TREND ANALYSIS IN TWO STANDARD GROWTH MODELS*

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Abstract

This paper analyzes the trend processes characterized by two standard growth models using simple econometrics. The first model is the basic neoclassical growth model that postulates a deterministic trend for output. The second model is the Uzawa-Lucas model that postulates a stochastic trend for output. The aim is to understand how the different trend processes for output assumed by these two standard growth models determine the ability of each model to explain the observed trend processes of other macroeconomic variables such as consumption and investment. The results show that the two models reproduce the output trend process. Moreover, the results show that the basic growth model captures properly the consumption trend process, but fails in characterizing the investment trend process. The reverse is true for the Uzawa-Lucas model.

Key words: basic neoclassical growth model, Uzawa-Lucas model, trend process, cointegration

1   INTRODUCTION

For long there has been a considerable debate on whether GNP of industrialized countries fluctuates around a deterministic or a stochastic long-run trend (for instance, Nelson and Plosser (1982) and Hansen (1997)).[1] This paper adopts an eclectic approach in order to analyze econometrically the trend processes characterized by two standard growth models. The first model is the basic neoclassical growth model (studied, among others, by King, Plosser and Rebelo (1988a) that postulates a deterministic linear trend for output. The second model studied in this paper is a generalized version of the Uzawa-Lucas endogenous growth model that postulates a stochastic trend for output (Uzawa (1965), Lucas (1988)). In this way, we try to understand how the different trend processes assumed by these two standard growth models determine the ability of these models to explain the observed trend processes of other macroeconomic variables such as aggregate consumption and aggregate investment.

We focus our attention on the time series of three aggregate variables: output, consumption and investment. For each model, the synthetic time series of output, consumption and investment used in the analysis of the trend process are constructed based on a time series of the technological shock that itself is constructed as a Solow residual (that is, based on the model considered and observed data). In this sense, we say that synthetic data used to analyze the trend process is consistent with both observed data and the model considered.

The two growth models studied in this paper postulate different growth processes and, then, the kind of analysis of the trend process must be different for each model. Since the basic neoclassical growth model assumes a linear deterministic trend, the analysis of the trend process characterized by this model consist in testing whether the difference between the observed and synthetic time series of a given variable is stationary. If this null hypothesis is not rejected, we would conclude that model characterized properly the trend process observed in the data for this particular variable since the discrepancy between the observed and the synthetic variable must be attributed to limitations of the model to characterize other components of data (for instance, cycle and irregular components) different from the trend component.

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[1]Hansen (1997) gives a clear exposition of the two alternative views postulated in the literature.

Since the synthetic time series of output, consumption and investment obtained from the generalized version of the Uzawa-Lucas model are integrated processes of order one, the analysis of the trend processes in this model is carried out by determining whether an observed time series and its associated synthetic time series are cointegrated. If this null hypothesis is not rejected, one would obtain evidence that the deviations between the observed and synthetic time series are stationary and, then, attributable to limitations of the model to characterize other components than the trend component.

The results found suggest that the two models reproduce adequately the trend process of output. The results also show that the basic neoclassical growth model captures properly the trend process of aggregate consumption, but fails in characterizing the trend process of aggregate investment. The reverse is true for the Uzawa-Lucas endogenous growth model. This model reproduces appropriately the trend process for investment, but fails to reproduce the trend process for consumption since this model characterizes a much smoother path for consumption than the path observed in the data.

The rest of the paper is organized as follows. Section 2 briefly describes the two standard growth models analyzed in this paper. Moreover, this section describes the procedure to obtain the synthetic time series that are consistent with the model and the observed data. Section 3 analyzes the trend processes postulated by the two growth models. Section 4 presents the main results. Moreover, in this section, we analyze the Solow residual obtained from Uzawa-Lucas model. Finally, section 6 concludes.

2 DESCRIPTION OF THE MODELS

2.1 The basic neoclassical model

The exogenous growth model considered in this paper is the basic neoclassical growth model studied by King, Plosser and Rebelo (1988a). This model is a one-sector model of physical capital accumulation and labor input is a choice variable. As is well known, the competitive equilibrium of this model can be characterized by solving the following intertemporal optimization problem faced by a benevolent social planner:

$$\underset{C_t, K_t, N_t, I_t, Y_t}{Max} E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + A \log(1 - N_t) \right], \right\} \tag{1}$$

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subject to the following constraints

$$C_t + I_t = Y_t,$$
$$K_{t+1} = (1 - \delta)K_t + I_t,$$
$$Y_t = Z_t K_t^\rho (X_t N_t)^{1-\rho}, \tag{2}$$
$$\log(\tfrac{Z_t}{\overline{Z}}) = \psi \log(\tfrac{Z_{t-1}}{\overline{Z}}) + \varepsilon_t,$$

where K_0, Z_0 and X_0 are given. The notation is standard. C_t, N_t, I_t, Y_t, K_t, and Z_t denote consumption, labor input, investment, output, capital stock and technological shock at time t, respectively. β and δ are the discount factor and the depreciation rate, respectively. A, ρ, ψ and $\overline{Z}$ are parameters. The variable X_t is exogenously determined and capture the growth process of the economy. It is assumed that X_t follows the process $X_t = \theta X_{t-1}$. Therefore, the growth process is deterministic. The parameter θ determines the steady-state rate of growth.

In the steady state, the variables Y_t, C_t, I_t, and K_t grow at rate θ whereas N_t and the 'gross real return' of capital, denoted by R_t, are constants. In order to facilitate the use of computational techniques, it is convenient to rewrite the model in terms of stationary variables by dividing all steady-state growing variables by X_t. Then, the model can be written in terms of the variables $c_t = C_t/X_t$, $k_t = K_t/X_t$, $i_t = I_t/X_t$, and $y_t = Y_t/X_t$ that are stationary. The necessary and sufficient first-order conditions for an (interior) optimum are then given by

$$Ac_t N_t = (1 - \rho)(y_t - y_t N_t), \tag{3}$$

$$1 = \frac{\beta}{\theta} E_t(\frac{c_t}{c_{t+1}} R_{t+1}), \tag{4}$$

$$R_t = \rho \frac{y_t}{k_t} + (1 - \delta), \tag{5}$$

$$c_t + i_t = y_t, \tag{6}$$

$$y_t = Z_t k_t^\rho N_t^{1-\rho}, \tag{7}$$

$$\theta k_{t+1} = (1 - \delta)k_t + i_t, \tag{8}$$

$$\lim_{t \to \infty} E_t \beta^t \frac{k_{t+1}}{c_t} = 0,$$

$$z_t = \psi z_{t-1} + \varepsilon_t, \tag{9}$$

where $z_t = \log(\frac{Z_t}{\overline{Z}})$.

By using the log-linear method suggested by Uhlig (1999), the following laws of motion for the stationary variables of the model are obtained[2]

$$
\begin{aligned}
\widetilde{k}_{t+1} &= 0.95294889\widetilde{k}_t + 0.13673068z_t, \\
\widetilde{c}_t &= 0.61705058\widetilde{k}_t + 0.29805711z_t, \\
\widetilde{y}_t &= 0.24941890\widetilde{k}_t + 1.60765210z_t, \\
\widetilde{N}_t &= -0.29410535\widetilde{k}_t + 1.04767600z_t, \\
\widetilde{R}_t &= -0.03046639\widetilde{k}_t + 0.06225525z_t, \\
\widetilde{i}_t &= -0.62455367\widetilde{k}_t + 4.72095840z_t.
\end{aligned}
\tag{10}
$$

Moreover, it is assumed that

$$
z_t = 0.90z_{t-1} + \varepsilon_t,
\tag{11}
$$

where $\varepsilon_t \sim iid(0, \sigma_\epsilon^2)$ and $\sigma_\epsilon = 0.009982$. A tilde ($\widetilde{}$) on a variable denotes its log-deviation from the steady-state value.

Using the system of equations (10) and initial values for $\widetilde{k}_t$ and z_t, we can obtain synthetic time series for the log-deviations from the steady-state values of the ratios $k_t = \frac{K_t}{X_t}$, $c_t = \frac{C_t}{X_t}$, $y_t = \frac{Y_t}{X_t}$, and $i_t = \frac{I_t}{X_t}$, and the levels of N_t and R_t. Using the initial value X_0 and the law of motion $X_t = \theta X_{t-1}$ the time series X_t is obtained. Next, the levels of all variables displaying a deterministic trend (Y_t, C_t, I_t, and K_t) can be easily obtained using X_t.

2.1.1 Obtaining the Solow Residual from the basic neoclassical model

We use US time series in order to evaluate the trend processes characterized by the two standard growth models considered in this paper. US time series are quarterly data for the period from the third quarter of 1955 to the first quarter of 1984. All series are per-capita and are described in detail in Christiano (1988). The time series for the technological shock are calculated from the residual obtained by substituting the actual US time series in the aggregate production function: $Y_t = Z_t K_t^\rho (X_t N_t)^{1-\rho}$. Denoting this residual by A_t, we have that $A_t = Z_t X_t^{1-\rho}$. Since this model assumes that growth is

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[2]Appendix 1 provides a discussion of how parameter values for this model are chosen and a description of Uhlig's method.

characterized by a linear deterministic trend, $\log Z_t$ is obtained as the least squares residual from the regression of $\log A_t$ on $\log(X_t)$. Formally,

$$\log(A_t) = \log(Z_t) + (1 - \rho) \log(X_t), \tag{12}$$

where $\log(A_t) \equiv \rho \log(K_t) + (1-\rho) \log(N_t)$. Using the time series of the technological shock $(z_t \equiv \log(Z_t))$, the system of equations (10) and the initial values of K_0, Z_0 and X_0, it is obtained the synthetic time series of output, consumption and investment that are consistent with the basic neoclassical model and US time series.

