proof of Theorem 3.3.5. Table 3.1 contains the list due to Will [127] of the rank-one Einstein 7-dimensional solvable Lie algebras, each of them defined in terms of a basis $\{e_1, \ldots, e_7\}$ which is orthonormal with respect to the Einstein inner product. In Table 3.2 we consider those Lie algebras admitting closed G_2 form (but clearly not inducing the Einstein inner product) and such a G_2 form is also given.

Table 3.1: **Rank-one Einstein 7-dimensional solvable Lie algebras.**

$\mathfrak{g}_1$	$(\frac{a}{2}e^{17}, ae^{27}, \sqrt{13}ae^{12} + \frac{3}{2}ae^{37}, 4ae^{13} + 2ae^{47}, 2\sqrt{3}e^{14} + 2ae^{23} + \frac{5}{2}ae^{57},$ $-\sqrt{13}ae^{25} + 2\sqrt{3}ae^{34} + \frac{7}{2}ae^{67}, 0)$
$\mathfrak{g}_2$	$(-\frac{\sqrt{21}}{42}ae^{17}, -\frac{\sqrt{21}}{14}ae^{27}, ae^{12} - 2\frac{\sqrt{21}}{21}ae^{37}, -2\frac{\sqrt{3}}{3}ae^{13} - 5\frac{\sqrt{21}}{42}ae^{47},$ $ae^{14} - \frac{\sqrt{21}}{7}ae^{57}, ae^{34} - ae^{25} - 3\frac{\sqrt{21}}{14}ae^{67}, 0)$
$\mathfrak{g}_3$	$(-\frac{\sqrt{14}}{56}ae^{17}, -9\frac{\sqrt{14}}{56}ae^{27}, ae^{12} - 5\frac{\sqrt{14}}{28}ae^{37}, \frac{\sqrt{6}}{2}ae^{13} - 11\frac{\sqrt{14}}{56}ae^{47}, \frac{\sqrt{6}}{2}ae^{14} - 3\frac{\sqrt{14}}{14}ae^{57},$ $ae^{15} - 13\frac{\sqrt{14}}{56}ae^{67}, 0)$
$\mathfrak{g}_4$	$(ae^{17}, 2ae^{27}, 2\sqrt{7}e^{12} + 3ae^{37}, \frac{6\sqrt{154}}{11}ae^{13} + 4ae^{47}, 2\sqrt{7}ae^{14} + 2\frac{\sqrt{1155}}{11}ae^{23} + 5ae^{57},$ $2\frac{\sqrt{1155}}{11}ae^{15} + 5\frac{\sqrt{154}}{11}ae^{24} + 6ae^{67}, 0)$
$\mathfrak{g}_5$	$\left(ae^{17}, 3ae^{27}, 2\sqrt{14}ae^{12} + 4ae^{37}, 2\sqrt{15}ae^{13} + 5ae^{47}, 6\sqrt{2}ae^{14} + 6ae^{57}, 4\sqrt{2}ae^{15} + 2\sqrt{15}ae^{23} + 7ae^{67}, 0\right)$
$\mathfrak{g}_6$	$\left(\frac{a}{2}e^{17}, ae^{27}, \sqrt{10}ae^{12} + \frac{3}{2}ae^{37}, \sqrt{10}ae^{13} + 2ae^{47}, \sqrt{10}ae^{23} + \frac{5}{2}ae^{57}, \sqrt{10}ae^{14} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_7$	$\left(ae^{17}, ae^{27}, 4ae^{12} + 2ae^{37}, 2\sqrt{5}ae^{13} + 3ae^{47}, 2\sqrt{5}ae^{23} + 3ae^{57}, 4ae^{14} - 4ae^{24} + 4ae^{67}, 0\right)$
$\mathfrak{g}_8$	$\left(ae^{17}, ae^{27}, 4ae^{12} + 2ae^{37}, 2\sqrt{5}ae^{13} + 3ae^{47}, 2\sqrt{5}ae^{23} + 3ae^{57}, 4ae^{14} + 4ae^{24} + 4ae^{67}, 0\right)$
$\mathfrak{g}_9$	$\left(\frac{-3}{14}ae^{17}, \frac{-11}{28}ae^{27}, \frac{-3}{7}ae^{37}, \frac{\sqrt{5}}{2}ae^{12} - \frac{17}{28}ae^{47}, ae^{14} - ae^{23} + \frac{-23}{28}ae^{57}, ae^{34} + \frac{\sqrt{5}}{2}ae^{15} - \frac{29}{28}ae^{67}, 0\right)$
$\mathfrak{g}_{10}$	$\left(\frac{4}{9}ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{2\sqrt{114}}{9}ae^{12} + \frac{13}{9}ae^{47}, \frac{2}{9}\sqrt{190}ae^{14} + \frac{17}{9}ae^{57}, \frac{2\sqrt{114}}{9}ae^{15} + \frac{2\sqrt{114}}{9}ae^{23} + \frac{7}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{11}$	$(\frac{a}{3}e^{17}, \frac{2}{3}ae^{27}, \frac{10\sqrt{7}}{21}ae^{12} + ae^{37}, \frac{4\sqrt{42}}{21}ae^{12} + ae^{47}, \frac{4\sqrt{105}}{21}ae^{13} + \frac{2\sqrt{70}}{21}ae^{14} + \frac{4}{3}ae^{57},$ $\frac{2\sqrt{6}}{3}ae^{15} + \frac{2\sqrt{7}}{3}ae^{24} + \frac{5}{3}ae^{67}, 0)$
$\mathfrak{g}_{12}$	$\left(\frac{a}{2}e^{17}, ae^{27}, \frac{11}{6}ae^{37}, \frac{2\sqrt{21}}{3}ae^{12} + \frac{3}{2}ae^{47}, \frac{2\sqrt{21}}{3}ae^{14} + 2ae^{57}, \frac{2\sqrt{14}}{3}ae^{15} + \frac{2\sqrt{14}}{3}ae^{24} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{13}$	$\left(\frac{2}{9}ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{2\sqrt{93}}{9}ae^{12} + \frac{33}{27}ae^{47}, \frac{4\sqrt{31}}{9}ae^{14} + \frac{39}{27}ae^{57}, \frac{2\sqrt{93}}{9}ae^{15} + \frac{5}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{14}$	$\left(\frac{a}{2}e^{17}, ae^{27}, \frac{3}{4}ae^{37}, \frac{\sqrt{21}}{2}ae^{12} + \frac{3}{2}ae^{47}, \frac{\sqrt{14}}{2}ae^{13} + \frac{5}{4}ae^{57}, \frac{\sqrt{14}}{2}ae^{14} + \frac{\sqrt{21}}{2}ae^{35} + 2ae^{67}, 0\right)$
$\mathfrak{g}_{15}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{10}ae^{12} + 2ae^{47}, \sqrt{10}ae^{23} + 2ae^{57}, \sqrt{10}ae^{14} + \sqrt{10}ae^{35} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{16}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{10}ae^{12} + 2ae^{47}, \sqrt{10}ae^{23} + 2ae^{57}, \sqrt{10}ae^{14} - \sqrt{10}ae^{35} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{17}$	$\left(ae^{17}, ae^{27}, \frac{12}{5}ae^{37}, \frac{4}{5}\sqrt{31}ae^{12} + 2ae^{47}, \frac{2}{5}\sqrt{93}ae^{14} + 3ae^{57}, \frac{2}{5}\sqrt{93}ae^{24} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{18}$	$\left(ae^{17}, ae^{27}, 2ae^{37}, 4ae^{12} + 2ae^{47}, 2ae^{13} - 2\sqrt{3}ae^{24} + 3ae^{57}, 2\sqrt{3}ae^{14} + 2ae^{23} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{19}$	$\left(5ae^{17}, 6ae^{27}, 12ae^{37}, 2\sqrt{134}ae^{12} + 11ae^{47}, \sqrt{402}ae^{14} + 16ae^{57}, \sqrt{134}ae^{13} - \sqrt{402}ae^{24} + 17ae^{67}, 0\right)$
$\mathfrak{g}_{20}$	$\left(ae^{17}, ae^{27}, 2ae^{12} + 2ae^{37}, 2\sqrt{3}ae^{12} + 2ae^{47}, 4ae^{14} + 3ae^{57}, 2ae^{24} + 2\sqrt{3}ae^{23} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{21}$	$\left(3ae^{17}, 5ae^{27}, 6ae^{37}, 2\sqrt{42}ae^{12} + 8ae^{47}, 2\sqrt{21}ae^{13} + 9ae^{57}, 2\sqrt{42}ae^{14} + 2\sqrt{21}ae^{23} + 11ae^{67}, 0\right)$
$\mathfrak{g}_{22}$	$\left(6ae^{17}, 5ae^{27}, 9ae^{37}, 2\sqrt{93}ae^{12} + 11ae^{47}, 2\sqrt{93}ae^{13} + 15ae^{57}, 4\sqrt{31}ae^{24} + 16ae^{67}, 0\right)$
$\mathfrak{g}_{23}$	$\left(ae^{17}, \frac{5}{2}ae^{27}, 3ae^{37}, \sqrt{37}ae^{12} + \frac{7}{2}ae^{47}, \frac{\sqrt{74}}{2}ae^{13} + 4ae^{57}, \sqrt{37}ae^{14} + \frac{9}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{24}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{6}ae^{12} + 2ae^{47}, \sqrt{6}ae^{13} + 2ae^{57}, \sqrt{6}ae^{23} + 2ae^{67}, 0\right)$
$\mathfrak{g}_{25}$	$\left(\frac{5\sqrt{31}}{124}ae^{17}, \frac{2\sqrt{31}}{31}ae^{27}, \frac{9\sqrt{31}}{124}ae^{37}, \frac{9\sqrt{31}}{124}ae^{47}, ae^{12} + \frac{13\sqrt{31}}{124}ae^{57}, -\frac{\sqrt{3}}{2}ae^{34} - \frac{\sqrt{3}}{2}ae^{15} + \frac{9\sqrt{31}}{62}ae^{67}, 0\right)$
$\mathfrak{g}_{26}$	$\left(ae^{17}, 2ae^{27}, 3ae^{37}, 3ae^{47}, 4\sqrt{2}ae^{12} + 3ae^{57}, 4\sqrt{2}ae^{15} + 4ae^{67}, 0\right)$
$\mathfrak{g}_{27}$	$\left(ae^{17}, \frac{3}{4}ae^{27}, \frac{7}{4}ae^{37}, \frac{3}{2}ae^{47}, \frac{\sqrt{148}}{4}ae^{12} + \frac{7}{4}ae^{57}, \frac{\sqrt{74}}{4}ae^{14} + \frac{\sqrt{37}}{2}ae^{25} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{28}$	$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, 2ae^{13} - 2ae^{24} + 2ae^{57}, 2ae^{14} + 2ae^{23} + 2ae^{67}, 0)$
$\mathfrak{g}_{29}$	$\left(ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{4}{3}ae^{47}, \sqrt{6}ae^{12} + 2ae^{57}, \sqrt{6}ae^{14} + \sqrt{6}ae^{23} + \frac{7}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{30}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{a}{2}e^{37}, \frac{a}{2}e^{47}, \sqrt{2}ae^{12} + ae^{57}, \sqrt{2}ae^{34} + ae^{67}, 0\right)$
$\mathfrak{g}_{31}$	$\left(\frac{\sqrt{11}}{11}ae^{17}, \frac{3\sqrt{11}}{22}ae^{27}, \frac{3\sqrt{11}}{22}ae^{37}, \frac{2\sqrt{11}}{11}ae^{47}, ae^{12} + \frac{5\sqrt{11}}{22}ae^{57}, ae^{13} + \frac{5\sqrt{11}}{22}ae^{67}, 0\right)$
$\mathfrak{g}_{32}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{a}{2}e^{37}, \frac{a}{2}e^{47}, \frac{2}{3}ae^{57}, \frac{\sqrt{11}}{3}ae^{12} + \frac{\sqrt{11}}{3}ae^{34} + ae^{67}, 0\right)$
$\mathfrak{g}_{33}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{3}{4}ae^{37}, \frac{3}{4}ae^{47}, \frac{3}{4}ae^{57}, \frac{\sqrt{34}}{4}ae^{12} + ae^{67}, 0\right)$

Table 3.2: **Closed G_2-structures on rank-one Einstein solvable Lie algebras.**

$\mathfrak{s}_7$	Closed G_2-structure
$\mathfrak{g}_1$	$\varphi = \frac{1}{432}\left(1440 - 128\sqrt{3}\right)e^{123} + \frac{\sqrt{13}}{2}e^{125} - \frac{(13312\sqrt{3}-748800)}{44928\sqrt{3}}e^{127}$ $+\frac{8}{9}e^{135} - 2e^{137} - \frac{1}{\sqrt{3}}e^{146} - \sqrt{3}e^{147} + 10e^{157} - e^{167} - \frac{1}{3}e^{236} + e^{237}$ $+\frac{1}{576}\left(1440 + 128\sqrt{3}\right)e^{247} + \frac{1}{\sqrt{3}}e^{267} + \frac{1}{\sqrt{3}}e^{345} - e^{357} - e^{457} + e^{567}$
$\mathfrak{g}_4$	$\varphi = -\frac{7}{2\sqrt{5}}e^{125} + e^{137} - \frac{7}{13}e^{146} - e^{147} + \frac{1}{2}e^{167} + \frac{7}{13}e^{236} - e^{237}$ $+2e^{247} - e^{267} + \frac{7}{13}e^{345} + \frac{1}{2}e^{357} - e^{457} - e^{567}$
$\mathfrak{g}_9$	$\varphi = -\frac{7}{2\sqrt{5}}e^{125} + e^{137} - \frac{7}{13}e^{146} - e^{147} + \frac{1}{2}e^{167} + \frac{7}{13}e^{236}$ $-e^{237} + 2e^{247} - e^{267} + \frac{7}{13}e^{345} + \frac{1}{2}e^{357} - e^{457} - e^{567}$
$\mathfrak{g}_{18}$	$\varphi = e^{123} - e^{127} - e^{136} + \sqrt{3}e^{145}$ $+3e^{167} + e^{235} + \sqrt{3}e^{246} - \frac{1}{2}e^{347} + 3e^{567}$
$\mathfrak{g}_{28}$	$\varphi = -2e^{127} - 2e^{347} - e^{136} + e^{145} + e^{235} + e^{246} + 2e^{567}$

3.3.2 Coclosed G_2-structures

In this section we study the existence of coclosed G_2-structures φ on 7-dimensional solvable Lie algebras whose underlying Riemannian metric g_φ is Einstein. We will use the classification of the 7-dimensional Einstein solvable Lie algebras contained in the previous section as well as the following results.

Lemma 3.3.8. *Let* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ *be a* 7*-dimensional metric Lie algebra. If there is a non-zero vector* $X \in \mathfrak{g}$ *such that* $(\iota_X\phi)^3 = 0$ *for every coclosed 3-form* ϕ *on* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$*, then* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ *has no coclosed* G_2*-structure.*

Proof. Suppose that $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ has a coclosed G_2 form φ inducing the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then, according with [38], for any $X \in \mathfrak{g}$, we know that $\iota_X\varphi$ is the two-form of an SU(3)-structure and therefore it is non-degenerate or, equivalently, $(\iota_X\varphi)^3 \neq 0$. In view of this contradiction we conclude that $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ does not admit coclosed G_2 forms.

$\square$

Lemma 3.3.9. *Let* $\mathfrak{g}$ *be a* 7*-dimensional solvable Lie algebra with a coclosed* G_2 *form* φ *inducing the inner product* $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{g}$*. Denote by* $*$ *the Hodge star operator on* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$*. Then, there exists a real number* λ *such that*

$$\varphi \wedge \tau = 0 \quad \textit{and} \quad *\varphi \wedge \tau = 0,$$

where τ *is the 4-form on* $\mathfrak{g}$ *given by*

$$\tau = -*(d\varphi - \lambda * \varphi).$$

Proof. Suppose that φ is a coclosed G_2 form on $\mathfrak{g}$. By Proposition 1.1.13, we know that $d\varphi$ and $d*\varphi$ are expressed in terms of the torsion forms as

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3,$$
$$d*\varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi,$$

where $\tau_0 \in \Lambda^0\mathfrak{g}^* = \mathbb{R}$, $\tau_1 \in \Lambda^1\mathfrak{g}^* = \mathfrak{g}^*$, $\tau_2 \in \Lambda^2_{14}\mathfrak{g}^*$ and $\tau_3 \in \Lambda^3_{27}\mathfrak{g}^*$. The coclosed condition ($d*\varphi = 0$) implies that $\tau_1 = \tau_2 = 0$. Thus,

$$d\varphi = \tau_0 * \varphi + *\tau_3.$$

This implies that

$$\tau_3 = - * (d\varphi - \tau_0 * \varphi).$$

But, by (1.12), we know that

$$\Lambda^3_{27}\mathfrak{g}^* = \{\rho \in \Lambda^3\mathfrak{g}^* \mid \rho \wedge \varphi = 0 = \rho \wedge *\varphi\}.$$

As τ_3 belongs to the space $\Lambda^3_{27}\mathfrak{g}^*$, the conditions

$$\varphi \wedge \tau_3 = 0 \quad \text{and} \quad * \varphi \wedge \tau_3 = 0$$

must be fulfilled. Thus, it is sufficient to take $\lambda = \tau_0$ and $\tau = \tau_3$. $\square$

We recall that a 5-dimensional manifold N has an SU(2)-*structure* if there exists a quadruplet of differential forms $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form and ω_i are 2-forms on N, satisfying $\omega_i \wedge \omega_j = \delta_{ij} v$, $v \wedge \eta \neq 0$, for some nowhere vanishing 4-form v, and

$$\iota_X\omega_3 = \iota_Y\omega_1 \implies \omega_2(X, Y) \geq 0,$$

for any vector fields X and Y on N. Such structures were introduced by Conti and Salamon in [39].

