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Reference Points and Optimal Management in Stochastic Age-Structured Fisheries Models

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ABSTRACT: The purpose of this article is to characterize dynamic optimal harvesting trajectories that maximize discounted utility assuming an age-structured population model, in the same line as Tahvonen (2009). The main novelty of our study is that uses as an age-structured population model the standard stochastic cohort framework applied in Virtual Population Analysis for fish stock assessment. This allows us to compare optimal harvesting in a discounted economic context with standard reference points used by fisheries agencies for long term management plans (e.g. F_{msy}). Our main findings are the following. First, optimal steady state is characterized and sufficient conditions that guarantees its existence and uniqueness for the general case of n cohorts are shown. It is also proved that the optimal steady state coincides with the traditional target F_{msy} when the utility function to be maximized is the yield and the discount rate is zero. Second, an algorithm to calculate the optimal path that easily drives the resource to the steady state is developed. And third, the algorithm is applied to the Northern Stock of hake. Results show that management plans based exclusively on traditional reference targets as F_{msy} may drive fishery economic results far from the optimal.

Key Words: fisheries management, age-structured models, reference point, F_{msy}, F_{pa}, Northern Stock of Hake.

JEL Classification: Q22.

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1 Introduction

The main characteristic of age-structured population models is that fish age distribution determines the number of spawning fish and, given a stock-recruitment relationship, the size of the recruits that enter in the exploitable population in the next period. This kind of population models has been the centerpiece of fisheries management for long time. From Baranov's seminal article (1918) to subsequent developments by Beverton and Holt (1957), Ricker (1975) or Shepherd (1982), there has been a vast amount of literature showing the advances of this approach in fishing population models.[1]

Recent studies have addressed a great variety of relevant empirical issues in fisheries applying age-structured population models. Kulmala, Laukkanenc and Michielsens (2008) build an age-structured population dynamics model for the Atlantic salmon fishery in the Baltic Sea which is compatible with the seasonal harvest and competing harvesting by commercial and recreational fishermen. Smith, Zhang and Coleman (2008) also use an age-structured model with 60 cohorts to model the Gulf of Mexico gag fishery. They suggests that the seasonal spawning closure caused an increase in fishing effort. Massey, Newbold and Gentner (2006) develop a three age structure model for the Atlantic Coast Summer Flounder Fishery. They find that improving water quality conditions throughout the range of the species could lead to substantial increases in fish abundance and higher benefits to recreational anglers. Nielsen (2006) studies the effects of trade liberalization in the East Baltic cod market by developing an age-structured fishery model with 8 cohorts combined with basic theory of trade between two countries. He finds liberalizing trade might cause welfare reductions in the supplier countries, but those reductions are small compared to the welfare gains from a hypothetical change to optimal fisheries management.

In parallel to these age-structured population developments, during the last decades most of the fisheries economics has been based on biomass models[2]. This biomass approach abstracts from the age-structured behind the biological dynamics of fish populations. Gordon (1954) and Schaefer (1954) were the pioneers of this approach which has been extensively applied to analyze optimal harvesting in a synthetic manner.

During years many authors have considered that models with explicit age structure are very convenient for practical management problems but intractable analytically to analyzed optimal management issues (Wilen (1985), Quinn and Deriso (1999) or Hilborn and Walters (2001)). Very recently, it has

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[1] A very good reflection of the past and future of population dynamics models in fisheries can be found in Quinn (2003).

[2] Some authors denominates this the surplus approach.

been shown that optimal harvesting may be different if age-structured information is ignored and optimization is based on traditional biomass variables. For instance, for the widow rockfish, Atlantic menhaden and Pacific halibut fisheries Tahvonen (2008) shows that avoiding age-structure information and applying biomass models may lead to major deviations between expected and actual outcomes especially under multiple steady states and nonlinearities. This result may appear contradictory at first glance with Moxnes (2005) results. Moxnes(2005) analyzes the Northeast Arctic cod fishery in the Barents Sea and he finds that a simple aggregated biomass model and a more complex cohort model lead to quite similar quota policy recommendations. But as Tahvonen (2008) points out Moxnes' results are highly influenced by the fact that in his age-structured model, optimal harvest is restricted to be a linear function of aggregate biomass. This *ad hoc* harvesting policy may remove some essential differences between the two approaches and cannot recognize problems such as growth and recruitment overfishing.

As far as we know Kulmala, Laukkanen and Michielsens (2008) and Tahvonen (2009) are the first successful attempts to analyze optimal harvesting in age-structured framework. Kulmala, Laukkanen and Michielsens (2008) solve numerically the optimal harvesting for the age-structure population of the Atlantic salmon fishery in the Baltic Sea using Bellman's (1957) principle of optimality. However, Tahvonen (2009) characterizes optimal harvesting in a generic age-structured model. His main finding is that given two age classes, knife-edge selectivity, and no stock-dependent harvesting cost, the steady state is a unique saddlepoint. He also proves that under specific conditions it may appear a stationary cycle that represents pulse fishing. Furthermore, for conditions such as low interest rate and knife-edge selectivity, he shows that optimal harvesting converges toward a unique saddle point independently of the number of age classes. This study opens lines of research which may be very fruitful in shedding light on many unanswered questions related to fisheries management.

Our work is on the same line that Tahvonen (2009). We aim to characterize dynamic optimal harvesting that maximizes discounted utility assuming a stochastic age-structured framework which has been extensively used for fish stock assessment. The population model we assumed is based on Baranov's catch equation (1918) and can be considered the common element of all Virtual Population Analysis (VPA) methods[3]. At first sight this choice

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[3] Virtual population analysis is a general method for fish stock assessment that it was introduced by Gulland (1965) based on older works. At present it is widely used. For instance the USA and Canada use the Adaptive Framework (ADAPT) that is a VPA variety based on minimizing the sum of squares over any number of indices of abundance to find best fit parameters. However the European Commission relies on the Extended

can be seen as a restriction in the sense that some particular specification is introduced in the age-structured model. However this allows us to compare optimal harvesting in a discounted economic context with standard reference points used by fisheries agencies for long term management plans (e.g. F_{msy} or F_{pa}).

The main findings of our research are the following. First, optimal steady state is characterized. We demostrate sufficient conditions guaranteeing its existence and uniqueness for the general case of n cohorts. It is also proved that the optimal steady state coincides with the traditional target F_{msy} when the utility function to be maximized is the yield and the discount rate is zero. Second, an algorithm to calculate the optimal path that drives the resource to the steady state in an easy manner is introduced. Finally, the algorithm is applied in the search for optimal harvesting in the Northern Stock of hake. Results show that management plans based exclusively on traditional reference targets as F_{msy} may drive fishery economic results far from the optimal.

The paper proceeds as follows. In the next section the age-structured population model is presented and stationary population structures are analyzed. Section 3 shows the traditional reference points used by stock assessment methods associated to the age-structured model. In particular, F_{msy} is characterized. Section 4 shows optimal harvesting in a discounted utility framework assuming a stochastic age-structure dynamics for the population. Subsection 4.1 characterizes the optimal stationary path and Subsection 4.2 presents an algorithm that derives numerically the optimal trajectories. The algorithm is applied to the Northern Stock of hake in Section 5 and optimal trajectories are compared with traditional management plans. Finally, Section 6 concludes the paper with a policy recommendation discussion.

2 A Stochastic Age-Structured Model

Age-structured models are the common population structure used in VPA for fish stock assessment. The population structure is applied to a group of fish that has the same life cycle, similar growth rates and can be considered a single biological unit. This unit stock is broken in cohorts, i.e. in groups of fish that have the same age and probably the same size, weight and will mature at the same time.

Lets assume that the fish stock is broken into A cohorts. That is in each period t, there are $A-1$ initial old cohorts and a new cohort is born. Let

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Survivor Analysis (XSA) which is a method that does not include biomass indices for fitting the VPA. See Lassen and Medley (2000) for an extensive survey on VPA.

z_t^a be the mortality rate that affects to the population of fish in the a^{th} age during the t^{th} period. This mortality rate can be decomposed into fishing mortality, F_t^a, and natural mortality (non-human predation, disease and old age), m^a,

$$z_t^a = F_t^a + m^a.$$

While fishing mortality rate may vary throughout periods and ages, natural mortality is constant among periods. Moreover, it is assumed that the fishing mortality over each age is given by stationary selection patterns, p^a, i.e.

$$F_t^a = p^a F_t.$$

Suppose that fish population is continuous and the mortality rate acts on the fish stock continuously throughout the period. Then the size of a cohort varies according to

$$N_{t+1}^{a+1} = e^{-z_t^a} N_t^a, \tag{1}$$

where N_t^a is the number of fish in the a^{th} age at the beginning of the t^{th} period.

The size of a new cohort (recruitment), N_{t+1}^1 , depends on the spawning stock biomass of the previous year, SSB_t,

$$N_{t+1}^1 = \Psi(SSB_t), \tag{2}$$

where Ψ denotes the stock recruits (S-R) relationship. Moreover, the spawning stock biomass, SSB, is a function of the stock weight distribution, ω, and the maturity fraction, μ, of each age,

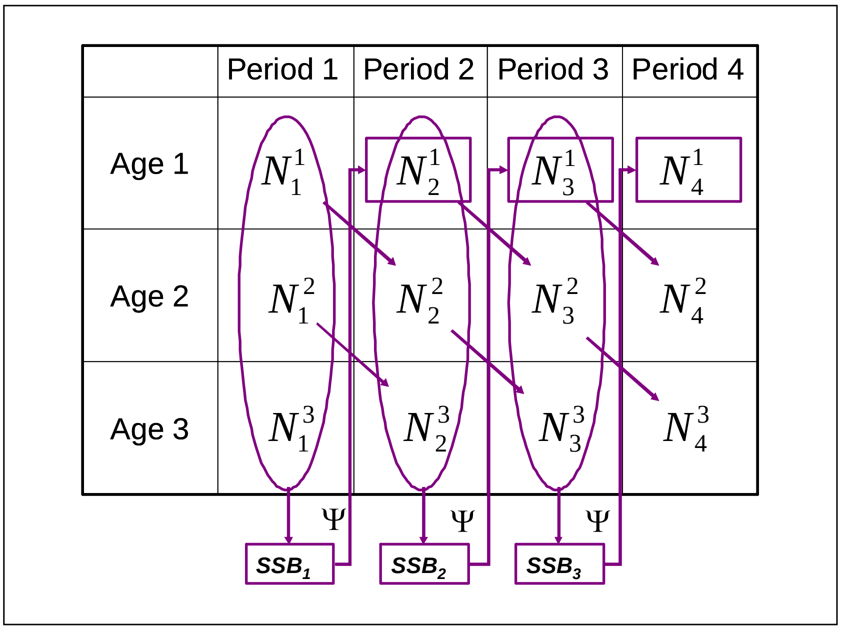
$$SSB_t = \sum_{a=1}^{A} \mu^a \omega^a N_t^a. \tag{3}$$

Finally, the oldest age group is assumed to be a true age group, i.e. $N_{t+1}^{A+1} = 0$.

