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THE COINCIDENCE OF THE KERNEL AND NUCLEOLUS OF A CONVEX GAME: AN ALTERNATIVE PROOF

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The coincidence of the kernel and nucleolus of a convex game: An alternative proof

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Abstract

In 1972, Maschler, Peleg and Shapley proved that in the class of convex games the nucleolus and the kernel coincide. The only aim of this note is to provide a shorter, alternative proof of this result.

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1 Introduction

Coalitional game theory deals with the problem of how to divide the outcome obtained by players that may coordinate their actions and form coalitions. Assuming transferable utility (TU games) the problem is solved by proposing real-valued vectors where each component is the payoff of a player. One of the most important solution concepts for TU games is the nucleolus (Schmeidler, 1969). The nucleolus of a game belongs to the kernel (Davis and Maschler, 1965) of the game and in some cases they coincide. Convex games were defined by Shapley (1971) and the coincidence of the nucleolus with the kernel (and the prekernel) for this class of games was established by Maschler, Peleg and Shapley in 1972. The proof of that result is quite long and complex. In this note we provide an alternative.

2 TU games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. Singletons are coalitions that contain only one player. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players -an allocation- is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum\limits_{i \in N} x_i = v(N)$ is called an *efficient allocation*. The set of efficient allocations is denoted by $X(v)$. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. The core of a game is the set of allocations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Games with nonempty core are called balanced games. We say that a game (N, v) is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. Convex games are balanced.

A solution φ on a class of games Γ_0 is a correspondence that associates a set $\varphi(N,v)$ in $\mathbb{R}^N$with each game (N,v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N,v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* allocation for each game in the class.

We say that the vector x *weakly lexicographically dominates* the vector y (denoted by $x \preceq_L y$) if either $\widetilde{x} = \widetilde{y}$ or there exists k such that $\widetilde{x}_i = \widetilde{y}_i$ for all $i \in \{1, 2, ..., k-1\}$ and $\widetilde{x}_k > \widetilde{y}_k$ where $\widetilde{x}$ and $\widetilde{y}$ are the vectors with the same components as the vectors $\widetilde{x}$, $\widetilde{y}$, but rearranged in a non decreasing order $(i > j \Rightarrow \widetilde{x}_i \leq \widetilde{x}_j)$.

Given $x \in \mathbb{R}^N$ the *satisfaction of a coalition S with respect to x* in a game v is defined as $e(S,x) := x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in non decreasing order. Schmeidler (1969) introduced the *prenucleolus* of a game (N,v), denoted by $PN(v)$, as the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of allocations. In formula:

$$PN(N,v) = \{x \in X(N,v) \,|\theta(x) \preceq_L \theta(y) \text{ for all } y \in X(N,v)\}.$$

Analogously, the *nucleolus* (Schmeidler, 1969) of a game (N,v), denoted by $N(v)$, is the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of *imputations*. An imputation is an allocation that satisfies *individual rationality*, i.e each player i receives at least his/her individual worth $v(\{i\})$.

Given a TU game (N,v) and an allocation x we denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction according to x.

Given a TU game (N,v) and an allocation $x \in X(N,v)$ the complaint of player i against player j is defined as follows:

$$s_{ij}(x) = \min_{S: i \in S, j \notin S} F(S,x).$$

The prekernel of a TU game (N,v) is:

$$PK(N,v) = \{x \in X(N,v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\}$$

For any game (N,v) the prenucleolus is single-valued, is contained in the prekernel and lies in the core whenever the core is nonempty. In general, the prekernel and the prenucleolus do not coincide[1].

The *Davis-Maschler reduced game property* is defined as follows:

Let (N,v) be a game, $T\subset N$, $T\neq N,\emptyset$ and $x\in\mathbb{R}^T$. Then the *Davis-Maschler reduced game* with respect to $N\backslash T$ and x is the game $(N\backslash T, v_x)$ where

$$v_x^{N\backslash T}(S):=\begin{cases} 0 & \text{if } S=\emptyset \\ v(N)-\sum\limits_{i\in T}x_i & \text{if } S=N\backslash T \\ \max\limits_{R\subset T}\left\{v(S\cup R)-\sum\limits_{i\in R}x_i\right\} & \text{for all } S\subset N\backslash T \end{cases}$$

The prekernel satisfies the *Davis-Maschler reduced game property*: Let (N,v) be a game and let S be a nonempty coalition. If $x\in PK(N,v)$ then $x^S\in PK(S,v_x)$.

