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A NONCOOPERATIVE VIEW OF TWO CONSISTENT AIRPORT COST SHARING RULES

by

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A noncooperative view on two consistent airport cost sharing rules

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Abstract

This paper provides a noncooperative understanding of the nucleolus and the egalitarian allocation for airport cost problems. We find that every Nash equilibrium of the noncooperative game has the nucleolus as outcome while the egalitarian allocation is just one of the Nash outcomes.

Keywords: Airport games, egalitarian allocation, nucleolus, Nash outcomes.
JEL Classification: C71.

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1 Introduction

Airport problems model situations where different agents interested in a common project can profit from cooperation. In many cases, the distribution of the gains of cooperation among the agents is analyzed following an axiomatic approach or coalitional approach, adapting to the model well-known solution concepts defined for coalitional games such as the nucleolus or the Shapley value.

This paper provides a noncooperative interpretation of two normative solutions in airport problems: the nucleolus and the egalitarian allocation. The noncooperative understanding of these two normative solutions is grasped by using a simple noncooperative game[1], in which one of the players performs a special role. He makes a proposal and the rest of the players in a given order, accept or reject that proposal sequentially. In case of rejection the conflict is solved bilaterally, applying a normative solution concept to a special two-agent problem. Therefore for any solution defined in the class of two-agent problems a noncooperative game can be formed.

We analyze this mechanism with respect to the nucleolus and the egalitarian allocation. For the nucleolus, we provide a noncooperative game form whose equilibria yield the nucleolus. For the egalitarian allocation we provide a noncooperative game whose Nash outcomes coincide with the elements of a certain set (the set of core allocations for which the last player receives as payoff the payoff provided by the egalitarian allocation). Obviously, among those elements we find the egalitarian allocation.

The paper is organized as follows: Section 2 introduces the preliminaries and the noncooperative game. Section 3 relates the noncooperative game and the nucleolus and the last section studies the noncooperative game and the egalitarian allocation.

2 Preliminaries

2.1 The model

Airport cost problems were introduced by Littlechild and Owen [6]. These authors analyze how to distribute the cost of a landing strip among "agents"

[1] Dagan, Serrano and Volij [3] introduce similar mechanisms on the context of bankruptcy problems.

who use runways of different lengths. This problem illustrates a class of cost sharing problems in which agents are ordered according to their needs for a public project, and if one agent's need is met then all agents with smaller needs are also satisfied. (See Potters and Sudhölter [7] for a detailed study of airport cost problems and consistent allocation rules.) Formally this problem can be defined as follows.

The tuple $(N, \preceq, C)$ is an airport cost problem if:

a) N is a finite nonempty set of agents.

b) $\preceq$ is an order relation on N where $i \preceq j$ means that agent j does not precede agent i.

c) $C : N \to R_{++}$ is non-decreasing cost function so that $i \preceq j$ implies $C(i) \leq C(j)$. It is assumed that $C(i) > 0$ for all $i \in N$.

Hereafter $1, 2, ..., n$ will denote agents' order in any instance, hence $\preceq$ may be replaced by $\leq$.

In general, an airport cost problem can be interpreted as follows. Each agent i wants to carry out a project that generates a cost $C(i)$. If $i \leq j$ then project j is considered an extension of project i and every agent located earlier than agent j may be a user of that project. Accordingly the last player's project is the one that should be implemented and its cost $C(n)$ distributed among all agents.

Let $(N, \preceq, C)$ be an airport cost problem then the associated airport cost game is the TU cooperative cost game (N, c) where

$$c(S) = \max_{i \in S} C(i).$$

Notice that the values of the airport cost problem can be derived from the airport cost game (N, c) because $C(i) = c(i)$. Therefore an airport cost problem and an airport cost game are frequently identified as being the same.

A distribution among the players is represented by a real valued vector $x \in R^N$ where x_i is the payoff assigned by x to player i. A distribution of an amount higher than or equal to $C(n)$ is called a feasible allocation.