Figure 1 illustrates the cohorts in a matrix where ages forming the rows and years forming the columns. The diagonal arrows represent the cohort dynamics shown in equation (1). The figure also shows the interconnections between the SSB of a period and next period recruitment given by equations (2) and (3).

Notice that if the initial (final) age distribution of the fish stock is given, $N_0^a \quad \forall a$, and the path of the mortality rate, $\{z_t^a\}_{t=0}^{\infty} \ \forall a$, is known, then the evolution of the the fish age structure can be calculated forward (backwards) using (1), (2) and (3).

Figure 1: Fishery Age Structured Model: Cohort Dynamics



Let D_t^a and C_t^a denote the number of fish dying from natural causes and from fishery (catches), respectively. Then the dynamics of the cohort can be expressed as

$$N_t^a - N_{t+1}^{a+1} = D_t^a + C_t^a.$$

Taking into account equation (1) and the definitions of natural and fishing mortality, D_t^a and C_t^a can be expressed as

$$\begin{aligned} D_t^a &= \frac{m^a}{z_t^a}\left(N_t^a - N_{t+1}^{a+1}\right) = \frac{m^a}{z_t^a}\left(1 - e^{-z_t^a}\right) N_t^a, \\ C_t^a &= \frac{F_t^a}{z_t^a}\left(N_t^a - N_{t+1}^{a+1}\right) = \frac{F_t^a}{z_t^a}\left(1 - e^{-z_t^a}\right) N_t^a. \qquad (4) \end{aligned}$$

This last equation is known as the Baranov catch equation (Baranov (1918)).

Generally in stock fishery assessment, catch data, C_t^a, is available for the different age classes. With this information and assuming that the natural mortality rate, m^a, is known and constant, equations (1) and (4) can be used to solve for the size of the fish stock, N_t^a, and the fishing mortality, F_t^a, from $a = 1, ...A$ age classes and from $t = 1, ...T$ periods. This can be forward solved when the initial distribution of the age structure is taken as given or backward if the final age population is known.

In most empirical studies, stock estimations generated by VPA methods are used as initial age distribution. Due to this, the population model we propose includes uncertainty about the initial age distribution and recruitment. In particular, the following lognormal distributions are used to describe the initial conditions of the state and the future number of recruits

$$N_0^a = e^{\sigma_a \varepsilon_a} EN_0^a, \quad \forall a, \tag{5}$$

$$N_{t+1}^1 = e^{\sigma_1 \varepsilon_t} \Psi(SSB_t), \quad \forall, \tag{6}$$

where ε_a is a random variable affecting the initial size of cohort of age a that follows a normal distribution with mean 0 and standard deviation σ_a. ε_t is a random variable affecting the size of recruitment period $t+1$ that follows a normal distribution with mean 0 and standard deviation σ_1. Therefore the mean of the initial distribution is given by EN_0^a and the mean of recruitment in period $t+1$ is $\Psi(SSB_t)$.

2.1 Stationary Population Structure

In a deterministic scenario, for example assuming $\sigma_a = 0, \quad \forall a$, a stationary path of fishing mortality, $F = F_t = F_{t-1}$, generates a stationary age structured population characterized by

1. The stock population for each age $a = 1, ...A$, is given by

$$N^a = N^1 \phi^a(F), \tag{7}$$

   where

$$\phi^a(F) = \begin{cases} 1 & \text{for } a = 1, \\ \Pi_{i=1}^{a-1} e^{-p^i F - m^i} & \text{for } a = 2, ...A, \end{cases}$$

   can be interpreted as the the accumulated probability of a recruit to reach age a for that stationary fishing mortality rate F.

2. The stationary recruit population N^1 satisfies the stationary S-R relationship, that is

$$N^1 = \Psi\left(\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F) N^1\right).$$

In order to guarantee the existence of a unique stationary population associated to a stationary fishing mortality, F, it is necessary the S-R relationship be invertible. This is,

$$\frac{\Psi^{-1}[N^1]}{N^1} = \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F).$$

This property is satisfied by the S-R relationship proposed by Shepherd (1982),

$$N^1 = \frac{\alpha SSB}{1 + \left(\frac{SSB}{K}\right)^b}, \tag{8}$$

where parameter $\alpha > 0$ is the maximum recruitment attainable when the SSB is very low, $K > 0$ is a threshold of SSB below which the likelihood of population collapse is increased and $b > 0$ measures the power of the density-dependent effects[4]. This S-R relationship is a mimic of those proposed by Cushing (1973), Beverton and Holt (1957) and Ricker (1954) for the cases of b greater, equal and lower than unity, respectively. The following proposition characterizes recruitment as a function of the fishing mortality in the stationary population structure for the case of the Shepherd S-R relationship.

Proposition 1 *For the Shepherd S-R relationship, (8), the stationary recruitment is given by*

$$N^1 = K \frac{\left(\alpha \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F) - 1\right)^{1/b}}{\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)},$$

which is well defined and unique whenever fishing mortality (SSB per recruit) is lower (higher) enough and $\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F) > 1/\alpha$. Furthermore,

i) If $b \leq 1$, $\partial N^1 / \partial F < 0$.

ii) If $b > 1$, $\partial N^1 / \partial F \gtreqless 0$ for values of F such that $\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F) \gtreqless b / \alpha (b-1)$.

Proof. See Appendix.■

Figure 2 illustrates the relationships between the SSB, recruitment and fishing mortality for the Northern Stock of Hake[5]. The right plot shows the relationship between recruitment and fishing mortality. We observe a bell shape which is consistent with Proposition 1 given that we calibrate $b = 1.7602$ for this fishing ground[6]. The left plot illustrates the relationship between SSB and recruitment associated to different stationary fishing mortality rates. Notice that the value of SSB per recruit may be displayed as a straight line through the origin on the plot of recruitment against SSB.

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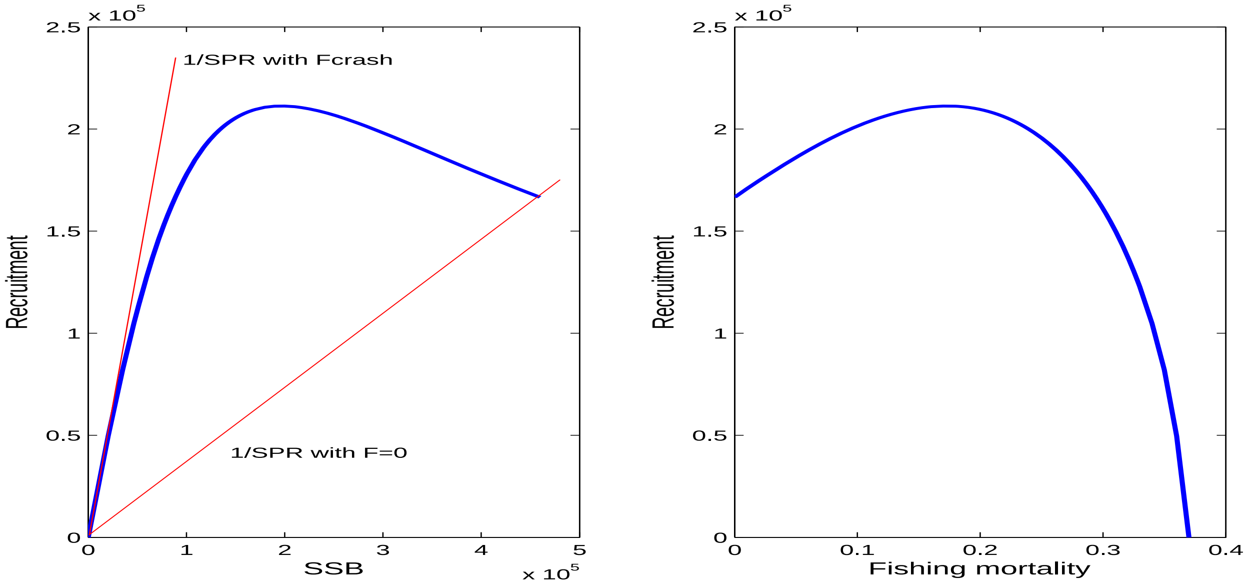
[4] We expect different values for b for stocks where density dependence is due to cannibalism, compared with those where it is due to competition for limited resources.

[5] In Section 5 we show in detail how the S-R relationship is calibrated for this fishery.

[6] Notice that $\partial \phi^a(F) / \partial F < 0$. Then, low (high) values of F are associated to high (low) values of $\phi^a(F)$ and therefore $\partial N^1 / \partial F > 0$ (<0).

Figure 2: SSB, SSB per recruiment and Fishing Mortality in the stationary population



The slope of such a line is just the reciprocal of the value of SSB per recruit (indicated as 1/SPR). Therefore there is a one-to-one correspondence between SSB per recruit and fishing mortality, F, so that the SSB per recruit lines may be labelled with the appropriate values of F. Notice that fishing mortality is bounded between not fishing at all ($F = 0$) and that maximum value, F_{crash}, that leads to the minimum SSB per recruitment[7].

3 Management Reference Points

Fishery management advice consists of the evaluation of scenarios associated to a desired stock status defined through reference points. These reference points are used to indicate changes in management controls to improve the status of the stock. Reference points fall into two groups: limit reference points and target reference points.[8]

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[7] F_{crash} is the fishing mortality rate that satisfies $SSB/N^1 = \sum_{a=1}^{A} \mu^a \omega^a \phi^a (F_{crash}) = 1/\alpha$. In this fishery $F_{crash} = 0.37$.

[8] See Caddy and Mahon (1995) for an extensive survey on reference points used in fisheries management.

Limit reference points are limits on the exploitation that should not be approached. It may be either some minimum levels as for example a dangerously low spawning biomass or some maximum conditions such as a maximum fishing mortality rate.

Target reference points are levels of exploitation which are considered to be desirable and at which management action should aim. The implicit fishing mortality target of many regional and national fishery management authorities and organizations is F_{msy}. The FAO Fishery Glossary defines F_{msy} as the fishing mortality rate which, if applied constantly, would result in Maximum Sustainable Yield. Other targets reference points are $F_{\max}$, $F_{0.1}$ or F_{med} which are applied in different contexts. For instance, $F_{\max}$ is defined as the fishing mortality rate that maximizes equilibrium yield per recruit and is used as a target in those fisheries in which the S-R relationship is not well defined. In some occasions it is not even clear where the maximum of yield per recruit is. For in these cases it is considered the $F_{0.1}$ which is the fishing mortality rate at which the slope of the yield per recruitment is 10% of its value at the origin. F_{med} corresponds to the fishing rate that guarantees the inverse of the median observed survival ratio (ratio of recruits to SSB). A fishery exploited continually at F_{med} should be able to replace itself with an abundance close to the observed historical median.

In general, long term fisheries management plans use a combination of limit and target reference points.[9] For example, ICES advice is based on stationary sustainable yield where F_{msy} is considered the main target reference point to be reached by the fishery in the long term as long as the SSB is above the limit reference point B_{pa}, where subindex pa stands for precautionary approach. However, for fishing grounds under recovery plans ICES target reference point is F_{pa} which is the fishing mortality that guarantees that the SSB reaches the SSB_{pa} level.

The next subsection describes how to obtain F_{msy} and F_{pa} in the age structured model described above.