3 Antipartitions and partitions

The notion of antipartitions (Arin and Inarra, 1998) plays a central role in the main result of this note.

A collection of sets $\mathcal{C}=\left\{\dot{S}:S\subset N\right\}$ is called a *partition* if $\underset{S\in\mathcal{C}}{\cup}S=N$ and for any $S,T\in\mathcal{C}$ it holds that $S\cap T=\varnothing$.

A collection of sets $\mathcal{C}=\left\{\dot{S}:S\subset N\right\}$ is called an *antipartition* if the collection of sets $\{N\backslash S:S\in\mathcal{C}\}$ is a partition of N. For any game (N,v) the *satisfaction of an antipartition* $\mathcal{C}$ is defined by

$$F(\mathcal{C}):=\frac{v(N)-\sum_{S\in\mathcal{C}}\frac{1}{|\mathcal{C}|-1}v(S)}{|\mathcal{C}|}.$$

[1] Consider a 4-person game (N,v) defined as follows:

$$v(S)=\begin{cases} 4 & \text{if } S=N \\ 0 & \text{if } |S|=1 \text{ or } S\in\{\{1,2\},\{3,4\}\} \\ 2 & \text{otherwise.}\end{cases}$$

It is immediately apparent that $(2,2,0,0)$ is an element of the prekernel (kernel) of the game and that the prenucleolus (nucleolus) of (N,v) is $(1,1,1,1)$.

For any game (N,v) the *satisfaction of a partition* $\mathcal{C}$ is defined by

$$F(\mathcal{C}) := \frac{v(N) - \sum_{S\in\mathcal{C}} v(S)}{|\mathcal{C}|}.$$

Lemma 1 *(Arin and Inarra, 1998) Let $(N,v\ )$ be a game and x be an allocation:*

a) if the collection of sets with minimal satisfaction $\mathcal{B}(x)$ contains an antipartition $\mathcal{C}$ then $F(S,x) = F(\mathcal{C})$ for all S belonging to $\mathcal{B}(x)$.

b) if the collection of sets with minimal satisfaction $\mathcal{B}(x)$ contains a partition $\mathcal{C}$ then $F(S,x) = F(\mathcal{C})$ for all S belonging to $\mathcal{B}(x)$.

Note that if the set of coalitions with minimal satisfaction with respect to an allocation contains an antipartition or a partition then the satisfaction of those coalitions depends only on the characteristic function of the game.

Let (N,v) be a TU game and x be an allocation. Denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction at x.

In our paper we also use Theorem 5.2. and Lemma 5.7 in Maschler et al. (1972), which we now restate without the proof[2].

Theorem 2 *For any convex game (N,v), coalition S and core allocation x, the reduced game (S,v_x) is convex.*

Lemma 3 *Let (N,v) be a convex game, and consider a core allocation x, a coalition S in $\mathcal{B}(x)$ and the reduced game (S,v_x). Then for all $R\subset S$ it follows that*

$$v_x(R) = \max_{Q\in N\backslash S}(v(R\cup Q) - x(Q)) = \max\left\{v(R), v(R\cup(N\backslash S)) - x(N\backslash S)\right\}.$$

4 Convex games: The kernel and the nucleolus coincide

In what follows we prove that in the class of convex games the prekernel and the prenucleolus coincide.

[2]The proofs of the statements are not long and complex.

A collection $\mathcal{C}$ of subsets of a set N is called *near-ring* if

$A, B \in \mathcal{C}$ then either $A \cup B = N$, or $A \cap B = \emptyset$ or both $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$.

It is immediately apparent that if (N, v) is convex and x is an allocation then $\mathcal{B}(x)$ is a near-ring[3].

Theorem 4 *Let (N, v) be a convex game and let $x \in PK(N, v)$. Then, either $\mathcal{B}(x)$ contains a partition or $\mathcal{B}(x)$ contains an antipartition*

Proof. Take any maximal coalition S in $\mathcal{B}(x)$[4] and define $\mathcal{K}$ as the subset of coalitions in $\mathcal{B}(x)$ which are contained in $N \backslash S$ and are maximal under these conditions. Two cases may occur:

a) $\mathcal{K} \neq \emptyset$

Then S together with coalitions in $\mathcal{K}$ form a partition. By maximality different coalitions in $\mathcal{K}$ do not intersect. It suffices to show that for every $i \in N \backslash S$ there is $R \in \mathcal{K}$ satisfying $i \in R$. Since $x \in PK(v)$ there is a coalition in $\mathcal{B}(x)$ containing i. Not every T of this type intersects S since we would contradict separability[5] observing that $T \supset N \backslash S$ by maximality of S.

