ISSN 1988-088X

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http://www.dfaeii.ehu.es

DFAE-II WP Series

2013-04

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Abstract

In this work we clarify the relationships between riskiness, risk acceptance and bankruptcy avoidance. We distinguish between the restriction on the current wealth required to make a gamble acceptable to the decision maker and the restriction on the current wealth required to guarantee no-bankruptcy if a gamble is accepted. We focus on the measure of riskiness proposed by Foster and Hart.

JEL classification codes: G11, G32, G33.

Keywords: riskiness measure, objective measures, risk acceptance, bankruptcy avoidance.

*Financial support from the Ministerio de Ciencia e Innovación (Chamorro: ECO2009-11213 and Usategui: ECO2012-35820) and from the Departamento de Educación, Universidades e Investigación del Gobierno Vasco (Usategui: IT-313-07 and IT-783-13) is gratefully acknowledged. ADDRESS: Dep. Fundamentos del Análisis Económico II; Facultad de CC. Económicas y Empresariales; Universidad del País Vasco; Avda. Lehendakari Agirre, 83; 48015 Bilbao; Spain. Tel. (34) 94-6013771, Fax (34) 94-6017123. Email: arri.chamorro@ehu.es; josemaria.usategui@ehu.es.

1 Introduction

The measure of riskiness has interested researchers for its theoretical appeal and its relevance to real-world decision-making. A useful measure of riskiness should not depend on the preferences of the decision maker. A measure of riskiness dependent on the preferences of the decision maker may imply an ordering of two gambles according to their riskiness that would depend on those preferences. Foster and Hart (2009) (hereafter $[FH]$) propose an objetive measure of riskiness that is independent of the preferences of the decision maker. They measure the riskiness of a gamble as the minimum wealth that would guarantee no-bankruptcy if that gamble is accepted (within a very general process of sequential offers of gambles). That minimum wealth level is, in general, different from the minimum wealth level required to make the gamble acceptable to the decision maker. A distinction should be made between the restriction on the current wealth required to make a gamble acceptable to the decision maker and the restriction on the current wealth required to guarantee no-bankruptcy if the gamble is accepted.

There might be another source of confusion with respect to the $[FH]$ measure of riskiness. $[FH]$ provide an interpretation of their measure of riskiness in terms of the behavior of a decision maker with logarithmic utility. They show that the behavior of rejecting a gamble when its riskiness is greater than the current wealth of the decision maker can be restated as rejecting any gamble that the logarithmic utility decision maker rejects. This interpretation of their measure by $[FH]$, however, does not imply that their measure of riskiness depends on the preferences of the decision maker. $[FH]$ show that a decision maker that accepts a gamble only when her wealth is greater than the level of riskiness of that gamble will guarantee no-bankruptcy, even if the preferences of that decision maker cannot be represented by a logarithmic function.

In this work we elaborate on these ideas and explain that proposals to extend the measure of riskiness of $[FH]$ must be independent of the preferences of the decision maker. Otherwise the order of gambles according to their riskiness would depend on those preferences. To illustrate our analysis we study a recent proposal of generalization of the $[FH]$ measure of riskiness.

The presentation is organized as follows: The model and the riskiness measure of $[FH]$ are presented in Section 2. In Section 3 we analyze and discuss the relationships between riskiness, risk acceptance and bankruptcy avoidance. Section 4 concludes.

2 The Model and the Riskiness Measure of Foster and Hart

Following $[FH]$, a gamble g is a real-valued discrete random variable having some negative values -losses are possible- and with positive expected value, i.e., $P[g<0]>0$ and $E[g]>0$. Gamble g takes finite values, say, x_1, x_2, ..., x_n with respective probabilities p_1, p_2, ..., p_n ($p_i>0$ and $\sum_{i=1}^{n} p_i = 1$). Let φ denote the collection of all such gambles.