3.1 F_{msy}

Among all the fishing mortality stationary paths, F_{msy} can be defined as the fishing mortality rate where the stationary yield is at its maximum. Formally,

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[9] Alternatively the limit reference points can be considered only as a management tool. The pulse fishing approach consists of allowing fishing in unregulated area as a limit reference point is reached. Then fishing is stopped, when the ecosystem recovers, fishing can begin again. See Quinn *et al.* (1990)

F_{msy} is the mortality rate that maximizes

$$\begin{aligned} \max_{\{F,N^1\}} \sum_{a=1}^{A} \omega^a C^a &= \sum_{a=1}^{A} y^a(F)\phi^a(F)N^1, \\ s.t. \quad N^1 &= \Psi\left(N^1 \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)\right), \end{aligned} \tag{9}$$

where $y^a(F) = \omega^a \frac{F}{p^a F + m}\left(1 - e^{-p^a F - m^a}\right)$. Notice that this problem maximizes a combination of the yield per recruit $\sum_{a=1}^{A} y^a(F)\phi^a$ and the SSB per recruit, $\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)$ through the S-R relationship, $\Psi()$.

The solution of problem (9) is characterized by the following first order conditions,

$$\sum_{a=1}^{A} \frac{\partial y^a(F)}{\partial F}\phi^a(F)N^1 + \sum_{a=1}^{A} y^a(F)\frac{\partial \phi^a(F)}{\partial F}N^1 + \lambda\Psi'(.)N^1 \sum_{a=1}^{A} \mu^a \omega^a \frac{\partial \phi^a(F)}{\partial F} = 0, \tag{10}$$

$$\sum_{a=1}^{A} y^a(F)\phi^a(F) - \lambda[1 - \Psi'(.)\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)] = 0, \tag{11}$$

$$N^1 = \Psi\left(N^1 \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)\right), \tag{12}$$

where $\Psi'(.)$ is valuated at $N^1 \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)$ and λ is the associated Lagrange multiplier that measures the impact of new recruits on the total yield.

Substituting equation (11) into (10) and taking into account total differentiation of (12), the following expression is obtained,

$$\sum_{a=1}^{A} \frac{\partial y^a(F)}{\partial F}\phi^a(F) + \sum_{a=1}^{A} y^a(F)\frac{\partial \phi^a(F)}{\partial F} + \frac{1}{N^1(F)}\frac{\partial N^1(F)}{\partial F}\sum_{a=1}^{A} y^a(F)\phi^a(F) = 0, \tag{13}$$

where

$$\frac{\partial N^1(F)}{\partial F} = \frac{N^1\Psi' \sum_{a=1}^{A} \mu^a \omega^a \frac{\partial \phi^a(F)}{\partial F}}{1 - \Psi' \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)}. \tag{14}$$

Equation (13) determines F_{msy} and its interpretation is clear. Variations on the fishing mortality rate, affect the stationary yield through three elements: the weighted catches (first sum), the accumulated probability of a recruit to reach age a (second sum) and the number of recruits (third sum). F_{msy} is chosen so that the three sources of variations cancel out.

Notice that an increase of the mortality rate has an ambiguous effect over the stationary recruits. In particular, this effect is negative (positive) whenever an increase in the number of recruits leads to a lower (larger) increase in the SSB.

The following statements characterize the existence and uniqueness of F_{msy}.

Lemma 1 *Yield per abundance,* $y^a(F) = \omega^a \frac{F}{p^aF+m^a}\left(1 - e^{-p^aF-m^a}\right)$, *is increasing in* F *and concave in* F *for any age,* a.

Proof. See Appendix.■

Lemma 2 *If* $\frac{m^ap^a}{F(p^aF+m^a)}+\frac{p^{a\,2}e^{-p^aF-m^a}}{1-e^{-p^aF-m^a}} > \sum_{k=1}^{a-1}(p^k)\ \ \forall a$, *then* $\sum_{1=2}^{A} y^a(F)\phi^a(F)$ *is increasing and concave in* F *for any age,* a

Proof. See Appendix.■

Lemma 3 *Suppose that* Ψ *is an invertible function. Then for each* F *there exists a unique* N^1, *that satisfies* $N^1 = \Psi(\sum_{a=1}^{A} \mu^a\omega^a\phi^a(F)N^1)$. *Moreover, if* $0 < \Psi' < \sum_{a=1}^{A} \mu^a\omega^a\phi^a(F)$ *then* N^1 *is decreasing in* F;*otherwise* N^1 *is increasing in* F.

Proof. Straightforward from (14) taking into account that $\partial\phi^a(F)/\partial F = \sum_{i=1}^{a-1}(-p^i)\phi^a(F) < 0$.■

Lemma 4 *If* $\sum_{1=2}^{A} y^a(F)\phi^a(F)$ *is increasing and concave in* F *and* N^1 *is decreasing and concave in* F *then* $\sum_{1=2}^{A} y^a(F)\phi^a(F)N^1$ *is concave in* F.

Proof. See Appendix.■

Proposition 2 *If i)* $\sum_{1=2}^{A} y^a(F)\phi^a(F)N^1$ *is concave in* F *and ii) For each* F *there exists a unique* N^1, *that satisfies* $N^1 = \Psi(\sum_{a=1}^{A} \mu^a\omega^a\phi^a(F)N^1)$, *then there exists a unique* F_{msy} *that solves the maximization problem (9)*

Proof. Condition *ii)* guarantees that the set of F over which the objective function is maximized is compact. Since the objective function is continuous, the maximization problem has at least a solution. Furthermore, since $\sum_{1=2}^{A} y^a(F)\phi^a(F)N^1$ is concave, the maximum is unique.■

Corollary 1 *If i)* $\frac{m^ap^a}{F(p^aF+m^a)}+\frac{p^{a\,2}e^{-p^aF-m^a}}{1-e^{-p^aF-m^a}} > \sum_{k=1}^{a-1}(p^k)\ \ \forall a$ *and ii)* Ψ *is an invertible function such that* $0 < \Psi' < \sum_{a=1}^{A} \mu^a\omega^a\phi^a(F)$, *then there exists a unique* F_{msy} *that solves the maximization problem (9).*

Proof. Straightforward for Lemmas 2, 3, 4 and Proposition 2.■

Notice that the above corollary shows sufficient conditions to guarantee the existence and uniqueness of F_{msy}. However, we may find a well defined F_{msy} in a different context. For instance, for the Northern Sock of Hake we find that recruitment shows a bell shape against fishing mortality (see Figure 2). In spite of the fact that N^1 is not decreasing for all F, however F_{msy} is found without major calculation problems (see section 5).

3.2 F_{pa}

F_{pa} is the fishing mortality rate that guarantees that the SSB reaches a particular level SSB_{pa}. Formally, F_{pa} is the fishing level that generates stationary recruitment N^1_{pa} such that the following set of equation holds

$$\begin{aligned} N^1_{pa} &= \Psi\left(N^1_{pa}\sum_{a=1}^{A}\mu^a\omega^a\phi^a(F_{pa})\right), && (15)\\ SSB_{pa} &= N^1_{pa}\sum_{a=1}^{A}\mu^a\omega^a\phi^a(F_{pa}). && (16)\end{aligned}$$

4 Optimal Management

Most of the reference points used for long term management plans are based on finding stationary mortality rates that maximize yield implied by the stationary population structure (F_{msy}). An alternative to fisheries management based on this kind of reference points is to look for optimal trajectories associated to some economic and/or biological criteria. In this section we show how to characterize the optimal harvesting trajectories for different economic criteria.

Lets assume that our aim is to find for a given discount factor[10], β, the optimal path of fishing mortality, $\{F_t\}_{t=0}^{\infty}$, that maximizes the expected present value of discounted profits of the fishery taking into account that the spawning stock biomass is always greater than the precautionary level, SSB_{pa}, and the dynamics described by equations (1) to (4).

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[10] Most of the times discount is introduced in fisheries economics using the discount rate, r, instead of discount factor, β. The former uses to be applied in continuous time frameworks while the latter is more commonly used in discrete set up. The inverse relationship between both terms is given by $\beta = (1+r)^{-1}$.

Formally, the optimal management path is the solution to the following discounted maximization problem

$$\max_{\left\{F_t, N_{t+2}^a\right\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{a=1}^{A} pr^a y^a(F_t) N_t^a - TC(F_t) \right\},$$

$$s.t. \begin{cases} N_{t+1}^{a+1} = e^{-z^a(F_t)} N_t^a \quad \forall t \;\; \forall a = 1, A-1 \\ \\ N_{t+1}^1 = e^{\sigma_1 \varepsilon_1} \Psi \left(\sum_{a=1}^{A} \mu^a \omega^a N_t^a \right) \quad \forall t \\ \\ N_0^a = e^{\sigma_a \varepsilon_a} E N_0^a, \;\; \forall a, \qquad N_0^a \text{ given}, \\ \\ SSB_{pa} \leq \sum_{a=1}^{A} \mu^a \omega^a N_t^a \quad \forall t \end{cases} \tag{17}$$

where $y^a(F_t) = \omega^a \frac{F_t}{p^a F_t + m} \left(1 - e^{-p^a F_t - m^a}\right)$ and pr and TF represent the price and the total cost function which depends positively on fishery mortality and it is convex, respectively. E_0 is the expectation operator conditioned on the information at period 0.

Notice the versatility of the objective function maximized in problem (17). It can be interpreted in several ways. For instance if $pr = 1$ and the marginal cost is zero, the objective function represents the present value of yield. When the marginal cost is zero and $pr \neq 1$, the objective function coincides with the revenues of the fishery. In the case of $pr \neq 1$, marginal cost different from zero and total cost equal to the cost of oil and other running costs, the objective function is equal to the added value of the yield. Finally if the total cost also includes the labor cost, then the objective function can be understood as the profits of the fishery.

By backwards substitution in the first restriction, the size of cohort age $a > 1$ in period t, N_t^a, can be expressed as a function of the past mortality rates and initial recruitment,

$$N_t^a = e^{-z_{t-1}^{a-1}(F_{t-1})} N_{t-1}^{a-1} = e^{-z_{t-1}^{a-1}(F_{t-1})} e^{-z_{t-2}^{a-2}(F_{t-2})} N_{t-2}^{a-2} = ... = \Pi_{i=1}^{a-1} e^{-z_{t-i}^{a-i}(F_{t-i})} N_{t-(a-1)}^1.$$

Therefore we can express N_t^a as

$$N_t^a = \phi_t^a N_{t-(a-1)}^1, \qquad \text{for } a = 1, ...A, \tag{18}$$

where

$$\phi_t^a = \phi(F_{t-1}, F_{t-2}, ...F_{t-(a-1)}) = \begin{cases} 1 & \text{for } a = 1, \\ \Pi_{i=1}^{a-1} e^{-z_{t-i}^{a-i}(F_{t-i})} & \text{for } a = 2, ...A, \end{cases}$$

can be understood as the survival function that shows the probability of a recruit born in period $t - (a-1)$ to reach age $a > 1$ for a given fishing mortality path $\left\{F_{t-1}, F_{t-2}, ...F_{t-(a-1)}\right\}$. Notice that the survival function in any period depends upon the $a - 2$ next past mortality rates.