b) $\mathcal{K} = \emptyset$

In this case every coalition in $\mathcal{B}(x)$ which intersects $N \backslash S$ also intersects S, hence it contains $N \backslash S$ by maximality of S. Let $\mathcal{F}$ be the subset of maximal coalitions of $\mathcal{B}(x)$ which contain $N \backslash S$. By maximality the union of two different coalitions in $\mathcal{F}$ coincides with N.

To show that S together with all coalitions of $\mathcal{F}$ form an antipartition it remains to be demonstrated that every player i of S is a member of precisely $\mid \mathcal{F} \mid -1$ coalitions of $\mathcal{F}$. By separability there is a coalition in $\mathcal{F}$ not containing i and the union of two different coalitions in $\mathcal{F}$ coincides with N. ■

The alternative proof is based in the following facts:

[3] The result arises by applying the definition of convexity to the sets in $\mathcal{B}(x)$.

[4] There is no coalition T in $\mathcal{B}(x)$ such that $S \subset T$.

[5] Separability implies that if there exists $S \in \mathcal{B}(x)$ such that $i \in S$ and $j \notin S$ then there exists $T \in \mathcal{B}(x)$ such that $j \in T$ and $i \notin T$. Clearly, if x is an allocation of the prekernel this condition of separability must hold in $\mathcal{B}(x)$.

1.- The prekernel and the prenucleolus satisfy the DM-RGP and they coincide if the game has three or fewer players.

2.- Let x be a prekernel allocation of a convex game (N, v). The set of coalitions with minimal satisfaction according to x contains a partition or an antipartition.

3.- Let (N, v) be a convex game and let $x, y \in PK(N, v)$. There exists a partition or antipartition, say Q, that must be included in $\mathcal{B}(x)$ and in $\mathcal{B}(y)$. The satisfaction of coalitions in Q depends only on the characteristic function of the game. Therefore, for any $S \in Q$ it must be the case that $x(S) = y(S)$.

4.- Let (N, v) be a convex game, S a nonempty coalition of N and x a prekernel allocation. In order to compute the DM-reduced game (S, v_x) only the total payoff of coalition $N \backslash S$, $x(N \backslash S)$, is needed.

5.- Combining facts 3 and 4 it must be concluded that the reduced games with respect to a coalition in Q and allocations x and y are identical. These reduced games are convex and therefore the arguments can be repeated until games of three or fewer players are obtained.

Theorem 5 *Let (N, v) be a convex game. Then $PK(N, v) = \{PN(N, v)\}$.*

Proof. *Let $x = PN(N, v)$ and assume that there exists y, $y \neq x$, such that $y \in PK(N, v)$. First of all note that $\mathcal{B}(y)$ and $\mathcal{B}(x)$ contain either an antipartition or a partition, denoted by* Q. *Since the satisfaction of an antipartition or a partition only depends on the characteristic function $\mathcal{B}(x)$ and $\mathcal{B}(y)$ should contain the same antipartition or partition. Therefore, for any coalition S in the antipartition (partition) it must hold that $y(S) = x(S)$ and consequently $y(N \backslash S) = x(N \backslash S)$. Finally, DM-reduced games (S, v_y) and (S, v_x) are convex and identical (by applying Lemma 3). The prekernel satisfies DM-reduced game property. If $(x_i)_{i \in S}$ and $(y_i)_{i \in S}$ are different we repeat the argument with the games (S, v_y) and (S, v_x). At some point the reduced games should have three or fewer players and in this cases the prekernel contains only one allocation. Therefore,it must be concluded that for all $S \in$*Q *and for all $i \in S$ it holds that $x_i = y_i$. Since* Q *is either a partition or an antipartition $\bigcup\limits_{S \in \mathrm{Q}} = N$ and therefore for all $i \in N$ it holds that $x_i = y_i$.* ■

In the class of balanced games the prenucleolus and the nucleolus coincide. The nucleolus of a game belongs to the kernel of the game Therefore,

Corollary 6 *Let* (N, v) *be a convex game. Then* $PK(N, v) = K(N, v) = \{\mathrm{N}(N, v)\}$.

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