Proposition 3.3.10. *Let $(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, \langle\cdot,\cdot\rangle)$ be a 7-dimensional rank-two Einstein Lie algebra, and let $\{e_1, \dots, e_7\}$ be an orthonormal basis of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ such that $\mathfrak{a} = \mathbb{R}\langle e_6, e_7\rangle$. Then, any* G_2*-structure φ on $\mathfrak{g}$, with $g_\varphi = \langle\cdot,\cdot\rangle$, induces an* SU(2)*-structure on $\mathfrak{n}$ such that the associated metric h is the restriction of g_φ to $\mathfrak{n}$.*

Proof. By [115, Proposition 4.5] we know that the 2-form $F = \iota_{e_7}\varphi$ and the 3-form ρ given by $\rho = \varphi - F \wedge e^7$ determine an SU(3)-structure on $\mathbb{R}\langle e_1, \dots, e_6\rangle$ whose associated metric is the restriction of g_φ to $\mathbb{R}\langle e_1, \dots, e_6\rangle$. Now we can write $F = f^{12} + f^{34} + f^5 \wedge e^6$ and $\rho + \imath\bar{\rho} = (f^1 + i f^2) \wedge (f^3 + i f^4) \wedge (f^5 + i\, e^6)$, where $f_i \in \mathbb{R}\langle e_1, \dots, e_5\rangle$ and $\{f_1, \dots, f_5, e_6\}$ is orthonormal. Then by [39, Proposition 1.4] the forms

$$\eta = f^5, \quad \omega_1 = f^{12} + f^{34}, \quad \omega_2 = f^{13} + f^{42}, \quad \omega_3 = f^{14} + f^{23}$$

define an SU(2)-structure on $\mathfrak{n}$. The basis $\{f_1, \ldots, f_5\}$ is orthonormal with respect to the metric h induced by the SU(2)-structure. So, h coincides with the restriction of g_φ to $\mathfrak{n}$. $\square$

Corollary 3.3.11. *Let* $(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, \langle \cdot, \cdot \rangle)$ *be a 7-dimensional rank-two Einstein Lie algebra, and let* $\{e_1, \ldots, e_7\}$ *be an orthonormal basis such that* $\mathfrak{a} = \mathbb{R}\langle e_6, e_7 \rangle$. *If for any coclosed 3-form*φ *one of the two following conditions*

- $(\omega_i^2 - \omega_j^2) \wedge \eta \neq 0$ *for some* $i \neq j$,
- $\omega_i \wedge \eta \neq *_h \omega_i$ *for some* i

is satisfied, where $(\omega_1, \omega_2, \omega_3, \eta)$ *is the* SU(2)*-structure as in Proposition 3.3.10, and* h *its corresponding metric, then,* $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ *does not admit any coclosed* G_2*-structure* φ *such that* $g_\varphi = \langle \cdot, \cdot \rangle$.

Proof. By Proposition 3.3.10, a G_2-structure φ on $(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, \langle \cdot, \cdot \rangle)$, such that $g_\varphi = \langle \cdot, \cdot \rangle$ and $\mathfrak{a} = \mathbb{R}\langle e_6, e_7 \rangle$, induces an SU(2)-structure $(\omega_1, \omega_2, \omega_3, \eta)$ on $\mathfrak{n}$. By definition of SU(2)-structure, the forms $(\omega_1, \omega_2, \omega_3, \eta)$ have to satisfy the conditions $(\omega_i^2 - \omega_j^2) \wedge \eta = 0$ for all i, j, and $\omega_i \wedge \eta = *_h \omega_i$ for all $i = 1, 2, 3$. $\square$

We already know that 7-dimensional Einstein solvable Lie algebras cannot admit nearly-parallel G_2-structures since the scalar curvature of such a structure has to be positive. Next, we show that in general those metric Lie algebras do not admit coclosed G_2 form whose induced metric is Einstein.

Theorem 3.3.12. *A* 7*-dimensional solvable Lie group cannot admit any left invariant coclosed* G_2*-structure* φ *such that its induced metric* g_φ *is Einstein, unless* g_φ *is flat.*

Proof. According with Theorem 3.1.4, an Einstein solvable Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ is necessarily standard, so one has the orthogonal decomposition (with respect to $\langle \cdot, \cdot \rangle$)

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},$$

where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is a nilpotent Lie algebra and $\mathfrak{a}$ Abelian. We will consider separately the different possibilities according to the rank of $\mathfrak{s}$, that is, the dimension of $\mathfrak{a}$.

Rank one

If $\mathfrak{n}$ is Abelian, then by [80, Proposition 6.12], we know that $\mathfrak{s}$ has the structure equations

$$\mathfrak{s} = (ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where a is a non-zero real number. A generic coclosed 3-form φ on $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ has the following expression

$$\begin{aligned}\varphi =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,3,4}e^{134} + \rho_{1,3,5}e^{135} + \rho_{1,3,6}e^{136}\\ &+ \rho_{1,4,5}e^{145} + \rho_{1,4,6}e^{146} + \rho_{1,5,6}e^{156} + \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,4,5}e^{245}\\ &+ \rho_{2,4,6}e^{246} + \rho_{2,5,6}e^{256} + \rho_{3,4,5}e^{345} + \rho_{3,4,6}e^{346} + \rho_{3,5,6}e^{356} + \rho_{4,5,6}e^{456}.\end{aligned}$$

We apply the aforementioned result [115, Proposition 4.5]. For example, for $X = e_1$, we have

$$\begin{aligned}\alpha = \iota_{e_1}\varphi =&\rho_{1,2,3}e^{23} + \rho_{1,2,4}e^{24} + \rho_{1,2,5}e^{25} + \rho_{1,2,6}e^{26} + \rho_{1,3,4}e^{34} + \rho_{1,3,5}e^{35} + \rho_{1,3,6}e^{36}\\ &+ \rho_{1,4,5}e^{45} + \rho_{1,4,6}e^{46} + \rho_{1,5,6}e^{56}\end{aligned}$$

and therefore,

$$\alpha \wedge \alpha \wedge \alpha = 0.$$

If $\mathfrak{n}$ is (non-Abelian) nilpotent, then $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i$ $(i = 1, \ldots, 33)$ in Table 3.1, endowed with the Riemannian metric such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal. To study the Lie algebras $\mathfrak{g}_3$, $\mathfrak{g}_{13}$, $\mathfrak{g}_{23}$ and $\mathfrak{g}_j$ $(25 \leq j \leq 33)$, we apply Lemma 3.3.8 with $X = e_7$, and we see in this way that they do not admit any coclosed G_2-structure φ such that $g_\varphi = \langle \cdot, \cdot \rangle$.

For the Lie algebras

$$\mathfrak{g}_1, \quad \mathfrak{g}_2, \quad \mathfrak{g}_4, \quad \mathfrak{g}_5, \quad \mathfrak{g}_6, \quad \mathfrak{g}_{20},$$

we write, for every of these algebras, the expression of a generic coclosed 3-form φ, and then we determine the torsion forms τ_0 and τ_3. In all the cases, we obtain that $\tau_3 \wedge \varphi \neq 0$ unless φ vanishes. So applying Lemma 3.3.9 we have that these Lie algebras do not admit a coclosed G_2-structure.

For the Lie algebras

$$\mathfrak{g}_j, \quad 7 \leq j \leq 24, \quad j \neq 13, 20, 23, \tag{3.10}$$

we first determine a generic coclosed 3-form φ and then, by applying [115, Proposition 4.5] with $\eta = e^1, \ldots, e^7$, we impose the conditions (3.6) for the corresponding forms α and β, which are defined as in (3.5). Moreover, we have that the coclosed 3-form φ defines a G_2-structure if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the orthonormal basis $\{e_1, \ldots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a coclosed G_2-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the basis $\{e_1, \ldots, e_7\}$, is a multiple of the identity matrix. By a direct computation we have thus that the Lie algebras (3.10) cannot admit any coclosed G_2-structure inducing an Einstein metric.

Rank two

For the 7-dimensional rank-two Einstein solvable Lie algebras, using the same notation as for the closed case (Theorem 3.3.5), we obtain the result for $\mathfrak{k}_1, \mathfrak{k}_3, \mathfrak{k}_5, \mathfrak{k}_{6,1}$ and $\mathfrak{k}_{6,2}$ by the first condition of Corollary 3.3.11. For the other Lie algebras, that is, $\mathfrak{k}_4, \mathfrak{k}_{7,1}, \mathfrak{k}_{7,2}, \mathfrak{k}_{7,3}, \mathfrak{k}_{7,4}, \mathfrak{k}_8$, as well as for the extension of the Abelian one, the result is obtained using the second condition of Corollary 3.3.11.

Rank three

As we did in the subsection concerning closed G_2-structures, we have to consider the rank-three solvable Einstein extensions of the two (non-Abelian) nilpotent 4-dimensional Lie algebras, as well as of the Abelian one.

From (3.7), we know that the 4-dimensional nilpotent Lie algebra with structure equations

$$\mathfrak{n} = (0, 0, e^{12}, e^{13})$$

does not admit a rank-three Einstein solvable extension.

For the nilpotent Lie algebra described by

$$\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R} = (0, 0, e^{12}, 0),$$

the rank-three Einstein solvable extension is given by (3.8). Then, we can take the expression of a 3-form φ which is coclosed. The, we calculate the metric g_φ induced by φ, and consider the matrix system

$$S = G - kI_7,$$

where G is the matrix corresponding to g_φ with respect to $\{e_1, \ldots, e_7\}$, and k a non-zero real number. Thus, if we try to solve the system $S = 0$, which is equivalent to the inner product to be Einstein, we obtain that $S_{6,6} = k$. Therefore, the system $S = G - kI_7$ does not have solution, and so the rank-three Einstein extension of $\mathfrak{h}_3 \oplus \mathbb{R}$ does not admit coclosed G_2-structure inducing the Einstein metric of (3.8).

Finally we consider the rank-three Einstein extension of the 4-dimensional Abelian Lie algebra. Its corresponding structure equations are described in (3.9). As before, we take the general expression of a coclosed 3-form φ and we calculate its induced metric. Imposing that the matrix system S (defined as before) must vanish, we obtain again that $S_{6,6} = k$. Therefore, the system $S = 0$ does not have solution or, equivalently, the rank-three Einstein extension of the 4-dimensional Abelian Lie algebra does not admit coclosed G_2-structure inducing the Einstein metric of (3.9).

□

3.4 An Einstein cocalibrated G_2 manifold from warped products

In this section, using warped products, we show an explicit example of a (non-nearly parallel) coclosed G_2 form inducing an Einstein metric on a non-compact manifold. We point out that, according with Chapter 1 (see (1.11)), the Riemannian product of a half-flat manifold by $\mathbb{R}$ (or by S^1 in the compact case) has a coclosed G_2 form but the induced metric is not Einstein. Moreover, Agricola and Friedrich in [1] prove that any 7-dimensional 3-Sasakian manifold $(M, \phi_i, \xi_i, \eta_i, g;\ i = 1, 2, 3)$ has a canonical coclosed G_2 form φ such that the induced metric g_φ is $g_\varphi = g$ and so is Einstein. But we will see that the metric of our example is not 3-Sasakian.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds, and let $f > 0$ be a real differentiable function on B. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. The warped product, namely $M = B \times_f F$, is the product manifold $M = B \times F$ endowed with the metric g given by

$$g = \pi^*(g_B) + f^2\sigma^*(g_F).$$

The manifold B is called the base of M, F the fibre, and the warped product is called trivial if f is a constant function. We denote by Ric^B the lift (pullback by π) of the Ricci curvature of B, similarly for Ric^F; and let $Hess(f)$ be the lift to M of the Hessian of f. By [109, p. 211] the warped product (M, g), where $M = B \times_f F$, is Einstein with $Ric = \lambda g$ if and only if (F, g_F) is Einstein ($Ric^F = \mu g_F$), with Einstein constant μ, and the following conditions are satisfied

$$\begin{aligned} &\lambda g_B = Ric^B - \tfrac{d}{f} Hess(f), \\ &\lambda = \tfrac{\mu}{f^2} - \tfrac{\Delta f}{f} - (d-1)\left|\tfrac{\nabla f}{f}\right|^2_{g_B}, \end{aligned} \tag{3.11}$$

where $\Delta f = tr\big(Hess(f)\big)$, ∇f denotes the gradient of f and $d = \dim(F)$.

Moreover, when the base space has dimension 1 ($\dim(B) = 1$), then equations (3.11) reduce to

$$(f')^2 + \frac{\lambda}{d} f^2 = \frac{\mu}{d-1}. \tag{3.12}$$

The behavior of the solutions of (3.12) depends on the signs of λ and μ. Nevertheless, up to homotheties, those solutions (besides the constant case) are given in the following table (Table 3.3) (see also [16]).

Table 3.3: **Solutions of the system** (3.12)

μ	$-(d-1)$	0	$d-1$	$d-1$	$d-1$
λ	$-d$	$-d$	$-d$	0	d
f	$\cosh t$	e^t	$\sinh t$	t	$\sin t$

From this table follows the next result.

Theorem 3.4.1 (Theorem 9.110, [16])**.** *Let* (M,g) *be a warped product, where* $M = B \times_f F$, $\dim(B) = 1$ *and* $\dim(F) = d > 1$. *If* (M,g) *is a complete Einstein manifold, then either* M *is a Ricci-flat Riemannian product, or* $B = \mathbb{R}$, F *is Einsten with non-positive scalar curvature and* M *has negative scalar curvature.*

Now, let $M = I \times_f N$ be a warped product, where I is an open interval, $f : I \to \mathbb{R}$ is a differentiable function, $f > 0$, and (N, g_N) is a 6-dimensional Riemannian manifold. Then, by [16], the metric g on M given by

$$g = dt^2 + f^2\, g_N$$

is Einstein if and only if g_N is Einstein and $(f')^2 + \rho_M f^2 = \rho_N$, where

$$\rho_M = \frac{scal(g)}{42}, \quad \rho_N = \frac{scal(g_N)}{30},$$

or equivalently,

$$\rho_M = \frac{\lambda}{6}, \quad \rho_N = \frac{\mu}{5},$$

where, as we mentioned before, λ and μ are the Einstein constant of (M,g) and (N, g_N), respectively.

From now on, we suppose that the 6-dimensional manifold N has an SU(3)-structure (ω, ψ_+, ψ_-) which belongs to a one-parameter family of SU(3)-structures $\big(\omega(t), \psi_+(t), \psi_-(t)\big)$ such that

$$\omega(t) = f_1(t)\,\omega, \qquad \psi_+(t) = f_2(t)\,\psi_+, \qquad \psi_-(t) = f_3(t)\,\psi_-,$$

where the functions $f_i = f_i(t)$ $(i = 1,2,3)$ are real differentiable functions on an open interval I. We will show conditions for (ω, ψ_+, ψ_-) and for the functions $f_i = f_i(t)$ under which the 3-form

$$\varphi = \omega(t) \wedge dt + \psi_+(t) \tag{3.13}$$

is either a closed or a coclosed G_2 form on the warped product $I \times_f N$ inducing an Einstein metric $g_\varphi = f^2 g_N + dt^2$, for some real differentiable function f on I, and where g_N is the Riemannian metric on N determined by (ω, ψ_+, ψ_-) . We obtain some results taking certain functions $f = f(t)$ and particular types of SU(3)-structures (ω, ψ_+, ψ_-) on N.

First, we consider a family of SU(3)-structures $\big(\omega(t), \psi_+(t), \psi_-(t)\big)$ on N, for t in some open interval I, such that

$$\boxed{\begin{aligned} \omega(t) &= f^2(t)\,\omega, \\ \psi_+(t) &= f^3(t)\,\psi_+, \\ \psi_-(t) &= f^3(t)\,\psi_-, \end{aligned}}$$

where $f = f(t) > 0$ is a real differentiable function on I. By (1.9) we know that, for any $t \in I$, the 3-form $\varphi(t)$ on $N \times \mathbb{R}$ given by

$$\varphi(t) = \omega(t) \wedge dt + \psi_+(t) \tag{3.14}$$

is a G_2 form inducing the metric

$$g_{\varphi(t)} = dt^2 + g_N(t),$$

where $g_N(t)$ denotes the Riemannian metric on N determined by the SU(3)-structure $\big(\omega(t), \psi_+(t), \psi_-(t)\big)$. Thus, the metric $g_{\varphi(t)}$ is the warped product metric

$$g_{\varphi(t)} = dt^2 + f^2(t) g_N \tag{3.15}$$

on $I \times_f N$ if and only if

$$g_N(t) = f^2(t) g_N.$$

In these conditions, the forms $\varphi(t)$ and $*_{\varphi(t)}\,\varphi(t)$ can be expressed, in terms of the function f and the SU(3)-structure (ω, ψ_+, ψ_-) on N, as follows

$$\begin{aligned} \varphi(t) &= f^2(t)\,\omega \wedge dt + f^3(t)\,\psi_+, \\ *_{\varphi(t)}\,\varphi(t) &= \frac{1}{2} f^4(t)\,\omega \wedge \omega + f^3(t)\,\psi_- \wedge dt. \end{aligned}$$

Therefore,

$$\begin{aligned} d\varphi(t) &= f^2(t)\,\widehat{d}\omega \wedge dt - 3f^2(t)f'(t)\,\psi_+ \wedge dt + f^3(t)\,\widehat{d}\psi_+, \\ d(*_{\varphi(t)}\,\varphi(t)) &= f^4(t)\,\widehat{d}\omega \wedge \omega + 2f^3(t)f'(t)\,\omega \wedge \omega \wedge dt + f^3(t)\,\widehat{d}\psi_- \wedge dt, \end{aligned} \tag{3.16}$$

where, as it is usual in this context, d denotes the differential on $M = I \times N$, and $\widehat{d}$ denotes the differential on N.

Taking into account the first equation of (3.16), the **closed condition** for $\varphi(t)$ ($d\varphi(t) = 0$), is equivalent to the following conditions on the SU(3)-structure (ω, ψ_+, ψ_-) and on the function $f = f(t)$

$$\begin{cases} \widehat{d}\psi_+ = 0, \\ \widehat{d}\omega = 3f'(t)\,\psi_+, \end{cases} \tag{3.17}$$

since $f(t) > 0$, for all $t \in I$. To study this system we use that the scalar curvature of the underlying metric g_φ of a closed G_2-structure φ is non-positive. Then, from Table 3.3, we see that the function $f(t) = \sin t$ can be discarded in order to have an Einstein warped product $M = I \times_f N$. Concretely, from Table 3.3, can be checked that the Einstein constants μ and λ of (N, g_N) and (M, g), respectively, are such that λ and μ are > 0. Thus, (M, g) has positive scalar curvature and therefore the metric g cannot be induced by a closed G_2-structure. Considering the function $f(t) = \cosh(t)$ the previous system (3.17) has not solution. This can be checked by taking into account the fact that ω and ψ_+ do not depend on the parameter t. Thus, for $f(t) = \cosh(t)$, the second equation of (3.17) does not admit solution. The same happens if $f(t) = e^t$ or if $f(t) = \sinh(t)$. However, a solution of (3.17) is obtained if $f(t) = t$ ($t \in \mathbb{R}^+$). In this case, we get the following well known result of Bryant and Salamon [23, 26] on the cone metric over a 6-dimensional nearly Kähler manifold.