2.2 Uzawa-Lucas model

This section describes a stochastic discrete time version of the generalized Uzawa-Lucas framework. One of the modifications, used by Bean (1990), King, Plosser and Rebelo (1988b) and Gomme (1993), is that physical capital is included as an input in the human capital production function. The second modification is that leisure is assumed to have a positive effect on agents' welfare. The economy is inhabited by a large number of identical households. The size of the population is assumed to be constant.

The representative household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t h_t^\lambda), \tag{13}$$

where E_0 denotes the conditional expectation operator, $0 < \beta < 1$ is the discount factor, c_t is consumption, and $l_t h_t$ is qualified leisure, and this captures Becker(1965)'s idea that the utility of a given amount of leisure increases with the stock of human capital. In particular, we assume that the preferences of the representative household are described by the following utility function:

$$U(c_t, l_t h_t^\lambda) = \frac{[c_t^\omega (l_t h_t^\lambda)^{1-\omega}]^{1-\gamma} - 1}{1 - \gamma}, \tag{14}$$

where $0 \leq \omega \leq 1$, $\gamma > 0$ and $0 \leq \lambda \leq 1$. This type of utility function guarantees the existence of a balanced growth path for the economy, where the fraction of time allocated to each activity remains constant and all per-capita variables grow at the same rate.[3]

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[3]See King et al. (1988 a, pp. 201-202) for an exposition of the conditions one should impose in order to guarantee a constant growth rate in a steady state.

There are two productive activities in this economy: the production of the final good (market sector) and the accumulation of human capital (human capital sector). At any point in time, a household has to decide what portion of its time is allocated to each of these activities, apart from the time allocated to leisure. The production function of the representative household is a production function with constant returns to scale with respect to physical capital and efficient labor. Formally,

$$y_t = F^m(\phi_t k_t, n_t h_t, z_t) = A_m Z_t (\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha}, \tag{15}$$

where A_m is a technology parameter, ϕ_t is the fraction of physical capital stock allocated to the market sector, n_t is the fraction of time allocated to the market sector, h_t denotes the stock of human capital at the beginning of time t (therefore, $n_t h_t$ denotes efficient labor), α is the share of physical capital in final good production and Z_t is a technology shock which follows a first-order autoregressive process $\log(Z_t) - \log(\overline{Z}) = \rho[\log(Z_{t-1}) - \log(\overline{Z})] + \epsilon_t$, where $0 \leq \rho \leq 1$, $\overline{Z}$ denotes the unconditional mean of the random variable Z_t and ϵ_t is a white noise with standard deviation σ_ϵ. The law of motion for physical capital is

$$k_{t+1} + c_t = A_m Z_t (\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha} + (1 - \delta_k) k_t, \tag{16}$$

where δ_k is the depreciation rate of physical capital.

The human capital sector is characterized as follows:

$$h_{t+1} = F^h[(1 - \phi_t)k_t, (1 - l_t - n_t)h_t] + (1 - \delta_h)h_t =$$

$$A_h[(1 - \phi_t)k_t]^\theta[(1 - l_t - n_t)h_t]^{1-\theta} + (1 - \delta_h)h_t, \tag{17}$$

where A_h is a technology parameter, θ is the share of physical capital stock in human capital production and δ_h is the rate of depreciation of human capital.

As is well known, the competitive equilibrium can be characterized through the first-order conditions derived from a benevolent social planner's problem in the absence of externalities and public goods. The social planner maximizes (13) subject to (16)-(17) with $k_0 > 0$ and $h_0 > 0$ given. In the steady state, the variables c_t, k_t and y_t grow at a constant rate which is equal to the rate of accumulation of human capital, and n_t, l_t and ϕ_t are constant. Therefore, the time series c_t, k_t and y_t obtained from the first-order conditions characterizing the social planner problem are non-stationary. In order

to facilitate the use of computational techniques, it is convenient to write the first-order conditions in terms of the ratios $\hat{c}_t = c_t/h_t$, $\hat{k}_t = k_t/h_t$, thus reducing the number of state variables. The necessary and sufficient first-order conditions for an (interior) optimum are then given by

$$U_1(\hat{c}_t, l_t) = \beta(\frac{h_{t+1}}{h_t})^\tau E_t\{U_1(\hat{c}_{t+1}, l_{t+1})[F_1^m(\phi_{t+1}\hat{k}_{t+1}, n_{t+1}) + 1 - \delta_k]\}, \quad (18)$$

$$U_1(\hat{c}_t, l_t) = \frac{U_2(\hat{c}_t, l_t)}{F_2^m(\phi_t\hat{k}_t, n_t)}, \quad (19)$$

$$\frac{U_2(\hat{c}_t, l_t)}{F_2^h[(1-\phi_t)\hat{k}_t, 1 - l_t - n_t]} = \beta(\frac{h_{t+1}}{h_t})^\tau E_t\{\frac{U_2(\hat{c}_{t+1}, l_{t+1})}{F_2^h[(1-\phi_{t+1})\hat{k}_{t+1}, 1 - l_{t+1} - n_{t+1}]}$$

$$[1 - \delta_h + (1 - l_{t+1} + \lambda l_{t+1})F_2^h((1-\phi_{t+1})\hat{k}_{t+1}, 1 - l_{t+1} - n_{t+1})]\}, \quad (20)$$

$$\frac{F_1^m}{F_2^m} = \frac{F_1^h}{F_2^h}, \quad (21)$$

$$\frac{h_{t+1}}{h_t} = A_h[(1-\phi_t)\hat{k}_t]^\theta(1 - l_t - n_t)^{1-\theta} + 1 - \delta_h, \quad (22)$$

$$\hat{c}_t + \hat{k}_{t+1}\frac{h_{t+1}}{h_t} = A_m Z_t(\phi_t\hat{k}_t)^\alpha n_t^{1-\alpha} + (1 - \delta_k)\hat{k}_t, \quad (23)$$

$$\lim_{t\to\infty} E_t\beta^t U_1\hat{k}_{t+1}\frac{h_{t+1}}{h_t} = 0,$$

$$\lim_{t\to\infty} E_t\beta^t U_2\frac{h_t^{\lambda-1}}{F_2^h}\frac{h_{t+1}}{h_t} = 0.$$

where $\tau = [\omega + \lambda(1-\omega)](1-\gamma) - 1$.

Next, Uhlig's method is used to solve the model. The solution can be written in matrix form as follows[4]

$$\mathbf{x}_t = P\mathbf{x}_{t-1} + Qz_t \quad (24)$$

$$\mathbf{y}_t = R\mathbf{x}_{t-1} + Sz_t \quad (25)$$

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[4]Appendix 2 describes the choice of parameter values and how Uhlig's log-linear method is implemented to solve Uzawa-Lucas model. Moreover, the numerical values of matrices P, Q, R and S in equations (24)-(25) are displayed in Appendix 2.

where $\mathbf{x}_t = (\widetilde{k}_{t+1}, hh_t)'$, $\mathbf{y}_t = (\widetilde{c}_t, \widetilde{\phi}_t, \widetilde{n}_t, \widetilde{l}_t)'$. As above, a tilde ($\tilde{\ }$) on a variable denotes its log-deviation from the steady-state value.

Next, it is described how one can obtain the synthetic technology shock time series using US time series and Uzawa-Lucas model. From this synthetic time series of the technology shock, one can derive the synthetic time series of output, consumption and investment that will be used in the evaluation of the trend processes characterized by the generalized version of Uzawa-Lucas model.

2.2.1 Obtaining the Solow Residual from the Uzawa-Lucas model

In this subsection, we describe how the technological shock is obtained using US data and some of the equations that characterize the competitive equilibrium in the Uzawa-Lucas model. From equation (24), we have that

$$\widetilde{k}_{t+1} = p_{11}\widetilde{k}_t + q_{11}z_t. \tag{26}$$

Using the lag operator, L, this equation can be rewritten as

$$\widetilde{k}_{t+1} = \frac{q_{11}z_t}{1 - p_{11}L}, \tag{27}$$

where $\widetilde{k}_{t+1} = \ln(\widehat{k}_{t+1}/\overline{\widehat{k}})$ and $\widehat{k}_{t+1} = k_{t+1}/h_t$. Similarly, we use the following notation $\widetilde{c}_{t+1} = \ln(\widehat{c}_{t+1}/\overline{\widehat{c}})$ with $\widehat{c}_{t+1} = c_t/h_t$, $hh_t = \ln((h_{t+1}/h_t)/hh)$, $\widetilde{\phi}_t = \ln(\phi_t) - \ln(\overline{\phi})$, and $\widetilde{n}_t = \ln(n_t) - \ln(\overline{n})$. As usual an upper bar on a variable denotes its a steady-state value.

