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THE SD-PRENUCLEOLUS FOR TU GAMES

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The SD-prenucleolus for TU games

J. Arin* and I. Katsev†

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Abstract

We introduce and characterize a new solution concept for TU games. The new solution is called SD-prenucleolus and is a lexicographic value although is not a weighted prenucleolus. The SD-prenucleolus satisfies several desirable properties and is the only known solution that satisfies core stability, strong aggregate monotonicity and null player out property in the class of balanced games. The SD-prenucleolus is the only known solution that satisfies core stability, continuity and is monotonic in the class of veto balanced games.

Keywords: TU games, prenucleolus, per capita prenucleolus

1. Introduction

This paper introduces and characterizes a new solution concept for coalitional games with transferable utility (TU games). The new solution is a lexicographic value, so its name (SD-prenucleolus) reflects its strong connection with the classic, widely-analyzed prenucleolus. The solution also has a strong relationship with the

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family of weighted prenucleoli although it is not a member of this family. In particular, the new solution shares some similarities with the per capita prenucleolus.

Given a TU game the prenucleolus, defined as a lexicographic value, selects the vector of excesses of coalitions that lexicographically dominates any other vector of excesses of coalitions. When this vector is selected its associated allocation is automatically selected and this proves to be the prenucleolus of the game. When the excesses of coalitions are weighted by using a system of weights for the size of the coalitions this procedure will generate the different weighted prenucleoli. In the per capita prenucleolus excesses are divided by the size (cardinality) of the coalition.

In this paper we propose a different way of computing the excesses of coalitions given an allocation. This is the main contribution of the paper[1] since whenever the vector of excesses is computed for any allocation the SD-prenucleolus arises as the lexicographic optimal value in the set of vectors of excesses of coalitions.

With the new definition we proceed as follows. We prove several interesting properties of the new solution. In particular, we show that in the class of balanced games the SD-prenucleolus satisfies core stability, the Null Player Out property and strong aggregate monotonicity. To our knowledge there is no other solution that satisfies these properties. We characterize the solution in terms of balanced collections of coalitions, the equivalent of Kohlberg's classic theorem of the prenucleolus (Kohlberg, 1971). This characterization is the main tool for checking whether an allocation is the SD-prenucleolus of the game. After introducing the SD-reduced game property we provide the characterizations of the SD-prenucleolus (the equivalent of Sobolev's characterization for the prenucleolus) and the SD-prekernel (the equivalent of Peleg's characterization for the prekernel). In Section 6 we provide a simple formula for computing the SD-prenucleolus of monotonic games with veto players. As a corollary of this result we show that in the class of veto monotonic games the SD-prenucleolus satisfies coalitional monotonicity. Among the solutions defined in the class of all balanced games the SD-prenucleolus is the only known solution satisfying core stability, continuity

[1] In Section 3 we argue broadly why we consider the new vector of excesses to be necessary.

and coalitional monotonicity in the class of veto balanced games.

2. Preliminaries

2.1. TU Games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. Singletons are coalitions that contain only one player. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an allocation, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum\limits_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(v)$. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

It has been shown that a game with a non-empty core is balanced[2] and therefore games with non-empty core are called balanced games. *Player i is a veto player if* $v(S) = 0$ for all S where player i is not present. A balanced game with at least one veto player is called a veto balanced game. We denote by Γ_{VB} the class of balanced games and by Γ_{VB} the class of veto balanced games.

A solution φ on a class of games Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* element for each game in the class.

[2] See Peleg and Südholter (2007).

Given $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game v is defined as $e(S,x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order. The weak lexicographic order $\preceq_L$ between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$ or $x = y$.

Schmeidler (1969) introduced the *prenucleolus* of a game v, denoted by $PN(v)$, as the unique allocation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of allocations. In formula:

$$\{PN(N,v)\} = \{x \in X(N,v) \,|\theta(x) \preceq_L \theta(y) \text{ for all } y \in X(N,v)\}.$$

For any game v the prenucleolus is a single-valued solution, is contained in the prekernel and lies in the core provided that the core is non-empty.