For each coalition of agents $S \subseteq N$ we use $x(S)$ to denote $\sum_{i \in S} x_i$. A distribution satisfying $x(N) = C(n)$ is called an efficient allocation. An efficient allocation satisfying $x_i \leq C(i)$ for all $i \in N$ is called an imputation and the set of imputations is denoted by $I(N, c)$. The set of non negative imputations is denoted by $D(N, c)$ and defined as:

$$D(N, c) = \left\{ x \in \mathbb{R}^N : x(N) = C(n) \text{ and } 0 \leq x_i \leq C(i) \text{ for all } i \in N \right\}.$$

The set of core allocations for the game (N, c) is defined as:

$$Core(N, c) = \left\{ x \in D(N, c) : \sum_{1 \leq i \leq j} x_i \leq C(j) \text{ for all } j \in N \right\}.$$

We denote $\max\{a, 0\}$ by a_+.

2.2 Solutions for two-person airport cost problems

Given a two-person airport cost problem $(\{i, j\}, C(i) \leq C(j))$ we define the standard solution of this problem as:

$$\begin{aligned} y_i &= \frac{C(i)}{2} \\ y_j &= C(j) - y_i. \end{aligned}$$

Notice that if agents i and j cooperate they should pay $C(j)$ otherwise each agent should pay for his own project. Therefore the amount $C(i)$ represents the savings from cooperation and the standard solution can be interpreted as an equal division of savings.

The constrained egalitarian solution[2] for the two-person airport cost problem $(\{i, j\}, C(i) \leq C(j))$ is defined as:

$$\begin{aligned} y_i &= \min\left\{ C(i), \frac{C(j)}{2} \right\} \\ y_j &= C(j) - y_i. \end{aligned}$$

This solution gives an equal division of $C(j)$ with the proviso that agent i will not pay more than the cost of his own project $C(i)$.

[2] See Dutta and Ray [4] and Arin and Iñarra [1].

2.3 The noncooperative game

An airport cost problem $(N, \preceq, C)$ is associated with a noncooperative game denoted by $G(N, \preceq, C)$. This game has n stages where only one player is playing in each stage. In the first one, the last player announces as proposal a non negative efficient allocation. In the subsequent stages each responder has two choices: to accept or reject that proposal. If a player accepts the proposal, he pays and leaves the game. In this case for the next stage the proposal will coincide with the one at the preceding stage. If a player rejects the proposal then an airport cost problem for the proposer and the responder is reformulated. The responder will pay the amount assigned by a normative solution to the this problem. Once all responders have played and paid, the last player´s payoff is determined.

Formally, the outcome of playing this game when the standard solution is considered can be described by the following algorithm.

Input: An airport cost problem $(N, \preceq, C)$.
Output: An efficient, non negative allocation x.

1. Stage 1. The proposer, player n, makes an efficient and non negative proposal x^1. (The superscript denotes the stage at which the allocation is considered as the proposal in force.)

2. Given the allocation x^{t-1} let t be the stage where the i^{th} responder plays[3]. If player i says yes he receives the payoff x_i^{t-1} and leaves the game; then $x^t = x^{t-1}$. If player i says no, a two-person airport cost problem is defined: $(\{i,n\}, \preceq, C')$ where

$$\begin{aligned} C'(n) &= x_n^{t-1} + x_i^{t-1} = C(n) - \sum_{j<i} x_j^{t-1} - \sum_{n>j>i} x_j^{t-1} \\ C'(i) &= (C(i) - \sum_{j<i} x_j^{t-1} - \max_{n-1 \geq l \geq i+1} (\sum_{l \geq j > i} x_j^{t-1} - (C(l) - C(i)))_+)_+ \end{aligned}$$

Now, $x^t = \begin{cases} x_n^{t-1} + x_i^{t-1} - y_t & \text{for player } n \\ y_t & \text{for player } i \\ x_l^{t-1} & \text{if } l \neq i, n \end{cases}$

[3] For the sake of simplicity of the model we assume that the responders are ordered according to their costs. That is, the first responder is the agent $n-1$ and the $i-th$ responder is the player $n-i$. The results do not change if we assume any other order in the set of responders whenever the proposer is the last player. In this case some proofs would have to be modified slightly.

where $y_t = C'(i)/2$. That is, (y_t, y_n) is the standard solution of $(\{i, n\}, \preceq, C')$.

1. The game ends when stage n is played. The payoff vector $x^n = x$ with coordinates $(x_j)_{j \in N}$ is the outcome of the game.

The definition of $C'(n)$ and $C'(i)$ can be explained as follows.