From equation (25), we have that

$$\widetilde{\phi}_t = R_{21}\widetilde{k}_t + S_{21}z_t. \tag{28}$$

Substituting equation (27) into equation (28) and taking into account that $\widetilde{\phi}_t = \ln(\phi_t) - \ln(\overline{\phi})$, we obtain

$$\ln(\phi_t) = \ln(\overline{\phi}) + R_{21}\frac{q_{11}z_{t-1}}{1 - p_{11}L} + S_{21}z_t. \tag{29}$$

Proceeding in the same way, we obtain the following expressions

$$\ln(n_t) = \ln(\overline{n}) + R_{31}\frac{q_{11}z_{t-1}}{1 - p_{11}L} + S_{31}z_t, \tag{30}$$

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$$\ln(\widehat{k}_{t+1}) = \ln(\overline{\overline{k}}) + \frac{q_{11}}{1 - p_{11}L} z_t, \tag{31}$$

$$\ln(\widehat{c}_t) = \ln(\overline{\overline{c}}) + \frac{R_{11}q_{11}}{1 - p_{11}L} z_{t-1} + S_{11}z_t. \tag{32}$$

Using the following identity

$$\ln\left(\frac{y_t}{c_t}\right) \equiv \ln\left(\frac{y_t}{h_t}\right) - \ln\left(\frac{c_t}{h_t}\right) = \ln(\widehat{y}_t) - \ln(\widehat{c}_t),$$

and taking into account equation (15), we have that

$$\ln\left(\frac{y_t}{c_t}\right) = \ln A_m + z_t + \alpha \ln \phi_t + \alpha \ln \widehat{k}_t + (1 - \alpha) \ln n_t - \ln \widehat{c}_t. \tag{33}$$

Substituting equations (29)-(32) into (33), we obtain

$$(1 - p_{11}L) \ln\left(\frac{y_t}{c_t}\right) = (1 - p_{11})A + B(1 - p_{11}L)z_t + Cq_{11}z_{t-1}, \tag{34}$$

where A, B and C are constants defined by the following expressions

$$A = \ln A_m + \alpha \ln \overline{\phi} + \alpha \ln \overline{\overline{k}} + (1 - \alpha) \ln \overline{n} - \ln \overline{\overline{c}},$$
$$B = 1 + \alpha S_{21} + (1 - \alpha)S_{31} - S_{11},$$
$$C = \alpha(1 + R_{21}) + (1 - \alpha)R_{31} - R_{11}.$$

Given an initial value for the technology shock, z_1, and the parameter values of the model, we can calculate recursively time series for the technology shock from equation (34) using US output and consumption time series. The value of z_1 chosen is one consistent with the steady state.[5]

Finally, the synthetic time series of output, consumption and investment are calculated using US data, the synthetic time series of the technology shock, z_t, equations (29)-(32), equations (15)-(17) and the initial values for k_t and h_t.[6]

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[5]More precisely, we choose the values of z_1 and z_2 by solving the sytem of equations formed by $z_t = \rho z_{t-1} + \varepsilon_t$ and (34), evaluated at $t = 2$ and imposing the condition $\epsilon_2 = 0$. Once z_2 is obtained, using (34) the time series for z_t, for $t = 3, 4, ...$ is calculated.

[6]Apendix 3 shows an alternative method to obtain a time series for the tecnology shock, z_t. This alternative method uses US work effort time series. Some synthetic time

3   TREND ANALYSIS

In this section, we study the trends characterized by the two alternative models of growth considered in this paper. The analysis of trends is carried out using directly US and synthetic time series since other components, such as cyclical and irregular components, do not interfere in the econometric analysis of trends. As pointed out above, the analysis of trends in each model considered is carried out in a different way.

3.1   The basic growth model

In the basic neoclassical growth model, the study of the trend processes is based on the analysis of whether the differences between US and synthetic time series are stationary. In other words, we study whether the actual time series are the same as the corresponding synthetic time series, except for a stationary component that must be then attributable to cyclical or irregular components. More precisely, the evaluation of whether this basic growth model characterized the trend process observed in US data lies in testing whether the following differences are stationary processes

$$\log(c_t) - \log(c_t^*) \equiv v_{1t}, \tag{35}$$

$$\log(i_t) - \log(i_t^*) \equiv v_{2t}, \tag{36}$$

$$\log(y_t) - \log(y_t^*) \equiv v_{3t}, \tag{37}$$

where c_t, i_t and y_t are the actual time series of consumption, investment and output, respectively; and c_t^*, i_t^* and y_t^* are the corresponding synthetic time series.

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series obtained through this method displays odd behavior. A possible explanation for this odd behavior is that, as most growth models used in the literature, this model presents some limitations to reproduce the dynamic behavior of labor market variables such as work effort. Therefore, the use of observed labor input time series in constructing the synthetic time series may induce some distorsions. For this reason, in the following analysis, we focus on the time series for the technology shock, z_t, obtained through the method described in the main text.

Figures 1-3 plot the time series for v_{1t}, v_{2t} and v_{3t}, respectively. Figure 1 shows that the difference between (the logs of) US and synthetic consumption looks non-stationary. However, if we omit the first 25 observations this time series looks stationary. Figure 2 suggests that the difference between (the logs of) actual and synthetic investment looks non-stationary since this time series exhibit strong persistence. From this picture inspection, as a first approach, one may conclude that the model do not capture appropriately the trend processes of investment. On the contrary, Figure 3 shows that the difference between (the logs of) actual and synthetic output looks stationary and, therefore, the basic growth model seems to capture the trend process of output.
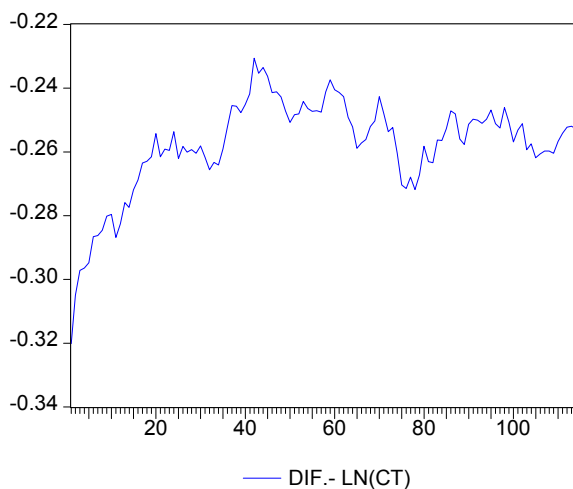

Figure 1: Difference between the (logs) of actual and synthetic consumption

Analyzing the process of v_{1t}, v_{2t} and v_{3t} we find that v_{1t} follow an $AR(1)$ process with an estimated autoregressive coefficient equal to $\widehat{\rho}_1 = 0.897$. v_{2t} and v_{3t} follow $AR(3)$ processes and the estimated first-order autoregression coefficients are $\widehat{\rho}_1 = 0.917$ and $\widehat{\rho}_1 = 0.895$, respectively. In sum, these results show that the differences between (logs of) actual and synthetic time series exhibit strong persistence that means that v_{1t}, v_{2t} and v_{3t} still preserve part of the trend components present in US time series.

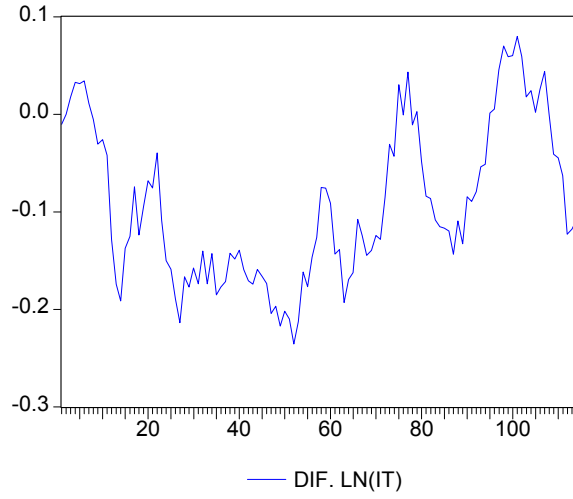In order to determine more rigorously whether the differences between

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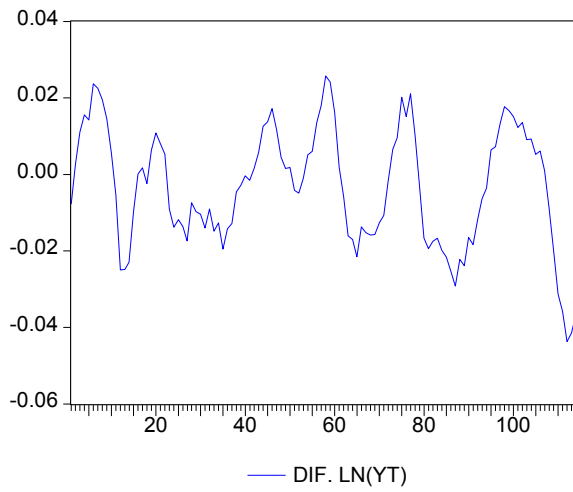Figure 2: Difference between the (logs) of actual and synthetic investment

Figure 3: Difference between the (logs) of actual and synthetic output

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(the logs of) actual and synthetic data for output, consumption and investment are stationary, we implement the augmented Dickey-Fuller (ADF) unit root test on time series v_{1t}, v_{2t} and v_{3t} . The test results are displayed in Table 1.

Table 1[7]:
ADF test for the differences between (the logs of) actual and synthetic data

time series	ADF statistic
$\log(c_t) - \log(c_t^*) = v_{1t}$	$DF(b) = -2.965277$
$\log(i_t) - \log(i_t^*) = v_{2t}$	$DF(b) = -2.177585$
$\log(y_t) - \log(y_t^*) = v_{3t}$	$DF(a) = -3.654430$

Table 1 shows that we cannot reject that v_{1t} and v_{3t} are stationary. Therefore, the basic growth model seems to reproduce quite well the trend component observed in the data. On the contrary, v_{2t} is not stationary. This result indicates that the model do not capture the trend component of investment. In sum, we may conclude that the basic growth model only reproduces partially the trend components of US data.

Figures 4-6 plot US and synthetic time series of output, consumption and investment, respectively. Independently of the tests implemented, these figures show that the model has an important element of truth since it captures relatively well the dynamic behavior of output, consumption and investment observed in post-war US data.