The per capita prenucleolus (Groote, 1970) is defined analogously by using the concept of per capita excess instead of excess. Given S and x the per capita excess of S at x is

$$e^{pc}(S,x) := \frac{v(S) - x(S)}{|S|}$$

Other weighted prenucleoli can be defined in a similar way whenever a weighted excess function is defined. The same solution concepts can be analogously defined using the notion of satisfaction instead of excess. Given $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game (N,v) is defined as $f(S,x) := x(S) - v(S)$. In this paper we use the notion of satisfaction in defining the new solution.

2.2. Properties

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following.

- φ satisfies **anonymity** if for each (N,v) in Γ_0 and each bijective mapping $\tau : N \longrightarrow N$ such that $(N, \tau v)$ in $\Gamma\Gamma_0$ it holds that $\varphi(N,\tau v) = \tau(\varphi(N,v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ $(x \in R^N, j \in N, T \subseteq N)$). In this case v and τv are equivalent games.

- φ satisfies **equal treatment property (ETP)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ interchangeable players i, j are treated equally, i.e. $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

- φ satisfies **desirability** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$, $x_i \geq x_j$ if i is more desirable than j in v. We say that in a game v a player i is more desirable than a player j if $v(S \cup i) \geq v(S \cup j)$ for all $S \subset N \setminus \{i, j\}$.

- φ satisfies **covariance** if $(N, v), (N, \alpha v + \beta) \in \Gamma_0$ for any $\alpha > 0$ and any $\beta \in R^N$ implies that $\varphi(N, \alpha v + \beta) = \alpha \varphi(N, v) + \beta$ holds.

- φ satisfies **null player property** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ null players receive 0. Here, a player is a null player if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$.

- φ satisfies **null player out property (NPO)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ it holds that $(x_i)_{i \in N \setminus T} \in \varphi(N \setminus T, v)$. Here T is the set of null players in game (N, v).

The NPO property implies the Null Player property. Both properties try to capture the idea that null players should not influence the allocations selected by a solution. However, only the NPO property captures entirely this idea. If the payoff of some players (different than the null player) can be affected for the presence of null players is difficult to conclude that null players are irrelevant players.

- φ satisfies **core stability** if it selects core allocations whenever the game is balanced.

Note that desirability implies ETP. The following two properties are defined for single-valued solutions.

- φ satisfies **coalitional monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i \in T$, $\varphi_i(v) \leq \varphi_i(w)$.

- φ satisfies **aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) \geq 0$.

- φ satisfies **strong aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) = \varphi_j(w) - \varphi_j(v) \geq 0$.

Young (1985) proves that no solution satisfies coalitional monotonicity and core stability. However, there are solutions satisfying core stability and the strong aggregate monotonicity. Meggido (1974) proves that the nucleolus does not satisfy aggregate monotonicity. Clearly, strong aggregate monotonicity implies aggregate monotonicity.

3. A new vector of satisfactions

3.1. Introduction

The prenucleolus is a lexicographic value that selects a maximal element in the set of vectors of excesses of coalitions. The solution does not change if the vector of satisfaction is taken instead of vectors of excesses. In the definition of the new lexicographic value we use the notion of satisfaction instead of excess. The main change with respect to the classic prenucleolus, the per capita prenucleolus and any other weighted prenucleolus lies in how the vector of satisfactions is defined. The main idea of the new vector of satisfactions is to identify how a coalition divides its surplus (the difference between the payoff received by the coalition and the worth of the coalition) among its members. We also require this distribution to keep some consistency. We argue that the classic prenucleolus has no answer to this question while the per capita prenucleolus provides an unsatisfactory answer.

Before introducing the new vector of satisfactions we illustrate by means of an example the two ideas that support the new solution.

Consider the following 4-player game[3] (N, v):

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,4\},\{1,2,4\}\} \\ 4 & \text{if } S = \{1,2,3\} \\ 8 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Consider the prenucleolus of the game, the allocation $x = (2, 2, 2, 2)$. The satisfaction of coalition $\{1, 2, 3\}$ is 2 and players 1, 2 and 3 share this surplus. That is, if the surplus obtained by player 1 in coalition $\{1, 2, 3\}$ at x is 2 then the surplus obtained by players 2 and 3 in coalition $\{1, 2, 3\}$ at x is 0. However, player 4 owns the entire satisfaction obtained by coalition $\{4\}$ at x. From the point of view of the coalitions it can be asserted that coalitions $\{1, 2, 3\}$ and $\{4\}$ have been treated equally at x but this assertion is not so evident from the point of view of the players.