Proposition 3.4.2 [23, 26]**.** *The cone metric over a Riemannian 6-manifold* N *has holonomy contained in* G_2 *if and only if* N *is a nearly Kähler manifold.*

Proof. Suppose that (ω, ψ_+, ψ_-) is a nearly Kähler structure on N, that is, the forms ω, ψ_+ and ψ_- satisfy $d\omega = 3\psi_+$ and $d\psi_- = -2\omega^2$. Clearly, (ω, ψ_+, ψ_-) belongs to the family of SU(3)-structures $\big(\omega(t), \psi_+(t), \psi_-(t)\big)$ on N such that

$$\omega(t) = t^2\,\omega, \qquad \psi_+(t) = t^3\,\psi_+, \qquad \psi_-(t) = t^3\,\psi_-,$$

and this family satisfies the system (3.17) for $f(t) = t$, where $t \in \mathbb{R}^+$. Therefore, according with (3.13), the 3-form φ is defined by

$$\varphi = t^2\,\omega \wedge dt + t^3\,\psi_+ \tag{3.18}$$

is a closed G_2 form on $M = \mathbb{R}^+ \times N$. Using that $f(t) = t$ and (3.15), we have that the induced metric g_φ is the cone metric of g_N. So $g_\varphi = dt^2 + t^2 g_N$ is Ricci-flat since $(N, \omega, \psi_+, \psi_-)$ is a nearly Kähler manifold and so g_N is Einstein and with Einstein constant 5. Now, by (3.3), both properties (φ closed and g_φ Ricci-flat) imply that the 3-form φ is also coclosed and so parallel with respect to the Levi-Civita connection of the cone metric of g_N.

Conversely, suppose that (ω, ψ_+, ψ_-) is the SU(3)-structure on N inducing the metric g_N such that the cone metric $g_\varphi = dt^2 + t^2 g_N$ has holonomy contained in G_2. This means that the G_2 form φ defined as in (3.18) is closed and coclosed. But the closedness of φ and $*_\varphi \varphi = -t^3\, dt \wedge \psi_- + \frac{1}{2} t^4\, \omega^2$ imply that $d\omega = 3\psi_+$ and $d\psi_- = -2\omega^2$, respectively. Thus, (ω, ψ_+, ψ_-) is a nearly Kähler structure on N. □

From the second equation of (3.16), and using that $f(t) \neq 0$ since $f(t) > 0$, the **coclosed condition** for $\varphi(t)$ $(d *_{\varphi(t)} \varphi(t) = 0)$ is equivalent to the following system of conditions involving the SU(3)-structure and the function $f(t)$

$$\begin{cases} \widehat{d}\omega \wedge \omega = 0, \\ \quad \widehat{d}\psi_- = -2f'(t)\omega \wedge \omega. \end{cases} \tag{3.19}$$

Regarding the possible functions $f(t)$, from Table 3.3 we obtain that the previous system only admits solution if $f(t) = t$ and $\widehat{d}\psi_- = -2\omega \wedge \omega$. Recall that, as we mentioned in Chapter 1, an SU(3)-structure (ω, ψ_+, ψ_-) satisfying $\widehat{d}\psi_- = -2\omega \wedge \omega$ is called nearly half-flat.

Proposition 3.4.3. *Let N be a differentiable manifold, of dimension 6, endowed with a nearly half-flat* SU(3)*-structure* (ω, ψ_+, ψ_-) $(\widehat{d}\psi_- = -2\omega^2)$ *whose underlying metric g_N is Einstein and with Einstein constant 5. Then, the 3-form φ on $M = \mathbb{R}^+ \times N$ defined by*

$$\varphi = t^2\, \omega \wedge dt + t^3\, \psi_+ \tag{3.20}$$

is a coclosed G_2 *form on $M = \mathbb{R}^+ \times N$ inducing the Ricci-flat metric given by the cone metric of g_N, that is,*

$$g_\varphi = dt^2 + t^2 g_N.$$

Proof. From (3.13) and (3.19), the G_2-structure on $M = \mathbb{R}^+ \times N$ determined by the 3-form given by (3.20) is coclosed. Using (3.15), we have that the induced metric g_φ is the cone metric of g_N and so g_φ is Ricci-flat since g_N is Einstein and with Einstein constant 5. □

Remark 3.4.4. *Note that in general a coclosed* G_2 *form inducing a Ricci-flat metric is not closed, and so (in opposite with closed* G_2 *forms inducing Ricci-flat metric) it is not parallel with respect to the Levi-Civita connection of the metric. However, as far as we know, examples of non-parallel coclosed* G_2 *forms defining a Ricci-flat metric are not known.*

Now we consider the family of SU(3)-structures $\big(\omega(t), \psi_+(t), \psi_-(t)\big)$ on N, for t in some open interval I, such that

$$\boxed{\begin{aligned} \omega(t) &= f^2(t)\,\omega, \\ \psi_+(t) &= f^3(t)(\cos t\,\psi_+ - \sin t\,\psi_-), \\ \psi_-(t) &= f^3(t)(\sin t\,\psi_+ + \cos t\,\psi_-). \end{aligned}}$$

where $f = f(t) > 0$ is a real differentiable function on I. From (3.14), for any $t \in I$, we have the forms $\varphi(t)$ and $*_\varphi\varphi(t)$ on $N \times \mathbb{R}$ given by

$$\begin{aligned} \varphi(t) &= f^2(t)\,\omega \wedge dt + f^3(t)\,(\cos t\,\psi_+ - \sin t\,\psi_-), \\ *_{\varphi(t)}\varphi(t) &= \frac{1}{2}f^4(t)\,\omega \wedge \omega + f^3(t)(\sin t\,\psi_+ + \cos t\,\psi_-\big) \wedge dt. \end{aligned} \tag{3.21}$$

Thus,

$$\begin{aligned} d\varphi(t) =& f^2(t)\,\widehat{d}\omega \wedge dt - 3f^2(t)f'(t)(\cos t\,\psi_+ \wedge dt - \sin t\,\psi_- \wedge dt) \\ &+ f^3(t)\,\sin t\,\psi_+ \wedge dt + f^3(t)\,\cos t\,\psi_- \wedge dt \\ &+ f^3(t)\,\cos t\,\widehat{d}\psi_+ - f^3(t)\,\sin t\,\widehat{d}\psi_-, \\ d\big(*_\varphi\,\varphi(t)\big) =& f^4(t)\,\widehat{d}\omega \wedge \omega + 2f^3(t)f'(t)\,\omega \wedge \omega \wedge dt \\ &+ f^3(t)\,\cos t\,\widehat{d}\psi_- \wedge dt + f^3(t)\,\sin t\,\widehat{d}\psi_+ \wedge dt. \end{aligned} \tag{3.22}$$

Therefore, according with the first equation of (3.22), the **closed condition** for $\varphi(t)$ is equivalent to the system

$$\begin{cases} \widehat{d}\psi_+ = 0, \\ \widehat{d}\psi_- = 0, \\ \quad \widehat{d}\omega = \big(3f'(t)\cos t - f(t)\sin t\big)\psi_+ \\ \qquad\quad - \big(3f'(t)\sin t + f(t)\cos t\big)\psi_-. \end{cases}$$

Now, Proposition 1.1.6 and the conditions $\widehat{d}\psi_+ = 0 = \widehat{d}\psi_-$ imply that:

$$\widehat{d}\omega = \nu_1 \wedge \omega + \nu_3,$$

with $\nu_1 \in \Omega^1(N^6)$ and $\nu_3 \in \Omega^3_{12}(N^6)$. Therefore the previous system does not admit any solution.

Taking into account the second equation of (3.22), the **coclosed condition** of $\varphi(t)$ is equivalent to the following system

$$\begin{cases} \qquad \widehat{d}\omega \wedge \omega = 0, \\ 2f'(t)\,\omega \wedge \omega = -\sin t\,\widehat{d}\psi_+ - \cos t\,\widehat{d}\psi_-. \end{cases}$$

Considering the possible functions $f = f(t)$ making Einstein the warped product metric, Table 3.3 implies that the only possibility is $f(t) = \sin t$. Therefore, the SU(3)-structure must satisfy the conditions

$$\widehat{d}\psi_+ = 0 \quad \text{and} \quad \widehat{d}\psi_- = -2\,\omega\wedge\omega. \tag{3.23}$$

An SU(3)-structure satisfying (3.23) is called *double half-flat* SU(3)*-structure* in [115, page 56].

Remark 3.4.5. *Note that the existence of an* SU(3)*-structure* (ω, ψ_+, ψ_-) *on a manifold* N *satisfying* (3.23) *is equivalent to the existence of an* SU(3)*-structure* $(\widetilde{\omega}, \widetilde{\psi}_+, \widetilde{\psi}_-)$ *on* N *such that*

$$\widehat{d}\,\widetilde{\psi}_+ = 0 \quad \textit{and} \quad \widehat{d}\,\widetilde{\psi}_- = \lambda\,\widetilde{\omega}\wedge\widetilde{\omega}, \tag{3.24}$$

where λ *is a non-zero real number. In fact, if* (ω, ψ_+, ψ_-) *satisfies* (3.23)*, then the* SU(3)*-structure* $(\widetilde{\omega}, \widetilde{\psi}_+, \widetilde{\psi}_-)$ *given by*

$$\widetilde{\omega} = (2/\lambda)^2\,\omega, \qquad \widetilde{\psi}_+ = -(2/\lambda)^3\,\psi_+, \qquad \widetilde{\psi}_- = -(2/\lambda)^3\,\psi_-$$

satisfies (3.24)*. Conversely, if* $(\widetilde{\omega}, \widetilde{\psi}_+, \widetilde{\psi}_-)$ *is an* SU(3)*-structure satisfying* (3.24)*, then the* SU(3)*-structure* (ω, ψ_+, ψ_-) *given by*

$$\omega = (\lambda/2)^2\,\widetilde{\omega}, \qquad \psi_+ = -(\lambda/2)^3\widetilde{\psi}_+, \qquad \psi_- = -(\lambda/2)^3\widetilde{\psi}_-$$

satisfies (3.23)*. Conditions* (3.24) *are given in [115, page 56] to define double half-flat* SU(3)*-structures. Note also that in the changes from* $(\widetilde{\omega}, \widetilde{\psi}_+, \widetilde{\psi}_-)$ *to* (ω, ψ_+, ψ_-)*, and conversely from* (ω, ψ_+, ψ_-) *to* $(\widetilde{\omega}, \widetilde{\psi}_+, \widetilde{\psi}_-)$*, we preserve the almost complex structure but we change the metric.*

For a double half-flat SU(3)-structure, we have the following result.

Proposition 3.4.6. *Let* $(N, \omega, \psi_+, \psi_-)$ *be a 6-dimensional differentiable manifold endowed with a double half-flat* SU(3)*-structure (in the sense of* (3.23)*) whose underlying metric* g_N *is Einstein and with positive scalar curvature. Then, the 3-form* φ *on* $(0,\pi)\times N$ *defined by*

$$\varphi = \sin^2 t\,\omega\wedge dt + \sin^3 t\,(\cos t\,\psi_+ - \sin t\,\psi_-)$$

is a coclosed G_2 *form on* $(0,\pi)\times N$ *inducing the well-known Einstein metric with positive scalar curvature*

$$g = dt^2 + \sin^2 t\, g_N.$$

Proof. A warped product (M, g), with $M = (0,\pi)\times N$ is Einstein with $Ric = \lambda g$ if and only if (N, g_N) is Einstein (with $Ric^N = \mu g_N$) and equation (3.12) is satisfied.

From the possible solutions of system (3.12) (which are described in Table 3.3) we see that the solution $f(t) = \sin t$ requires μ to be positive, or equivalently $Scal(g_N) > 0$. Therefore, the G_2-structure

$$\varphi = \sin^2 t\, \omega \wedge dt + \sin^3 t\, \left(\cos t\, \psi_+ - \sin t\, \psi_-\right)$$

defines the metric

$$g_\varphi = dt^2 + \sin^2 t\, g_N,$$

which is Einstein if and only if g_N is Einstein with positive scalar curvature.

On the other hand, if $(N, \omega, \psi_+, \psi_-)$ is double half-flat, that is

$$\widehat{d}\psi_+ = 0 \quad \text{and} \quad \widehat{d}\psi_- = -2\,\omega\wedge\omega,$$

from equation (3.22) we have that

$$\begin{aligned} d(*_{\varphi(t)}\varphi(t)) =& \sin^4 t\, \widehat{d}\omega \wedge \omega + 2\sin^3 t \cos t\, \omega\wedge\omega\wedge dt \\ &+ \sin^3 t\cos t\, \widehat{d}\psi_- \wedge dt + \sin^4 t\, \widehat{d}\psi_+ \wedge dt \\ =& 2\sin^3 t\cos t\, \omega\wedge\omega\wedge dt - 2\sin^3 t\cos t\, \omega\wedge\omega\wedge dt = 0. \end{aligned}$$

□

For the particular case of $(N, \omega, \psi_+, \psi_-)$ being nearly Kähler the following known result holds.

Theorem 3.4.7. *[57, Theorem 5.3] Let $(N, \omega, \psi_+, \psi_-)$ be a nearly Kähler manifold. Then, the 3-form φ on $(0,\pi)\times N$ defined by*

$$\varphi = \sin^2 t\, \omega \wedge dt + \sin^3 t\, \left(\cos t\, \psi_+ - \sin t\, \psi_-\right)$$

is a nearly parallel G_2 *form on $(0,\pi)\times N$ inducing the well-known Einstein metric*

$$g = dt^2 + \sin^2 t\, g_N,$$

where g_N is the nearly Kähler metric on N.

If $(N, \omega, \psi_+, \psi_-)$ is compact, then $(N \times S^1, \varphi)$ is a compact nearly parallel G_2 *manifold with two conical singularities at $t = 0$ and $t = \pi$.*

Proof. Recall, that $(N, \omega, \psi_+, \psi_-)$ is a nearly Kähler manifold if and only if

$$\widehat{d}\omega = 3\psi_+ \quad \text{and} \quad \widehat{d}\psi_- = -2\,\omega\wedge\omega.$$

Thus, from (3.22) we have that

$$\begin{aligned} d(\varphi(t)) =& 3\sin^2 t\,\psi_+ \wedge dt - 3\sin^2 t\cos^2 t\,\psi_+ \wedge dt + 3\sin^3 t\cos t\,\psi_- \wedge dt \\ &+ \sin^4 t\,\psi_+ \wedge dt + \sin^3 t\cos t\,\psi_- \wedge dt + 2\sin^4 t\,\omega\wedge\omega \\ =& 4\sin^4 t\,\psi_+ \wedge dt + 4\sin^3 t\cos t\,\psi_- \wedge dt + 2\sin^4 t\,\omega\wedge\omega. \end{aligned}$$

Then, using (3.21), we obtain that the G_2-structure $\varphi(t)$ is nearly parallel since

$$d(\varphi(t)) = 4 *_\varphi \varphi(t).$$

□

Next, we construct a 7-dimensional (non-compact) manifold M with a coclosed G_2 form φ which induces an Einstein metric on M which is not 3-Sasakian. We also show that φ does not define a nearly parallel G_2-structure.

Let us consider the sphere S^3, viewed as the Lie group SU(2), with the basis of left invariant 1-forms $\{e^1, e^2, e^3\}$ satisfying

$$de^1 = e^{23}, \quad de^2 = -e^{13}, \quad \text{and} \quad de^3 = e^{12}.$$

Hence, the Lie algebra of $S^3 \times S^3$ is $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and its structure equations are

$$\mathfrak{g} = (e^{23}, -e^{13}, e^{12}, f^{23}, -f^{13}, f^{12}),$$

where $\{f^i\}$ denotes the basis of 1-forms on the second sphere.

Now, we consider the basis $\{h^1, \dots, h^6\}$ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ given by

$$h^1 = \frac{1}{4}(e^1+f^1),\ h^2 = \frac{1}{4}(-e^1+f^1),\ h^3 = \frac{\sqrt{2}}{4}e^2,\ h^4 = \frac{\sqrt{2}}{4}f^2,\ h^5 = \frac{\sqrt{2}}{4}e^3,\ h^6 = \frac{\sqrt{2}}{4}f^3.$$

With respect to this basis, the structure equations of the Lie algebra $\mathfrak{g}$ of $S^3 \times S^3$ turn into

$$\mathfrak{g} = (2h^{35}+2h^{46}, -2h^{35}+2h^{46}, -2h^{15}+2h^{25}, -2h^{16}-2h^{26}, 2h^{13}-2h^{23}, 2h^{14}+2h^{24}).$$

We define the SU(3)-structure (ω, ψ_+, ψ_-) on $S^3 \times S^3$ by

$$\begin{aligned} \omega &= h^{12} + h^{34} + h^{56}, \\ \psi_+ &= h^{135} - h^{146} - h^{236} - h^{245}, \\ \psi_- &= -h^{246} + h^{235} + h^{145} + h^{136}. \end{aligned} \tag{3.25}$$

Then, an easy calculation shows that

$$\begin{aligned} \widehat{d}\omega &= 3\psi_+ + \nu_3, \\ \widehat{d}\psi_+ &= 0, \\ \widehat{d}\psi_- &= -2\omega\wedge\omega, \end{aligned} \tag{3.26}$$

which means that (ω, ψ_+, ψ_-) is a double half-flat structure (in the sense of (3.23)) on $S^3 \times S^3$, where $\nu_3 \in \Omega^3_{12}(S^3 \times S^3)$ is given by

$$\nu_3 = -h^{135} + h^{146} - h^{236} - h^{245} + 2h^{235} + 2h^{246}. \tag{3.27}$$

The second and third equations of (3.26) imply that the $\mathrm{SU}(3)$-structure defined by (3.25) is double half-flat but it is not nearly Kähler because $\nu_3 \neq 0$, and hence $\widehat{d\omega} \neq 3\psi_+$. We have that the metric g induced by $(\omega, \Psi = \psi_+ + i\psi_-)$ on $S^3 \times S^3$ is Einstein with positive scalar curvature. Indeed, g is given by

$$g = (h^1)^2 + (h^2)^2 + (h^3)^2 + (h^4)^2 + (h^5)^2 + (h^6)^2, \tag{3.28}$$

and its Ricci curvature tensor is

$$Ric = 4\,(h^1 \otimes h^1 + h^2 \otimes h^2 + h^3 \otimes h^3 + h^4 \otimes h^4 + h^5 \otimes h^5 + h^6 \otimes h^6).$$

Thus, g is Einstein with Einstein constant $\mu = 4$.