Let players $1, ..., n$ be placed sequentially on the nodes of a line graph where the cost of player i is given by the distance from the root to node i and denoted by $C(i)$.

Assume player i is facing the proposal x, Player $n-1$ ($(n-1) \neq i$), the first responder, pays x_{n-1}. This amount is represented on the line graph by the stretch that goes from the node $n-1$ towards the root. It may happen that $x_{n-1} > C(n-1) - C(n-2)$.

Player $n-2$ ($(n-2) \neq i$) pays the amount x_{n-2} represented by the stretch that goes either from his own location $n-2$ or from the location determined by the point $C(n-1) - x_{n-1}$. The latter will occur whenever $x_{n-1} > C(n-1) - C(n-2)$.

If for an agent l it turns out that x_l is higher than the cost of the location at which he starts to pay we assume that the remaining amount is paid starting from the first location for which there is still any cost to pay.

Thus, in general, any player k located before player i pays the amount x_k represented by the stretch that goes either from his own location or from the point determined by $C(k) - \max_{n-1 \geq l \geq k+1} (\sum_{l \geq j > k} x_j^{t-1} - (C(l) - C(k))_+)$.

Accordingly, we have that player n always profits from the payoffs made by the remaining players since he is located on the final node. However, player i profits from the payoffs made by players located before him, but he could also be profited from payoffs that will be made by some players behind him $\{i+1, 1+2, ..., k\}$, $k < n$, whenever they jointly pay more than $C(k) - C(i)$. This interpretation of the payments of players i and n explains the construction of $C'(n)$ and $C'(i)$. ($C'(i) \geq 0$ has been assumed.)

3 The nucleolus

The nucleolus (Schmeidler [8]) of an airport cost game (N, c) derived from airport cost problem $(N, \preceq, C)$ can be computed using the following formula provided by Littlechild [5]

Input: An airport cost problem $(N, \preceq, C)$

Output: The nucleolus η.

1. Start with stage 1. Define $\eta^1 = \min_{1 \leq j < n} \frac{C(j)}{j+1}$. Let $j = argmin \frac{C(j)}{j+1}$ and allocate $\eta_i = \eta^1$ for all agents $i \leq j$.

2. Consider the stage t where the first l agents have been allocated and define
$$\eta^t = \min_{l < j \leq n-1} \left\{ \frac{C(j) - \sum\limits_{1 \leq i \leq l} \eta_i}{j - l + 1} \right\}.$$
Let $j = argmin \frac{C(j) - \sum\limits_{1 \leq i \leq l} \eta_i}{j - l + 1}$ and allocate $\eta_i = \eta^t$ for all $l < i \leq j$.

3. The algorithm ends when the first $n-1$ agents are assigned. The last agent's payoff is determined by $C(n) - \sum\limits_{1 \leq i \leq n-1} \eta_i$.

Notice that all agents assigned at the same stage have identical payoff. Hereafter, we can rename[4] the nucleolus payoffs by adding a superscript so that η_i^j may be read as the nucleolus of agent i assigned at stage j.

Remark 1 *The inequality $\eta_i^j > \eta_k^{j-1}$ is a direct consequence of the algorithm above.*

Lemma 2 *Let $\{i, ..., m\}$ be the group of agents assigned at stage t where $C(i) \leq ... \leq C(m)$. Then*
$$|\{i, ..., m\}| \, \eta_k^t \leq C(m) - C(i-1).$$

Proof. Let h be the last agent assigned at stage $t-1$ $(t \neq 1)$. Let j be the last agent assigned at stage t. And let p be the last agent assigned at stage $t+1$. Therefore
$$\frac{C'(j)}{j^* + 1} < \frac{C'(p)}{p^* + 1}$$
where $C'(j) = C(j) - \sum_{i=1}^{h} \eta_i$ and

$C'(p) = C(p) - \sum_{i=1}^{h} \eta_i$. Also $p^* = p - h$ and $j^* = j - h$ and consequently $p^* > j^*$.

Assume that the lemma is not true. That is,
$$\frac{C'(p) - j^* \frac{C'(j)}{j^*+1}}{p^* + 1 - j^*} (p^* - j^*) > C(p) - C(j) = C'(p) - C'(j)$$

[4] This notation can help the reading of the proofs.