The per capita prenucleolus apparently solves this question. Consider the per capita prenucleolus of the game, the allocation $y = (2.6, 2.2, 2.2, 1)$.

The per capita satisfaction of coalition $\{1, 2, 3\}$ is 1 and players 1, 2 and 3 share a total surplus of 3. That is, the per capita satisfaction can be seen as how much each player receives from the total surplus. Now the assertion that players in coalition $\{1, 2, 3\}$ and player 4 have been equally treated at y can be justified. But consider now the situation of coalition $\{2, 4\}$. According to the per capita satisfaction it must be concluded that each player in the coalition receives a surplus of 1.6, i.e. more than the total payoff received by player 4. It seems incorrect to allocate a surplus of 1.6 to player 4 in coalition $\{2, 4\}$ at y. It seems more correct to consider that the total surplus of coalition $\{2, 4\}$ at y has been distributed as follows: player 2 gets 2.2 and player 4 gets 1.

These ideas motivate the definition of a new vector of satisfactions (and therefore a new lexicographic value) and the name of the new solution concept: Surplus Distributor Prenucleolus.

[3]In Section 5 we compute the SD-prenucleolus of this game, i.e. $(3, 2, 2, 1)$.

3.2. The Algorithm

Consider a game (N, v) and an allocation x. Our goal is to calculate a satisfaction vector $\{F(S,x)\}_{S\subsetneq N}$. We define the components of this vector recursively by defining an algorithm.

The algorithm has several steps (at most $2^n - 2$) and at each step we identify the collection of coalitions that has obtained the satisfaction. We denote by $\mathcal{H}$ this collection of coalitions. In the first step this collection $\mathcal{H}$ is empty. The algorithm ends when $\mathcal{H} = 2^N$.

For a collection $\mathcal{H}$ and a function $F : \mathcal{H} \to \mathbb{R}$ the function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$ is defined. To this end, we introduce some notation.

For $\mathcal{H} \subset 2^N$ we denote

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T\in\mathcal{H}, T\subset S} T$$

and also for a collection $\mathcal{H} \subset 2^N$ and a function $F : \mathcal{H} \to \mathbb{R}$ we denote by $f_{\mathcal{H},F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathcal{H}$ $(i \in \sigma_{\mathcal{H}}(S))$:

$$f_{\mathcal{H},F}(i, S) = \min_{T:T\in\mathcal{H}, i\in T\subset S} F(T)$$

Note that this definition can only be used in a situation when the function $F(S)$ is defined for all $S \in \mathcal{H}$.

Now we define a function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subset S$):

1. Relevant coalitions. $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum_{i\in\sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S)}{|S| - |\sigma_{\mathcal{H}}(S)|}$$

Note that if the collection $\mathcal{H}$ is empty then the current satisfaction of the coalition S coincides with its *per capita satisfaction*:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}$$

2. Non relevant coalitions. $\sigma_{\mathcal{H}}(S) = S$. In this case the current satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i, S) + \max_{i \in S} f_{\mathcal{H},F}(i, S)$$

Therefore for any function $F : \mathcal{H} \to \mathbb{R}$ the value $f_{\mathcal{H},F}(i, S)$ can be calculated for every coalition S and player $i \in \sigma_{\mathcal{H}}(S)$. Also if a function $f_{\mathcal{H},F}(\cdot, \cdot)$ is defined for each $S \subsetneq N$ and $i \in \sigma_{\mathcal{H}}(S)$ then the function $F_{\mathcal{H}}$ can be defined.

The algorithm for the satisfaction vector is defined as follows:.

Consider a game (N, v) and an allocation $x \in X(N, v)$.

Step 1: Set $k = 0$, $\mathcal{H}_0 = \emptyset$. Go to Step 2.

Step 2: Set

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}$$

Step 3: Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:

$$F(S) = F_{\mathcal{H}_k}(S)$$

Step 4: If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k + 1$ and go to Step 2, else go to Step 5.