Proposition 3.4.8. *Let* (ω, ψ_+, ψ_-) *be the double half-flat* $\mathrm{SU}(3)$*-structure on* $S^3 \times S^3$ *defined by* (3.25)*. Then, the 3-form* φ *on* $M = (0, \pi) \times_{\sin t} (S^3 \times S^3)$ *given by*

$$\begin{aligned} \varphi = {} & \sin^2 t\,(h^{12} + h^{34} + h^{56}) \wedge dt + \sin^3 t\, \cos t\,(h^{135} - h^{146} - h^{236} - h^{245}) \\ & - \sin^4 t\,(h^{136} + h^{145} + h^{235} - h^{246}) \end{aligned} \tag{3.29}$$

is a (non-nearly parallel) coclosed G_2 *form whose underlying metric is Einstein but not a 3-Sasakian metric.*

Proof. By Proposition 3.4.6, we know that the 3-form φ defined by (3.29) is a coclosed G_2 form on $M = (0, \pi) \times_{\sin t} (S^3 \times S^3)$. Moreover, from (3.29) (or interchangeably from (3.22)) we have

$$\begin{aligned} d\varphi &= -4 *_\varphi \varphi(t) + \ sin^2(t)\nu_3 \wedge dt, \\ d\big(*_\varphi \varphi\big) &= 0, \end{aligned}$$

where ν_3 is defined by (3.27). These equations not only imply that φ is coclosed but also they imply that the G_2-structure defined by φ is not nearly-parallel since $\nu_3 \wedge dt \neq 0$. Also, by Proposition 3.4.6, we know that φ induces the Einstein metric

$$g_\varphi = dt^2 + \sin^2 t\, g$$

where g is the Einstein metric on $S^3 \times S^3$ given by (3.28). Now, a direct calculation shows that the Einstein constant of g_φ is $\lambda = \frac{24}{5}$. Thus, the metric g_φ is not 3-Sasakian since, according with Proposition 1.1.18, the Einstein constant of a 3-Sasakian metric on a 7-dimensional manifold is 6. $\square$

Chapter 4

Closed G_2 forms inducing nilsoliton metrics

"La matematica è l'alfabeto in cui Dio ha scritto l'Universo."
Galileo Galilei

According to the previous chapter, we know that simply connected solvable Lie groups can have an Einstein metric but they do not carry any closed G_2 form inducing the Einstein metric, unless the induced metric is flat. For nilsoliton metrics a natural question is the following:

Do there exist 7-dimensional simply connected (non-Abelian) nilpotent Lie groups with nilsoliton metric determined by a closed G_2 form?

In this Chapter, we answer this question in the affirmative. Using the classification in [38] and in [47], we show that, up to isomorphism, there is a unique nilpotent Lie algebra (denoted by $\mathfrak{n}_9$ in Theorem 4.2.1 of section 4.2) with a closed G_2 form which does not admit nilsoliton metrics. It turns out that all the other ten (non-Abelian) nilpotent Lie algebras have a nilsoliton. However, in Proposition 4.2.3 we prove that the Lie algebra $\mathfrak{n}_i$ $(i = 3, 5, 7, 8, 11)$ (defined in Theorem 4.2.1 of section 4.2) has a nilsoliton but does not carry closed G_2-structures inducing the nilsoliton. For the Lie algebra, namely $\mathfrak{n}_{10}$ in Theorem 4.2.1, the existence of a nilsoliton was shown in [47, Example 2] but we do not know whether or not there is a closed G_2 form inducing it since we cannot explicit the nilsoliton. Since $\mathfrak{n}_{10}$ is 4-step nilpotent, the result of Theorem 4.3.1 is restricted to s-step nilpotent Lie algebras, with $s = 2, 3$. In fact, in Theorem 4.3.1, we show that, up to isomorphism, there are exactly four s-step nilpotent Lie algebras $(s = 2, 3)$ with a closed G_2 form inducing a nilsoliton metric.

For each one of those closed G_2 forms inducing nilsoliton metrics, we study in section 4.4 the Laplacian flow introduced by Bryant in [24] for any closed G_2 form.

The short time existence and uniqueness of solution for this flow, on a compact manifold M, has been proved by Bryant and Xu in [27]. In Theorem 4.4.2, Theorem 4.4.5, Theorem 4.4.8 and Theorem 4.4.10, we show the first examples of compact manifolds with a closed G_2 form whose Laplacian flow has long time existence of solution. The results of this Chapter can be found in [54].

4.1 Nilsoliton metrics. Einstein nilradicals

This section is devoted to recall some definitions and results about *nontrivial homogeneous Ricci soliton metrics* and, in particular, on *nilsolitons* (see for instance [34], [90] and [85]).

A complete Riemannian metric g on a manifold M is said to be *Ricci soliton* if its Ricci curvature tensor $Ric(g)$ satisfies the following condition

$$Ric(g) = \lambda g + \mathcal{L}_X g, \tag{4.1}$$

for some real constant λ and some complete vector field X on M, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. If in addition X is the gradient vector field of a smooth function $f\colon M \to \mathbb{R}$, then the Ricci soliton is said to be of *gradient type*. Ricci solitons are called *expanding*, *steady* or *shrinking* depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

We also need to distinguish the following types of Ricci soliton metrics. A Ricci soliton metric g on M is called *trivial* if g is an Einstein metric or g is the product of a homogeneous Einstein metric with the Euclidean metric; and g is said to be *homogeneous* if its isometry group acts transitively on M, and hence g has bounded curvature [94].

In order to characterize the nontrivial homogeneous Ricci soliton metrics, we note that any homogeneous steady or shrinking Ricci soliton metric g of gradient type is trivial. Indeed, if g is steady, one can check that g is Ricci flat, and so by [4] g must be flat. If g is shrinking, then by the results in [105, Theorem 1.2] and in [111], (M, g) is isometric to a quotient of $P \times \mathbb{R}^k$, where P is some homogeneous Einstein manifold with positive scalar curvature. Now, we should notice that this last result for shrinking homogeneous Ricci soliton metrics is also true for homogeneous Ricci solitons of gradient type [111]. Moreover, if a homogeneous Ricci soliton g on a manifold M is expanding, then by [84] M must be non-compact; and from [110] all Ricci solitons (homogeneous or non-homogeneous) on a compact manifold are of gradient type. Conversely, as was proved by Lauret in [94] we have the following:

Lemma 4.1.1 [94]**.** *Let g be a nontrivial homogeneous Ricci soliton on a manifold M. Then, g is expanding and it cannot be of gradient type. Moreover, M is non-compact.*

All known examples of nontrivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups whose Ricci operator is a multiple of the identity modulo derivations, and they are called *solsolitons* or, in the nilpotent case, *nilsolitons*, which are defined as follows.

Definition 4.1.2. *Let N be a simply connected nilpotent Lie group, and denote by $\mathfrak{n}$ its Lie algebra. A left invariant metric g on N is called a* Ricci nilsoliton metric *(or simply* nilsoliton metric*) if its Ricci endomorphism $Ric(g)$ differs from a derivation D of $\mathfrak{n}$ by a scalar multiple of the identity map I, that is, if there exists a real number λ such that*

$$Ric(g) = \lambda I + D.$$

Lauret in [94] shows that if g is a nilsoliton metric on a simply connected nilpotent Lie group N, then g is a Ricci soliton, that is, its Ricci curvature tensor $Ric(g)$ satisfies (4.1). In fact, since N is simply connected and D is a derivation of the Lie algebra $\mathfrak{n}$ of N, we can consider the flow $\phi_t : N \longrightarrow N$, $t \in \mathbb{R}$, where ϕ_t is the unique automorphism of N such such that the morphism $(\phi_t)_* : \mathfrak{n} \longrightarrow \mathfrak{n}$ is given by

$$(\phi_t)_* = e^{tD} \in Aut(\mathfrak{n}).$$

Now, let X_D be the left invariant vector field on N determined by ϕ_t, that is,

$$X_D(p) = \frac{d}{dt}_{|0} \phi_t(p),$$

where $p \in N$. Then

$$\mathcal{L}_{X_D} g = \frac{d}{dt}_{|0} \phi_t^* g = \frac{d}{dt}_{|0} g(e^{-tD}\cdot, e^{-tD}\cdot) = -2g(D\cdot, \cdot).$$

This implies that the Ricci tensor is such that $Ric(g) = cg - \frac{1}{2}\mathcal{L}_{X_D} g$, and henceforth g is a Ricci soliton.

From now on, we will always identify a left invariant metric on a Lie group N with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ on the Lie algebra $\mathfrak{n}$ of N. Then, according to Definition 4.1.2, we say that $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ is a *nilsoliton inner product* (or simply *nilsoliton*) on $\mathfrak{n}$ if there exists a real number λ and a derivation D of $\mathfrak{n}$ such that

$$Ric(\mathfrak{n}, \langle \cdot, \cdot \rangle_{\mathfrak{n}}) = \lambda I + D. \tag{4.2}$$

Nilsoliton metrics have properties that make them preferred left invariant metrics on nilpotent Lie groups in the absence of Einstein metrics. Indeed, non-Abelian nilpotent Lie groups do not admit left invariant Einstein metrics. Nevertheless, not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to automorphism and scaling [90].

By Lauret's results it turns out that nilsoliton metrics on simply connected nilpotent Lie groups N are strictly related to Einstein metrics on the so-called solvable rank-one extensions of N which were defined in section 3.1 (Definition 3.1.2).

Theorem 4.1.3 [92, 93]**.** *A simply connected nilpotent Lie group N admits a nilsoliton metric if and only if its Lie algebra $\mathfrak{n}$ is an Einstein nilradical, that is, $\mathfrak{n}$ possesses an inner product $\langle\cdot,\cdot\rangle$ such that $(\mathfrak{n},\langle\cdot,\cdot\rangle)$ has a metric solvable extension which is Einstein.*

4.2 Nilsoliton metrics not induced by closed G_2 forms

We determine the nilpotent Lie algebras with closed G_2 form and nilsoliton inner product which is not induced by any closed G_2 form. We also show that, up to isomorphism, there is a unique 7-dimensional nilpotent Lie algebra with a closed G_2 form but not having nilsolitons.

The classification of nilpotent Lie algebras admitting a closed G_2-structure is given in [38] as follows.

Theorem 4.2.1. *Up to isomorphism, there are exactly 12 nilpotent Lie algebras that admit a closed G_2-structure. They are:*

$$
\begin{aligned}
\mathfrak{n}_1 &= (0,0,0,0,0,0,0),\\
\mathfrak{n}_2 &= (0,0,0,0,e^{12},e^{13},0),\\
\mathfrak{n}_3 &= (0,0,0,e^{12},e^{13},e^{23},0),\\
\mathfrak{n}_4 &= (0,0,e^{12},0,0,e^{13}+e^{24},e^{15}),\\
\mathfrak{n}_5 &= (0,0,e^{12},0,0,e^{13},e^{14}+e^{25}),\\
\mathfrak{n}_6 &= (0,0,0,e^{12},e^{13},e^{14},e^{15}),\\
\mathfrak{n}_7 &= (0,0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}),\\
\mathfrak{n}_8 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}),\\
\mathfrak{n}_9 &= (0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25}),\\
\mathfrak{n}_{10} &= (0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}),\\
\mathfrak{n}_{11} &= (0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26}),\\
\mathfrak{n}_{12} &= (0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-2e^{16}+2e^{25}).
\end{aligned}
$$

Using Table 1 in [47] we can determine which indecomposable Lie algebras $\mathfrak{n}_i$ ($4 \leq i \leq 12$) do not have nilsoliton inner products. Note that the existence of nilsolitons on $\mathfrak{n}_2$ and $\mathfrak{n}_3$ is not studied in [47] since they are decomposable. Concretely, the correspondence between the indecomposable Lie algebras of Theorem 4.2.1 and Table 1 in [47] is the following:

$$\mathfrak{n}_4 \cong 3.8, \qquad \mathfrak{n}_5 \cong 3.11, \qquad \mathfrak{n}_6 \cong 3.20, \qquad \mathfrak{n}_7 \cong 2.39,$$
$$\mathfrak{n}_8 \cong 2.5, \qquad \mathfrak{n}_9 \cong 1.1(iv), \qquad \text{and} \qquad \mathfrak{n}_{10} \cong 1.3(i_1).$$

Moreover, $\mathfrak{n}_{11}$ and $\mathfrak{n}_{12}$ are respectively isomorphic to the real form of $1.2(i_{-3})$ and $3.1(i_2)$. In particular, we have that $\mathfrak{n}_9$ is the only 7-dimensional nilpotent Lie algebra with a closed G_2 form but not admitting a nilsoliton.

Remark 4.2.2. *Note that the Abelian Lie algebra* $\mathfrak{n}_1$ *admits as rank-one Einstein solvable extension the Lie algebra* $\mathfrak{s}_1$ *with structure equations*

$$(ae^{18}, ae^{28}, ae^{38}, ae^{48}, ae^{58}, ae^{68}, ae^{78}, 0),$$

for some real number $a \neq 0$*, and the nilsoliton on* $\mathfrak{n}_1$ *is trivial because it is flat.*

Since we are interested in non-trivial nilsolitons inner products, in the sequel when we refer to a nilpotent Lie algebra we will mean a non-Abelian nilpotent Lie algebra.

In order to classify the Lie algebras $\mathfrak{n}_i$ admitting a (non-trivial) nilsoliton but with no closed G_2 forms inducing it, we will use Proposition 1.4.5 (see Chapter 1, section 1.4) which shows an obstruction to the existence of closed G_2-structures on 7-dimensional Lie algebras.

As we mentioned in Chapter 3, by [115, Proposition 4.5] we know that if φ is a G_2-structure on a 7-dimensional Lie algebra and we choose a non-zero vector $X \in \mathfrak{g}$ of length one with respect to g_φ, then on the orthogonal complement of the span of X one has an SU(3)-structure given by the 2-form $\alpha = \iota_X\varphi$ and the 3-form $\beta = \varphi - \alpha \wedge \eta$, where $\eta = \iota_X(g_\varphi)$. So in particular $\alpha \wedge \beta = 0$.

By using these results we can prove the following proposition

Proposition 4.2.3. *The Lie algebra* $\mathfrak{n}_i$ ($i = 3, 5, 7, 8, 11$) *has a nilsoliton inner product but no closed* G_2*-structure inducing it.*

Proof. To prove that $\mathfrak{n}_3$ has a nilsoliton, we consider the **Lie algebra** $\mathfrak{n}_3$ defined by the equations given in Theorem 4.2.1. Let $\langle \cdot, \cdot \rangle_{\mathfrak{n}_3}$ be the inner product on $\mathfrak{n}_3$ such that $\{e^1, \ldots, e^7\}$ is orthonormal. Then, $\langle \cdot, \cdot \rangle_{\mathfrak{n}_3}$ is a nilsoliton because its Ricci tensor

$$Ric = diag\left(-1, -1, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0 \right)$$

satisfies (4.2), for $\lambda = -5/2$ and

$$D = diag\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 3, 3, 3, \frac{5}{2}\right).$$

Since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed G_2 form inducing such an inner product.

Suppose that $\mathfrak{n}_3$ has a closed G_2 form ϕ such that

$$g_\phi = \langle \cdot, \cdot \rangle_{\mathfrak{n}_3} = \sum_{i=1}^{7} (e^i)^2. \tag{4.3}$$

Thus, g_ϕ has to satisfy

$$\prod_{i=1}^{7} g_\phi(e_i, e_i) = 1. \tag{4.4}$$

A generic closed 3-form γ on $\mathfrak{n}_3$ has the following expression

$$\begin{aligned}\gamma =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,2,7}e^{127} + \rho_{1,3,4}e^{134} + \rho_{1,3,5}e^{135}\\ &+ \rho_{1,3,6}e^{136} + \rho_{1,3,7}e^{137} + \rho_{1,4,5}e^{145} + \rho_{1,4,6}e^{146} + \rho_{1,4,7}e^{147} + \rho_{1,5,6}e^{156} + \rho_{1,5,7}e^{157}\\ &+ \rho_{1,6,7}e^{167} + \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,3,7}e^{237} + \rho_{1,4,6}e^{245} + \rho_{2,4,6}e^{246}\\ &+ \rho_{2,4,7}e^{247} + \rho_{2,5,6}e^{256} + \rho_{2,5,7}e^{257} + \rho_{2,6,7}e^{267} + \rho_{1,5,6}e^{345} + \rho_{2,5,6}e^{346}\\ &+ \left(\rho_{2,5,7} - \rho_{1,6,7}\right) e^{347} + \rho_{3,5,6}e^{356} + \rho_{3,5,7}e^{357} + \rho_{3,6,7}e^{367},\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary real numbers. Now, we show conditions on the coefficients $\rho_{i,j,k}$ so that $\phi = \gamma$ is a closed G_2 form such that g_ϕ satisfies (4.3). To this end, we apply the aforementioned result of [115, Proposition 4.5] for $X = e_i$ $(1 \leq i \leq 7)$ and so $\eta = e^i$ by (4.3). For $X = e_1$, thus $\eta = e^1$, we have

$$\begin{aligned}\alpha_1 =&\iota_{e_1}\phi = \rho_{1,2,3}e^{23} + \rho_{1,2,4}e^{24} + \rho_{1,2,5}e^{25} + \rho_{1,2,6}e^{26} + \rho_{1,2,7}e^{27} + \rho_{1,3,4}e^{34} + \rho_{1,3,5}e^{35}\\ &+ \rho_{1,3,6}e^{36} + \rho_{1,3,7}e^{37} + \rho_{1,4,5}e^{45} + \rho_{1,4,6}e^{46} + \rho_{1,4,7}e^{47} + \rho_{1,5,6}e^{56} + \rho_{1,5,7}e^{57}\\ &+ \rho_{1,6,7}e^{67},\end{aligned}$$

and

$$\begin{aligned}\beta_1 =&\phi - \iota_{e_1}\phi \wedge e^1 = \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,3,7}e^{237} + \rho_{1,4,6}e^{245}\\ &+ \rho_{2,4,6}e^{246} + \rho_{2,4,7}e^{247} + \rho_{2,5,6}e^{256} + \rho_{2,5,7}e^{257} + \rho_{2,6,7}e^{267} + \rho_{1,5,6}e^{345}\\ &+ \rho_{2,5,6}e^{346} + \left(\rho_{2,5,7} - \rho_{1,6,7}\right) e^{347} + \rho_{3,5,6}e^{356} + \rho_{3,5,7}e^{357} + \rho_{3,6,7}e^{367}.\end{aligned}$$

But, $\alpha_1 \wedge \beta_1 = 0$ describes a system of 6 equations. Hence, after apply the result of [115, Proposition 4.5] for $X = e_2, \ldots, e_7$, we obtain a system of 42 equations.