Therefore

$$(C'(p) - j^* \frac{C'(j)}{j^*+1})(p^* - j^*) > (C'(p) - C'(j))(p^* + 1 - j^*),$$

$$((j^*+1)C'(p) - j^* C'(j))(p^* - j^*) > (C'(p) - C'(j))(p^* + 1 - j^*)(j^* + 1).$$

After some simplifications we get

$$((j^* - p^*)(j^* + 1))C'(p) > (-p^* - 1)C'(j)$$

$$((p^* - j^*)(j^* + 1))C'(p) < (p^* + 1)C'(j)$$

Since $p^* - j^* \geq 1$

$(j^* + 1))C'(p) \leq ((p^* - j^*)(j^* + 1))C'(p) < (p^* + 1)C'(j)$

And we obtain

$$(j^* + 1))C'(p) < (p^* + 1)C'(j)$$

or equivalently

$$\frac{C'(j)}{j^*+1} > \frac{C'(p)}{p^*+1}.$$

Therefore for any two stages $t, t+1$ we prove that there is no group of agents that pay more than their marginal contribution. ■

The proof of the next lemma is almost identical to the proof of the previous one and therefore omitted.

Lemma 3 *Let $\{i, ..., m\}$ be the group of agents assigned at stage t where $C(i) \leq ... \leq C(m)$. Then for any $l \in \{i, ..., m-1\}$,*

$$\sum_{l<j\leq m} \eta_j = |\{l+1, ..., m\}| \, \eta_m^t > C(m) - C(l).$$

Lemma 4 *Let k be an agent whose nucleolus is determined by player j's location. Then $\eta_n - \eta_k = C(n) - C(j) - \sum_{i=j+1}^{n-1} \eta_i$.*

Proof. Assume that every player of the order $1, 2, ..., p$ has a payoff lower than η_k as the nucleolus, while every player of the order $p+1, ..., j$ has exactly η_k as the nucleolus. Then

$$\eta_k = \frac{C(j) - \sum_{i=1}^{p} \eta_i}{j - p + 1},$$

which is equivalent to

$$C(j) = \sum_{i=1}^{p} \eta_i + (j - p + 1)\eta_k = \sum_{i=1}^{j} \eta_i + \eta_k.$$

Therefore

$$C(j) = \sum_{i=1}^{j} \eta_i + \eta_k = \sum_{i=1}^{j} \eta_i + \eta_k + \sum_{i=j+1}^{n} \eta_i - \sum_{i=j+1}^{n} \eta_i.$$

Since the nucleolus is an efficient allocation, $\sum_{i=1}^{j} \eta_i + \sum_{i=j+1}^{n} \eta_i = C(n)$, therefore

$$C(j) = C(n) + \eta_k - \sum_{i=j+1}^{n} \eta_i = C(n) + \eta_k - \eta_n - \sum_{i=j+1}^{n-1} \eta_i.$$

■

Lemma 5 *Let $(N, \preceq, C)$ be an airport cost problem and $G(N, \preceq, C)$ its associated noncooperative game. If the initial proposal is the nucleolus then the final outcome will be the nucleolus.*

Proof. Assume that the lemma is not true.

Let the nucleolus be the initial proposal denoted by $\eta = (\eta_1, ...\eta_k, \eta_{k+1}, ..., \eta_n)$ and let k be the first responder that can change this proposal. Assume that agent k rejects the proposal η. Then

$$\begin{aligned} C'(n) &= \eta_n + \eta_k = C(n) - \sum_{j<k} \eta_j - \sum_{h \geq j > k} \eta_j - \sum_{n>j>h} \eta_j \\ C'(k) &= C(k) - \sum_{j<k} \eta_j - (\sum_{h \geq j > k} \eta_j - (C(h) - C(k))) \end{aligned}$$

where h is the agent that determines the nucleolus for player k. Note that by lemmas 2 and 3 we have

$$\max_{n>l>i} (\sum_{l \geq j > i} \eta_j - (C(l) - C(i)))_+ = \sum_{h \geq j > k} \eta_j - (C(h) - C(k))).$$

Denote by (y_n, y_k) the standard solution of this two-agent airport cost problem then

$$y_n - y_k = C'(n) - C'(k) =$$

$$C(n) - C(k) - (C(h) - C(k)) - \sum_{n>j>h+1} \eta_j = \eta_n - \eta_k.$$

This last equality results from applying the previous lemma. Therefore $(y_n, y_k) = (\eta_n, \eta_k)$ and agent k cannot change the initial proposal by rejecting the proposal. ■

The main result of this section is presented in the following theorem.