Step 5: Stop. Return the vector

$$\{F(S), S \subsetneq N\}$$

For simplicity we use the notation $F(S)$ instead of $F(S, x)$.

Note that according to this algorithm if the game introduced in this section and the allocation y (the per capita prenucleolus of the game) are considered it holds that $F(\{2, 4\}, y) = 2.2 > 1.6$.

The outcome provided by the algorithm satisfies several interesting properties, which are pointed out in the following lemmas.

Lemma 3.1. *Let (N, v) be a TU game and x be an allocation. Let function F be the result of Algorithm 3.2 and let $\{\mathcal{H}_i\}_{i=1..k}$ be the associated collections of sets. Then*

1. the function F is defined for every $S \subsetneq N$

2. the function F is continuos.

Proof. 1. It holds that $\mathcal{H}_0 = \emptyset$, $\mathcal{H}_k = 2^N \setminus \{N\}$. In the i-th stage of the algorithm the function F is defined for all coalitions from $\mathcal{H}_i \setminus \mathcal{H}_{i-1}$. Therefore at the end this function is defined for all coalitions in

$$\bigcup_{i=1..k} (\mathcal{H}_i \setminus \mathcal{H}_{i-1}) = \mathcal{H}_k \setminus \mathcal{H}_0 = 2^N \setminus \{N\}.$$

2. This is immediately apparent. ■

Lemma 3.2. *Let (N, v) be a TU game and x be an allocation. Let function F be the result of Algorithm 3.2 and let $\{\mathcal{H}_i\}_{i=1..k}$ be the associated collections of sets. If $S \in \mathcal{H}_i$, $T \notin \mathcal{H}_i$ then $F(T) > F(S)$.*

Proof. Assume that the lemma is not true. Consider the minimal number k such that there exist coalitions S, T with $S \in \mathcal{H}_k$, $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$, and $F(T) \leq F(S)$. Consider the k-th stage of the algorithm where the collection $\mathcal{H}_{k-1}$ was fixed. It holds that $S \in \mathcal{H}_k$ and therefore

$$F_{\mathcal{H}_{k-1}}(S) = \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)$$

It is also known that $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$, so $F(T) = F_{\mathcal{H}_k}(T)$.

Note that because of the assumption of the minimality of k

1. for every $i \in \sigma_{\mathcal{H}_{k-1}}(T)$ it holds that $f_{\mathcal{H}_k,F}(i,T) = f_{\mathcal{H}_{k-1},F}(i,T) < F(S)$

2. for every $i \in \sigma_{\mathcal{H}_k}(T) \setminus \sigma_{\mathcal{H}_{k-1}}(T)$ it holds that $f_{\mathcal{H}_k,F}(i,T) = F(S)$

Consider three cases:

A. $\sigma_{\mathcal{H}_k}(T) \neq T$. Then

$$F_{\mathcal{H}_k}(T) = \frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_k}(T)} f_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|\sigma_{\mathcal{H}_k}(T)| - |\sigma_{\mathcal{H}_{k-1}}(T)|)}{|T| - |\sigma_{\mathcal{H}_k}(T)|}$$

From the assumption that $F_{\mathcal{H}_k}(T) = F(T) \leq F(S)$

$$\frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|\sigma_{\mathcal{H}_k}(T)| - |\sigma_{\mathcal{H}_{k-1}}(T)|)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} \leq F(S) \Leftrightarrow$$

$$\Leftrightarrow x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) \leq F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) \Leftrightarrow$$

$$\Leftrightarrow \frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T)}{|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|} \leq F(S) \Leftrightarrow F_{\mathcal{H}_{k-1}}(T) \leq F(S)$$

But $F(S) = F_{\mathcal{H}_{k-1}}(S) = \min\limits_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)-$ Therefore $F_{\mathcal{H}_{k-1}}(T) = \min\limits_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)$ and from the Algorithm 3.2 it holds that $T \in \mathcal{H}_k$. This is in contradiction with the assumption that $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$.