3.2 Uzawa-Lucas growth model

A simple approach for analyzing the trend process characterized by the generalized version of Uzawa-Lucas model lies in testing whether US time series

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[7]Critical values for Dickey-Fuller test are: DF(b) at 10%, -2.5811 and at 5%, -2.8887; and for DF(a) at 10%, -2.5805 and at 5%, -2.8877.
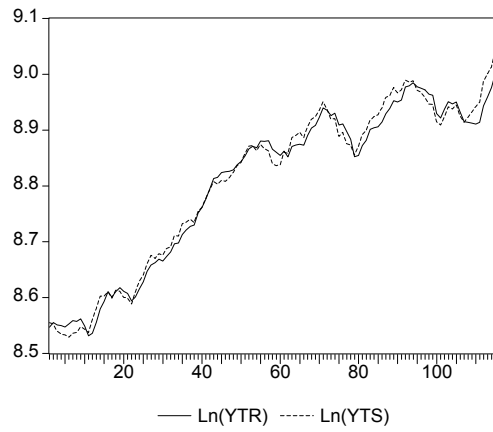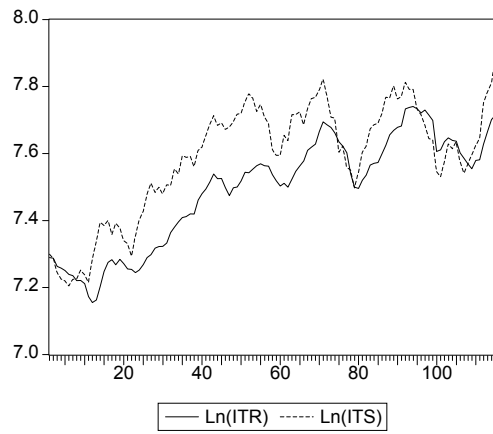
Figure 4: Logs of actual (YTR) and synthetic (YTS) output

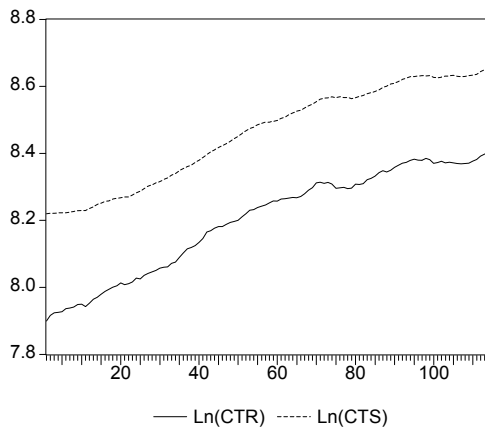Figure 5: Logs of actual (ITR) and synthetic (ITS) investment

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Figure 6: Logs of actual (CTR) and synthetic (CTS) consumption

and the corresponding synthetic data are cointegrated. The intuition is simple. If Uzawa-Lucas model reproduces the trend components observed in US time series, actual and synthetic time series must be then integrated processes of order one and the existence of a cointegration relationship between a particular US time series (for instance, output, consumption and investment) and the corresponding synthetic time series will indicate that the trend component observed in US data is reproduced properly by the trend component characterized by the model.

The cointegration test use in this paper is the one suggested by Engle and Granger (1987). This test is based on a Dickey-Fuller unit root test on the cointegration residuals. Formally, the cointegration test starts running the following least squares regression

$$w_t = \beta_0 + \beta_1 w_t^* + e_t, \tag{38}$$

where w_t denote a particular US time series (of output, consumption or investment) and w_t^* is the corresponding synthetic time series. The parameter β_1 is known as the cointegration parameter. The existence of a cointegration relationship between a pair of variables (for instance, actual and synthetic consumption) can be understood as the existence of a long-run linear relationship, in the sense that the two time series share a common trend. If there is cointegration, the deviations from the long-run equilibrium (common

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trend), measured by e_t, must follow a stationary process. For this to happen, it is required that the stochastic trends present in the two time series (actual and synthetic) are common and then they must cancel in the cointegration regression (38). Therefore, if there exists a cointegration relationship between a particular US time series and the corresponding synthetic time series, we may conclude that the model reproduces adequately the observed trend process for this variable.

The trend analysis is carried out in three steps:

i) test whether w_t and w_t^* are integrated processes of order 1,

ii) run the regression (38) by least squares,

iii) implement Dickey-Fuller (DF) or augmented Dickey-Fuller (ADF) (model a: neither constant nor time trend is included in the regression) on regression residuals, e_t. More precisely, the following regression is carried out

$$\triangle e_t = \rho_1 e_{t-1} + \sum_{i=1}^{p-1} \gamma_i \triangle e_{t-i} + \zeta_t. \tag{39}$$

The term $\sum_{i=1}^{p-1} \gamma_i \triangle e_{t-i}$ is added into the equation in order to guarantee that the error term ζ_t is white noise. The test is the following:

$$\begin{cases} H_0 : \rho_1 = 0, \\ \qquad \text{vs} \\ H_a : \rho_1 < 0. \end{cases}$$

The critical values for this cointegration test are different from the usual Dickey-Fuller critical values and they are tabulated in Engle and Granger (1987) or Engle and Yoo (1987).

Table 2 shows the ADF test results for actual and synthetic time series of output, consumption and investment.

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<div align="center">

Table 2.[8]
ADF tests: US and synthetic time series

Series	US	Synthetic
$\log(c_t)$	$ADF(b) = -1.999437$	$ADF(b) = -1.760002$
$\log(i_t)$	$ADF(c) = -2.722955$	$ADF(c) = -2.617276$
$\log(y_t)$	$ADF(c) = -2.051041$	$ADF(b) = -2.149128$

</div>

Critical values at 5%: ADF(a) -1.9430, ADF(b) -2.8897,
and ADF(c) -3.4501

The results shown in Table 2 clearly point out that US and synthetic time series are individually integrated of order one process. Moreover, the results show that the model reproduces the structure of consumption and investment. Thus, actual and synthetic consumption are non-stationary around a constant (ADF (b)), and actual and synthetic investment are non-stationary around a deterministic trend (ADF (c)). However, the model does not reproduce the structure for output since actual output is non-stationary around a deterministic trend, whereas synthetic output is non-stationary around a constant.

Figures 7-9 plot (the logs of) US and synthetic time series of output, investment and consumption, respectively. By looking at these figures, we can observe that the generalized version of Uzawa-Lucas model captures relatively well the temporal evolution of the three variables. However, the model generates a smoother pattern for consumption than the one observed.

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[8]ADF(c), ADF(b) and ADF(a) denote the augmented Dickey-Fuller statistics for a model that includes constant and time trend, without time trend, and without constant and time trend, respectively.
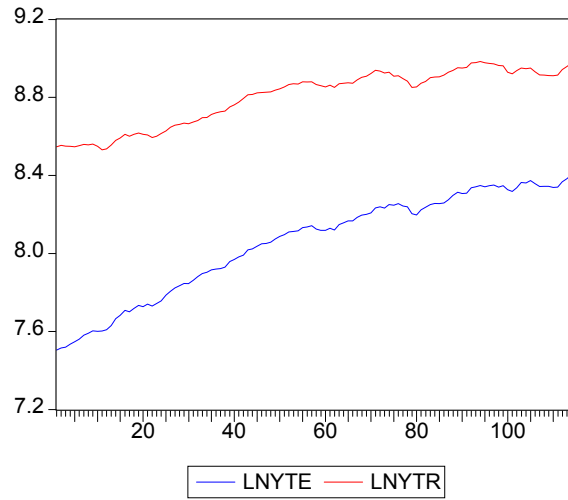
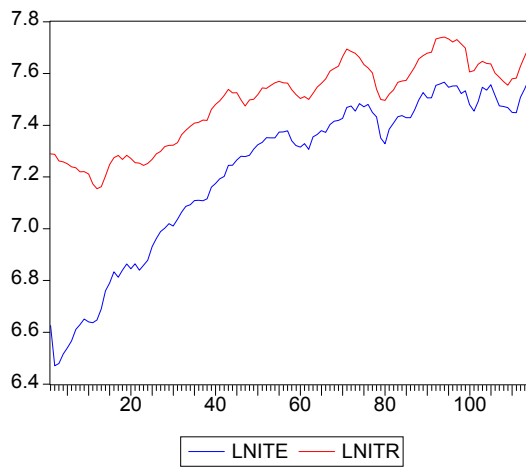Figure 7: Time series (logs) of U.S. (LNYTR) and synthetic (LNYTE) output

Figure 8: Time series (logs) of U.S. (LNITR) and synthetic (LNITE) investment
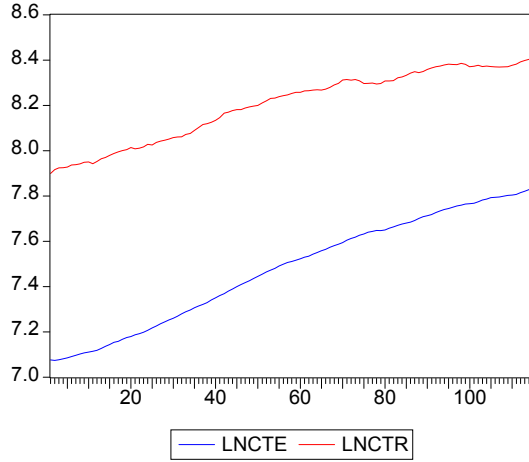
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Figure 9: Time series (logs) of U.S. (LNCTR) and synthetic (LNCTE) consumption

Once it has been tested that actual and synthetic times series are I(1) processes, we next study whether actual and synthetic time series are cointegrated. We carried the following regressions

$$\log(c_t) = \beta_{0c} + \beta_{1c} \log(c_t^*) + e_{1t}, \tag{40}$$

$$\log(i_t) = \beta_{0i} + \beta_{1i} \log(i_t^*) + e_{2t}, \tag{41}$$

$$\log(y_t) = \beta_{0y} + \beta_{1y} \log(y_t^*) + e_{3t}, \tag{42}$$

where c_t, i_t and y_t denote US time series for consumption, investment and output, respectively; and c_t^*, i_t^* and y_t^* are the corresponding synthetic time series. Table 3 shows the regression results.

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Table 3.[9]
Cointegration regressions

Dependent variable	Constant	Slope
$\log(C_t)$	3.398163 (0.0000)	0.642428 (0.0000)
$\log(It)$	3.864088 (0.0000)	0.502178 (0.0000)
$\log(Yt)$	4.461669 (0.0000)	0.539056 (0.0000)

Finally, we carry out ADF tests on cointegration residuals. The results of these tests are displayed in Table 3.