Theorem 6 *Let $(N, \preceq, C)$ be an airport cost problem and $G(N, \preceq, C)$ its associated noncooperative game. Every Nash equilibrium strategy profile has the nucleolus as outcome.*

Proof. Assume that there is a Nash outcome $(z_1, ..., z_n)$ which is not the nucleolus. Let k be the first player for whom $z_k > \eta_k$. Player k should be a responder. The proposer can guarantee for himself the payoff provided by the nucleolus by just proposing it. Therefore if z is a Nash outcome then $z_n \leq \eta_n$.

Assume that the proposal faced by player k when playing is

$$x = (x_1, x_2, ...x_k, z_{k+1}, ..., z_{n-1}, x_n).$$

Assume that agent k rejects the proposal x. Therefore we have the problem $(\{k, n\}, \preceq, C')$ where $C'(k) \geq 0$. If $C'(k) = 0$ it is immediate that after rejecting x the payoff of player k is $0 \leq \eta_k \leq z_k$ contradicting that z is a Nash outcome. Therefore we only focus in the case $C'(k) > 0$. Then

$$\begin{aligned} C'(n) &= x_n + x_k = C(n) - \sum_{j<k} x_j - \sum_{h \geq j > k} z_j - \sum_{n>j>h} z_j \\ C'(k) &\leq C(k) - \sum_{j<k} x_j - (\sum_{h \geq j > k} z_j - (C(h) - C(k)))_+ \\ &\leq C(k) - \sum_{j<k} x_j - (\sum_{h \geq j > k} z_j - (C(h) - C(k)) \end{aligned}$$

where h is the agent whose location determines the nucleolus of agent k. Note that $\sum_{n>j>h} z_j \leq \sum_{n>j>h} \eta_j <$ [5] $C(n-1) - C(h)$ and these agents do not contribute to the reduction of the cost of agent k. Therefore if there are agents located after agent k that contribute to the reduction of the cost of agent k those players belong to $\{k+1, ..., h\}$. And it is clear that

$$\max_{h \geq l \geq k+1} (\sum_{l \geq j > k} z_j^{t-1} - (C(l) - C(k))) \geq \sum_{h \geq j > k} z_j^{t-1} - (C(h) - C(k)).$$

The standard solution implies that $y_n - y_k = C'(n) - C'(k)$.

Therefore,

$$C'(n) - C'(k) \geq C'(n) - C(k) - \sum_{j<k} x_j - (\sum_{h \geq j > k} z_j - (C(h) - C(k)) \geq$$

$$C(n) - C(k) - (C(h) - C(k)) - \sum_{n>j>h+1} z_j.$$

If $\sum_{h \geq j > k} z_j \geq C(h) - C(k)$ the last inequality becomes an equality and otherwise it is strict.

[5] By applying Lemma 2.

Finally, we obtain

$$y_n - y_k \geq (C(n) - C(h)) - \sum_{n>j>h} z_j$$

$$\geq (C(n) - C(h)) - \sum_{n>j>h} \eta_j = \eta_n - \eta_k.$$

This last equality results from applying Lemma 4. Since $y_n \leq$ [6] $z_n \leq \eta_n$ we conclude that $y_k \leq \eta_k < z_k$.

Therefore after the optimal answer of agent k the payoff is not higher than the nucleolus. ■

Remark 7 *Given an airport cost problem and its associated noncooperative game consider the following profile of strategies: the nucleolus is offered by the proposer and the responders respond to any proposal by rejecting it if and only if after rejection they increase their payoff. Otherwise they accept. It is immediately apparent that this profile is a Nash equilibrium (indeed it is a subgame perfect equilibrium) and the final payoff vector is the nucleolus.*

The following example shows that the result holds whenever the proposer is the last player. Otherwise, it is not necessarily true that the only Nash outcome of the game is the nucleolus.