B. $\sigma_{\mathcal{H}_{k-1}}(T) \neq T$, $\sigma_{\mathcal{H}_k}(T) = T$. Then

$$F_{\mathcal{H}_k}(T) = x(T) - v(T) - \sum_{i \in T} f_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f_{\mathcal{H},F}(i,T)$$

By using the fact that $\sigma_{\mathcal{H}_{k-1}}(T) \neq T$ it can be concluded that

$$F_{\mathcal{H}_k}(T) = x(T) - v(T) - \sum_{i \in T} f_{\mathcal{H},F}(i,T) + F(S) =$$

$$= x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) + F(S)$$

From the assumption that $F_{\mathcal{H}_k}(T) = F(T) \leq F(S)$ it can be obtained that

$$x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) \leq F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) \Leftrightarrow$$

$$\Leftrightarrow \frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T)}{|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|} \leq F(S) \Leftrightarrow F_{\mathcal{H}_{k-1}}(T) \leq F(S)$$

and as in the previous case the contradiction is obtained.

C. $\sigma_{\mathcal{H}_{k-1}}(T) = \sigma_{\mathcal{H}_k}(T) = T$. Then

$$F_{\mathcal{H}_{k-1}}(T) = F_{\mathcal{H}_k}(T)$$

Because of the fact that $T \notin \mathcal{H}_k$ it can be concluded that

$$F_{\mathcal{H}_k}(T) = F_{\mathcal{H}_{k-1}}(T) > \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U) = F_{\mathcal{H}_{k-1}}(S).$$

■

This lemma implies that for a relevant coalition ($\sigma_{\mathcal{H}}(S) \neq S$) it holds that

$$x(S) - v(S) = (|S| - |\sigma_{\mathcal{H}}(S)|)F_{\mathcal{H}}(S) + \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S) = \sum_{i \in S} f_{\mathcal{H},F}(i,S)$$

which can be interpreted as a distribution of the total surplus of coalition S among its members. The following 3-person game is used to illustrate how this algorithm works. Let (N, v) be a game where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1,3\}, \{1,2\}\} \\ -10 & \text{if } S = \{2,3\} \\ 6 & \text{if } S = N. \end{cases}$$

Consider the allocation $x = (5, 1, 0)$. Applying the algorithm the following is obtained:

Coalition	Satisfaction
$\{3\}$	0
$\{2\}\ \{1,2\}\ \{1,3\}$	1
$\{1\}$	5
$\{2,3\}$	11.

Coalition $\{2, 3\}$ is a non relevant coalition. The rest of the coalitions are relevant coalitions. Consider the satisfaction of coalition $\{1, 3\}$. This coalition has a subset (coalition $\{3\}$) that has already obtained its satisfaction. This fact is incorporated into the computation of the satisfaction of coalition $\{1,3\}$ since $\sigma_{\mathcal{H}}(\{1,3\}) = \{3\}$. Therefore

$$F_{\mathcal{H}}(\{1,3\}\,,x) = \frac{x(\{1,3\}) - v(\{1,3\}) - \sum_{i \in \sigma_{\mathcal{H}}(\{1,3\})} f_{\mathcal{H},F}(i,\{1,3\})}{|\,\{1,3\}\,| - |\sigma_{\mathcal{H}}(\{1,3\})|} = \frac{5-4-0}{2-1}.$$

The total surplus of the coalition is divided as follows: player 1 gets 1 and player 3 gets 0.

The case of non relevant coalitions is different. If a coalition is non relevant for any player in the coalition there exists a subset of the coalition with a lower satisfaction and that subset determines the individual satisfaction of the player in the non relevant coalition. Note that

$$x(\{2,3\}) - v(\{2,3\}) = 11 > \sum_{i \in \sigma_{\mathcal{H}}(\{2,3\})} f_{\mathcal{H},F}(i,\{2,3\}) = 1 + 0.$$

4. The SD-prenucleolus

4.1. Definition

We define the new solution concept (the SD-prenucleolus) as a lexicographic value in the set of vectors of the new satisfactions. We denote the SD-prenucleolus of game (N,v) by $SD(N,v)$.

The definition of the SD-prenucleolus coincides with the definition of the classic prenucleolus, except that we use the vector of negative satisfactions $\{-F(S,x)\}$ instead of the vector of excesses. Therefore the SD-prenucleolus is a lexicographic value that selects from a set a vector that lexicographically dominates the other vectors of the set.