After substituting the survival function (18), the maximization problem (17) can be rewritten as

$$\max_{\left\{F_t, N^1_{t+2}\right\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ pr^1 y^1(F_t) N^1_t + \sum_{a=2}^{A} pr^a y^a(F_t) \phi^a_t N^1_{t+1-a} - TC(F_t) \right\},$$

$$s.t. \begin{cases} N^1_{t+1} = e^{\sigma_1 \varepsilon_1} \Psi \left(\mu^1 \omega^1 N^1_t + \sum_{a=2}^{A} \mu^a \omega^a \phi^a_t N^1_{t+1-a} \right) \quad \forall t, \\ N^a_0 = e^{\sigma_a \varepsilon_a} E N^a_0, \ \ \forall a, \qquad N^a_0 \text{ given}, \\ SSB_{pa} \leq \mu^1 \omega^1 N^1_t + \sum_{a=2}^{A} \mu^a \omega^a \phi^a_t N^1_{t+1-a} \ \ \forall t. \end{cases} \tag{19}$$

In the appendix we show how to find the first order conditions that solve this problem. Formally, the optimal paths can be characterized by the following set of dynamic equations

$$\sum_{a=1}^{A} pr^a \frac{\partial y^a(F_t)}{\partial F_t} N^a_t - \frac{\partial TC_t}{\partial F_t} = E_t \sum_{a=1}^{A-1} p^a \left\{ \sum_{j=1}^{A-a} \beta^j \left[pr^a y^{a+j}(F_{t+j}) + (\Psi'_{t+j} \lambda_{t+j} + \theta_{t+j}) \mu^{a+j} \omega^{a+j} \right] N^{a+}_{t+j} \right. \tag{20}$$

$$E_t \sum_{a=1}^{A} \beta^a pr^a y^a(F_{t+1+a}) \phi^a_{t+1+a} = \lambda_{t+1} - E_t \sum_{a=1}^{A} \beta^a (\Psi'_{t+1+a} \lambda_{t+1+a} + \theta_{t+1+a}) \mu^a \omega^a \phi^a_{t+1+a}, \tag{21}$$

$$N^{a+1}_{t+1} = e^{-z^a(F_t)} N^a_t, \qquad \forall t \quad \ \forall a = 1, A-1 \tag{22}$$

$$N^1_{t+2} = \Psi \left(\sum_{a=1}^{A} \mu^a \omega^a N^a_{t+1} \right), \qquad \forall t \tag{23}$$

$$\theta_{t+1} \left[\sum_{a=1}^{A} \mu^a \omega^a N^a_{t+1} - SSB_{pa} \right] = 0, \qquad \forall t \tag{24}$$

where λ_t and θ_t are the Lagrange multiplier associated to the first and third restriction of the maximization problem (19), respectively.

Condition (20) shows how the mortality rate, F_t, is selected. The insight is the following. In the optimal path, an increase in current mortality rate leads

Table 1: Age Structure and the Intertemporal Maximization Problem

	t	t+1	t+2	...	t+A-2	t+A-1	t+A	t+A+1
a=1	N_t^1		N_{t+2}^1	...				
a=2	N_t^2	N_{t+1}^2		...	N_{t+A-2}^2			
...	...	...	...	...		...	...	...
a=A-1	N_t^{A-1}	N_{t+1}^{A-1}	N_{t+2}^{A-1}	...	N_{t+A-2}^{A-1}		N_{t+A}^{A-1}	
a=A	N_t^A	N_{t+1}^A	N_{t+2}^A	...	N_{t+A-2}^A	N_{t+A-1}^A		N_{t+A+1}^A

to an increase in current fishery profits (left hand side) that is compensated with the decrease of future profits derived from reductions in future stock (right hand side). In particular, the left hand side represents the effects of changes in fishing mortality over the current profit of the fishery. However, the right hand size shows the effect on the future size of the alive cohorts, $t+1$ to $t+A-1$ (first sum) and on the future stock recruitments from periods $t+2$ to $t+A$ (second sum). This can be visualized also looking at age structure in Table 1. The left hand side represents the effects of F_t on the structure of the fishery in period t (column t). The first sum of the right hand side shows the effects of F_t on the structure of future size of the alive cohorts (lower triangle matrix) and the second sum illustrates the effects of F_t on future stock recruitments (row $a=1$).

Equation (21) indicates that the optimal path recognized that the effects of an increase in the stock recruitment, N_{t+2}^1, is two fold. On the one hand, the abundance in periods $t+2$ to $t+2+A-1$ goes up and this leads to an increase in catches (left hand side). On the other hand, the SSB for periods $t+3$ to$t+3+A-1$ also increase, (right hand side).

Equations (22) and (23) show the dynamics of the population cohorts. Finally, equation (24) indicates if SSB is under the precautionary level, SSB_{pa}. The Lagrange multiplier θ_t shows the effects over the mortality if the precautionary principle is not binding. If at some period t, the SSB is below the precautionary level, SSB_{pa}, then θ_t indicates how much we should modify the fishing mortality rates between periods $t-A$ and $t-1$.

4.1 Optimal Stationary Solution

If the precautionary restriction is not binding, $\theta = 0$. In this context, a stationary solution is defined as an optimal solution characterized by a vector $(F_{ss}, N_{ss}^1, N_{ss}^2, ...N_{ss}^A, \lambda_{ss})$ such that for any future period t

$$\begin{aligned} F_{ss} &= F_t = F_{t+1}, \\ N_{ss}^a &= N_t^a = N_{t+1}^a, \qquad \forall a = 1,..,A, \\ \lambda_{ss} &= \lambda_t = \beta^j \lambda_{t+j}, \qquad \forall j = 1,..,A+1. \end{aligned}$$

The first order conditions (20)-(21) valued at the steady sate can be written as the following $A+2$ equation system,

$$\sum_{a=1}^{A} pr^a \frac{\partial y^a(F_{ss})}{\partial F_{ss}} N_{ss}^a - \frac{\partial TC}{\partial F_{ss}} = \sum_{a=1}^{A-1} p^a \left\{ \sum_{j=1}^{A-a} \left[\beta^j pr^a y^{a+j}(F_{ss}) + \Psi' \lambda_{ss} \mu^{a+j} \omega^{a+j} \right] N_{ss}^{a+j} \right\}, \tag{25}$$

$$\sum_{a=1}^{A} \beta^{1+a} pr^a y^a(F_{ss}) \phi^a(F_{ss}) = \lambda_{ss} \left[1 - \Psi' \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F_{ss}) \right], \tag{26}$$

$$N_{ss}^{a+1} = e^{-z^a(F_{ss})} N_{ss}^a, \qquad \forall a = 2,..,A, \tag{27}$$

$$N_{ss}^1 = \Psi \left(\sum_{a=1}^{A} \mu^a \omega^a N_{ss}^a \right), \tag{28}$$

where the expectational operator does not appear because all random variables are independently distributed.

Notice that using the survival functions (18) valued in the steady state, $N_{ss}^a = \phi^a(F_{ss}) N_{ss}^1$, this $A+2$ equation system can be reduced to a 3 equation system that solves out $(F_{ss}, N_{ss}^1, \lambda_{ss})$. Once N_{ss}^1 is known the cohort size of any age can be calculated using the survival function.

The next proposition shows sufficient conditions that guarantee the existence of the F_{ss}. Furthermore, we show that the relationship between fishing mortality and discount factor is negative. Furthermore, it is proven that the relationship between recruitment and discount factor is positive. The significance of these two facts is clear. The more impatient we are, the less we care about the future (i.e. the lower is the discount factor, β). And this leads to a higher stationary fishing mortality and lower recruitment.

Proposition 3 *If i) $\sum_{1=2}^{A} y^a \phi^a$ is increasing in F, ii) For each F there exists a unique N^1, that satisfies $N^1 = \Psi(\sum_{a=1}^{A} \mu^a \omega^a \phi^a(F) N^1)$ and such that N^1 is decreasing and concave in F, and iii) For $F = 0$ the marginal profit per recruitment is positive, then* *a) There exists a unique F_{ss} that solves the optimal stationary conditions (25) to (28).*

b) F_{ss} depends negatively on β.

c) N^1_{ss} depends positively on β.

Proof. See Appendix. ■

An important issue to be analyzed is the relationship between the solutions of the discounted maximization problem (17) and the stationary maximization problems (9). We can prove that the optimal stationary mortality rate, F_{ss}, is just a generalization of F_{msy}. In particular, we show that F_{msy} coincides with F_{ss} for the case of in which future is not discounted and all periods are treated equally. The following proposition formalize this result.

Proposition 4 *If $\beta = 1$, $pr^a = 1$ and $\partial TC/\partial F = 0$, then $F_{ss} = F_{msy}$.*

Proof. See Appendix.■

This result may become relevant because only when the future is not discounted the F_{msy} is a good target reference point in terms of guarantee that present value of yield is at its maximum.

Finally, it may be that if the future is significatly discounted and we only care the immediately following periods, the optimal stationary fishing mortality may be so high that the SSB may fall below the precautionary level. In this case, the stationary solution is given by the biological conditions of SSB_{pa}. Notice that the more the future is discounted, the higher the likelihood this situation will happen.

The following proposition sets the optimal stationary solution for the case in which this corresponds to the corner solution associated to SSB_{pa} .

Proposition 5 *Let β_{pa} the discount factor for which the optimal stationary fishing mortality solving (25) to (28) generates a SSB equal to SSB_{pa}. Then for any discount factor $\beta < \beta_{pa}$ the optimal stationary fishing rate that solves the discounted maximization problem (19) is given by F_{pa} characterized by (15)-(16). Furthermore, whenever $0 < \Psi' < \sum_{a=1}^{A} \mu^a \omega^a \phi^a(F)$, F_{pa} is the maximum optimal fishing rate among the optimal solutions generated by different β.*

Proof. See Appendix. ■

It is worth mentioning that the above proposition leads to similar results as those in Proposition 1 in Tahvonen (2009). Notice that if for a particular discount factor the optimal solution is F_{pa}, any reduction of the discount factor would imply an increase in the effort that leads to a SSB under SSB_{pa}; so F_{pa} continous to be the optimal fishing rate for the new discount rate. In summary as in Tahvonen (2009) we find a range of optimal solutions for which the optimal stationary fishing mortality does not depend on the discount factor.

4.2 Finding Numerical Optimal Trajectories

The optimal trajectories derived from maximization problem (17) or (19) are the optimal paths for $\{F_t\}_{t=1}^{\infty}$, $\{\lambda_t\}_{t=2}^{\infty}$, and $\left\{N_{t+2}^1\right\}_{t=1}^{\infty}$ that satisfy the infinity set of equations that characterizes the first order conditions (20) to (24).

To make tractable the computation of the optimal trajectories, we assume that these converge to the stationary solution, $(F_{ss}, N_{ss}^1, \lambda_{ss})$, in a finite number of periods, T. That is, we truncate the first order conditions using that $F_T = F_{ss}$, $N_{T+2}^1 = N_{ss}$, and $\lambda_{T+1} = \lambda_{ss}$. Taking into account this, solving the model consist of choosing F_1,F_1, F_2, ..., $F_T = F_{ss}$ such that the system of equations implied by the first order condition (20) is satisfied. This system of $(T-1)$ nonlinear equations with $(T-1)$ unknowns can be solved relatively quickly using standard numerical methods following the next algorithm.