The decision maker's initial wealth is w_0. At every period $t = 0, 1, ...,$ the decision maker, whose current wealth we denote w_t, is offered a gamble $g_t \epsilon \varphi$ that she may accept or reject. If she accepts g_t, then her wealth at time $t+1$ will be $w_{t+1} = w_t + g_t$; and if she rejects g_t, then $w_{t+1} = w_t$.

Let G denote the process $(g_t)_{t=0,1,...}$. $[FH]$ make no restrictions on the random variable g_t and no assumptions about the underlying probability distribution on the space of processes from which G is drawn. At time t, the decision maker has no information about future gambles and does not have information on how her current decision will affect future gambles. Assume that G is generated by a finite set of gambles.

Let R^{FH} be the measure of riskiness introduced by $[FH]$: for each $g \in \varphi$, the riskiness $R^{FH}(g)$ is given by

$$E\left[\log\left(1+\frac{1}{R^{FH}(g)}g\right)\right]=0. \tag{1}$$

The measure R^{FH} is objective, as it only depends on the distribution of outcomes of the gamble and not on the decision maker's preferences.

The measure R^{FH} identifies for every gamble the critical wealth level below which it becomes "risky" to accept a gamble. The behavior of a decision maker that accepts a gamble g only when the current wealth is above $R^{FH}(g)$ guarantees no-bankruptcy, where no-bankruptcy requires that $\lim_{t\to\infty} w_t \geq 0$. Instead, bankruptcy may occur if the strategy of the decision maker is such that there are gambles g which would be accepted when the current wealth is below $R^{FH}(g)$. These characteristics of $R^{FH}(g)$ imply that not accepting gambles g when the current wealth is below $w(g)$ also guarantees no-bankruptcy if $w(g) > R^{FH}(g)$. Moreover, if $w(g) > R^{FH}(g)$ bankruptcy is prevented not only if gambles g are accepted only when the current wealth is above $w(g)$ but also if they are also accepted when the current wealth is below $w(g)$ and above $R^{FH}(g)$: the only requirement to avoid bankruptcy is that gambles g should be rejected whenever the current wealth is below $R^{FH}(g)$. Thus, $R^{FH}(g)$ is the minimal current wealth level at which gamble g may be accepted by the decision maker to guarantee no-bankruptcy. When g is a discrete variable, $R^{FH}(g)$ is greater than minus the maximum loss under g.

As (1) implies

$$E\left[\log\left(R^{FH}(g)+g\right)\right]=\log(R^{FH}(g)),$$

we have that a decision maker with logarithmic utility function is indifferent between accepting and rejecting gamble g if her current wealth is $R^{FH}(g)$. Hence, that decision maker rejects g if her current wealth is smaller than $R^{FH}(g)$ and accepts g if her current wealth is greater than $R^{FH}(g)$. To avoid bankruptcy a decision maker should, thus, reject at least as much as a decision maker with logarithmic utility function. Nevertheless, the definition of $R^{FH}(g)$ as the minimum wealth level at which the gamble g may be accepted by the decision maker to guarantee no-bankruptcy is independent of the preferences of the decision maker. A decision maker that, according to her preferences, accepts a gamble only when her wealth is greater than the level of riskiness of that gamble will guarantee no-bankruptcy, even if the preferences of the decision maker cannot be represented by a logarithmic function.[1]

$[FH]$ prove that their measure of riskiness is monotonic with respect to first and second-order stochastic dominance.[2] Although R^{FH} cannot be calculated for some gambles with continuos distributions, Riedel and Hellmann (2013) propose a consistent extension of the $[FH]$ measure of riskiness to continuous random variables.