This system and condition (4.3) imply that any closed G_2 form on $\mathfrak{n}_3$ satisfying (4.4) is expressed as follows

$$\begin{aligned}\phi =&\rho_{1,2,3}e^{123} + \rho_{1,4,5}e^{145} + \rho_{1,6,7}e^{167} + \rho_{2,4,6}e^{246} + \rho_{2,5,7}e^{257}\\ &+ (\rho_{2,5,7} - \rho_{1,6,7})\, e^{347} + \rho_{3,5,6}e^{356}.\end{aligned} \tag{4.5}$$

Because ϕ should be a closed G_2 form on $\mathfrak{n}_3$, at least for certain coefficients $\rho_{i,j,k}$, Proposition 1.4.5 implies that the coefficients appearing on (4.5) cannot vanish. In particular, $\rho_{2,5,7} - \rho_{1,6,7} \neq 0$. Now, denote by G_ϕ the matrix associated to the inner product on $\mathfrak{n}_3$ induced by the 3-form ϕ given by (4.5). Then, (4.3) implies that $G_\phi = I_7$, for some $\rho_{i,j,k}$ and then

$$S = G_\phi - I_7 = 0, \tag{4.6}$$

for those coefficients. From now on, we denote by $S_{i,j}$ the (i,j) entry of the matrix S. One can check that equations $S_{1,1} = S_{2,2} = S_{5,5} = 0$ imply that

$$\rho_{3,5,6} = \frac{1}{\rho_{1,4,5}\rho_{2,5,7}}, \quad \rho_{2,4,6} = -\frac{\rho_{1,4,5}\,\rho_{1,6,7}}{\rho_{2,5,7}} \quad \text{and} \quad \rho_{1,2,3} = \frac{1}{\rho_{1,4,5}\rho_{1,6,7}}.$$

Therefore the expression of $S_{6,6}$ becomes

$$S_{6,6} = \frac{(\rho_{1,6,7} - \rho_{2,5,7})(\rho_{1,6,7} + \rho_{2,5,7})}{\rho_{2,5,7}^2},$$

and hence $\rho_{1,6,7} = \pm\rho_{2,5,7}$. But we know that $\rho_{1,6,7} \neq \rho_{2,5,7}$, and for $\rho_{1,6,7} = -\rho_{2,5,7}$, we have that $S_{3,3} = -\rho_{1,2,3}(\rho_{1,6,7} - \rho_{2,5,7})\rho_{3,5,6}$ and so $S \neq 0$, which is a contradiction with (4.6). This means that $\mathfrak{n}_3$ does not admit a closed G_2 form inducing the nilsoliton given by (4.3).

To prove that $\mathfrak{n}_5$ has a nilsoliton, we consider the Lie algebra $\mathfrak{n}_5$ defined by the structure equations

$$\mathfrak{n}_5 = (0, 0, \sqrt{3}e^{12}, 0, 0, 2e^{13}, e^{14} + \sqrt{3}e^{25}).$$

Consider the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$ such that the basis $\{e^1, \ldots, e^7\}$ is orthonormal. Then, its Ricci tensor satisfies

$$Ric = diag\Big(-4, -3, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, 2, 2\Big).$$

Actually, $Ric = -\frac{13}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_5$ given by

$$D = diag\left(\frac{5}{2}, \frac{7}{2}, 6, 6, 5, \frac{17}{2}, \frac{17}{2}\right)$$

and so $\langle\cdot,\cdot\rangle_{\mathfrak{n}_5} = \sum_{i=1}^{7}(e^i)^2$ is a nilsoliton inner product.

Since the nilsoliton inner product is unique (up to isometry and scaling) it is sufficient to prove that there is no closed G_2 form on $\mathfrak{n}_5$ inducing such an inner product. Suppose that $\mathfrak{n}_5$ has a closed G_2 form ϕ such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$.

A generic closed 3-form γ on $\mathfrak{n}_5$ has the following expression

$$\begin{aligned}\gamma =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,2,7}e^{127} + \rho_{1,3,4}e^{134} + \rho_{1,3,5}e^{135}\\ &+ \rho_{1,3,6}e^{136} + \rho_{1,3,7}e^{137} + \rho_{1,4,5}e^{145} + \rho_{1,4,6}e^{146} + \rho_{1,4,7}e^{147} + \rho_{1,5,6}e^{156}\\ &+ \rho_{1,5,7}e^{157} + \rho_{1,6,7}e^{167} + \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,3,7}e^{237}\\ &+ \rho_{2,4,5}e^{245} + \frac{1}{2}\rho_{2,3,7}e^{246} + \rho_{2,4,7}e^{247} - \frac{1}{2}\sqrt{3}\rho_{1,3,7}e^{256} + \sqrt{3}(\rho_{3,4,5} - \rho_{1,4,7})e^{257}\\ &+ \rho_{3,4,5}e^{345} - \rho_{1,6,7}e^{356} + \rho_{4,5,7}e^{457},\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary real numbers. Now we show conditions on the coefficients $\rho_{i,j,k}$ so that $\phi = \gamma$ is a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$. Proposition 1.4.5 (applied for $X = e_7$) implies that

$$\rho_{1,6,7}\,\rho_{2,3,7}\,\rho_{4,5,7} \neq 0. \tag{4.7}$$

Now, we denote by G_ϕ the matrix associated to the inner product on $\mathfrak{n}_5$ induced by the generic closed 3-form ϕ. Then the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_5}$ implies (4.6) for some coefficients $\rho_{i,j,k}$. From the equations $S_{6,6} = S_{7,7} = S_{6,7} = S_{3,7} = S_{4,6} = S_{3,3} = S_{3,6} = S_{4,7} = 0$ we have that

$$\begin{aligned}&\rho_{2,3,7} = \frac{2}{\rho_{1,6,7}^2}, \quad \rho_{4,5,7} = \frac{1}{2}\rho_{1,6,7}, \quad \rho_{2,3,6} = -2\rho_{2,4,7}, \quad \rho_{1,3,6} = -2\rho_{1,4,7},\\ &\rho_{3,4,5} = 0, \qquad \rho_{1,3,4} = \frac{1}{2}\rho_{1,6,7}, \qquad \rho_{1,3,7} = 2\rho_{1,4,6}, \qquad \rho_{2,3,4} = 0.\end{aligned}$$

Therefore, $S_{4,4} = -\frac{3}{8}\,\rho_{1,6,7}^2\rho_{2,3,7}$ which by (4.7) cannot vanish and so $S \neq 0$, which is a contradiction with (4.6).

Consider now the **Lie algebra** $\mathfrak{n}_7$ defined by the structure equations

$$\mathfrak{n}_7 = \Big(0, 0, 0, e^{12}, \frac{\sqrt{6}}{2}e^{13}, e^{14} + \frac{\sqrt{6}}{2}e^{23}, \sqrt{2}e^{15}\Big).$$

Let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ be the inner product on $\mathfrak{n}_7$ such that the basis $\{e^1,\dots,e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_7} = \sum_{i=1}^{7}(e^i)^2$ is a nilsoliton since

$$Ric = \bigg(-\frac{11}{4}, -\frac{5}{4}, -\frac{3}{2}, 0, -\frac{1}{4}, \frac{5}{4}, 1 \bigg) = -4I_7 + D,$$

where

$$D = diag\left(\frac{5}{4}, \frac{11}{4}, \frac{5}{2}, 4, \frac{15}{4}, \frac{21}{4}, 5\right),$$

is a derivation of $\mathfrak{n}_7$. As before, since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed G_2 form on $\mathfrak{n}_7$ inducing such an inner product.

Suppose that $\mathfrak{n}_7$ has a closed G_2 form ϕ such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$. A generic closed 3-form γ on $\mathfrak{n}_7$ has the following expression

$$\begin{aligned}\gamma =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,2,7}e^{127} + \rho_{1,3,4}e^{134} + \rho_{1,3,5}e^{135}\\ &+ \rho_{1,3,6}e^{136} + \rho_{1,3,7}e^{137} + \rho_{1,4,5}e^{145} + \rho_{1,4,6}e^{146} + \rho_{1,4,7}e^{147} + \rho_{1,5,6}e^{156} + \rho_{1,5,7}e^{157}\\ &+ \rho_{1,6,7}e^{167} + \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \left(\frac{\sqrt{6}}{2}\rho_{2,4,5} - \frac{\sqrt{6}}{2}\rho_{1,4,6}\right)e^{236} + \rho_{2,3,7}e^{237}\\ &+ \rho_{2,4,5}e^{245} + \rho_{2,4,6}e^{246} + \frac{\sqrt{2}}{2}\rho_{2,5,6}e^{247} + \rho_{2,5,6}e^{256} + \left(\rho_{1,6,7} + \frac{\sqrt{6}}{3}\rho_{3,4,7}\right)e^{257}\\ &+ \left(\frac{\sqrt{6}}{2}\rho_{1,5,6} + \sqrt{2}\rho_{2,3,7}\right)e^{345} + \frac{\sqrt{6}}{2}\rho_{256}e^{346} + \rho_{3,4,7}e^{347} + \sqrt{2}\rho_{3,4,7}e^{356}\\ &+ \rho_{3,5,7}e^{357},\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary real numbers. Now, we show conditions on the coefficients $\rho_{i,j,k}$ so that $\phi = \gamma$ be a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$. Proposition 1.4.5 applied for $X = e_7$ implies that

$$\rho_{1,6,7} \neq 0. \tag{4.8}$$

Now we apply the result of [115, Proposition 4.5] for $X = e_i$ $(1 \leq i \leq 7)$ and so $\eta = e^i$ by (4.3). For $X = e_1$, we have

$$\begin{aligned}\alpha_1 =&\iota_{e_1}\phi = \rho_{1,2,3}e^{23} + \rho_{1,2,4}e^{24} + \rho_{1,2,5}e^{25} + \rho_{1,2,6}e^{26} + \rho_{1,2,7}e^{27} + \rho_{1,3,4}e^{34} + \rho_{1,3,5}e^{35}\\ &+ \rho_{1,3,6}e^{36} + \rho_{1,3,7}e^{37} + \rho_{1,4,5}e^{45} + \rho_{1,4,6}e^{46} + \rho_{1,4,7}e^{47} + \rho_{1,5,6}e^{56} + \rho_{1,5,7}e^{57}\\ &+ \rho_{1,6,7}e^{67}\end{aligned}$$

and

$$\begin{aligned}\beta_1 =&\phi - \iota_{e_1}\phi \wedge e^1 = \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \left(\frac{\sqrt{6}}{2}\rho_{2,4,5} - \frac{\sqrt{6}}{2}\rho_{1,4,6}\right)e^{236} + \rho_{2,3,7}e^{237}\\ &+ \rho_{2,4,5}e^{245} + \rho_{2,4,6}e^{246} + \frac{\sqrt{2}}{2}\rho_{2,5,6}e^{247} + \rho_{2,5,6}e^{256} + \left(\rho_{1,6,7} + \frac{\sqrt{6}}{3}\rho_{3,4,7}\right)e^{257}\\ &+ \left(\frac{\sqrt{6}}{2}\rho_{1,5,6} + \sqrt{2}\rho_{2,3,7}\right)e^{345} + \frac{\sqrt{6}}{2}\rho_{256}e^{346} + \rho_{3,4,7}e^{347} + \sqrt{2}\rho_{3,4,7}e^{356}\\ &+ \rho_{3,5,7}e^{357}.\end{aligned}$$

Therefore, $\alpha_1 \wedge \beta_1 = 0$ describes a system of 6 equations. Hence, after apply the result of [115, Proposition 4.5] for $X = e_2, \dots, e_7$, we obtain a system of 42 equations. This system together with the fact that $\rho_{1,6,7} \neq 0$ and the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ imply that any closed G_2 form on $\mathfrak{n}_7$ satisfying (4.4) is expressed as follows

$$\begin{aligned}\phi =&\rho_{1,2,3}e^{123} + \rho_{1,4,5}e^{145} + \rho_{1,6,7}e^{167} + \rho_{2,4,6}e^{246} + \left(\rho_{1,6,7} + \frac{\sqrt{6}}{3}\rho_{3,4,7}\right)e^{257}\\ &+ \rho_{3,4,7}e^{347} + \sqrt{2}\rho_{3,4,7}e^{356}.\end{aligned} \tag{4.9}$$

Now we denote by G_ϕ the matrix associated to the inner product on $\mathfrak{n}_7$ induced by the 3-form ϕ given by (4.9). Then, the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_7}$ implies (4.6) is satisfied for some coefficients $\rho_{i,j,k}$. From equations $S_{1,1} = S_{3,3} = S_{4,4} = S_{6,6} = 0$ we have

$$\rho_{1,2,3} = \frac{\sqrt{2}}{2\rho_{3,4,7}^3}, \quad \rho_{1,4,5} = -\sqrt{2}\rho_{3,4,7}, \quad \rho_{1,6,7} = -\rho_{3,4,7}, \quad \text{and} \quad \rho_{2,4,6} = \frac{\sqrt{2}}{2\rho_{3,4,7}^3}.$$

Therefore $S_{5,5} = 1$ and so $S \neq 0$ which is a contradiction with (4.6).

The metric **Lie algebra** $\mathfrak{n}_8$ with structure equations Let **the Lie algebra** $\mathfrak{n}_8$ be described by the structure equations

$$\mathfrak{n}_8 = (0, 0, e^{12}, -e^{13}, -e^{23}, e^{15} + e^{24}, -e^{16} - e^{34}),$$

and let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$ be the inner product on $\mathfrak{n}_8$ such that $\{e^1, \dots, e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_8} = \sum_{i=1}^7 (e^i)^2$ is a nilsoliton because its Ricci tensor

$$Ric = diag\left(-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right)$$

satisfies (4.2), for $\lambda = -\frac{5}{2}$ and

$$D = diag\left(\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\right).$$

The nilsoliton inner product is unique (up to isometry and scaling) therefore it suffices to prove that there is no closed G_2 form on $\mathfrak{n}_8$ inducing such an inner product. A generic closed 3-form γ on $\mathfrak{n}_8$ has the following expression

$$\begin{aligned}\gamma =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,2,7}e^{127} + \rho_{1,3,4}e^{134} + \rho_{1,3,5}e^{135}\\ &+ \rho_{1,3,6}e^{136} + \rho_{1,3,7}e^{137} + \left(-\rho_{1,2,7} - \rho_{1,3,6}\right)e^{145} + \rho_{1,4,6}e^{146} + \rho_{1,4,7}e^{147} + \rho_{1,5,6}e^{156}\\ &+ \rho_{1,5,7}e^{157} + \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,3,7}e^{237} + \rho_{2,3,6}e^{245}\\ &+ \left(\rho_{1,5,6} - \rho_{2,3,7}\right)e^{246} + \rho_{1,5,7}e^{247} + \rho_{2,5,6}e^{256} + \rho_{2,6,7}e^{267} + \left(\rho_{2,3,7} - 2\rho_{1,5,6}\right)e^{345}\\ &+ \rho_{1,5,7}e^{346} - \rho_{2,6,7}e^{357} + \rho_{2,6,7}e^{456},\end{aligned}$$

where $\rho_{i,j,k}$ are real numbers. Now, we show conditions on the coefficients $\rho_{i,j,k}$ so that $\phi = \gamma$ is a closed G_2 form such that $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$. We apply the result previously mentioned [115, Proposition 4.5] for $X = e_i$ $(1 \leq i \leq 7)$ and so $\eta = e^i$ by the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_8}$. After solving the system of 42 equations we have that any closed G_2 form on $\mathfrak{n}_8$ satisfying (4.4) is expressed as follows

$$\begin{aligned}\phi =&\rho_{1,2,3}e^{123} + \rho_{1,2,4}e^{124} + \rho_{1,2,5}e^{125} + \rho_{1,2,6}e^{126} + \rho_{1,3,5}e^{135} + \rho_{1,3,6}e^{136} - \rho_{1,3,6}e^{145}\\ &+ \rho_{2,3,4}e^{234} + \rho_{2,3,5}e^{235} + \rho_{2,3,6}e^{236} + \rho_{2,3,6}e^{245} + \rho_{2,5,6}e^{256}.\end{aligned} \tag{4.10}$$

Now denote by G_ϕ the matrix associated to the inner product on $\mathfrak{n}_8$ induced by the 3-form ϕ given by (4.10). Then $G_\phi = 0$ obtaining a contradiction with (4.6).