Example 8 *Consider the following airport cost problem* $(N, \preceq, C)$ *where* $C = (C(1), C(2), C(3)) = (8, 16, 24)$ *and its associated noncooperative game where the proposer is player* 1*.*

The nucleolus of this game is $\eta = (4, 6, 14)$. Assume that proposal $(4, 4, 16)$ is made by player 1 and further assume the optimal behavior of the responders to any proposal. After the optimal response of players 2 and 3 the final outcome will coincide with the initial proposal. That means that the nucleolus is not the unique Nash outcome of the game. It is not difficult to check that both outcomes considered are Nash outcomes.

4 The egalitarian allocation

In Subsection 2.3 we have introduced a noncooperative game where the standard solution for a two person airport cost problem. In this section the conflict between the proposer and the responder is solved by applying we modify that

[6] Given the Nash behavior of the responders the payoff of the proposer cannot decrease.

noncooperative game slightly: The conflict is solved by applying the egalitarian allocation to the same two person airport cost problem. Therefore we omit the formal presentation of the new noncooperative game, which hereafter is denoted by $G^e(N, \preceq, C)$.

The egalitarian allocation can be computed by applying the following algorithm (see Chun-Hsien [2]):

Input: An airport cost problem $(N, \preceq, C)$.
Output: the egalitarian allocation e.

1. Start with stage 1. Define $e^1 = \min_{1\leq j\leq n} \frac{C(j)}{j}$. Let $j = arg \min_{1\leq j\leq n} \frac{C(j)}{j}$ and allocate $e_i = e^1$ for all $i \leq j$.

2. Consider stage t where the first l players have been allocated and define
$$e^t = \min_{l<j\leq n} \left\{ \frac{C(j) - \sum_{1\leq i\leq l} e_i}{j-l} \right\}.$$
Let $j = arg \min_{l<j\leq n} \frac{C(j) - \sum_{1\leq i\leq l} e_i}{j-l}$ and allocate $e_i = e^t$ for all $l < i \leq j$.

3. The algorithm ends when the n players are assigned.

The proof of the following lemma is immediate.

Lemma 9 *Let $(N, \preceq, C)$ be an airport cost problem and $G^e(N, \preceq, C)$ its associated noncooperative game. Let z be a final outcome of the game where the responders have played optimally. Then $z_i \leq z_n$ for all $i \in N$.*

The following lemmas investigate the features of the Nash outcomes.

Lemma 10 *Let $(N, \preceq, C)$ be an airport cost problem and $G^e(N, \preceq, C)$ its associated noncooperative game. Let z be a final outcome of the game where the responders have played optimally. Then $\sum_{1\leq i\leq p} z_i \leq C(p)$ for all $p \in \{1, ..., n-1\}$.*

Proof. Assume that there exists a player p, $p \neq n$, such that $\sum_{1\leq i\leq p} z_i > C(p)$. Therefore $z_p > C(p) - \sum_{1\leq i\leq p-1} z_i$. Let y be the proposal faced by player p.

Since all the responders play optimally, clearly $y_i \geq z_i$ for all $i \in \{1,...,p\}$ and therefore

$$C(p) - \sum_{1 \leq i \leq p-1} y_i \leq C(p) - \sum_{1 \leq i \leq p-1} z_i < z_p.$$

If player p rejects proposal y he will receive a payoff lower than or equal to $C(p) - \sum\limits_{1 \leq i \leq p-1} y_i$ contradicting the fact that z is an outcome derived from the optimal behavior of the responders. ■

Lemma 11 *Let $(N, \preceq, C)$ be an airport cost problem and $G^e(N, \preceq, C)$ its associated noncooperative game. Let z be the final outcome of the game where the responders have played optimally. Then $z_n \geq e_n$.*

Proof. We consider two cases:

a) $e_n = \frac{C(n)}{n.}$. Since $z_i \leq z_n$ for all $i \in N$ it is clear that $z_n \geq e_n$.

b) $e_n = \frac{C(n)-C(j)}{n-j}$. By Lemma 10 if z is a final outcome of the game where the responders have played optimally then $\sum\limits_{1 \leq i \leq j} z_i \leq C(j)$ and therefore $\sum\limits_{j+1 \leq i \leq n} z_i \geq C(n) - C(j)$. Notice also that $z_n \geq z_i$ for all $i \in N$. Combining the two inequalities $z_n \geq e_n$ is obtained. ■