1. Assume that the fishery is above the precautionary level. That is $SSB_t > SSB_{pa}$ $\forall t$, and therefore $\theta_t = 0$, $t = 2, .., T$.

2. Compute the stationary solution, $(F_{ss}, N_{ss}^1, \lambda_{ss})$.

3. Guess a trajectory for the fishing mortality rate path, $\{F_t\}_{t=1}^{T-1}$. And assume that in period T the stationary state has been reached, i.e. $F_T = ... = F_{T+A+1} = F_{ss}$.

4. Project the future age cohort structure for periods $1, ..., T+A+1$, $\{N_t^a\}_{t=1}^{T+A+1}$ using the mean of the initial age structure, EN_0^a, and the mean of the S-R relationship. In order to do this we use the cohort dynamic population (1), and the recruitment relationship, (2) and (3).

5. Compute Ψ_t' using the the recruitment relationship, (2) and (3), associated with $\{N_t^a\}_{t=1}^{T+A+1}$.

6. Using λ_{ss} compute λ_T from equation (21) valued at $t = T-1$,

$$\lambda_T = \sum_{a=1}^{A} \beta^a pr^a y^a(F_{ss})\phi_{ss}^a(F_{ss}) + \sum_{a=1}^{A} \Psi_{T+a}' \lambda_{ss} \mu^a \omega^a \phi_{ss}^a(F_{ss}).$$

Note that Ψ'_{T+a} is a function of N^1_{T+a} which depends on the guess $\{F_t\}_{t=1}^{T+a-2}$.

7. Given λ_T, compute backwards recursively $\{\lambda_{t+1}\}_{t=1}^{T-1}$, using equation (21).

8. Using the values of $\{\lambda_{t+1}\}_{t=1}^{T-1}$, the guess of $\{F_t\}_{t=1}^{T-1}$ and the cohort projections $\{N_t^a\}_{t=1}^{T+A+1}$, we can compute how far we are from the first order condition (20). Formally we calculate, $\forall t = 1, ...T$,

$$
\begin{aligned}
f_t = & \sum_{a=1}^{A} pr^a \frac{\partial y^a(F_t)}{\partial F_t} N_t^a - \frac{\partial TC_t}{\partial F_t} \\
& - \sum_{a=1}^{A-1} p^a \left\{ \sum_{j=1}^{A-a} \beta^j \left[pr^a y^{a+j}(F_{t+j}) + (\Psi'_{t+j}\lambda_{t+j} + \theta_{t+j})\mu^{a+j}\omega^{a+j} \right] N_{t+j}^{a+j} \right\}.
\end{aligned}
$$

   Using a Newton-Raphson algorithm, a new guess for the mortality rate path is obtained.

9. Repeat the procedure for step 1 to 8 until f_t is low enough.

10. Finally check that $\left\{ \sum_{a=1}^{A} \mu^a \omega^a N_t^a \right\}_{t=1}^{T} > SSB_{pa}$. If the restriction is not satisfied, we should guess a new set of positive[11] $\{\theta_t\}_{t=2}^{T}$.

Notice that optimal trajectories are not contingent to the shocks affecting the initial conditions. This is because we are assuming that when decisions are taken policy makers only know the mean of the initial population distribution. In the case of shocks affecting the initial distribution could be known when taking decisions, then the optimal trajectories should take them into account. In order to do this, it would be enough to use as initial population distribution the observed realization instead of the mean. Since at any period recruitment is affected by a shock, the optimal trajectory should recalculated each period once the drawn is known.[12]

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[11] In long-run management plans, the stock uses to be far from SSB_{pa}. So for those cases, the best initial guess is $\theta_t = 0$.

[12] This numerical method is known as the Model Predictive Control (see Garcia, Prett and Morari (1989) and Mayne, Rawlings, Rao and Scokaert (2000)).

Table 2: Parameters by age

	Initial conditions										
	Age 0	Age 1	Age 2	Age 3	Age 4	Age 5	Age 6	Age 7	Age 8	Age 9	Age 10
$N^{a\ (1)}$	186213	152458	123457	100213	67409	35551	19674	10206	9147	4078	1819
	Population dynamics										
	Age 0	Age 1	Age 2	Age 3	Age 4	Age 5	Age 6	Age 7	Age 8	Age 9	Age 10
p^a	0,00	0,06	0,54	1,15	1,03	1,52	2,09	2,43	2,43	2,43	2,43
$\omega^{a\ (2)}$	0.06	0.13	0.22	0.34	0.60	0.98	1.44	1.83	2.68	2.68	2.68
μ^a	0.00	0.00	0.00	0.23	0.60	0.90	1.00	1.00	1.00	1.00	1.00
	Stochastic shocks										
	Age 0	Age 1	Age 2	Age 3	Age 4	Age 5	Age 6	Age 7	Age 8	Age 9	Age 10
$\sigma logN$	0.200	0.200	0.166	0.086	0.061	0.063	0.076	0.084	0.084	0.084	0.084
	Prices										
	Age 0	Age 1	Age 2	Age 3	Age 4	Age 5	Age 6	Age 7	Age 8	Age 9	Age 10
pr^a	2.36	2.93	3.42	3.85	4.55	5.22	5.81	6.22	6.92	6.92	6.92

Source: Meeting on Northern Hake Long-Term Management Plans (STECF/SGBRE-07-03) and ICES Report (2007)
$^{(1)}$ Thousand;$^{(2)}$ kg ; $^{(3)}$ €/kg

5 The Northern Stock of Hake

In this section we illustrate the use of the algorithm developed to find the optimal trajectories for the Northern Stock of Hake[13]. In order to calibrate the age structured model for this fishery two data sources have been used. First, the information regarding the biological parameters of the fishery comes from expert working group meeting on Northern Hake Long-Term Management Plans (STECF/SGBRE-07-03) held in Lisbon, June 4-8 2007. Most of the parameters come from the summary of XSA results from the 2006 update (ICES (2007)). Secondly, economic data of the fishery emanate of expert working group meeting on Northern Hake Long-Term Management Plan Impact Assessment (STECF/SGBRE-07-05) held in Brussels, December 3-6 2007.

Table 2 shows, for each age, the number of fishes at the initial conditions, the parameters of the population dynamics (selection pattern, weight

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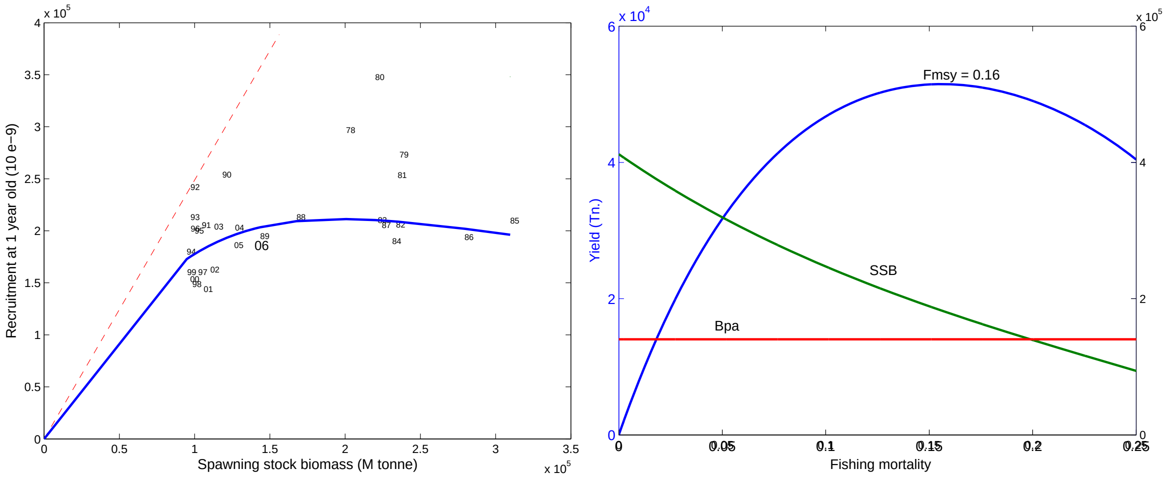
[13] STEFC advice for the Long Term Plan of this fishery was established comparing the benefits of gradual changes of the current level of fishing mortality to a new target, F_{max}, in steps 5% per year, 10% per year and 15% per year.

Figure 3: Calibration of the model under Shepherd S-R relationship



and maturity), the stochastic structure about the initial conditions and the prices[14]. Following Pontual, Groison, Piñeiro and Bertignac (2006) we consider that $A = 11$. The 8(plus) age-group is disaggregated assuming that the sum of the abundance of the new age-groups $(8- > 11)$ is equal to the 8 (plus) age-group. We also use the values of the STECF group for $SSB_{pa} = 140,000$ t.

As S-R relationship we use the Shepherd relationship (1982) described by (8). To calibrate this function recruitment and SSB data for the period 1978-2006 are used. Since parameter α represents the slope of the S-R relationship at the origin, it is calibrated as the maximum value of N_t^1/SSB_t. This calibration implies $\alpha = 2.4879$ which means that 2.4879 recruits are generated by each kilo of biomass in the case this is close to zero. Given α, the Shepherd S-R relationship implies $K = SSB/(\alpha SSB/N^1 - 1)^{1/b}$. In order to calibrate K and b, first for each possible value of b, we calculate the value of K that reproduces the mean of SSB and N^1. Then, among all the pairs of (b, K) we select the one that minimizes the sum of the square errors by comparing the model calibrated and the data. This calibration implies $K = 168270$ and $b = 1.7602$.

The left plot in Figure 3 displays the accuracy of the Shepherd S-R calibration. Also from this calibration of the S-R relationship, stationary yield

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[14] To calculate prices as a function of ages we have used data on 2007 daily sales for the trawl, gill nets and long lines Galician fleets.

Table 3: Economic Parameters Calibration

Cost structure		Macro magnitudes		
	Data per vessel		Data	Model
Fuel per day (€)	471.39	Landings (t)	54,889	54,889
Other costs per day (€)	444.48	Income (thousand €)	301,551	301,560
Total cost per day (€)	915.87	Total cost (thousand €)	73,576	73,576
Total days	80,335	Value Added (thousand €)	227,975	227,984
Total cost (thousand €)	73,576	Wages (thousand €)	120,620	120,624
Wages (thousand €)	120,620	Profits (thousand €)	107,355	107,360

Own calculations from the Spanish data fleet (2006) and French data fleet (2004)

and SSB can be calculated as functions of the stationary fishing mortality. Right plot in Figure 3 shows the shape of those relationships. We observe that the model implies as stationary targer reference points $F_{msy} = 0.156$.

Finally, the model uses the fact that catches were equal to 54,889 t in 2007 with a fishing mortality rate of $F_{sq} = 0.25$.This situation represents the so called *status quo*.