We now formulate it in detail.

We say that the satisfaction vector $F^x = \{F(S,x)\}_{S \subset N}$ *dominates* the satisfaction vector $F^y = \{F(S,y)\}_{S \subset N}$ if there is $k \geq 1$ such that

1. $\tilde{F}_i^x = \tilde{F}_i^y$ for all $i < k$
2. $\tilde{F}_k^x > \tilde{F}_k^y$,

where $\tilde{F}^x$ and $\tilde{F}^y$ are the vectors with the same components as the vectors F^x, F^y, but rearranged in a non decreasing order ($i > j \Rightarrow \tilde{F}^x_i \leq \tilde{F}^x_j$).

We say that the vector x belongs to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector.

Definition 4.1. *Let (N, v) be a TU game. Then $x \in SD(N, v)$ if and only if for any $y \in X(N, v)$ it holds that $F^x \succeq_L F^y$.*

Similarly to the prenucleolus, the SD-prenucleolus satisfies nonemptiness and single-valuedness on the class of all TU games.

Proposition 4.2. *Let (N, v) be a TU game. Then $|SD(N, v)| = 1$.*

Proof. The standard proof of the nonemptiness of the prenucleolus can be repeated in this case with no changes. The proof of single-valuedness is also very close to the standard one but has some differences. Assume that there is a pair of vectors $x, y \in X(N, v)$ such that both vectors $\{-F(S, x)\}_{S \subsetneq N}$,$\{-F(S, x)\}_{S \subsetneq N}$ dominates a vector $\{-F(S, z)\}_{S \subsetneq N}$ for every $z \in X(N, v)$.

Consider the allocation $t = \frac{x+y}{2}$ and the vector $\{-F(S, t)\}_{S \subsetneq N}$. Because $x \neq y$ the number k can be chosen such that for every $i < k$ it holds that $\mathcal{H}_i(x) = \mathcal{H}_i(y)$ and that $\mathcal{H}_k(x) \neq \mathcal{H}_k(y)$.

Assume that because of the linearity of functions f and F it can be concluded that for $i < k$ it is also true that $\mathcal{H}_i(x) = \mathcal{H}_i(t)$.

Consider the k-th stage of the algorithm for all three vectors (x, y, t). We can note that functions f and F are the same for these vectors. Denote $F^x_{\mathcal{H}_k(S)}$ for $S \in \mathcal{H}^x_k \setminus \mathcal{H}_{k-1}$ by G_k. Because of the coincidence of the satisfaction vectors for x and y it also holds that $G_k = F^y_{\mathcal{H}_k(T)}$ for $T \in \mathcal{H}^y_k \setminus \mathcal{H}_{k-1}$.

With no loss of generality it can be assumed that there exists $T \in H_{\{k\}}(x) \setminus H_{\{k\}}(y)$. By the linearity of the function F it can be concluded that

$$F^t_{\mathcal{H}_{k-1}}(T) = \frac{F^x_{\mathcal{H}_{k-1}}(T) + F^y_{\mathcal{H}_{k-1}}(T)}{2} = \frac{G_k + F^y_{\mathcal{H}_{k-1}}(T)}{2}$$

Because of $T \notin \mathcal{H}_k(y)$ we get $F^y_{\mathcal{H}_{k-1}}(T) > G_k$. Therefore

$$F^t_{\mathcal{H}_{k-1}}(T) = \frac{G_k + F^y_{\mathcal{H}_{k-1}}(T)}{2} > G_k$$

The same conclusions can be used for an arbitrary coalition U which belongs to $\mathcal{H}_k(y)$ but not to $\mathcal{H}_k(x)$. Therefore the collection of coalitions with satisfaction less than or equal to G_k for the vector t is equal to the intersection of such collections for vectors x and y. It means that the satisfaction vector for t dominates the satisfaction vectors for x and y and this contradicts the assumption. ■

4.2. Properties

The new solution shares other interesting properties with the classic prenucleolus and the per capita prenucleolus. For example, it is not difficult to prove that the SD-prenucleolus also satisfies desirability (and therefore the equal treatment property), anonymity, covariance and efficiency.