Before presenting the main theorem of this section we introduce some notation. Let $B(N, \preceq, C)$ be the following set:

$$B(N, \preceq, C) = \{x \in Core(N, c) : \; x_i \leq x_n \text{ for all } i \in N\}.$$

Lemma 12 *Let $(N, \preceq, C)$ be an airport cost problem and $G^e(N, \preceq, C)$ its associated noncooperative game. Let $z \in B(N, \preceq, C)$ be an initial proposal. Then if all responders have played optimally, z will be the final outcome of the game.*

Proof. Assume that the lemma is not true. Then there exists a responder that by rejecting optimally proposal z gets a lower payoff. Let i be the first responder doing this. The following two person airport cost problem results:

$$\begin{aligned} C'(n) &= z_n + z_i = C(n) - \sum_{j<i} z_j - \sum_{n>j>i} z_j \\ C'(i) &= C(i) - \sum_{j<i} z_j - \max_{n-1 \geq l \geq i+1} \Big(\sum_{l \geq j > i} z_j - (C(l) - C(i)) \Big)_+ \end{aligned}$$

Let (y_i, y_n) be the egalitarian allocation of this problem. Since $y_i < z_i \leq \frac{z_n+z_i}{2}$ we obtain $y_i = C'(i) = C(i) - \sum\limits_{j<i} z_j - \max\limits_{n-1 \geq l \geq i+1} \Big(\sum\limits_{l \geq j > i} z_j - (C(l) - C(i)) \Big)_+ < z_i$.

There are two cases:

a) $\max\limits_{n-1\geq l\geq i+1}(\sum\limits_{l\geq j>i} z_j - (C(l) - C(i))) \leq 0$.

In this case we have

$C(i) - \sum\limits_{j<i} z_j < z_i$ and consequently $\sum\limits_{j\leq i} z_j > C(i)$. But this contradicts the fact that $z \in B(N, \preceq, C)$.

b) $\max\limits_{n-1\geq l\geq i+1}(\sum\limits_{l\geq j>i} z_j - (C(l) - C(i))) > 0$. Let $h = \arg\max\limits_{n-1\geq l\geq i+1}(\sum\limits_{l\geq j>i} z_j - (C(l) - C(i)))$.

In this case we have

$C(i) - \sum\limits_{j<i} z_j - (\sum\limits_{h+1>j>i} z_j - (C(h) - C(i))) < z_i$ and consequently $\sum\limits_{j\leq h} z_j > C(h)$, which it contradicts $z \in B(N, \preceq, C)$. ■

Theorem 13 *Let $(N, \preceq, C)$ be an airport cost problem and $G^e(N, \preceq, C)$ its associated noncooperative game. Then z is a Nash outcome if and only if $z \in B(N, \preceq, C)$ and $z_n = e_n$.*

Proof. a) Let $z \in B(N, \preceq, C)$ with $z_n = e_n$. Assume the following profile of strategies; z is offered by the proposer and the responders respond to any proposal by rejecting it if and only if after rejection they increase their payoff. By Lemma 12 the final outcome will be z. And by Lemma 11 the proposer, given the optimal behavior of the responders, cannot get a payoff lower than e_n. Therefore the outcome of this profile is z which is a Nash outcome (indeed it is a subgame perfect outcome).

b) Let z be a Nash outcome. Lemmas 10 and 11 imply that $z \in B(N, \preceq, C)$. If the initial proposal is the egalitarian allocation, the final outcome will be the egalitarian allocation because $e \in B(N, \preceq, C)$. Therefore $z_n = e_n$. ■

The following example shows that the result holds whenever the proposer is the last player. Otherwise, it is not necessarily true that the only Nash outcomes of the game belong to the set $B(N, \preceq, C)$.

Example 14 *Consider the following airport cost problem $(N, \preceq, C)$ where $C = (C(1), C(2), C(3)) = (8, 18, 24)$ and its associated noncooperative game where the proposer is player 1.*

Assume that the proposal $(8, 10, 6)$ is made by player 1 and players 2 and 3 play optimally to any proposal. After the optimal response of players 2 and 3 the final outcome will coincide with the initial proposal.

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