Table 3 illustrates the cost structure and the variables related with the output for the Northern Stock of Hake[15]. In the numerical simulations we assume that the cost of effort is proportional to the mortality rate, $TC = qF$, where $q = TC/F$ represents the marginal cost. It is worth mentioning that the valuation of total costs has to be consistent with the variable that is considered as output in the objective function. For instance, to obtain the optimal paths that maximize the added value of yield we use as value of cost the total operating costs, 73,576 Euros. This value is divided by the current mortality rate, $F = 0.25$, to calculate the marginal cost. When the variable to maximize correspond to the profits, the value of cost used is the sum of operating cost and labor cost (73,576 plus 120,620 Euros), which is divided by the current mortality rate, $F = 0.25$.

Once the model is calibrated Monte Carlo simulations are carried out using 20,000 replications of the fishery for 28 periods.

Figure 4 shows the optimal paths that maximize the economic indicators assuming the Shepherd S-R relationship for the Northern Stock of Hake. The solid blue and red lines display the optimal path assuming a discount factor of 0.90 and 0.95, respectively. These optimal paths are compared with strategies

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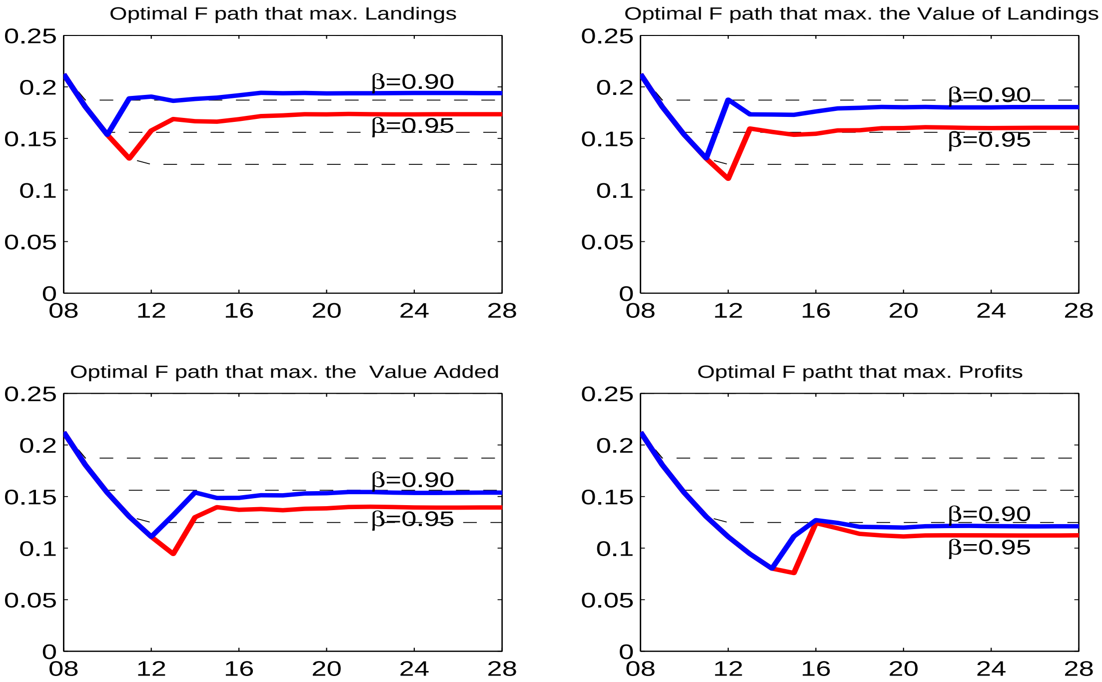
[15] To calculate the costs associated to each fleet we only consider the proportion of hake in relation to the total revenues.

Figure 4: Optimal Fishing Mortality for different Discount Factors



that consist of reaching target reference points related to F_{msy}. In concrete, taking into account STEFC advice for this fishery, we show the evolution of approaching at steps of 15% to $1.2 \times F_{msy} = 0.1872$, $F_{msy} = 0.1560$ and $0.8 \times F_{msy} = 0.1248$ which are displayed in shaded lines.

The main results we observe are the following. First, in all the cases the optimal paths consist of reducing drastically current mortality ($F_{sq} = 0.25$) up to values even lower than 0.10 in the short run. After this, fishing mortality recovers until it reaches the stationary values in the long run. Second, the level of the optimal stationary fishing mortality depends on which economic indicator we are interested in. For instance, when the aim is to maximize landings or the values of these, the stationary fishing rate is around 0.20 which can be identified with the classical target reference of $1.2 \times F_{msy}$. However, when valued added is the objective, the stationary fishing mortality fluctuates between 0.16 for and 0.19 depending on the discount factor. The optimal fishing mortality falls even more when we focus on maximizing profits, dropping up to 0.11 in the steady state. In this case the stationary state is very close to the classical target of $0.8 \times F_{msy}$.

Those results are very relevant because they indicate that management

Table 4: Discounted Economic Indicators under Several Management Scenarios

Discounted Indicators			F_{ss} with $\beta = 0.95$				F_{ss} with $\beta = 0.90$			
	F_{SQ}	F_{msy}	Yield	Income	VA	Profits	Yield	Income	VA	Profits
F target	0.250	0.156	0.173	0.160	0.139	0.112	0.194	0.180	0.154	0.121
$\sum_{t=1}^{\infty} \beta^{t-1} \sum_{a=1}^{A} y_t^a$										
mean	570	557	1176	1164	1131	1064	573	567	550	518
cv	3.83	3.26	3.35	3.27	3.18	3.09	3.43	3.35	3.21	3.07
$\sum_{t=1}^{\infty} \beta^{t-1} \sum_{a=1}^{A} pr^a y_t^a$										
mean	3045	3102	6517	6536	6404	6099	3136	3145	3076	2930
cv	3.69	3.19	3.36	3.29	3.21	3.14	3.34	3.27	3.15	3.02
$\sum_{t=1}^{\infty} \beta^{t-1} VA_t$										
mean	2309	2619	5527	5570	5578	5390	2587	2616	2618	2534
cv	4.87	3.78	3.97	3.85	3.69	3.55	4.07	3.92	3.70	3.49
$\sum_{t=1}^{\infty} \beta^{t-1} \pi_t$										
mean	1103	1829	3873	4038	4202	4226	1671	1762	1868	1886
cv	10.19	5.41	5.67	5.31	4.90	4.53	6.29	5.83	5.19	4.69

plans based exclusively on biological target references may drive the economic results of the fishery far from the optimal.

Table 4 reports the discounted value of several economic indicators for different scenarios. By rows, information on discounted yield, revenues, value added and profits are shown. For any of them, the mean and the coefficient of variation associated to the 20,000 simulations are displayed. Column 1 shows the information assuming that the fishery maintains the *status quo* harvesting policy, $F_{sq} = 0.25$, for ever. Column 2 shows the results when the target is to move from the actual situation to the $F_{msy} = 0.156$ in steps of 15% per year. Those values displayed for the *status quo* and the MSY scenarios are discounted with a discount factor $\beta = 0.90$.

Columns 3 to 6 show the results of optimal harvesting assuming a discount factor, $\beta = 0.95$, for the cases in which yield, revenues, value added and profits are used as objective function, respectively. Finally, columns 7 to 10 show the results of optimal management assuming a discount factor $\beta = 0.90$.

In light of these results we can observe that the *status quo* policy generates a higher discounted yield than the F_{msy} policy. However, optimal management implies a $F_{ss} = 0.1940$ which is a policy in between the *status quo* and the F_{msy}. So if the criteria used to select management policies is

Table 5: Annual Risk of SBB falling under SSB_{pa}

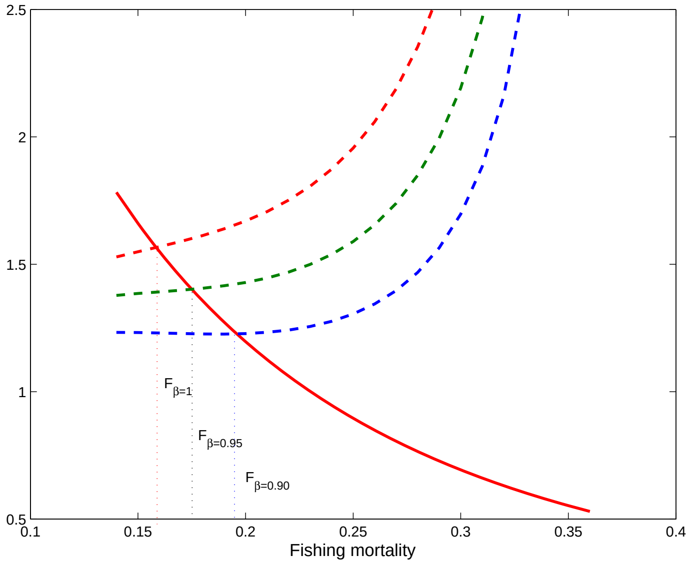
	SQ	F_{ss} (yield)	
		$\beta = 0.95$	$\beta = 0.90$
target	0.250	0.173	0.194
$t = 1$	0.00	0.00	0.00
$t = 2$	0.40	0.01	0.01
$t = 3$	0.83	0.00	0.00
$t = 4$	0.92	0.00	0.00
$t = 5$	0.92	0.00	0.00
$t = 6$	0.90	0.00	0.00
$t = 7$	0.86	0.00	0.00
$t = 8$	0.82	0.00	0.00
$t = 9$	0.82	0.00	0.00
$t = 10$	0.82	0.00	0.00

based exclusively on discounted indicators without taking into account optimal trajectories, the resource can be pushed to risk situations. As we prove in Section 4.1, F_{msy} only matches up with the optimal policy if the future is not discounted. This can be the reason why the socio-economic analysis carry out for the ICES working group advised no to change the actual policy instead to move towards a F_{msy} policy proposed an the first case by the biologist.

Our results also bear out Grafton, Kompas and Hilborn (2007) conclusions. They analyze the biomass associated to yield maximization and discounted profit maximization for the Western and Central Pacific big eye tuna and yellowfin tuna, the Australian northern prawn fishery and the Australian orange roughy fishery. Their main conclusion is that the stock associated to the maximization of yield is always lower than the stock derived from the maximization of discounted profits. This also happens for the Northern stock of hake since fishing mortality for profit maximization runs from 0.112 for $\beta = 0.95$ to 0.121 for $\beta = 0.90$. Whereas fishing mortality goes up to 0.156 for $\beta = 1$, 0.173 for $\beta = 0.95$, and 0.194 for $\beta = 0.90$ when yield is maximized.

Our results also highlight the relevance of optimal behavior with respect to the possibility of pressuring the stock towards a risk situation. Table 5 shows the annual probability of the SSB is under SSB_{pa} for the *status quo* scenario and for the case in which discounted yield is maximized. Probability of period t for a particular scenario is calculated as the ratio of number of simulations where SSB falls under SSB_{pa} in period t over 20000 which is the

Figure 5: Optimal Stationary Fishinf Rate and Discount Factor



total number of simulation run. We can see that the *status quo* policy which consists of keeping current fishing rate forever leads to a high probability of putting the stock at risk. However, the optimal trajectory that selects fishing mortality maximizing discounted yield reduces to the minimum the risk of the stock regardless of the discount factor applied. Similar results appears when the variable to maximize is discounted revenues, valued added or profits.