3 Riskiness, risk acceptance and bankruptcy avoidance

To clarify the relationships between riskiness, risk acceptance and bankruptcy avoidance we study the generalization of the measure of riskiness of $[FH]$ proposed in Bali, Cakici and Chabi-Yo (2011) (hereafter $[BCC]$). We show the differences between the minimum wealth level that makes a gamble acceptable to the decision maker and the minimum wealth level required to guarantee no-bankruptcy if the gamble is accepted. We also point out that when a riskiness

[1] Further explanations and clarifications on the properties and characteristics of the $[FH]$ measure of riskiness may be found in Hart (2011) and in Foster and Hart (2013). These works also discuss the relationships between R^{FH} and the index of riskiness proposed in Palacios-Huerta, Serrano and Volig (2004) and characterized in Aumann and Serrano (2008).

[2] First- and second-order stochastic dominance were introduced in Rothschild and Stiglitz (1970 and 1971) Hadar and Willian (1969) and Hanoch and Levy (1969).

measure dependent on the preferences of the decision maker is used the ordering of two gambles according to that measure of riskiness may depend on those preferences.

$[BCC]$ claim to progress in measuring riskiness through the development of a generalization of the measure of riskiness of $[FH]$. Their modelization is analogous to that in $[FH]$.

We know that a decision maker with logarithmic utility function has constant relative risk aversion (CRRA) with a coefficient of relative risk aversion equal to 1. $[BCC]$ propose a generalization of the measure of riskiness of $[FH]$ considering CRRA decision makers with coefficient of relative risk aversion γ greater or equal than 1. They use $u^{(\delta)}(x) = \frac{x^\delta}{\delta}$, with $\delta = 1 - \gamma$, for the utility function of a CRRA decision maker with coefficient of relative risk aversion γ greater than 1.[3] Hence, this decision maker is indifferent between accepting and rejecting g when her current wealth is $R_\delta(g)$ such that

$$E(\frac{(R_\delta(g)+g)^\delta}{\delta}) = \frac{(R_\delta(g))^\delta}{\delta} \Leftrightarrow \frac{E\left[(1+\frac{g}{R_\delta(g)})^\delta\right]-1}{\delta} = 0. \tag{2}$$

Moreover, a CRRA decision maker with coefficient of relative risk aversion γ greater than 1 will accept gamble g if her current wealth is greater than $R_\delta(g)$ and will reject gamble g if her current wealth is smaller than $R_\delta(g)$. $[BCC]$ call $R_\delta(g)$ the critical-wealth function for a given value of δ: it is given for each gamble by the minimum wealth that makes that gamble acceptable to the decision maker, under the types of decision makers considered in $[BCC]$.

$[BCC]$ prove that the solution $R_\delta(g)$ to (2) is unique and define the generalized measure of riskiness as follows: the riskiness measure of gamble g for a CRRA decision maker with coefficient of relative risk aversion greater than 1 is $R_\delta(g)$ and the measure of riskiness for CRRA decision makers with coefficient of relative risk aversion equal to 1 is R^{FH}. As $[BCC]$ show that $R_\delta(g)$ decreases with δ and as $R_0(g) = R^{FH}$, we have from above that a CRRA decision maker with coefficient of relative risk aversion greater or equal than 1 will avoid bankruptcy by means of her decisions of acceptance and rejection of gambles g.

Rejection of gamble g whenever the current wealth of the decision maker is below $R_\delta(g)$ is, nevertheless, not required to guarantee no-bankruptcy when $\gamma > 1$. To avoid bankruptcy gamble g must be rejected whenever the current wealth level of the decision maker is smaller than $R^{FH}(g)$. As $R^{FH}(g) < R_\delta(g)$

[3] Any positive linear transformation of that utility function represents the same preferences of the decision maker.

when $\gamma > 1$, the restriction on the current wealth required to make the gamble g acceptable to the decision maker in this case is not the same as the restriction on the current wealth required to guarantee no bankruptcy. The generalized measure of riskiness of $[BCC]$ informs us really on the minimum wealth level that will induce the decision maker to accept gamble g, but not on the minimum wealth level required to avoid bankruptcy.[4]

The generalized measure of riskiness of $[BCC]$ depends on the decision maker's preferences and it is, therefore, subjective. This dependency would not pose a problem to measure riskiness if the ordering of gambles were unaffected by the decision maker's preferences. Nevertheless, we can prove:

Proposition 1 *The ordering of a pair of gambles g and h under the generalized measure of riskiness of $[BCC]$ may differ for CRRA decision makers with different coefficients of relative risk aversion.*

Proof. $[BCC]$ prove that their generalized measure of riskiness is monotonic with respect to first and second-order stochastic dominance. For any $\delta \leq 0$, if g first order stochastically dominates h or if g second order stochastically dominates h, then $R_\delta(g) < R_\delta(h)$. As a consequence, we know that if gambles g and h are comparable in regard to stochastic dominance all CRRA decision makers with coefficients of relative risk aversion greater than (or equal to) 1 will order those gambles in the same way. To prove the result let us, thus, consider pairs of gambles that are not comparable in regard to stochastic dominance (stochastic dominance orders are partial orders).

As an example of a situation where the orderings of a pair of gambles g and h according to their generalized measure of riskiness differ for CRRA decision makers with different coefficients of relative risk aversion consider that gamble g results in a gain of 150 with probability 1/3 and in a loss of -50 with probability 2/3, and gamble h results in a gain of 300 with probability 0.7 and in a loss of -200 with probability 0.3. Notice that $E[g] = 16.67$ and $E[h] = 150$. Therefore, we know that $g, h \epsilon \varphi$. The following table shows the riskiness measure of $[FH]$ for gambles g and h and the generalized measure of riskiness of $[BCC]$ for those gambles considering different values of $\delta < 0$:

	$R^{FH}_{(\delta=0)}$	$R_{\delta=-0.1}$	$R_{\delta=-0.2}$	$R_{\delta=-0.5}$	$R_{\delta=-1}$	$R_{\delta=-5}$
g	215.139	238.38	261.71	332.06	450	1399.9
h	234.195	248.42	263.57	312.5	400	1148.98

[4] As a consequence, the only if statement in Theorem 1 of $[BCC]$ is not correct.

If we consider, for instance, $\delta = -1$ the ordering of gambles g and h given by $R_{\delta=-1}(g)$ and by $R_{FH}(g)$ is not the same since $R_{\delta=-1}(g) = 450 > R_{\delta=-1}(h) = 400$ and $R_{FH}(g) = 215.139 < R_{FH}(h) = 234.195$. Moreover, the ordering of g and h given by $R_{\delta}(g)$ is not the same if we consider different negative values of δ: $R_{\delta=-0.1}(g) = 238.38 < R_{\delta=-0.1}(h) = 248.42$ and $R_{\delta=-1}(g) = 450 > R_{\delta=-1}(h) = 400$. ■

Hence, the proposal of $[BCC]$ does not distinguish between the restriction on the current wealth required to make a gamble acceptable to the decision maker, according to her preferences, and the restriction on the current wealth required to guarantee no-bankruptcy if the gamble is accepted. We have shown that in the case of CRRA decision makers with coefficients of relative risk aversion greater than 1 the mimimum wealth levels required for risk acceptance of a gamble g are greater than the minimum wealth level that guarantees no-bankruptcy. Nevertheless, it is not required a critical wealth greater than to avoid bankruptcy when the decision maker has CRRA preferences and relative risk aversion larger than unity.

4 Conclusion

In this work we clarify the relationships between riskiness, risk acceptance and bankruptcy avoidance. We distinguish between the restriction on the current wealth required to make a gamble g acceptable to the decision maker and the restriction on the current wealth required to guarantee no-bankruptcy if g is accepted. Through this distinction we have shown that the generalization of the riskiness measure of $[FH]$ proposed by $[BCC]$ implies that the ordering of two gambles according to their riskiness may depend on the preferences of the decision maker. As a riskiness measure becomes useless if the ordering of gambles according to that measure depends on the preferences of the decision maker, our analysis may help in the development and application of general measures of riskiness.

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