Table 4. Engle and Granger tests[10]

Variable	Statistic
Consumtion	-1.720977
Investment	-4.070937
Ouput	-3.047022

Table 4 shows that US and synthetic output and US and synthetic investment are *vis à vis* cointegrated at 10% and 1% significance levels, respectively. On the contrary, US consumption and synthetic consumption are not cointegrated at any standard significance level. Therefore, we can conclude that the generalized version of Uzawa-Lucas model considered reproduces relatively well the trend processes of output and investment, but fails in reproducing the consumption trend process.

The analysis of the cointegration residuals shows that the residuals from the cointegration regressions of output and investment can be modeled as AR(2) processes. The inverse of the roots of the autoregressive polynomials are 0.87 and 0.51 for the case of output, and 0.83 and 0.25 for investment.

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[9]The p-values are in parentheses.

[10]Critical values at 10%, 5% and 1% significance levels are -3.03, -3.37 and -4.07, respectively.

However, the residuals from the cointegration regression of consumption cannot be modeled as an autoregressive process because they seem to be nonstationary. The estimated values of the first-order autocorrelation coefficients of the residuals from the cointegration regressions of investment, output and consumption are 0.929, 0.9625 and 0.9635, respectively. Therefore, the results show that consumption cointegration residuals are slightly more persistent than investment and output cointegration residuals.

3.2.1 Analysis of the technology shock

Figure 10 plots the time series of the technology shock, z_t, that is obtained from Uzawa-Lucas model and US data. This time series shows strong persistence.
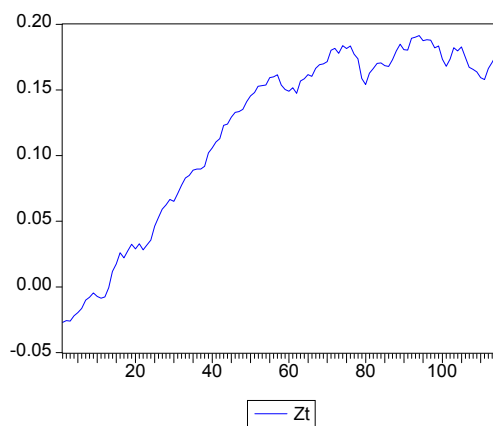
Figure 10: Time series of technology shock

In the calibration step it is assumed that $z_t \sim AR(1)$, with an autocorrelation coefficient $\rho = 0.95$. Next, we test whether the technology shock time series, z_t, obtained from Uzawa-Lucas model and US data follows this process. To do this, we run a first-order autoregression for this synthetic z_t. The regression results are[11]

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[11]The numbers in parentheses are the standard errors.

$$z_t = 0.222828 + 0.981920z_{t-1}.$$
$$(0.04393) \quad (0.00636)$$

The regression results confirm that z_t follows an $AR(1)$. However, the results also show that synthetic z_t exhibit more persistence than the one assumed in the model since the estimated autocorrelation coefficient is equal to 0.981920 and this value is slightly, but significantly, higher than the value assumed in the model that is equal to 0.95.

4   CONCLUSIONS

The question of whether GNP of industrialized countries fluctuates around a deterministic or a stochastic long-run trend has been debated for long. In this paper, we follow an eclectic approach in order to analyze econometrically the trend processes characterized by two standard growth models. The first model is the basic neoclassical growth model that postulates a deterministic linear trend for output. The second model studied in this paper is a generalized version of the Uzawa-Lucas endogenous growth model that postulates a stochastic trend for output. By using this approach, we try to understand how the different trend processes for output assumed by these two standard growth models determine the ability of these models to explain the observed trend processes of other macroeconomic variables such as aggregate consumption and aggregate investment.

The two growth models studied in this paper postulate different growth processes and, then, the kind of analysis of the trend process must be different for each model. On the one hand, since the basic neoclassical growth model assumes a linear deterministic trend, the analysis of the trend process characterized by this model lies in testing whether the differences between US and the corresponding synthetic time series are stationary. If this null hypothesis is not rejected, one may conclude that the model characterized properly the trend process observed in US data since the discrepancies between US and synthetic time series must be attributed to limitations of the model to characterize other components of data (for instance, cycle and irregular components) different from the trend component.

Since synthetic time series of output, consumption and investment obtained from the generalized version of the Uzawa-Lucas model are integrated processes of order one, the analysis of the trend processes in this model is carried out by determining whether an observed time series and its associated synthetic time series are cointegrated. If this null hypothesis is not rejected, one would obtain evidence that the deviations between the observed and synthetic time series are stationary and, then, attributable to limitations of the model to characterize other components than the trend components.

The results found in this paper show that the two models reproduce the trend process of output. Moreover, the results show that the basic neoclassical growth model captures properly the trend process of aggregate consumption, but fails in characterizing the trend process of aggregate investment. The reverse is true for the Uzawa-Lucas endogenous growth model.

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Appendix 1

This appendix describes how the basic neoclassical model is solved. We use the log-linear method suggested by Uhlig (1999) to solve the model. This method can be summarized in four steps:

Step 1. Find the necessary first-order conditions that characterize the competitive equilibrium. These conditions are (3)-(9).

Step 2. Find the steady-state. We firstly need to calibrate the model. We use King, Plosser and Rebelo (1988a) calibration of the parameter values. Thus, ρ (the share of capital in total production) is assumed to be equal to 0.48. Considering that the annual real return of capital is 6.5%, we have that the quarterly real gross return of capital is $\overline{R} = 1 + \frac{6.5\%}{4} = 1.01625$. The annual rate of depreciation is assumed equal to 10%, which implies $\delta = 0.025$ when evaluating the model using quarterly data. The fraction of time devoted to work $\overline{N}$ is assumed to be equal to 0.2. The parameter θ is 1 plus the rate of growth of output. In our exercise, this rate is assumed to be equal to growth rate of US output for the period, that is, 0.0040806697, thus $\theta = 1.0040806697$.[12] Finally, it is assumed that $\psi = 0.9$ and the standard deviation of the innovation in the first-order autoregressive process for the technology shock, ϵ_t, is adjusted in order that the standard deviation of per-capita US GNP be close to the standard deviation of the synthetic output times series. This condition imposes that $\sigma_\varepsilon = 0.0099819$.

Evaluating equations (3)-(9) at steady-state, we have that

$$A = \frac{(1-\rho)(1-\overline{N})\overline{y}}{\overline{c}\,\overline{N}}, \tag{43}$$

$$\theta = \beta\overline{R}, \tag{44}$$

$$\overline{R} = \rho\frac{\overline{y}}{\overline{k}} + (1-\delta), \tag{45}$$

$$\overline{c} + \overline{i} = \overline{y}, \tag{46}$$

$$\overline{y} = \overline{k}^\rho\overline{N}^{1-\rho}, \tag{47}$$

$$\theta\overline{k} = \overline{i} + (1-\delta)\overline{k}. \tag{48}$$

After some algebra, we easily obtain the steady-state values for all variables. The following table displayed the steady-state values.

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[12] The rate of growth is obtained by regressing the trend component, isolated via Hodrick and Prescott (1980, 1997) filter, on a deterministic trend.

<div align="center">

Parameter values and steady-state values[13]

</div>

Parameter	Value		Variable	Steady-state value
ρ	0.4200000		$\overline{k}$	10.931000
θ	1.0040807		$\overline{c}$	0.7556996
ψ	0.9000000		$\overline{y}$	1.0735803
β	0.9880253		$\overline{R}$	1.0162500
δ	0.0250000		$\overline{i}$	0.3178808
σ_ε	0.0099819		$\overline{N}$	0.2000000
			A	3.2958951

Step 3. The first-order conditions that characterize the competitive equilibrium are log-linearized around the steady-state in order to make all equations approximately linear in log-deviations from the steady state. After some simple, but tedious algebra, one can show that the log-linearized conditions are

$$0 = -A\overline{c}\,\overline{N}\widetilde{c}_t + (1-\rho)(1-\overline{N})\overline{y}\widetilde{y}_t - (A\overline{c} + (1-\rho)\overline{y})\overline{N}\widetilde{N}_t,$$

$$0 = -\rho\frac{\overline{y}}{\overline{k}}\widetilde{k}_t + \rho\frac{\overline{y}}{\overline{k}}\widetilde{y}_t - \overline{R}\widetilde{R}_t,$$

$$0 = \overline{c}\widetilde{c}_t - \overline{y}\widetilde{y}_t + \overline{i}\widetilde{i}_t,$$

$$0 = \rho\widetilde{k}_t - \widetilde{y}_t + (1-\rho)\widetilde{N}_t + z_t,$$

$$0 = -\theta\overline{k}\widetilde{k}_{t+1} + \overline{i}\widetilde{i}_t + (1-\delta)\overline{k}\widetilde{k}_t,$$

$$0 = E_t(-\widetilde{c}_{t+1} + \widetilde{c}_t + \widetilde{R}_{t+1}),$$

$$z_{t+1} = \psi z_t + \varepsilon_{t+1},$$

where $\widetilde{c}_t = Ln(c_t/\overline{c})$, $\widetilde{y}_t = Ln(y_t/\overline{y})$, $\widetilde{k}_t = Ln(k_t/\overline{k})$, $\widetilde{i}_t = Ln(i_t/\overline{i})$, $\widetilde{R}_t = Ln(R_t/\overline{R})$, $\widetilde{N}_t = Ln(N_t/\overline{N})$, and $z_t = Ln(Z_t/\overline{Z})$.