Finally, Figure 5 displays the relationship between fishing mortality and discount factor for the optimal policy that maximized present value of discounted yield. The solid red line shows the relationship between the marginal current yield, $\sum_{a=1}^{A} N_{ss}^{a} \partial y^{a}(F_{ss})/F_{ss}$, and the fishing mortality. We can see that this relationship is negative and it does not depend on the parameterization of the model[16]. The red, green and blue dashed lines show the effects that changes on fishing mortality have over the future yields, multiplied by less 1, for $\beta = 1$, $\beta = 0.95$ and $\beta = 0.90$, respectively. For any discount rate, an increase on the fishing mortality reduce the future yield because of the cut

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[16] Observe that the negative relationship always holds because the yield function is increasing and concave with respect to F (see Lemma 1) and $\partial N^{a}/F_{ss} = N^{1}\partial\phi^{a}(F_{ss})/F_{ss} < 0$.

in future recruitment.[17] We see that future marginal yield depends positively on the discount rate. And the intuition is clear, the more we care about the future (i.e. the higher β is), the more value has the future yield. Observe that all these plots are related with the optimality condition associated to the stationary solution, (25). Solid line corresponds to the right hand side of the optimal condition which does not depend on β.Dashed lines correspond with the right hand side which depend positively on β. Therefore any point in the solid line corresponds to an optimal solution for a particular discount factor. For instance, the fishing mortality associated to the point where red lines crossed represents the optimal stationary fishing rate when $\beta = 1$. In summary, it is clear from Figure 5 that the relationship between optimal stationary fishing rate and discount factor is negative. This empirical finding is in accordance with Proposition 3.

6 Conclusions

During the last decades, age-structured and biomass fisheries models have developed without much iteration. Age-structured model has been considered to complex to be included into optimal harvesting models. However, recent advances by and Kulmala, Laukkanen and Michielsens (2008) and Tahvonen (2008, 2009) have shown that the task is not impossible.

The study presented in this article follows the research line opened by Tahvonen (2009). Dynamic optimal harvesting that maximizes discounted utility is characterized. One of the difference between this study and Tahvonen' works is that we assume a stochastic age-structured framework similar to the one used by VPA methods. This allows us to compare optimal harvesting in a discounted economic context with standard target references used by fisheries agencies for long term management plans (e.g. F_{msy} or $F_{\max}$).

Our main findings are the following. First, we show sufficient conditions that guarantees the existence and uniqueness of the optimal steady state for the general case of n cohorts. It is also proved that the optimal steady state coincides with the traditional target F_{msy} whenever the utility function to be maximized is the yield and the discount rate is zero. Second, an algorithm is provided to calculate the optimal trajectories that drives the resource to the steady state. Finally, when the algorithm is applied to Northern Stock of hake, we observe that management plans based exclusively on biological target references may drive the fishery economic results far from the optimal.

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[17] Notice however that the dashed lines are increasing because we are displayed the reductions on future yield multiplied by less 1.

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A Appendix

Proof of Proposition 1

N^1 is obtained directly by substituting equations (3) and (7) into Shepherd's S-R relationship, (8).

On the other hand, taking derivatives over N^1 and after some manipulation it can be shown that

$$\frac{\partial N^1}{\partial F} = K \frac{\sum_{a=1}^{A} \mu^a \omega^a \frac{\partial \phi^a}{\partial F} \left[\alpha \sum_{a=1}^{A} \mu^a \omega^a \phi^a - 1\right]^{(1-b)/b}}{\sum_{a=1}^{A} \mu^a \omega^a \phi^a} \left[\alpha \frac{(1-b)}{b} \sum_{a=1}^{A} \mu^a \omega^a \phi^a + 1\right].$$

Notice that for well defined values of N^1, $\alpha \sum_{a=1}^{A} \mu^a \omega^a \phi^a - 1 > 0$. Furthermore, since $\phi^a(F) = \Pi_{i=1}^{a-1} e^{-p^i F - m^i}$,

$$\frac{\partial \phi^a(F)}{\partial F} = \sum_{i=1}^{a-1} (-p^i) \phi^a(F) < 0. \tag{29}$$

Therefore, the sign of $\partial N^1 / \partial F$ it is the opposite of the sign of the right brace.

It is straightforward that $\alpha \frac{(1-b)}{b} \sum_{a=1}^{A} \mu^a \omega^a \phi^a + 1 > 0$ whenever $b \leqq 1$. However for the case of $b > 1$,

$$\alpha \frac{(1-b)}{b} \sum_{a=1}^{A} \mu^a \omega^a \phi^a + 1 \gtreqless 0 \text{ if } \sum_{a=1}^{A} \mu^a \omega^a \phi^a \lesseqgtr \frac{b}{\alpha (b-1)}.$$

■

Proof of Lemma 1

The derivative of yield abundance respect to F is given by

$$\frac{\partial y^a(F)}{\partial F} = \omega^a \left\{ \frac{p^a m^a}{(p^a F + m^a)^2} \left(1 - e^{-p^a F - m^a}\right) + \frac{p^{a\,2} F}{p^a F + m^a} e^{-p^a F - m^a} \right\}, \tag{30}$$

which is positive because $1 - e^{-p^a F - m^a} > 0$.

The second derivative can be expressed as

$$\begin{aligned} \frac{\partial^2 y^a(F)}{\partial F^2} &= -\frac{2\omega^a\, p^{a\,2}}{(p^a F + m^a)^3} \left\{ m^a \left(1 - e^{-p^a F - m^a}\right) + \left[\frac{p^a F (p^a F + m^a)^2}{2} - m^a (p^a F + m^a)\right] e^{-p^a F -} \right. \\ &= -\frac{2\omega^a\, p^{a\,2}}{(p^a F + m^a)^3} \left\{ m^a \left[1 - e^{-p^a F - m^a} \left(1 + p^a F + m^a\right)\right] + \frac{p^a F (p^a F + m^a)^2}{2} e^{-p^a F - m^a} \right\} \end{aligned}$$

which is negative because $1 > e^{-z^a} (1 + z^a)$ for any z.

■

Proof of Lemma 2
First derivative of $y^a(F)\phi^a(F)$ is given by,

$$\frac{\partial y^a(F)\phi^a(F)}{\partial F} = \frac{\partial y^a(F)}{\partial F}\phi^a(F) + y^a(F)\frac{\partial \phi^a(F)}{\partial F}.$$

Substituting equations (29) and (30) in this expression,

$$\frac{\partial y^a(F)\phi^a(F)}{\partial F} = \left[\frac{m^a p^a}{F(p^a F + m^a)} + \frac{{p^a}^2\, e^{-p^a F - m^a}}{1 - e^{-p^a F - m^a}} - \sum_{k=1}^{a-1}(p^k)\right] y^a(F)\phi^a(F).$$

Therefore, a sufficient condition for $\sum_{a=1}^{A} y^a(F)\phi^a$ to be increasing is that $\forall a > 1$,

$$\frac{m^a p^a}{F(p^a F + m^a)} + \frac{{p^a}^2\, e^{-p^a F - m^a}}{1 - e^{-p^a F - m^a}} > \sum_{k=1}^{a-1}(p^k). \tag{31}$$

On the other hand, the second derivative of $y^a(F)\phi^a(F)$ is given by,

$$\frac{\partial^2 y^a(F)\phi^a(F)}{\partial F^2} = \frac{\partial^2 y^a(F)}{\partial F^2}\phi^a(F) + 2\frac{\partial y^a(F)}{\partial F}\frac{\partial \phi^a(F)}{\partial F} + y^a(F)\frac{\partial^2 \phi^a(F)}{\partial F^2}. \tag{32}$$

Derivating (29) we obtain that

$$\frac{\partial^2 \phi^a(F)}{\partial F^2} = \left(\sum_{i=1}^{a-1}(-p^i)\right)^2 \phi^a(F) > 0.$$

Substituting this and (30) into (32)

$$\frac{\partial^2 y^a(F)\phi^a(F)}{\partial F^2} =$$

$$\left[\frac{\partial^2 y^a(F)}{\partial F^2} + y^a(F)\left(\sum_{k=1}^{a-1}(-p^k)\right)\left(\frac{2m^a p^a}{F(p^a F + m^a)} + \frac{2\,{p^a}^2\, e^{-p^a F - m^a}}{1 - e^{-p^a F - m^a}} - \sum_{k=1}^{a-1}(p^k)\right)\right]\phi^a(F).$$

Since Lemma 1 shows that the yield function is concave, $\partial^2 y^a(F)/\partial F^2 < 0$. On the other hand, if the condition that guarantees $\sum_{a=1}^{A} y^a(F)\phi^a$ is increasing (31) holds, then the second sum inside the bracket also is negative.
■

Proof of Lemma 4
Second derivative of $\sum_{a=1}^{A} y^a(F)\phi^a(F)N^1(F)$ is given by,

$$\frac{\partial^2 \sum_{a=1}^{A} y^a\phi^a N^1}{\partial F^2} = \frac{\partial^2 \sum_{a=1}^{A} y^a\phi^a}{\partial F^2}N^1 + \sum_{a=1}^{A} y^a\phi^a\frac{\partial^2 N^1}{\partial F^2} + 2\frac{\partial \sum_{a=1}^{A} y^a\phi^a}{\partial F}\frac{\partial N^1}{\partial F}.$$

If $\sum_{a=1}^{A} y^a \phi^a$ and N^1 are concave, first and second term on the right hand side are negative. Moreover, if $\sum_{a=1}^{A} y^a \phi^a$ is increasing and N^1 is decreasing, the third term also is negative. Therefore, under those circumstances $\sum_{a=1}^{A} y^a(F)\phi^a(F)N^1(F)$ is concave.