It only remains to study the **Lie algebra** $\mathfrak{n}_{11}$. According to Theorem 4.2.1, $\mathfrak{n}_{11}$ is defined by the structure equations

$$\mathfrak{n}_{11} = (0, 0, f^{12}, 0, f^{13}, f^{24} + f^{23}, f^{25} + f^{34} + f^{15} + f^{16} - 3f^{26}).$$

We consider the new basis $\{e^j\}_{j=1}^7$ of $\mathfrak{n}_{11}^*$ with

$$\begin{aligned}&\{e^1 = f^2, e^2 = -\frac{\sqrt{3}}{3}f^1, e^3 = \frac{\sqrt{39}}{39}f^3 + \frac{\sqrt{39}}{78}f^4, e^4 = -\frac{\sqrt{78}}{78}f^4,\\ &e^5 = \frac{\sqrt{3}}{39}f^6, e^6 = -\frac{1}{3}f^5, e^7 = -\frac{\sqrt{3}}{1014}f^7\}.\end{aligned}$$

Thus, the Lie algebra $\mathfrak{n}_{11}$ can also be described by the structure equations

$$\begin{aligned}\mathfrak{n}_{11} =&\Big(0, 0, \frac{\sqrt{13}}{13}e^{12}, 0, \frac{\sqrt{13}}{13}e^{13} - \frac{\sqrt{26}}{26}e^{14}, \frac{\sqrt{26}}{26}e^{24} + \frac{\sqrt{13}}{13}e^{23},\\ &\frac{\sqrt{13}}{26}e^{25} + \frac{\sqrt{26}}{26}e^{34} + \frac{\sqrt{39}}{26}e^{15} + \frac{\sqrt{13}}{26}e^{16} - \frac{\sqrt{39}}{26}e^{26}\Big).\end{aligned}$$

Let $\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$ be the inner product on $\mathfrak{n}_{11}$ such that $\{e^1,\dots,e^7\}$ is orthonormal. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}=\sum_{i=1}^{7}(e^i)^2$ is a nilsoliton because its Ricci tensor

$$Ric=\frac{1}{52}diag(-7,-7,-3,-3,1,1,5)$$

satisfies $Ric=-\frac{11}{52}I_7+D$, where D is the derivation of the Lie algebra $\mathfrak{n}_{11}$ given by

$$D=\frac{1}{13}diag(1,1,2,2,3,3,4).$$

It suffices to prove that there is no closed G_2 form on $\mathfrak{n}_{11}$ inducing such an inner product. Lets suppose that $\mathfrak{n}_{11}$ has a closed G_2 form ϕ such that $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. A generic closed 3-form γ on $\mathfrak{n}_{11}$ has the following expression

$$\begin{aligned}\gamma=&\rho_{1,2,3}e^{123}+\rho_{1,2,4}e^{124}+\rho_{1,2,5}e^{125}+\rho_{1,2,6}e^{126}+\rho_{1,2,7}e^{127}+\rho_{1,3,4}e^{134}\\&+\rho_{1,3,5}e^{135}+\rho_{1,3,6}e^{136}+\rho_{1,3,7}e^{137}+\rho_{1,4,5}e^{145}+\rho_{1,4,6}e^{146}+\rho_{1,4,7}e^{147}\\&+\rho_{1,5,6}e^{156}-\sqrt{\frac{3}{2}}\rho_{3,4,7}e^{157}+\frac{\rho_{3,4,7}e^{167}}{\sqrt{2}}+\rho_{2,3,4}e^{234}+\rho_{2,3,5}e^{235}+\rho_{2,3,6}e^{236}\\&+\left(\frac{\rho_{1,3,7}}{\sqrt{3}}-\frac{2\rho_{1,5,6}}{\sqrt{3}}\right)e^{237}+\left(\frac{\rho_{1,2,7}}{\sqrt{2}}-\frac{\rho_{1,3,6}}{\sqrt{2}}+\rho_{1,4,6}-\frac{\rho_{2,3,5}}{\sqrt{2}}\right)e^{245}+\rho_{2,4,6}e^{246}\\&+\rho_{2,4,7}e^{247}+\left(\frac{\rho_{1,5,6}}{\sqrt{3}}-\frac{2\rho_{1,3,7}}{\sqrt{3}}\right)e^{256}+\frac{\rho_{3,4,7}e^{257}}{\sqrt{2}}+\sqrt{\frac{3}{2}}\rho_{3,4,7}e^{267}\\&+\left(\frac{1}{2}\rho_{1,4,7}-\frac{\rho_{1,5,6}}{\sqrt{2}}-\frac{1}{2}\sqrt{3}\rho_{2,4,7}\right)e^{345}\\&+\left(-\sqrt{\frac{2}{3}}\rho_{1,3,7}-\frac{1}{2}\sqrt{3}\rho_{1,4,7}+\frac{\rho_{1,5,6}}{\sqrt{6}}-\frac{1}{2}\rho_{2,4,7}\right)e^{346}+\rho_{3,4,7}e^{347}+\sqrt{2}\rho_{3,4,7}e^{356}\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary real numbers.

Now, we show conditions on the coefficients $\rho_{i,j,k}$ so that $\phi=\gamma$ is a closed G_2 form such that $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. We apply the result of [115, Proposition 4.5] for $X=e_i$ $(1\le i\le 7)$ and so $\eta=e^i$ by the condition $g_\phi=\langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$. After solving the system of 42 equations we have that any closed G_2 form on $\mathfrak{n}_{11}$ satisfying (4.4) is expressed as follows

$$\begin{aligned}\phi=&\rho_{1,2,3}e^{123}-\rho_{2,4,6}e^{145}-\sqrt{3}\rho_{2,4,6}e^{167}-\frac{\sqrt{6}}{2}\rho_{3,4,7}e^{157}+\frac{\sqrt{2}}{2}\rho_{3,4,7}e^{167}-\sqrt{3}\rho_{2,4,6}e^{245}\\&+\rho_{2,4,6}e^{246}+\frac{\sqrt{2}}{2}\rho_{3,4,7}e^{257}+\frac{\sqrt{6}}{2}\rho_{3,4,7}e^{267}+\rho_{3,4,7}e^{347}+\sqrt{2}\rho_{3,4,7}e^{356}.\end{aligned}\tag{4.11}$$

As before denote by G_ϕ the matrix associated to the inner product induced by the 3-form ϕ given by (4.11). Then, the condition $g_\phi = \langle\cdot,\cdot\rangle_{\mathfrak{n}_{11}}$ implies (4.6), for some $\rho_{i,j,k}$. Equations $S_{6,6} = S_{7,7} = 0$ imply that

$$\rho_{2,4,6} = -\frac{1}{2}\rho_{3,4,7}, \text{ and } \rho_{3,4,7} = 2^{-1/3}.$$

Therefore, $S_{4,4} = -\frac{1}{2}$ and so $S \neq 0$ which contradicts (4.6).

□

Remark 4.2.4. *Note that the 4-step nilpotent Lie algebra* $\mathfrak{n}_{10}$ *is isomorphic to the Lie algebra* $1.3(i)[\lambda = 1]$ *in the classification given in [49] and the existence of the nilsoliton was shown in [47, Example 2]. Since an explicit expression of the nilsoliton is not known, we cannot apply the argument used in the proof of Proposition 4.2.3. Thus, it remains open the question of whether the Lie algebra* $\mathfrak{n}_{10}$ *admits a closed* G_2 *form inducing a nilsoliton or not. Moreover, the explicit expression of the nilsolitons for* $\mathfrak{n}_{11}$ *and* $\mathfrak{n}_{12}$ *have been already determined in [49] (see there page 20, Remark 3.5), but our basis is different for the nilsoliton on the other Lie algebras.*

4.3 Nilsoliton metrics determined by closed G_2 forms

In this section we prove that, up to isomorphism, there are only four (non-Abelian) s-step nilpotent Lie groups ($s = 2, 3$) with a nilsoliton metric determined by a left invariant closed G_2 form.

Theorem 4.3.1. *Up to isomorphism,* $\mathfrak{n}_2$*,* $\mathfrak{n}_4$*,* $\mathfrak{n}_6$ *and* $\mathfrak{n}_{12}$ *are the unique s-step nilpotent Lie algebras* ($s = 2, 3$) *with a nilsoliton inner product determined by a closed* G_2*-structure.*

Proof. We will show that the Lie algebra $\mathfrak{n}_i$ ($i = 2, 4, 6, 12$) has a closed G_2 form, namely, φ_i such that the Ricci tensor of the inner product g_{φ_i} satisfies (4.2), for some derivation D of $\mathfrak{n}_i$ and some real number λ.

For the **Lie algebra** $\mathfrak{n}_2$ we consider the closed G_2 form φ_2 defined by

$$\varphi_2 = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}. \tag{4.12}$$

The inner product g_{φ_2} is the one making orthonormal the basis $\{e^1, \ldots, e^7\}$, and it is a nilsoliton since $Ric = -2I_7 + D$, where

$$D = diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right),$$

is a derivation of $\mathfrak{n}_2$.

On the **Lie algebra** $\mathfrak{n}_4$, we define the G_2 form φ_4 by

$$\varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236}. \tag{4.13}$$

Then, φ_4 is closed and the inner product g_{φ_4} makes the basis $\{e^1, \dots, e^7\}$ orthonormal. Therefore, g_{φ_4} is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_4$ given by

$$D = diag\left(1, \frac{3}{2}, \frac{5}{2}, 2, 2, \frac{7}{2}, 3\right).$$

For the **Lie algebra** $\mathfrak{n}_6$ we consider the closed G_2-structure defined by the 3-form

$$\varphi_6 = e^{123} + e^{145} + e^{167} + e^{257} - e^{246} + e^{347} + e^{356}. \tag{4.14}$$

Therefore, the inner product g_{φ_6} is such that the basis $\{e^1, \dots, e^7\}$ is orthonormal and it is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where D is the derivation of $\mathfrak{n}_6$ given by

$$D = diag\left(\frac{1}{2}, 2, 2, \frac{5}{2}, \frac{5}{2}, 3, 3\right).$$

Theorem 4.2.1 implies that the **Lie algebra** $\mathfrak{n}_{12}$ is defined by the equations

$$\mathfrak{n}_{12} = (0, 0, 0, h^{12}, h^{23}, -h^{13}, 2h^{26} - 2h^{34} - 2h^{16} + 2h^{25}).$$

If we consider the basis $\{e^1, \dots, e^7\}$ of $\mathfrak{n}_{12}^*$ given by

$$\Big\{e^1 = \frac{\sqrt{3}}{2}h^2, e^2 = h^1 - \frac{1}{2}h^2, e^3 = h^3, e^4 = -\frac{1}{4}h^4, e^5 = \frac{1}{4}h^5 + \frac{1}{4}h^6,$$
$$e^6 = -\frac{\sqrt{3}}{12}h^5 + \frac{\sqrt{3}}{12}h^6, e^7 = -\frac{\sqrt{3}}{48}h^7\Big\},$$

then, $\mathfrak{n}_{12}$ is described as follows

$$\begin{aligned}\mathfrak{n}_{12} =\Big(&0, 0, 0, \frac{\sqrt{3}}{6}e^{12}, -\frac{1}{4}e^{23} + \frac{\sqrt{3}}{12}e^{13}, -\frac{\sqrt{3}}{12}e^{23} - \frac{1}{4}e^{13},\\ &- \frac{\sqrt{3}}{6}e^{34} + \frac{\sqrt{3}}{12}e^{25} + \frac{1}{4}e^{26} + \frac{\sqrt{3}}{12}e^{16} - \frac{1}{4}e^{15}\Big).\end{aligned} \tag{4.15}$$

We define the G_2 form φ_{12} by

$$\varphi_{12} = -e^{124} + e^{135} + e^{167} - e^{236} + e^{257} + e^{347} - e^{456}. \tag{4.16}$$

Clearly φ_{12} is closed. Moreover, it defines the inner product $g_{\varphi_{12}}$ which makes the basis $\{e^1, \dots, e^7\}$ orthonormal, and thus $g_{\varphi_{12}}$ is a nilsoliton since $Ric = -\frac{1}{4}I_7 + \frac{1}{8}D$, where D is the derivation of $\mathfrak{n}_{12}$ given by

$$D = diag(1, 1, 1, 2, 2, 2, 3).$$

□

4.4 Solutions of Laplacian flow with long time existence

Let us consider the nilpotent Lie algebra $\mathfrak{n}_i$ $(i = 2, 4, 6)$ defined in Theorem 4.2.1, and the Lie algebra $\mathfrak{n}_{12}$ defined by (4.15). Let N_i be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}_i$ and closed G_2 form φ_i $(i = 2, 4, 6, 12)$ given by (4.12), (4.13), (4.14) and (4.16), for $i = 2, 4, 6$ and 12, respectively. By Mal'cev theorem [102], we know that N_i has a discrete subgroup $\Gamma_i \subset N_i$ such that the quotient space

$$M_i = \Gamma_i \backslash N_i$$

is a compact manifold. Since φ_i $(i = 2, 4, 6, 12)$ is a left invariant closed G_2 form on N_i, it descends to a closed G_2 form φ_i on M_i.

The purpose of this section is to prove long time existence and uniqueness of solution for the Laplacian flow of φ_i on M_i. Moreover, we show that the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in N_i, as t goes to infinity.

Let M be a 7-dimensional manifold with an arbitrary G_2 form φ. The *Laplacian flow* of φ, introduced by Bryant in [24], is given by

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ \varphi(0) = \varphi_0, \end{cases}$$

where Δ_t is the Hodge Laplacian of the metric g_t determined by the G_2 form $\varphi(t)$.

In general, the expression of the Laplacian of a G_2 form $\varphi(t)$ is described in terms of the torsion forms as:

$$\begin{aligned} \Delta_t \varphi(t) =& \tau_0(t)^2 \varphi(t) + 3\tau_0(t) * \big(\tau_1(t) \wedge \varphi(t)\big) + \tau_0(t)\tau_3(t) + 3 * d * \big(\tau_1(t) \wedge \varphi(t)\big) \\ &+ *d\tau_3(t) - 4d\big(* (\tau_1(t) \wedge *\varphi(t))\big) + d\tau_2(t), \end{aligned} \tag{4.17}$$

with $\tau_0(t) \in \Omega^0(M), \tau_1(t) \in \Omega^1(M), \tau_2(t) \in \Omega^2_{14}(M)$ and $\tau_3(t) \in \Omega^3_{27}(M)$.

Therefore, the behavior of the solution of the Laplacian flow is very different for the different types of G_2-structures. For example, the stable solutions of the Laplacian flow, that is, $\frac{d}{dt}\varphi(t) = 0$ are given by the G_2 manifolds (M, g) such that $Hol(M) \subseteq \mathrm{G}_2$.

The study of the Laplacian flow of a closed G_2 form φ_0 on a manifold M consists to study long time existence, convergence and formation of singularities for the system of differential equations

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = \Delta_t \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases} \tag{4.18}$$

By Stoke's theorem, the two first equations of the system (4.18) imply that all the G_2 forms $\varphi(t)$ are cohomologous for any t.

In the case of closed G_2 forms on compact manifolds, Bryant and Xu [27] gave a result of short time existence and uniqueness of solution.

Theorem 4.4.1 [27]**.** *If M is compact, then* (4.18) *has a unique solution for a short time* $0 \leq t < \epsilon$*, with ϵ depending on* $\varphi_0 = \varphi(0)$.

Therefore if the initial 3-form φ_0 is closed, taking (4.17) into account, the Laplacian flow is described by

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = d\tau_2(t), \\ \varphi(0) = \varphi_0, \end{cases}$$

In the following theorem we determine a global solution of Laplacian flow of the closed G_2 form φ_2 on M_2.

Theorem 4.4.2. *The family of closed* G_2 *forms* $\varphi_2(t)$ *on* M_2 *given by*

$$\varphi_2(t) = e^{147} + e^{267} + e^{357} + f(t)^3 e^{123} + e^{156} + e^{245} - e^{346}, \qquad t \in \left(-\frac{3}{10}, +\infty\right), \tag{4.19}$$

is the solution of the Laplacian flow (4.18) *of* φ_2*, where* $f = f(t)$ *is the function*

$$f(t) = \left(\frac{10}{3}t + 1\right)^{\frac{1}{5}}.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in M_2, as t goes to infinity.

Proof. First we study the Laplacian flow of φ_2 on N_2. Let $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_2 defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7. \tag{4.20}$$

From now on we write $f_{ij} = f_{ij}(t) = f_i(t)f_j(t)$, $f_{ijk} = f_{ijk}(t) = f_i(t)f_j(t)f_k(t)$, and so forth. Then, the structure equations of N_2 with respect to this basis are

$$dx^i = 0, \quad i = 1, 2, 3, 4, 7, \qquad dx^5 = \frac{f_5}{f_{12}}x^{12}, \qquad dx^6 = \frac{f_6}{f_{13}}x^{13}. \tag{4.21}$$

Now, for any $t \in I$, we consider the G_2 form $\varphi_2(t)$ on N_2 given by

$$\begin{aligned}\varphi_2(t) &= x^{147} + x^{267} + x^{357} + x^{123} + x^{156} + x^{245} - x^{346} \\ &= f_{147}e^{147} + f_{267}e^{267} + f_{357}e^{357} + f_{123}e^{123} + f_{156}e^{156} + f_{245}e^{245} - f_{346}e^{346}.\end{aligned} \tag{4.22}$$

Note that $\varphi_2(0) = \varphi_2$ and, for any t, the 3-form $\varphi_2(t)$ on N_2 determines the metric g_t such that the basis $\{x_i = \frac{1}{f_i}e_i;\ i = 1, \ldots, 7\}$ of $\mathfrak{n}_2$ is orthonormal. So, $g(t)(e_i, e_i) = {f_i}^2$.