It is convenient to rewrite the system of log-linearized first-order conditions in matrix form as follows

$$\mathbf{0} = \mathbf{A}\mathbf{X}_t + \mathbf{B}\mathbf{X}_{t-1} + \mathbf{C}\mathbf{Y}_t + \mathbf{D}\mathbf{Z}_t,$$
$$\mathbf{0} = E_t\{\mathbf{F}\mathbf{X}_{t+1} + \mathbf{G}\mathbf{X}_t + \mathbf{H}\mathbf{X}_{t-1} + \mathbf{J}\mathbf{Y}_{t+1} + \mathbf{K}\mathbf{Y}_t + \mathbf{L}\mathbf{Z}_{t+1} + \mathbf{M}\mathbf{Z}_t\}, \qquad (49)$$
$$\mathbf{Z}_{t+1} = \mathbf{N}\mathbf{Z}_t + \boldsymbol{\varepsilon}_t,$$

---

[13]King, Plosser and Rebelo (1988a) consider that $\sigma_z = 2.29\%$. Since $z_t = 0.9z_{t-1} + \varepsilon_t$, then, $\sigma_\varepsilon = \sigma_z(1 - 0.9^2)^{1/2} = 0.00998188$.

where $\mathbf{X}_t$ is $m \times 1$ vector of endogenous state variables, $\mathbf{Y}_t$ is an $n \times 1$ vector containing other endogenous variables, $\mathbf{Z}_t$ es a $k \times 1$ vector of exogenous stochastic variables. Matrices $\mathbf{A}$ and $\mathbf{B}$ are of size $l \times m$, where l denotes the number of deterministic equations. Matrix C is of size $l \times n$, with $l \geq n$ and rank n. Matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are of size $(m + n - l) \times m$, and matrices $\mathbf{J}$ and $\mathbf{K}$ are of size $(m + n - l) \times n$, where $(m + n - l)$ represents the number of equations involving conditional expectations. $\mathbf{N}$ has only stable eigenvalues. In the basic growth model, $\mathbf{X}_t = (\widetilde{k}_{t+1})$, thus $m = 1$. $\mathbf{Y}_t = (\widetilde{c}_t, \widetilde{y}_t, \widetilde{N}_t, \widetilde{R}_t, \widetilde{i}_t)'$, then $n = 5$. $\mathbf{Z}_t = (z_t)$ that implies $k = 1$. Moreover, $l = n = 5$ and, therefore, the number of equations involving expectations is the same as the number of endogenous state variables, that is, $m + n - l = m$. Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{J}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ are given by

$$
\mathbf{A} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\theta\overline{k} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ -\rho\frac{\overline{y}}{k} \\ 0 \\ 0 \\ (1-\delta)\theta\overline{k} \end{bmatrix} \ \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},
$$

$$
\mathbf{C} = \begin{pmatrix} -A\overline{c}\overline{N} & (1-\rho)(1-\overline{N})\overline{y} & -(A\overline{c}+(1-\rho)\overline{y})\overline{N} & 0 & 0 \\ 0 & \rho\frac{\overline{y}}{k} & 0 & -\overline{R} & 0 \\ \overline{c} & -\overline{y} & 0 & 0 & \overline{i} \\ 0 & -1 & (1-\rho) & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{i} \end{pmatrix},
$$

$$
\mathbf{J} = (-1, \ 0, \ 0, \ 1, \ 0), \ \mathbf{K} = (1, \ 0, \ 0, \ 0, \ 0), \ \mathbf{N} = (\psi),
$$
$$
\mathbf{F} = \mathbf{G} = \mathbf{H} = \mathbf{L} = \mathbf{M} = (0).
$$

Step 4. The log-linear solution method seeks a recursive law of motion of the following form

$$
\mathbf{X}_t = \mathbf{P}\mathbf{X}_{t-1} + \mathbf{Q}\mathbf{Z}_t, \tag{50}
$$

$$
\mathbf{Y}_t = \mathbf{R}\mathbf{X}_{t-1} + \mathbf{S}\mathbf{Z}_t, \tag{51}
$$

that is, finding $\mathbf{P}_{m\times m}$, $\mathbf{Q}_{m\times k}$, $\mathbf{R}_{n\times m}$, and $\mathbf{S}_{n\times k}$ so that the equilibrium described by these rules is stable. As pointed out above, for this model $l = n$ (see Corollary 1 of Uhlig (1999))

i) $\mathbf{P}$ satisfies the following quadratic equation:

$$
(\mathbf{F} - \mathbf{J}\mathbf{C}^{-1}\mathbf{A})\mathbf{P}^2 - (\mathbf{J}\mathbf{C}^{-1}\mathbf{B} - \mathbf{G} + \mathbf{K}\mathbf{C}^{-1}\mathbf{A})\mathbf{P} - \mathbf{K}\mathbf{C}^{-1}\mathbf{B} + \mathbf{H} = \mathbf{0}. \tag{52}
$$

ii) $\mathbf{R}$ is given by

$$\mathbf{R} = -\mathbf{C}^{-1}(\mathbf{AP} + \mathbf{B}). \tag{53}$$

iii) $\mathbf{Q}$ satisfies

$$
\begin{aligned}
vec(\mathbf{Q}) \;=\; & \mathbf{N}'\otimes((\mathbf{F} - \mathbf{JC}^{-1}\mathbf{A}) + \mathbf{I}_k\otimes(\mathbf{JR} + \mathbf{FP} + \mathbf{G} - \mathbf{KC}^{-1}\mathbf{A}))^{-1} \\
& vec((\mathbf{JC}^{-1}\mathbf{D} - \mathbf{L})\mathbf{N} + \mathbf{KC}^{-1}\mathbf{D} - \mathbf{M}).
\end{aligned} \tag{54}
$$

where $vec(.)$ denotes columnwise vectorization and I_k is the identity matrix of size k.

iv) $\mathbf{S}$ is given by

$$\mathbf{S} = -\mathbf{C}^{-1}(\mathbf{AQ} + \mathbf{D}). \tag{55}$$

In order to find $\mathbf{P}$, we rewrite equation (52) as

$$\mathbf{P}^2 - \mathbf{\Gamma P} - \mathbf{\Theta} = \mathbf{0}, \tag{56}$$

where

$$\mathbf{\Gamma} = (\mathbf{F} - \mathbf{JC}^{-1}\mathbf{A})^{-1}(\mathbf{JC}^{-1}\mathbf{B} - \mathbf{G} + \mathbf{KC}^{-1}\mathbf{A}),$$

$$\mathbf{\Theta} = (\mathbf{F} - \mathbf{JC}^{-1}\mathbf{A})^{-1}(\mathbf{KC}^{-1}\mathbf{B} - \mathbf{H}).$$

Solving for $\mathbf{P}$ in this equation requires the use of Theorem 2 in Uhlig (1999). If $m = 1$, as in this model, the solutions for $\mathbf{P}$ are given by

$$\mathbf{P}_{1,2} = \frac{\mathbf{\Gamma} \pm \sqrt{\mathbf{\Gamma}^2 + 4\mathbf{\Theta}}}{2}.$$

Using matrix definitions and the parameter values chosen in the calibration step, we have that

$$
\begin{aligned}
\mathbf{P} &= (0.95294889), \\
\mathbf{Q} &= (0.13673068),
\end{aligned}
$$

$$
\mathbf{R} = \begin{pmatrix} 0.61705058 \\ 0.24941890 \\ -0.29410535 \\ -0.03046639 \\ -0.62455367 \end{pmatrix}, \text{ and } \mathbf{S} = \begin{pmatrix} 0.29805711 \\ 1.60765210 \\ 1.04767600 \\ 0.06225525 \\ 4.72095840 \end{pmatrix}.
$$

Using these matrices and equations (50)-(51), we find the laws of motion of the log-deviations from the steady-state values for all variables of the model:

$$\widetilde{k}_{t+1} = 0.95294889\widetilde{k}_t + 0.13673068z_t,$$
$$\widetilde{c}_t = 0.61705058\widetilde{k}_t + 0.29805711z_t,$$
$$\widetilde{y}_t = 0.24941890\widetilde{k}_t + 1.60765210z_t,$$
$$\widetilde{N}_t = -0.29410535\widetilde{k}_t + 1.04767600z_t,$$
$$\widetilde{R}_t = -0.03046639\widetilde{k}_t + 0.06225525z_t,$$
$$\widetilde{i}_t = -0.62455367\widetilde{k}_t + 4.72095840z_t,$$

Appendix 2

This appendix describes how the generalized version of Uzawa-Lucas is solved by implementing Uhlig's (1999) log-linear method. First, we need to calibrate the model. The derivation of reasonable values for the parameters describing household preferences follows standard procedures. The discount factor, β, is chosen so that the annual real interest rate is equal to 4%. The value for β is obtained from the following equation, which characterizes the steady state given the homogeneity properties of the utility function:

$$1.01 \ \beta(\frac{h_{t+1}}{h_t})^\tau = 1.$$

Mehra and Prescott (1985) establish that a reasonable value for the relative risk aversion parameter, σ, lies in the interval $[1, 2]$. We consider $\sigma = 1.3$. As shown by Barañano, Iza and Vázquez (2001), the numerical solutions obtained with $\sigma = 1.3$ are similar to those found when $\sigma = 2$ in a model which exhibits a weak propagation mechanism of the technology shocks (that is, when θ is close to zero). Since the utility function is multiplicatively separable we have that $U(c, lh^\lambda) = u(c)v(lh^\lambda)$, where $u(c)$ is homogeneous of degree $1-\sigma$. Moreover, we follow the suggestion made by Gomme (1993) and Greenwood and Hercowitz (1991) that a reasonable value for the fraction of time allocated to the market sector is 0.24, and from this value we can derive reasonable parameter values for ω and γ using the homogeneity properties of the utility function. Finally, the choice of a parameter value for λ is not straightforward, because there is no empirical evidence. This paper considers

$\lambda = 1$ (qualified leisure).[14] Looking at the market sector, the value of α is chosen so that it equals the average share of physical capital in the US GNP over the period ($\alpha = 0.36$). Since we are using quarterly data, the rate of depreciation for physical capital, δ_k, has been fixed at 0.025, which is equivalent to the 10% annual rate used by Kydland and Prescott (1982). The value for A_m is normalized to unity.