■

Obtaining First Order Condition (20) to (24):
The Lagrangian associated to the maximization problem (19) is given by

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} pr^1 y^1(F_t) N_t^1 + \sum_{a=2}^{A} pr^a y^a(F_t) \phi_t^a N_{t-(a-1)}^1 - TC(F_t) \\ +\lambda_t \left[\Psi\left(\mu^1 \omega^1 N_t^1 + \sum_{a=2}^{A} \mu^a \omega^a \phi_t^a N_{t+1-a}^1 - N_{t+1}^1 \right) \right] \\ +\theta_t \left[\mu^1 \omega^1 N_t^1 + \sum_{a=2}^{A} \mu^a \omega^a \phi_t^a N_{t+1-a}^1 - SSB_{pa} \right] \end{array} \right\}.$$

At any period t, N_t^a and N_{t+1}^1 are given and the variables to solve are F_t, N_{t+1}^a $\forall a = 2, ...A$ and N_{t+2}^1. Notice that $\phi_t^a = \phi(F_{t-1}, F_{t-2}, ...F_{t-(a-1)}) = \Pi_{i=1}^{a-1} e^{-p^{a-i} F_{t-i} - m^{a-i}}$. This means that

$$\frac{\partial \phi_{t+j}^a}{\partial F_t} = \left\{ \begin{array}{ll} 0 & \text{for } j = 0, \\ -p^a \phi_{t+j}^a & \text{for } j = 1, ..., A-1. \end{array} \right.$$

Taking into account this fact it is easy to write the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial F_t} &= 0, \\ &\Longrightarrow \beta^t \left[\sum_{a=1}^{A} pr^a \frac{\partial y^a(F_t)}{\partial F_t} N_t^a - \frac{\partial TC_t}{\partial F_t} \right] \\ &= E_t \sum_{a=1}^{A-1} p^a \left\{ \sum_{j=1}^{A-a} \beta^{t+j} \left[pr^a y^{a+j}(F_{t+j}) + (\Psi'_{t+j} \lambda_{t+j} + \theta_{t+j}) \mu^{a+j} \omega^{a+j} \right] N_{t+j}^{a+j} \right\}, \\ \frac{\partial \mathcal{L}}{\partial N_{t+2}^1} &= 0, \\ &\Longrightarrow E_t \sum_{a=1}^{A} \beta^{t+1+a} pr^a y^a(F_{t+1+a}) \phi_{t+1+a}^a \\ &= \beta^{t+1} \lambda_{t+1} - E_t \sum_{a=1}^{A} \beta^{t+1+a} (\Psi'_{t+1+a} \lambda_{t+1+a} + \theta_{t+1+a}) \mu^a \omega^a \phi_{t+1+a}^a, \end{aligned}$$

which are completed with the restriction $N_{t+1}^{a+1} = e^{-z^a(F_t)} N_t^a \quad \forall t \quad \forall a = 1, ..., A-1$.

■

Proof of Proposition 3

F_{ss} is determined by the equation system (25) and (28). Using the survival functions (18) and substituting (26), (27) and (28) into (25) the following expression is obtained

$$\sum_{a=1}^{A} pr^a \frac{\partial y^a(F_{ss})}{\partial F}\phi^a(F)N^1_{ss} - \frac{\partial TC}{\partial F} = \sum_{a=1}^{A-1} p^a \left(\sum_{j=1}^{A-a}\beta^j pr^a y^{a+j}(F_{ss})\phi^{a+j}(F_{ss})N^1_{ss}\right) +$$

$$+\frac{\Psi'\sum_{a=1}^{A}\beta^{1+a}pr^a y^a(F_{ss})\phi^a(F_{ss})}{1-\Psi'\sum_{a=1}^{A}\mu^a\omega^a\phi^a(F_{ss})}\sum_{a=1}^{A-1}p^a\sum_{j=1}^{A-a}\mu^{a+j}\omega^{a+j}\phi^{a+j}(F)N^1_{ss} \tag{33}$$

On the one hand, we can see after some manipulation that

$$\sum_{a=1}^{A-1} p^a \left(\sum_{j=1}^{A-a}\beta^j pr^a y^{a+j}(F_{ss})\phi^{a+j}(F_{ss})N^1_{ss}\right) = \sum_{a=1}^{A} y^a(F_{ss})\phi^a(F_{ss})\sum_{i=1}^{a-1}\beta^{A-i}p^i pr^i N^1_{ss}. \tag{34}$$

On the other hand, taking into account that $\frac{\partial\phi^a(F)}{\partial F} = \phi^a(F)\sum_{i=1}^{a-1}(-p^i)$, then it is easy to show that

$$\sum_{a=1}^{A-1} p^a \sum_{j=1}^{A-a}\mu^{a+j}\omega^{a+j}\phi^{a+j}(F_{ss}) = -\sum_{a=1}^{A}\mu^a\omega^a\frac{\partial\phi^a(F_{ss})}{\partial F_{ss}}. \tag{35}$$

Moreover, from the first order condition (23) we obtain that

$$\frac{\partial N^1_{ss}}{\partial F_{ss}} = \frac{N^1_{ss}\Psi'\sum_{a=1}^{A}\mu^a\omega^a\frac{\partial\phi^a(F_{ss})}{\partial F_{ss}}}{1-\Psi'\sum_{a=1}^{A}\mu^a\omega^a\phi^a(F_{ss})}. \tag{36}$$

Substituting (34), (35) and (36) into (33)

$$\sum_{a=1}^{A} pr^a \frac{\partial y^a(F_{ss})}{\partial F_{ss}}\phi^a(F_{ss}) - \frac{1}{N^1_{ss}}\frac{\partial TC}{\partial F_{ss}} = \sum_{a=1}^{A} y^a(F_{ss})\phi^a(F_{ss})\left(\sum_{i=1}^{a-1}\beta^{A-i}p^i pr^i\right) -$$

$$-\frac{\partial N^1_{ss}}{N^1_{ss}\partial F_{ss}}\sum_{a=1}^{A}\beta^{1+a}pr^a y^a(F_{ss})\phi^a(F_{ss}). \tag{37}$$

To prove the existence and uniqueness of the F that solves (37), we prove that under the assumption of the proposition right (left) hand side is strictly increasing (decreasing). Then we show that both right and left hand side crosses each other.

Lets call LHS_1 and $LHS2$ to the two left hand side terms, respectively. In the same way, define $RHS_1$and RHS_2 as the two sums in the right hand side. First we show that the left hand side of this equation is decreasing in F. Notice that by Lemma 1 and taking into account equation (29) we know that

$$\frac{\partial LHS_1}{\partial F_{ss}} = \sum_{a=1}^{A} pr^a \left[\frac{\partial^2 y^a}{\partial F_{ss}^2} \phi^a + \frac{\partial y^a}{\partial F_{ss}} \frac{\partial \phi^a}{\partial F_{ss}} \right] < 0.$$

Also considering that the cost function, TC, is increasing and convex in F and assuming N^1 is decreasing in F, we have

$$\frac{\partial LHS_2}{\partial F_{ss}} = -\frac{\partial^2 TC}{\partial F_{ss}^2} \frac{1}{N_{ss}^1} + \frac{1}{(N_{ss}^1)^2} \frac{\partial N_{ss}^1}{\partial F_{ss}} \frac{\partial TC}{\partial F_{ss}} < 0.$$

Second we prove that the right hand side of expression (37) is decreasing on F. If $y^a(F_{ss})\phi^a(F_{ss})$ is increasing in F, $\forall a$, then easy to show that

$$\frac{\partial RHS_1}{\partial F_{ss}} = \sum_{a=1}^{A} pr^a \frac{\partial y^a \phi^a}{\partial F_{ss}} \left(\sum_{i=1}^{a-1} \beta^{A-i} p^i pr^i \right) > 0.$$

Moreover, if the $y^a \phi^a$ is increasing in F, $\forall a$, and N^1 is decreasing and concave in F

$$\frac{\partial RHS_2}{\partial F_{ss}} = \left[\frac{1}{(N^1)^2} \left(\frac{\partial N^1}{\partial F_{ss}} \right)^2 - \frac{1}{N^1} \frac{\partial^2 N^1}{\partial F_{ss}^2} \right] \sum_{a=1}^{A} \beta^{1+a} pr^a y^a \phi^a - \frac{1}{N^1} \frac{\partial N^1}{\partial F_{ss}} \sum_{a=1}^{A} \beta^{1+a} pr^a \frac{\partial y^a \phi^a}{\partial F_{ss}} > 0.$$

Notice that for the case of $F_{ss} = 0$ the left hand side is equal to zero because $y^a(0) = 0$, $\forall a$. Therefore, the left hand side of (37) is a strictly increasing function that crosses the origin. On the other hand, under the assumption that for $F_{ss} = 0$ current marginal profit is positive, the right hand side is a strictly decreasing function crossing the y-axis at a positive level. Therefore there is a unique F_{ss} that satisfies (37).

In order to prove that F_{ss} depends negatively on β, we totally differentiate expression (37)

$$\frac{\partial F_{ss}}{\partial \beta} = \frac{\frac{\partial RHS_1}{\partial \beta} + \frac{\partial RHS_2}{\partial \beta}}{\frac{\partial LHS_1}{\partial F_{ss}} + \frac{\partial LHS_2}{\partial F_{ss}} - \frac{\partial RHS_1}{\partial F_{ss}} - \frac{\partial RHS_2}{\partial F_{ss}}}.$$

Notice that the under the conditions of the proposition, the denominator of the expression is negative and

$$\begin{aligned} \frac{\partial RHS_1}{\partial \beta} &= \sum_{a=1}^{A} pr^a y^a(F_{ss}) \phi^a(F_{ss}) \left(\sum_{i=1}^{a-1} (A-i)\, \beta^{A-i-1} p^i pr^i \right) > 0, \\ \frac{\partial RHS_2}{\partial \beta} &= -\frac{\partial N_{ss}^1}{N_{ss}^1 \partial F_{ss}} \sum_{a=1}^{A} (1+a)\, \beta^a pr^a y^a(F_{ss}) \phi^a(F_{ss}) > 0. \end{aligned}$$

Therefore $\partial F_{ss}/\partial\beta < 0$.

To analyze the relationship between N^1_{ss} and β,we just take into account (36). Under the conditions of the proposition $\partial N^1_{ss}/\partial F_{ss} < 0$ and $\partial F_{ss}/\partial\beta < 0$ and therefore,

$$\frac{\partial N^1_{ss}}{\partial\beta} = \frac{\partial N^1_{ss}}{\partial F_{ss}}\frac{\partial F_{ss}}{\partial\beta} > 0.$$

■

Proof of Proposition 4

If $\beta = 1$, $pr^a = 1$ and $\partial TC/\partial F = 0$, the equation that determines F_{ss}, (37), can be expressed as

$$\sum_{a=1}^{A}\frac{\partial y^a(F_{ss})}{\partial F}\phi^a(F)N^1_{ss}+\sum_{a=1}^{A}y^a(F_{ss})\frac{\partial\phi^a(F_{ss})}{\partial F_{ss}}N^1_{ss}+\frac{\partial N^1(F_{ss})}{\partial F_{ss}}\sum_{a=1}^{A}y^a(F_{ss})\phi^a(F_{ss}) = 0.$$

Comparing this expression with the equation that determines F_{msy}, (13), it is clear that $F_{ss} = F_{msy}$.

■

Proof of Proposition 5

We prove that when $0 < \Psi' < \sum_{a=1}^{A}\mu^a\omega^a\phi^a(F)$, the SSB associated to the optimal fishing rate F_{ss} depends positively on β. Notice that

$$\frac{\partial SSB}{\partial\beta} = \frac{\partial SSB}{\partial F}\frac{\partial F}{\partial\beta} = \left[\frac{\partial N^1}{\partial F}\sum_{a=1}^{A}\mu^a\omega^a\phi^a(F) + N^1\sum_{a=1}^{A}\mu^a\omega^a\frac{\partial\phi^a(F)}{\partial F}\right]\frac{\partial F}{\partial\beta} > 0,$$

because $\partial\phi^a(F)/\partial F < 0$ by (29) and $0 < \Psi' < \sum_{a=1}^{A}\mu^a\omega^a\phi^a(F)$ guarantees a decrease N^1 and $\partial F/\partial\beta < 0$.

This implies that for any $\beta < \beta_{pa}$ the optimal mortality rate that solves (25) to (28) generates $SSB < SSB_{pa}$. For these cases, the optimal solution consists of selecting the corner solution F_{pa}.

■