Using (4.21), one can check that $d\varphi_2(t) = 0$ if and only if

$$f_{26}(t) = f_{35}(t), \tag{4.23}$$

for any t. Assuming $f_i(0) = 1$ and (4.23), to solve the Laplacian flow (4.18) of φ_2, we need to determine the functions f_i and the interval I so that $\frac{d}{dt}\varphi_2(t) = \Delta_t\varphi_2(t)$, for $t \in I$. Using (4.22) we have

$$\begin{aligned}\frac{d}{dt}\varphi_2(t) =& \Big(f_{147}\Big)'e^{147} + \Big(f_{267}\Big)'e^{267} + \Big(f_{357}\Big)'e^{357} + \Big(f_{123}\Big)'e^{123} \\ &+ \Big(f_{156}\Big)'e^{156} + \Big(f_{245}\Big)'e^{245} - \Big(f_{346}\Big)'e^{346}.\end{aligned} \tag{4.24}$$

Now, we calculate $\Delta_t\varphi_2(t) = -d *_t d *_t \varphi_2(t)$. On the one hand, we have

$$*_t\varphi_2(t) = x^{2356} - x^{1345} - x^{1246} + x^{4567} + x^{2347} - x^{1367} + x^{1257}. \tag{4.25}$$

So, x^{4567} is the unique nonclosed summand in $*_t\varphi_2(t)$. Then, taking into account (4.23), we obtain

$$d(*_t d *_t \varphi_2(t)) = \frac{f_6}{f_{13}}\left(-\frac{f_6}{f_{13}}x^{123} - \frac{f_5}{f_{12}}x^{123}\right) = -2\Big(\frac{f_6}{f_{13}}\Big)^2 x^{123}.$$

Therefore, in terms of the forms e^{ijk}, the expression of $-d(*_t d *_t \varphi_2(t))$ is

$$- d(*_t d *_t \varphi_2(t)) = 2f_{123}\Big(\frac{f_6}{f_{13}}\Big)^2 e^{123} = 2\Big(\frac{f_2(f_6)^2}{f_{13}}\Big)e^{123}. \tag{4.26}$$

Comparing (4.24) and (4.26) we see that, in particular, $f_{156}(t) = 1$, for any $t \in I$. Then, using (4.23), we have

$$\frac{f_2(f_6)^2}{f_{13}} = \frac{1}{(f_1)^2}.$$

This equality and (4.26) imply that $-d(*_t d *_t \varphi_2(t))$ can be expressed as follows

$$- d(*_t d *_t \varphi_2(t)) = 2\frac{1}{(f_1)^2}e^{123}. \tag{4.27}$$

Then, from (4.24) and (4.27) we have that $\frac{d}{dt}\varphi_2(t) = \Delta_t\varphi_2(t)$ if and only if the functions $f_i(t)$ satisfy the following system of differential equations

$$\begin{aligned} &\Big(f_{147}\Big)' = \Big(f_{267}\Big)' = \Big(f_{357}\Big)' = \Big(f_{156}\Big)' = \Big(f_{245}\Big)' = \Big(f_{346}\Big)' = 0,\\ &\Big(f_{123}\Big)' = 2\frac{1}{(f_1)^2}. \end{aligned} \tag{4.28}$$

Because $\varphi_2(0) = \varphi_2$, the equations in the first line of (4.28) imply

$$f_{147}(t) = f_{267}(t) = f_{357}(t) = f_{156}(t) = f_{245}(t) = f_{346}(t) = 1, \tag{4.29}$$

for any $t \in I$. From the equations (4.29) we obtain

$$f_1^2 = f_2^2 = f_3^2.$$

Let us consider $f = f_1 = f_2 = f_3$. Using again (4.29) we have

$$f_i(t) = \Big(f(t)\Big)^{-\frac{1}{2}}, \quad i = 4, 5, 6, 7.$$

Now, the last equation of (4.28) implies that $f^4 f' = \frac{2}{3}$. Integrating this equation, we obtain

$$f^5 = \frac{10}{3}t + B, \quad B = constant.$$

But $\varphi_2(0) = \varphi_2$ implies $f^3(0) = f_{123}(0) = 1$, that is, $B = 1$. Hence,

$$f(t) = \Big(\frac{10}{3}t + 1\Big)^{\frac{1}{5}},$$

and so the one parameter family of 3-forms $\{\varphi_2(t)\}$ given by (4.19) is the solution of the Laplacian flow of φ_2 on N_2, and it is defined for every $t \in (-\frac{3}{10}, +\infty)$.

Moreover, $\{\varphi_2(t)\}$ is also the solution of the Laplacian flow of φ_2 on M_2 since, for any $t \in \left(-\frac{3}{10}, +\infty\right)$, $\varphi_2(t)$ is a left invariant closed G_2 form on N_2.

To complete the proof, we study the behavior of the underlying metric $g(t)$ of such a solution in the limit for $t \to +\infty$. Indeed, if we think of the Laplacian flow as a one parameter family of G_2 manifolds with a closed G_2-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. For every $t \in \left(-\frac{3}{10}, +\infty\right)$, denote by $g(t)$ the metric on M_2 induced by the G_2 form $\varphi_2(t)$ given by (4.19). Then,

$$\begin{aligned} g(t) =& \left(\frac{10}{3}t+1\right)^{2/5} e^1 \otimes e^1 + \left(\frac{10}{3}t+1\right)^{2/5} e^2 \otimes e^2 + \left(\frac{10}{3}t+1\right)^{2/5} e^3 \otimes e^3 \\ &+ \left(\frac{10}{3}t+1\right)^{-1/5} e^4 \otimes e^4 + \left(\frac{10}{3}t+1\right)^{-1/5} e^5 \otimes e^5 + \left(\frac{10}{3}t+1\right)^{-1/5} e^6 \otimes e^6 \\ &+ \left(\frac{10}{3}t+1\right)^{-1/5} e^7 \otimes e^7. \end{aligned}$$

Concretely, taking into account the symmetry properties of the Riemannian curvature $R(t)$ we obtain

$$\begin{aligned} &R_{1212} = R_{1313} = -\frac{3}{4(1+\frac{10}{3}t)}, \\ &R_{1515} = R_{1616} = R_{3636} = R_{2525} = \frac{1}{4(1+\frac{10}{3}t)}, \\ &R_{2356} = -\frac{1}{4(1+\frac{10}{3}t)}, \quad R_{ijkl} = 0 \quad \text{otherwise}, \end{aligned}$$

where $R_{ijkl} = R(t)(e_i, e_j, e_k, e_l)$. Therefore, $\lim_{t\to+\infty} R(t) = 0$.

$\square$

Remark 4.4.3. *Note that, for every $t \in (-\frac{3}{10}, +\infty)$, the metric $g(t)$ is a nilsoliton on the Lie algebra $\mathfrak{n}_2$ of N_2 isometric to $g(0)$. In fact, taking into account* (4.20) *and* (4.22)*, it is sufficient to consider the isometry $F_t\colon (\mathfrak{n}_2, g(t)) \longrightarrow (\mathfrak{n}_2, g(0))$ such that, at the level of the dual space $\mathfrak{n}_2^*$ of $\mathfrak{n}_2$, it is given by $F_t^*(e^i) = x^i(t)$, that is,*

$$\begin{aligned} F_t^*(e^i) &= \left(\frac{10}{3}t+1\right)^{1/5} e^i \qquad \textit{if } i = 1,2,3 \textit{ and} \\ F_t^*(e^i) &= \left(\frac{10}{3}t+1\right)^{-1/10} e^i \qquad \textit{if } i = 4,5,6,7. \end{aligned}$$

Then, with respect to the orthonormal basis $(x_1(t), \dots, x_7(t))$ dual to $(x^1(t), \dots, x^7(t))$, we have

$$Ric(g(t)) = -\frac{6}{(3+10t)} Id + \frac{3}{(3+10t)} diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right) = \frac{3}{(3+10t)} Ric(g(0)),$$

where $\frac{3}{(3+10t)}diag\left(1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2\right)$ *is a derivation of the Lie algebra* $\mathfrak{n}_2$ *of* N_2*, for every* t*. Moreover, the Ricci tensor* $Ric(g(t))$ *of* $g(t)$ *is expressed as*

$$\begin{aligned} Ric(g(t)) = &-\Big(\frac{10}{3}t+1\Big)^{-3/5} e^1 \otimes e^1 - \frac{1}{2}\Big(\frac{10}{3}t+1\Big)^{-3/5} e^2 \otimes e^2 \\ &- \frac{1}{2}\Big(\frac{10}{3}t+1\Big)^{-3/5} e^3 \otimes e^3 + \frac{1}{2}\Big(\frac{10}{3}t+1\Big)^{-6/5} e^5 \otimes e^5 \\ &+ \frac{1}{2}\Big(\frac{10}{3}t+1\Big)^{-6/5} e^6 \otimes e^6. \end{aligned}$$

Remark 4.4.4. *Note that the limit can be also computed fixing the* G_2*-structure and changing the Lie bracket as in [95]. We evolve the Lie brackets* $\mu(t)$ *instead of the 3-form defining the* G_2*-structure and we can show that the corresponding bracket flow has a solution for every* t*. Indeed, if we fix on* $\mathbb{R}^7$ *the 3-form*

$$x^{147} + x^{267} + x^{357} + x^{123} + x^{156} + x^{245} - x^{346},$$

the basis $(x_1(t), \dots, x_7(t))$ *defines, for every positive* t*, a nilpotent Lie algebra with bracket* $\mu(t)$ *such that* $\mu(0)$ *is the Lie bracket of* $\mathfrak{n}_2$*. Moreover, the solution converges to the null bracket corresponding to the Abelian Lie algebra.*

In the following theorem we show a long time existence of solution for the Laplacian flow (4.18) of the closed G_2 form φ_4 on the compact manifold M_4.

Theorem 4.4.5. *There exists a solution,* $\varphi(t)$ *of the Laplacian flow of* $\varphi(0) = \varphi_4$ *with*

$$\varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236},$$

on M_4 *defined in the interval* $I = (t_{min}, +\infty)$*, where* t_{min} *is the negative real number given by the elliptic integral*

$$t_{min} = -\frac{3}{2}\int_0^1 x^{3/2}(2 - x^3)^{-5/2} dx.$$

Moreover, the underlying metrics $g(t)$ *of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in* M_4*, as* t *goes to infinity.*

Proof. We study now the Laplacian flow of φ_4 on N_4. Consider some differentiable real functions $f_i = f_i(t)$ $(i = 1, \dots, 7)$ and $h_j = h_j(t)$ $(j = 1, 2, 3)$ depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1, h_j(0) = 0$ and $f_i(t) \neq 0$, for any $t \in I$ and for any i and j. For each $t \in I$, we consider the basis $\{x^1, \dots, x^7\}$ of left invariant 1-forms on N_4 defined by

$$\begin{aligned} x^i = x^i(t) = f_i(t)e^i, \quad 1 \le i \le 4, \quad x^5 = x^5(t) = f_5(t)e^5 + h_1(t)e^1, \\ x^6 = x^6(t) = f_6(t)e^6 + h_2(t)e^2, \quad x^7 = x^7(t) = f_7(t)e^7 + h_3(t)e^4. \end{aligned}$$

From now on we write $f_{ij} = f_{ij}(t) = f_i(t)f_j(t)$ and $f_{ijk} = f_{ijk}(t) = f_i(t)f_j(t)f_k(t)$.

The structure equations of N_4 with respect to this basis are

$$\begin{aligned} dx^i = 0, \quad i = 1,2,4,5, \quad dx^3 = \frac{f_3}{f_{12}}x^{12}, \\ dx^6 = \frac{f_6}{f_{13}}x^{13} + \frac{f_6}{f_{24}}x^{24}, \quad \textit{and} \quad dx^7 = \frac{f_7}{f_{15}}x^{15}. \end{aligned} \tag{4.30}$$

For any $t \in I$, we define the G_2 form $\varphi(t)$ on N_4 by

$$\begin{aligned} \varphi(t) =& - x^{124} - x^{456} + x^{347} + x^{135} + x^{167} + x^{257} - x^{236} \\ =&\Big(- f_{124} - f_4h_{12} - f_2h_{13} + f_1h_{23}\Big)e^{124} - f_{456}e^{456} + f_{347}e^{347} \\ &+ f_{135}e^{135} + f_{167}e^{167} + f_{257}e^{257} - f_{236}e^{236} + \Big(f_{46}h_1 - f_{16}h_3\Big)e^{146} \\ &- \Big(f_{45}h_2 + f_{25}h_3\Big)e^{245} + \Big(- f_{27}h_1 + f_{17}h_2\Big)e^{127}. \end{aligned} \tag{4.31}$$

Clearly $\varphi(0) = \varphi_4$ since $f_i(0) = 1$ and $h_j(0) = 0$. Moreover, using (4.30) and (4.31), one can check that $d\varphi(t) = 0$ if and only if

$$f_{16}(t) = f_{34}(t), \quad f_{37}(t) = f_{56}(t),$$

for any t.

To study the Laplacian flow (4.18) of φ_4, we need to determine the functions f_i, h_j and the interval I so that $\frac{d}{dt}\varphi(t) = \Delta_t\varphi(t)$, for $t \in I$. On the one hand, using (4.31) we have

$$\begin{aligned} \frac{d}{dt}\varphi(t) =&\Big(- f_{124} - f_4h_{12} - f_2h_{13} + f_1h_{23}\Big)'e^{124} - \Big(f_{456}\Big)'e^{456} + \Big(f_{347}\Big)'e^{347} \\ &+ \Big(f_{135}\Big)'e^{135} + \Big(f_{167}\Big)'e^{167} + \Big(f_{257}\Big)'e^{257} - \Big(f_{236}\Big)'e^{236} \\ &+ \Big(f_{46}h_1 - f_{16}h_3\Big)'e^{146} - \Big(f_{45}h_2 + f_{25}h_3\Big)'e^{245} + \Big(- f_{27}h_1 + f_{17}h_2\Big)'e^{127}. \end{aligned} \tag{4.32}$$

On the other hand,

$$*_{\varphi(t)}\varphi(t) = x^{3567} + x^{1237} + x^{1256} - x^{2467} + x^{2345} + x^{1457} + x^{1346}.$$

So, x^{3567} and x^{2467} are the nonclosed summands in $*_{\varphi(t)}\varphi(t)$.

Then, for $\Delta_t\varphi(t) = -d *_{\varphi(t)} d *_{\varphi(t)} \varphi(t)$ we obtain

$$\begin{aligned} \Delta_t\varphi(t) \;\; = \;\; & -\Big(f_{124}\big(\tfrac{f_3^2}{f_1^2f_2^2} + \tfrac{f_6^2}{f_2^2f_4^2}\big) - \tfrac{f_{37}h_3}{f_{15}} - \tfrac{f_6^2h_1}{f_{13}}\Big)e^{124} \\ & +f_{135}\Big(\tfrac{f_6^2}{f_1^2f_3^2} + \tfrac{f_7^2}{f_1^2f_5^2}\Big)e^{135} + \tfrac{f_5f_6^2}{f_{13}}e^{245} + \tfrac{f_3f_7^2}{f_1f_5}e^{127}. \end{aligned} \tag{4.33}$$

Comparing (4.32) and (4.33) we see that the functions f_i, h_1 and h_3 satisfy

$$f_{167}(t) = f_{236}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \qquad f_{46}(t)h_1(t) - f_{16}(t)h_3(t) = 0,$$

for any $t \in I$. But these equations are satisfied if

$$f_1 = f_{23}^2, \qquad f_4 = f_2, \qquad f_5 = f_3, \qquad f_6 = f_7 = \frac{1}{f_{23}}, \qquad h_1 = f_2 f_3^2 h_3. \quad (4.34)$$

Using (4.34), we write (4.32) and (4.33) in terms of f_i, h_1 and h_3. Then, we see that $\frac{d}{dt}\varphi(t) = \Delta_t \varphi(t)$ if and only if

$$\begin{array}{l} f_1 = u \cdot v, \quad f_2 = f_4 = v^{1/2}, \quad f_3 = f_5 = u^{1/2}, \quad f_6 = f_7 = (uv)^{-1/2}, \\ h_1 = \frac{1}{2}u^{5/2}v - \frac{1}{2}u^{1/2}, \quad h_2 = 0, \quad h_3 = \frac{1}{2}u^{3/2}v^{1/2} - \frac{1}{2}(uv)^{-1/2}, \end{array} \quad (4.35)$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions satisfying the system of ordinary differential equations

$$\begin{cases} u' = +\dfrac{2}{3}\,\dfrac{2 - u^3}{u^3 v^3}, \\ v' = -\dfrac{2}{3}\,\dfrac{1 - 2u^3}{u^4 v^2}, \end{cases} \quad (4.36)$$

with initial conditions

$$u(0) = v(0) = 1. \quad (4.37)$$

Let us accept for the moment that (4.36)-(4.37) has a solution $u = u(t), v = v(t)$ defined in $I = (t_{min}, +\infty)$. Then, taking into account (4.31) and (4.35), the family of closed G_2 forms $\varphi(t)$ solving (4.18) for φ_4 is given by

$$\begin{aligned} \varphi(t) =& \frac{1}{4}e^{124}\left(-u^4v^2 + 2u^2v - 4uv^2 - 1\right) + \frac{1}{2}e^{127}\left(u^2v - 1\right) + u^2ve^{135} \\ &+ e^{167} - e^{236} + \frac{1}{2}e^{245}\left(u^2v - 1\right) + e^{257} + e^{347} - e^{456}, \end{aligned}$$

for $t \in (t_{min}, +\infty)$. Moreover, $\{\varphi(t)\}$ is also the solution of the Laplacian flow of φ_4 on M_4 since, for any $t \in (t_{min}, +\infty)$, $\varphi(t)$ is a left invariant closed G_2 form on N_4.

The underlying metric $g(t)$ of this solution converges to a flat metric. To check that the corresponding manifold in the limit is flat, we note that all non-vanishing coefficients of the Riemannian curvature $R(t)$ of $g(t)$ are proportional to the function $2u(t) - u^4(t)$.

Below (see Corollary 4.4.7) we show that the function $u(t)$ satisfies

$$\lim_{t \to +\infty} u(t) = 2^{1/3},$$

and so

$$\lim_{t \to +\infty} R(t) = 0.$$

To study the system (4.36)-(4.37) we note that the first equation of (4.36) implies that $u' > 0$ since $u(0) = 1$ and the functions $u = u(t)$, $v = v(t)$ are both positive. Moreover, we note also that the functions at the second member of (4.36) are C^∞ in the domain

$$\Omega = \big\{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2^{1/3},\ v > 0\big\},$$

in the phase plane. Then, for every point $(u_0, v_0) \in \Omega$, there exists a unique maximal solution (u, v), which has (u_0, v_0) as initial condition, and with existence domain a certain open interval I such that either

$$\lim_{t \to \inf I} \big(u(t)^2 + v(t)^2\big) = +\infty,$$

or

$$\lim_{t \to \inf I} \big(u(t), v(t)\big) \in \partial\Omega,$$

and either

$$\lim_{t \to \sup I} \big(u(t)^2 + v(t)^2\big) = +\infty,$$

or

$$\lim_{t \to \sup I} \big(u(t), v(t)\big) \in \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω. □

To complete the proof of Theorem 4.4.5, we study the system (4.36)–(4.37) proving the two following results.