Based on first moments from the Solow residual, we follow Prescott's (1986) suggestion for ρ: $\rho = 0.95$. Moreover, the standard deviation of the innovation in the first-order autoregressive process for the technology shock, ϵ_t, is adjusted; in order that the standard deviation of per-capita US GNP be close to the standard deviation of the synthetic output times series.

Since there is not enough empirical evidence to establish the parameter values characterizing the human capital sector, we have decided to choose parameter values in such a way that they guarantee reasonable values for steady state variables. In particular, A_h is chosen so that the growth rate of output in the steady state matches the average annual growth rate of per-capita US GNP, 1.4%. Moreover, we choose $\theta = 0.05$, which implies a weak internal propagation mechanism. A small value of θ is needed to mimic the cyclical features (characterized by the Hodrick and Prescott (1980, 1997) filter) displayed by standard real business cycle models. Proceeding in this way, the two models studied share similar cyclical properties that allow us to distinguish easily the trend features exhibited by the two models.

Once the parameter values of the model are chosen, we implement Uhlig's method to solve the generalized version of Uzawa-Lucas model as follows. In this model, we have six first-order conditions (18)-(23).

Substituting the functional forms chosen for the utility and production functions, we have the following expressions for the first-order conditions,

$$\hat{c}_t(1-w)n_t^\alpha = wA_m(1-\alpha)(\phi_t\hat{k}_t)^\alpha l_t Z_t,$$

$$\frac{\phi_t}{n_t} = \frac{(1-\theta)\alpha}{\theta(1-\alpha)}\frac{(1-\phi_t)}{(1-l_t-n_t)},$$

or equivalently,

---

[14]As shown by Ladrón-de-Guevara, Ortigueira and Santos (1997), a value of $\lambda = 1$ guarantees the existence of a unique balanced growth path.

$$\phi_t = \frac{an_t}{1 - l_t - n_t + an_t},$$

where $a = \frac{(1-\theta)\alpha}{\theta(1-\alpha)}$,

$$\frac{h_{t+1}}{h_t} = A_h[(1-\phi_t)\hat{k}_t]^\theta(1-l_t-n_t)^{1-\theta} + 1 - \delta_h,$$

$$\frac{h_{t+1}}{h_t}\hat{k}_{t+1} = A_m(\phi_t\hat{k}_t)^\alpha n_t^{1-\alpha}Z_t + (1-\delta_k)\hat{k}_t - \hat{c}_t,$$

$$\hat{c}_t^{\omega(1-\gamma)-1}l_t^{(1-\omega)(1-\gamma)} = \beta(\frac{h_{t+1}}{h_t})^{-\gamma}E_t[\hat{c}_{t+1}^{\omega(1-\gamma)-1}l_{t+1}^{(1-\omega)(1-\gamma)}$$

$$[A_m\alpha Z_{t+1}(\phi_{t+1}\hat{k}_{t+1}^{\alpha-1})n_{t+1}^{1-\alpha} + 1 - \delta_k],$$

$$\frac{\hat{c}_t^{\omega(1-\gamma)}l_t^{(1-\omega)(1-\gamma)-1}}{[(1-\phi_t)\hat{k}_t]^\theta(1-l_t-n_t)^{-\theta}} = \beta(\frac{h_{t+1}}{h_t})^{-\gamma}$$

$$E_t\{\frac{\hat{c}_{t+1}^{\omega(1-\gamma)}l_{t+1}^{(1-\omega)(1-\gamma)-1}}{[(1-\phi_{t+1})\hat{k}_{t+1}]^\theta(1-l_{t+1}-n_{t+1})^{-\theta}}$$

$$[A_h(1-\theta)((1-\phi_{t+1})\hat{k}_{t+1})^\theta(1-l_{t+1}-n_{t+1})^{-\theta} + 1 - \delta_h]\}.$$

Making the equations approximately linear in the log-deviations from the steady state as shown by Uhlig, and rearranging, we have that

$$0 = \bar{c}(1-w)\bar{n}^\alpha(\hat{cc}_t + \alpha nn_t) - wA_m(1-\alpha)\bar{\phi}^\alpha\bar{\hat{k}}^\alpha\bar{Z}\bar{l}[\alpha\phi\phi_t + \alpha k\hat{k}_t + z_t + ll_t],$$

where $\hat{cc}_t = log(\frac{\hat{c}_t}{\bar{c}})$, $nn_t = log(\frac{n_t}{\bar{n}})$, $\phi\phi_t = log(\frac{\phi_t}{\bar{\phi}})$, $k\hat{k}_t = log(\frac{\hat{k}_t}{\bar{\hat{k}}})$, and $ll_t = log(\frac{l_t}{\bar{l}})$;

$$0 = [a\bar{n} - (a-1)\bar{\phi}\bar{n}]nn_t + \bar{\phi}\bar{l}ll_t - \bar{\phi}[1 - \bar{l} + (a-1)\bar{n}]\phi\phi_t,$$

$$0 = -\bar{H}Hhh_{t+1} + A_h(\frac{\bar{\phi}\bar{\hat{k}}}{\bar{n}})^\theta[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta(1-\bar{l}-\bar{n})\theta[\phi\phi_t + k\hat{k}_t]$$

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$$-A_h(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}})^\theta[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta\bar{l}ll_t - A_h(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}})^\theta[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta[(1-\bar{l}-\bar{n})\theta+\bar{n}]nn_t,$$

$$0 = -\bar{\bar{k}}\bar{H}H[k\hat{k}_{t+1}+hh_{t+1}] + A_m\bar{Z}(\bar{\phi}\bar{\bar{k}})^\alpha\bar{n}^{1-\alpha}[\alpha\phi\phi_t+(1-\alpha)nn_t+z_t]$$

$$+[A_m\bar{Z}(\bar{\phi}\bar{\bar{k}})^\alpha\bar{n}^{1-\alpha}\alpha+(1-\delta_k)\bar{\bar{k}}]k\hat{k}_t-\bar{\bar{c}}c\hat{c}_t,$$

$$0 = E_t\{-(w(1-\gamma)-1)\hat{c}c_t-(1-w)(1-\gamma)ll_t$$

$$+\beta\bar{H}H^{-\gamma}A_m\alpha\bar{Z}(\bar{\phi}\bar{\bar{k}})^{\alpha-1}\bar{n}^{1-\alpha}[(\alpha-1)\phi\phi_{t+1}+(\alpha-1)kk_{t+1}+(1-\alpha)nn_{t+1}+z_{t+1}]$$

$$+\beta\bar{H}H^{-\gamma}(w(1-\gamma)-1)[A_m\alpha(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{n}})^{\alpha-1}\bar{Z}+(1-\delta_k)]c\hat{c}_{t+1}$$

$$+\beta\bar{H}H^{-\gamma}[(1-w)(1-\gamma)][A_m\alpha(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{n}})^{\alpha-1}\bar{Z}+(1-\delta_k)]ll_{t+1}$$

$$-\beta\bar{H}H^{-\gamma}\gamma[A_m\alpha(\frac{\bar{\phi}\bar{\bar{k}}}{\hat{\bar{n}}})^{\alpha-1}\bar{Z}+(1-\delta_k)]hh_{t+1}\},$$

$$0 = E_t\{-w(1-\gamma)\hat{c}c_t-((1-w)(1-\gamma)-1)ll_t+\theta\phi\phi_t+\theta k\hat{k}_t-\theta nn_t+$$

$$\beta\bar{H}H^{-\gamma}w(1-\gamma)[A_h(1-\theta)(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}})^\theta(\frac{\theta(1-\alpha)}{\alpha(1-\theta)})^\theta+(1-\delta_h)]c\hat{c}_{t+1}+$$

$$\beta\bar{H}H^{-\gamma}[(1-w)(1-\gamma)-1][A_h(1-\theta)(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}})^\theta(\frac{\theta(1-\alpha)}{\alpha(1-\theta)})^\theta+(1-\delta_h)]ll_{t+1}$$

$$-\beta\bar{H}H^{-\gamma}\theta(1-\delta_h)[\phi\phi_{t+1}+k\hat{k}_{t+1}-nn_{t+1}]$$

$$-\beta\bar{H}H^{-\gamma}\gamma[A_h(1-\theta)(\frac{\bar{\phi}\bar{\bar{k}}}{\bar{n}})^\theta(\frac{\theta(1-\alpha)}{\alpha(1-\theta)})^\theta+(1-\delta_h)]hh_{t+1}\}.$$

The steady-state values are denoted with an upper bar, e.g., $\bar{H}H$ is the steady state value for $\frac{h_{t+1}}{h_t}$, i.e. the endogenous growth rate.