Proposition 4.4.6. *The maximal solution* $\big(u(t), v(t)\big)$ *of* (4.36)*, satisfying the initial conditions* (4.37)*, belongs to the trajectory of equation*

$$v = \frac{1}{\sqrt{u(2 - u^3)}}. \tag{4.38}$$

Proof. From (4.36) we obtain

$$\frac{dv}{du} = -\frac{v(1 - 2u^3)}{u(2 - u^3)},$$

that is,

$$\frac{dv}{v} = -\frac{1 - 2u^3}{u(2 - u^3)}\, du.$$

Integrating this equation and using (4.37), we have

$$\log v = \log\bigl(u(2-u^3)^{-1/2}\bigr).$$

Therefore,

$$v = \frac{1}{\sqrt{u(2-u^3)}}. \qquad \square$$

As a consequence we have the following corollary, which completes the proof of Theorem 4.4.5.

Corollary 4.4.7. *The maximal solution of* (4.36)-(4.37)*,*

$$I \ni t \mapsto \bigl(u(t), v(t)\bigr) \in \Omega$$

parametrizes the whole curve (4.38)*. Moreover, the maximal solution is defined in the interval*

$$I = (t_{min}, +\infty),$$

where

$$t_{min} = -\frac{3}{2}\int_0^1 \frac{x^{3/2}}{(2-x^3)^{5/2}}dx, \tag{4.39}$$

and

$$\begin{cases} \lim_{t\to t_{min}} u(t) = 0, \\ \lim_{t\to t_{min}} v(t) = +\infty, \end{cases} \qquad \begin{cases} \lim_{t\to +\infty} u(t) = 2^{1/3}, \\ \lim_{t\to +\infty} v(t) = +\infty. \end{cases}$$

Proof. Let $I = (t_{min}, t_{max})$ the existence interval of the maximal solution $(u(t), v(t))$ of (4.36) satisfying the initial conditions (4.37). Using the previous proposition and the first equation of (4.36) we see that

$$v(t) = (2u(t) - u(t)^4)^{-1/2}, \qquad u'(t) = -\frac{2u(t)^3 - 4}{3u(t)^3 v(t)^3},$$

which imply

$$u'(t) = \frac{2}{3}\frac{(2-u(t)^3)^{\frac{5}{2}}}{u(t)^{\frac{3}{2}}}.$$

We define the functions $x(t)$ and $f(x)$ by

$$x(t) = u(t), \quad f(x) = \frac{2}{3}\frac{(2-x^3)^{\frac{5}{2}}}{x^{\frac{3}{2}}}.$$

In order to find t_{max}, we can use that $\frac{dx}{dt} = f(x(t))$ or, equivalently,

$$\frac{dx}{f(x)} = dt.$$

So, in particular, we have

$$\frac{dt}{dx} = \frac{3}{2} x^{\frac{3}{2}} (2 - x^3)^{-\frac{5}{2}}.$$

Note that the function $\frac{3}{2} x^{\frac{3}{2}} (2 - x^3)^{-\frac{5}{2}}$ is increasing from 0, for $x = 0$, to $+\infty$, for $x = 2^{\frac{1}{3}}$. Then, integrating $\frac{dx}{f(x)} = dt$ between t_{min} and 0, and using that $x(t_{min}) = 0$ and $x(0) = 1$, we have that t_{min} is finite and equal to the real number

$$t_{min} = -\frac{3}{2} \int_0^1 x^{3/2} (2 - x^3)^{-5/2} dx.$$

Similarly, in order to find t_{max} we integrate again $\frac{dx}{f(x)} = dt$ between 0 and t_{max}. Since $x(t_{max}) = 2^{\frac{1}{3}}$ we get

$$t_{max} = -\frac{3}{2} \int_1^{2^{\frac{1}{3}}} x^{\frac{3}{2}} (2 - x^3)^{-\frac{5}{2}} dx,$$

which implies that t_{max} is $+\infty$ because this integral is not defined in $x = 2^{\frac{1}{3}}$. □

Concerning the Laplacian flow (4.18) of the closed G_2 form φ_6 on M_6 we have the following.

Theorem 4.4.8. *There exists a solution, $\varphi(t)$ of the Laplacian flow of $\varphi(0) = \varphi_6$ with*

$$\varphi_6 = e^{123} + e^{347} + e^{356} + e^{145} + e^{167} - e^{246} + e^{257},$$

on M_6 defined in the interval $I = (t_{min}, +\infty)$, where t_{min} is the negative real number given by (4.39). Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in M_6, as t goes to infinity.

Proof. Consider the Laplacian flow of φ_6 on N_6. We take differentiable real functions $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ and $h_j = h_j(t)$ $(j = 1, 2)$ depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1, h_j(0) = 0$ and $f_i(t) \neq 0$, for any $t \in I$ and for any i and j. Now, for each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_6 defined by

$$\begin{aligned} x^i &= x^i(t) = f_i(t) e^i, \quad 1 \leq i \leq 5, \\ x^6 &= x^6(t) = f_6(t) e^6 + h_1(t) e^2, \\ x^7 &= x^7(t) = f_7(t) e^7 + h_2(t) e^3. \end{aligned}$$

For any $t \in I$, let $\varphi(t)$ the G_2 form on N_6 defined by

$$\varphi(t) = x^{123} + x^{145} + x^{167} + x^{257} - x^{246} + x^{347} + x^{356}. \tag{4.40}$$

In order to study the flow (4.18) of φ_6, we proceed as in the proof of Theorem 4.4.5. We see that the forms $\varphi(t)$ defined by (4.40) are a solution of (4.18) if and only if the functions f_i, h_1 and h_2 satisfy

$$f_1 = u \cdot v, \qquad f_2 = f_3 = v^{1/2}, \qquad f_4 = f_5 = u^{1/2},$$
$$f_6 = f_7 = (uv)^{-1/2}, \qquad h_1 = h_2 = -\frac{1}{2}(uv)^{-1/2} + \frac{1}{2}u^{3/2}v^{1/2},$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions satisfying the system of ordinary differential equations

$$\begin{cases} u' = \dfrac{2}{3}\dfrac{2-u^3}{u^3v^3}, \\ v' = -\dfrac{2}{3}\dfrac{1-2u^3}{u^4v^2}, \end{cases} \tag{4.41}$$

with initial conditions

$$u(0) = v(0) = 1. \tag{4.42}$$

Clearly, the systems (4.41)-(4.42) and (4.36)-(4.37) are the same. Thus, the maximal solution of (4.41)-(4.42) satisfies the properties expressed in Corollary 4.4.7 for the maximal solution of (4.36)-(4.37). Moreover, $\{\varphi(t)\}$ is also the solution of the Laplacian flow of φ_6 on M_6 since, for any $t \in (t_{min}, +\infty)$, $\varphi(t)$ is a left invariant closed G_2 form on N_6.

To finish the proof we see that, for $t \in (t_{min}, +\infty)$, the expression of $\varphi(t)$ is given by

$$\begin{aligned}\varphi(t) =& \frac{1}{4}(1 + 4uv^2 - 2u^2v + u^4v^2)e^{123} + e^{347} + e^{356} + e^{167} - e^{246} + e^{257} \\ &+ u^2ve^{145} + \frac{1}{2}(1 - u^2v)(e^{136} - e^{127}).\end{aligned}$$

The underlying metric $g(t)$ of this solution converges to a flat metric. To check that the limit metric is flat, we note that all non-vanishing coefficients of the Riemannian curvature $R(t)$ of $g(t)$ are proportional to the function

$$u^p(t)(2 - u^3(t))^q,$$

where p and q are real numbers satisfying that $q > 0$. According with Corollary 4.4.7), we have that the function $u(t)$ satisfies

$$\lim_{t\to+\infty} u(t) = 2^{1/3},$$

and so

$$\lim_{t\to+\infty} R(t) = 0.$$

□

Remark 4.4.9. *Note that surprising in the N_4 and N_6 cases we get the same system of equations.*

Finally, for the Laplacian flow of the closed G_2 form φ_{12} on M_{12} we have the following.

Theorem 4.4.10. *The family of closed* G_2 *forms* $\varphi(t)$ *on* M_{12} *given by*

$$\varphi(t) = -e^{124}+e^{167}+f(t)^6e^{135}-f(t)^6e^{236}+e^{257}+e^{347}-e^{456}, \quad t \in \Big(-3, +\infty\Big) \tag{4.43}$$

is the solution of the Laplacian flow of $\varphi(0) = \varphi_{12}$*, where* $f = f(t)$ *is the function*

$$f(t) = \Big(\frac{1}{3}t+1\Big)^{1/8}.$$

Moreover, the underlying metrics $g(t)$ *of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in* M_{12}*, as* t *goes to infinity.*

Proof. It is sufficient to prove that (4.43) solves the Laplacian flow of φ_{12} on N_{12}. Let $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is an open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on N_{12} defined by

$$x^i = x^i(t) = f_i(t)e^i, \;\; 1 \le i \le 7. \tag{4.44}$$

Then, from (4.15) the structure equations of N_{12} with respect to this basis are

$$\begin{aligned} &dx^i = 0, \;\; i = 1, 2, 3, && dx^4 = \frac{\sqrt{3}}{6}\frac{f_4}{f_{12}}x^{12}, \\ &dx^5 = -\frac{1}{4}\frac{f_5}{f_{23}}x^{23} + \frac{\sqrt{3}}{12}\frac{f_5}{f_{13}}x^{13}, && dx^6 = -\frac{\sqrt{3}}{12}\frac{f_6}{f_{23}}x^{23} - \frac{1}{4}\frac{f_6}{f_{13}}x^{13}, \\ &dx^7 = -\frac{\sqrt{3}}{6}\frac{f_7}{f_{34}}x^{34} + \frac{\sqrt{3}}{12}\frac{f_7}{f_{25}}x^{25} + \frac{1}{4}\frac{f_7}{f_{26}}x^{26} + \frac{\sqrt{3}}{12}\frac{f_7}{f_{16}}x^{16} - \frac{1}{4}\frac{f_7}{f_{15}}x^{15}. \end{aligned} \tag{4.45}$$

Now, for any $t \in I$, we consider the G_2 form $\varphi(t)$ on N_{12} given by

$$\begin{aligned} \varphi(t) &= -x^{124} + x^{167} + x^{135} - x^{236} + x^{257} + x^{347} - x^{456} = \\ &= -f_{124}e^{124} + f_{167}e^{167} + f_{135}e^{135} - f_{236}e^{236} + f_{257}e^{257} + f_{347}e^{347} - f_{456}e^{456}. \end{aligned} \tag{4.46}$$

Note that $\varphi(0) = \varphi_{12}$ and, for any t, the 3-form $\varphi(t)$ on N_{12} determines the metric g_t such that the basis $\{x_i = \frac{1}{f_i}e_i; i = 1, \ldots, 7\}$ of $\mathfrak{n}_{12}$ is orthonormal. So,

$g_t(e_i, e_i) = f_i^2$.
We need to determine the functions f_i and the interval I so that $\frac{d}{dt}\varphi(t) = \Delta_t\varphi(t)$, for $t \in I$. Using (4.46) we have

$$\begin{aligned} \frac{d}{dt}\varphi(t) = & -(f_{124})'e^{124} + (f_{167})'e^{167} + (f_{135})'e^{135} - (f_{236})'e^{236} \\ & + (f_{257})'e^{257} + (f_{347})'e^{347} - (f_{456})'e^{456}. \end{aligned} \tag{4.47}$$

Now, we calculate $\Delta_t\varphi(t) = -d *_{\varphi(t)} d *_{\varphi(t)} \varphi(t)$. On the one hand, we have

$$*_{\varphi(t)} \varphi(t) = x^{3567} - x^{2467} + x^{2345} + x^{1457} + x^{1346} + x^{1256} + x^{1237}. \tag{4.48}$$

So, x^{2467} and x^{1457} are the unique non closed summands in $*_{\varphi(t)}\varphi(t)$. Then, taking into account the structure equations (4.45) and that $x^i(t) = f_i(t)e^i, 1 \leq i \leq 7$ we obtain

$$\begin{aligned} \Delta_t\varphi(t) = & -\frac{(f_{15} + f_{26})(f_5^2 f_6^2 + f_3^2 f_7^2)}{16 f_1 f_2 f_3 f_5 f_6}(e^{236} - e^{135}) \\ & + \frac{(f_{15} + f_{26})(f_5^2 f_6^2 - f_3^2 f_7^2)}{16\sqrt{3} f_1 f_2 f_3 f_5 f_6}(e^{136} + e^{235}). \end{aligned} \tag{4.49}$$

Comparing (4.47) and (4.49), in particular, we have that

$$(f_{124})' = (f_{167})' = (f_{257})' = (f_{347})' = (f_{456})' = 0,$$

and since $\varphi(0) = \varphi_{12}$ this imply that

$$f_{124}(t) = f_{167}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \tag{4.50}$$

for any $t \in I$. From the equation (4.50) we obtain that the functions f_i, where $i \in \{3, 4, 5, 6, 7\}$, can be expressed in terms of f_1 and f_2 as follows

$$f_3 = (f_1 f_2)^2, \qquad f_4 = \frac{1}{f_1 f_2}, \qquad f_5 = f_1, \qquad f_6 = f_2, \qquad f_7 = \frac{1}{f_1 f_2}.$$

Let us consider $f = f_1 = f_2$. With these concrete values (4.47) and (4.49) become

$$\frac{d}{dt}\varphi(t) = (f^6(t))'(e^{135} - e^{236}), \tag{4.51}$$

and

$$\Delta_t\varphi(t) = \frac{f(t)^{-2}}{4}(e^{135} - e^{236}), \tag{4.52}$$

respectively. From (4.51) and (4.52) finding a solution of the Laplacian flow is equivalent to solve $f^7 f' = \frac{1}{24}$. Integrating this equation, we obtain

$$f^8 = \frac{1}{3}t + B, \quad B = constant.$$

But $\varphi(0) = \varphi_{12}$ implies that $f(0) = 1$, that is, $B = 1$. Hence

$$f(t) = \Big(\frac{1}{3}t + 1\Big)^{1/8},$$

and so the one parameter family of 3-forms $\{\varphi(t)\}$ given by (4.43) is the solution of the Laplacian flow of φ_{12} on N_{12}, and it is defined for every $t \in (-3, +\infty)$.
Finally, we study the behavior of the underlying metric $g(t)$ of such a solution in the limit. If we think of the Laplacian flow as a one parameter family of G_2 manifolds with a closed G_2-structure, it can also be checked that, in the limit, the resulting manifold has vanishing curvature. Denote by $g(t)$, $t \in (-3, +\infty)$, the metric on N_{12} induced by the G_2 form $\varphi_{12}(t)$ defined by (4.43). Then, $g(t)$ has the following expression

$$\begin{aligned} g(t) =&\Big(\frac{1}{3}t + 1\Big)^{1/4} e^1 \otimes e^1 + \Big(\frac{1}{3}t + 1\Big)^{1/4} e^2 \otimes e^2 + \Big(\frac{1}{3}t + 1\Big)^{-1} e^3 \otimes e^3 \\ &+ \Big(\frac{1}{3}t + 1\Big)^{-1/2} e^4 \otimes e^4 + \Big(\frac{1}{3}t + 1\Big)^{1/4} e^5 \otimes e^5 + \Big(\frac{1}{3}t + 1\Big)^{1/4} e^6 \otimes e^6 \\ &+ \Big(\frac{1}{3}t + 1\Big)^{-1/2} e^7 \otimes e^7. \end{aligned}$$

Concretely, every non vanishing coefficient appearing in the expression of the Riemannian curvature $R(t)$ of $g(t)$ is proportional to $(t + 3)^{-1}$. Therefore, $\lim_{t\to+\infty} R(t) = 0$. $\square$

Remark 4.4.11. *Note that, for every $t \in (-3, +\infty)$, the metric $g(t)$ is a nilsoliton on the Lie algebra $\mathfrak{n}_{12}$ of N_{12} isometric to $g(0)$. In fact, taking into account* (4.44) *and* (4.46)*, it is sufficient to consider the isometry $F_t \colon (\mathfrak{n}_{12}, g(t)) \longrightarrow (\mathfrak{n}_{12}, g(0))$ such that, at the level of the dual space $\mathfrak{n}_{12}^*$ of $\mathfrak{n}_{12}$, it is given by $F_t^*(e^i) = x^i(t)$, that is,*

$$\begin{aligned} F_t^*(e^i) &= \Big(\frac{1}{3}t + 1\Big)^{1/8} e^i && \textit{if } i = 1, 2, 5, 6, \\ F_t^*(e^i) &= \Big(\frac{1}{3}t + 1\Big)^{1/2} e^i && \textit{if } i = 3, \textit{ and} \\ F_t^*(e^i) &= \Big(\frac{1}{3}t + 1\Big)^{-1/4} e^i && \textit{if } i = 4, 7. \end{aligned}$$

Then, with respect to the orthonormal basis $(x_1(t), \dots, x_7(t))$ dual to $(x^1(t), \dots, x^7(t))$, we have

$$Ric(g(t)) = -\frac{3}{4(3+t)} Id + \frac{3}{8(3+t)} diag(1, 1, 1, 2, 2, 2, 3) = \frac{3}{(3+t)} Ric(g(0)),$$

where $\frac{3}{8(3+t)}diag(1,1,1,2,2,2,3)$ *is a derivation of the Lie algebra* $\mathfrak{n}_{12}$ *of* N_{12}*, for every* t*. Moreover, the Ricci tensor* $Ric(g(t))$ *of* $g(t)$*, is expressed as*

$$\begin{aligned} Ric(g(t)) = &-\frac{1}{8}\Big(\frac{1}{3}t+1\Big)^{-3/4} e^1 \otimes e^1 - \frac{1}{8}\Big(\frac{1}{3}t+1\Big)^{-3/4} e^2 \otimes e^2 \\ &- \frac{1}{8} e^3 \otimes e^3 + \frac{1}{8}\Big(\frac{1}{3}t+1\Big)^{-3/2} e^7 \otimes e^7. \end{aligned}$$

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