Using Uhlig's terminology, there is an endogenous state vector x_t, size 2x1, a list of other endogenous variables y_t, size 4x1, and a list of exogenous stochastic processes z_t, size 1x1. The equilibrium relationships between these variables can be expressed as follows

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t, \tag{57}$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t],$$

$$z_{t+1} = Nz_t + \epsilon_{t+1},$$

where $E_t(\epsilon_{t+1}) = 0$, $x'_t = (k\hat{k}_{t+1}, hh_{t+1})$, $y'_t = (c\hat{c}_t, \phi\phi_t, nn_t, ll_t)$, $z_t = z_t$, and matrices:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\overline{HH} \\ -\bar{\hat{k}}\overline{HH} & -\bar{\hat{k}}\overline{HH} \end{pmatrix},$$

$$B = \begin{pmatrix} -\alpha\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha\overline{Zl} & 0 \\ 0 & 0 \\ A_h(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta(1-\bar{l}-\bar{n})\theta & 0 \\ A_m\bar{Z}(\bar{\phi}\bar{\hat{k}})^\alpha\bar{n}^{1-\alpha}\alpha + (1-\delta_k)\bar{\hat{k}} & 0 \end{pmatrix}.$$

Denoting the generic element of C by C_{ij} we have that

$$C_{11} = \hat{c}(1-\omega)\overline{n}^\alpha$$

$$C_{12} = -\alpha\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha\overline{Zl},$$

$$C_{13} = \alpha\bar{\hat{c}}(1-\omega)\overline{n}^\alpha$$

$$C_{14} = -\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha\overline{Zl},$$

$$C_{21} = 0$$

$$C_{22} = -\overline{\phi}[1-\bar{l}+(a-1)\overline{n}],$$

$$C_{23} = a\overline{n} - \overline{\phi}(a-1)\overline{n},$$

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$$C_{24} = \overline{\phi l},$$

$$C_{31} = 0,$$

$$C_{32} = A_h(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta(1-\bar{l}-\bar{n})\theta,$$

$$C_{33} = -A_h(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta[(1-\bar{l}-\bar{n})\theta+\bar{n}],$$

$$C_{34} = -A_h(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}\bar{l}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta,$$

$$C_{41} = -\bar{\hat{c}},$$

$$C_{42} = \alpha A_m(\bar{\phi}\bar{\hat{k}})^\alpha \overline{n}^{1-\alpha}\overline{Z},$$

$$C_{43} = (1-\alpha)A_m(\bar{\phi}\bar{\hat{k}})^\alpha \overline{n}^{1-\alpha}\overline{Z},$$

$$C_{44} = 0,$$

$$D = \begin{pmatrix} -\omega A_m(1-\alpha)(\bar{\phi}\bar{\hat{k}})^\alpha \overline{Z}l \\ 0 \\ 0 \\ A_m(\bar{\phi}\bar{\hat{k}})^\alpha \overline{n}^{1-\alpha}\overline{Z} \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} (\alpha-1)\beta\overline{HH}^{-\gamma}(I51) & -\gamma\beta\overline{HH}^{-\gamma}(I51+I52) \\ -\theta\beta\overline{HH}^{-\gamma}(I62) & -\gamma\beta\overline{HH}^{-\gamma}(I61+I62) \end{pmatrix},$$

where

$$I_{51} = \alpha A_m(\bar{\phi}\bar{\hat{k}})^{\alpha-1}\overline{n}^{1-\alpha}\overline{Z},$$

$$I_{52} = (1-\delta_k),$$

$$I_{61} = A_h(1-\theta)(\bar{\phi}\bar{\hat{k}})^\theta n^{-\theta}[\frac{\theta(1-\alpha)}{\alpha(1-\theta)}]^\theta,$$

$$I_{62} = (1-\delta_h),$$

$$H = \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix}.$$

Denoting the generic element of J by J_{ij} we have that

$$J_{11} = \beta\overline{HH}^{-\gamma}[\omega(1-\gamma)-1](I51+I52),$$

$$J_{12} = \beta\overline{HH}^{-\gamma}(\alpha-1)I51,$$

$$J_{13} = \beta\overline{HH}^{-\gamma}(1-\alpha)I51,$$

$$J_{14} = \beta\overline{HH}^{-\gamma}(1-\omega)(1-\gamma)(I51+I52),$$

$$J_{21} = \beta\overline{HH}^{-\gamma}\omega(1-\gamma)(I61+I62),$$

$$J_{22} = -\theta\beta\overline{HH}^{-\gamma}(I62),$$

$$J_{23} = \theta\beta\overline{HH}^{-\gamma}(I62),$$

$$J_{24} = \beta\overline{HH}^{-\gamma}[(1-\omega)(1-\gamma)-1](I61+I62),$$

$$K = \begin{pmatrix} 1-\omega(1-\gamma) & 0 & 0 & -(1-\omega)(1-\gamma) \\ -\omega(1-\gamma) & \theta & -\theta & 1-(1-\omega)(1-\gamma) \end{pmatrix}.$$

$$L = \begin{pmatrix} \beta\overline{HH}^{-\gamma}(I51) \\ 0 \end{pmatrix}.$$

$$M' = (0,0).$$

Appendix 3

In this appendix, we describe an alternative procedure to obtain a time series of the technology shock using US data and the model. As pointed out above, this procedure make use of the work effort time series. The procedure is described as follows. Household production function is

$$y_t = A_m e^{z_t}(\phi_t k_t)^\alpha (n_t h_t)^{1-\alpha}. \tag{58}$$

Dividing this expression by c_t

$$\frac{y_t}{c_t} = A_m e^{z_t}(\phi_t \frac{k_t}{h_t})^\alpha n_t^{1-\alpha}(\frac{c_t}{h_t})^{-1}. \tag{59}$$

Use notation $\widehat{k}_t = k_t/h_t$ and $\widehat{k}_t = k_t/h_t$. Then, taking natural logs in equation (58), we have that

$$\ln\left(\frac{y_t}{c_t}\right) = \ln(A_m) + z_t + \alpha\ln(\phi_t) + \alpha\ln(\widehat{k}_t) + (1-\alpha)\ln(n_t) - \ln(\widehat{c}_t). \tag{60}$$

Using Uhlig's method, we find that

$$\mathbf{x}_t = P\mathbf{x}_{t-1} + Qz_t, \tag{61}$$

$$\mathbf{y}_t = R\mathbf{x}_{t-1} + Sz_t, \tag{62}$$

where $\mathbf{x}_t = (\widetilde{kk}_{t+1}, hh_t)'$, $\mathbf{y}_t = (\widetilde{cc}_t, \widetilde{\phi}_t, \widetilde{n}_t, \widetilde{l}_t)'$. Therefore,

$$\widetilde{kk}_{t+1} = P_{11}\widetilde{kk}_t + Q_{11}z_t. \tag{63}$$

Using lag operator L,

$$\widetilde{kk}_t = \frac{Q_{11}}{1 - P_{11}L}z_{t-1}, \tag{64}$$

and $\widetilde{kk}_t = \ln(\widehat{k}_t/\overline{\overline{k}})$, we have that

$$\ln(\widehat{k}_t) = \ln(\overline{\overline{k}}) + \frac{Q_{11}}{1 - P_{11}L}z_{t-1}. \tag{65}$$

Next, from equation (62)

$$\widetilde{cc}_{t+1} = R_{11}\widetilde{kk}_t + S_{11}z_t,$$

and $\widetilde{cc}_t = \ln(\widehat{c}_t/\overline{\overline{c}})$, we have that

$$\ln(\widehat{c}_t) = \ln(\overline{\overline{c}}) + \frac{R_{11}Q_{11}}{1 - P_{11}L}z_{t-1} + S_{11}z_t. \tag{66}$$

Similarly, using equations (62), (64), $\widetilde{\phi}_t = \ln(\phi_t/\overline{\phi})$ and $\widetilde{n}_t = \ln(n_t/\overline{n})$, we obtain

$$\ln(\phi_t) = \ln(\overline{\phi}) + \frac{R_{21}Q_{11}}{1 - P_{11}L}z_{t-1} + S_{21}z_t, \tag{67}$$

$$\ln(n_t) = \ln(\overline{n}) + \frac{R_{31}Q_{11}}{1 - P_{11}L}z_{t-1} + S_{31}z_t. \tag{68}$$

By substituting equations (65), (66) and (67) into (60), after some algebra, we have that

$$
\begin{aligned}
(1 - P_{11}L)z_t \;=\; & \frac{(1 - P_{11}L)}{B}\ln(\frac{y_t}{c_t}) - \frac{(1-\alpha)}{B}(1 - P_{11}L)\ln(n_t) \\
& -\frac{(1 - P_{11})}{B}C - \frac{Q_{11}D)}{B}z_{t-1},
\end{aligned} \tag{69}
$$

where B, C and D are constants defined by the following expressions

$$B = 1 + \alpha S_{21} - S_{11},$$
$$C = \ln(A_m) + \alpha \ln(\overline{\phi}) + \alpha \ln(\overline{\overline{k}}) - \ln(\overline{\overline{c}}),$$
$$D = \alpha R_{21} + \alpha - R_{11}.$$

Now, equation (68) can be rewritten as

$$(1 - P_{11}L)z_t = \frac{(1 - P_{11}L)}{S_{31}} \ln(n_t) - \frac{(1 - P_{11})\ln(\overline{n})}{S_{31}} - \frac{R_{31}Q_{11}}{S_{31}} z_{t-1}. \quad (70)$$

Finally, after some algebra, we obtain from equations (69) and (70) that

$$z_{t-1} = A^{-1} \left[B_1(1 - P_{11}L)\ln(n_t) - (1 - P_{11}L)\ln(y_t/c_t) + C_1 \right], \quad (71)$$

where A, B_1 and C_1 are constants defined by

$$A = \frac{BR_{31}Q_{11}}{S_{31}} - Q_{11}D,$$
$$B_1 = \frac{B}{S_{31}} + 1 - \alpha,$$
$$C_1 = (C - \frac{B}{S_{31}} \ln(\overline{n}))(1 - P_{11}).$$

Using equation (71) and US time series, one can calculate a time series for the technology shock, z_t, that is consistent with US data and Uzawa-Lucas model. The main difference between this procedure to obtain the time series of z_t and the procedure described in the main text is that the former procedure does not introduce an additional assumption on ε_t, but it takes into account the labor input time series. As explained above, real business cycle models shows some limitation in reproducing labor market dynamic features. Then, considering the labor input time series in computing z_t may introduce some distortions. Thus, the time series of z_t obtain from this procedure follows an $AR(2)$ process when, according to the model, z_t must follow an $AR(1)$.