Also the SD-prenucleolus is a core selector, i.e. if a game is balanced its SD-prenucleolus is a core allocation. This is so because any core allocation has a non negative vector of satisfactions.

Unlike the prenucleolus, the SD-prenucleolus satisfies strong aggregate monotonicity. This property is also satisfied by the per capita prenucleolus.

Proposition 4.3. *The SD-prenucleolus satisfies the strong aggregate monotonicity property.*

Proof. Consider games (N, v) and (N, v_A) where $v_A(N) = v(N) + A|N|$ and $v_A(S) = v(S)$ for $S \neq N$. Let $x \in X(v)$ and $y \in X(v_A)$ such that for each $i \in N$ $y_i = x_i + A$.

It is sufficient to show that if the following holds for any $k \geq 0$

1. $F_{(N,v_A)}(S, y) = F_{(N,v)}(S, x) + A$ for each $S \in \mathcal{H}_k$
2. The collections $\mathcal{H}_k$ for (N, v_A, y) and (N, v, x) coincide

then the two facts hold also for $k+1$ (for $k=0$ it is evident). This is shown below. Note that for every $T \subsetneq N$, $i \in \sigma_{\mathcal{H}_k}(T)$ it holds that

$$f^{v_A}_{\mathcal{H}_k, F_{(N,v_A)}}(i,T) = \min_{U: U \in \mathcal{H}_k, i \in U \subset T} (F_{(N,v)}(U) + A) = f^{v}_{\mathcal{H}_k, F_{(N,v)}}(i,T) + A.$$

Consider a coalition $T \subsetneq N$ and two possible variants:

1. $\sigma_{\mathcal{H}_k}(T) \neq T$

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = \frac{y(T) - v_A(T) - \sum_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v_A}_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) + A|T| - v(T) - \sum_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v}_{\mathcal{H}_k,F}(i,T) - A|\sigma_{\mathcal{H}_k}(T)|}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v}_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} + A = F^{(N,v)}_{\mathcal{H}_k}(T,x) + A$$

2. $\sigma_{\mathcal{H}_k}(T) = T$

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = y(T) - v_A(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f^{v_A}_{\mathcal{H},F}(i,T)$$

$$= x(T) + A|T| - v(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) - A|T| + \max_{i \in T} f_{\mathcal{H},F}(i,T) + A =$$

$$= x(T) - v(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f_{\mathcal{H},F}(i,T) + A = F^{(N,v)}_{\mathcal{H}_k}(T,x) - A$$

In this way, for every coalition $T \subsetneq N$ it holds that

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = F^{(N,v)}_{\mathcal{H}_k}(T,x) + A$$

and therefore collections $\mathcal{H}_{k+1}$ in both games (N, v_A) and (N, v) coincide. ■

We show that the SD-prenucleolus does not satisfy the null player property by showing that there is incompatibility between strong aggregate monotonicity, the null player property and core stability.

Proposition 4.4. *If a solution φ defined in the class of all TU games satisfies core stability and the null player property then φ does not satisfy the strong aggregate monotonicity property.*

Proof. Consider the following two games (N, v_1) and (N, v_2) where $N = \{1, 2, 3, 4\}$ and

$$v_1(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 0 & \text{if } |S| = 2 \text{ and } 4 \in S \\ 4 & \text{otherwise,} \end{cases}$$

$$v_2(S) = \begin{cases} 6 & \text{if } S = N \\ v_1(S) & \text{if } S \neq N. \end{cases}$$

In game (N, v_1) player 4 is a null player and therefore $\varphi_4(N, v_1) = 0$. In game (N, v_2) the core is $\{(2, 2, 2, 0)\}$ and therefore $\varphi_4(N, v_2) = 0$. It must be concluded that φ violates strong aggregate monotonicity. ■

Therefore, the SD-prenucleolus and the per capita prenucleolus do not satisfy the null player property on the class of all TU games.

Obviously, on the class of balanced games a solution that satisfies core stability must satisfy the null player property. But this is not necessarily true for the NPO property. For example, the per capita prenucleolus does not satisfy the NPO property. The result below reinforces the interest in the new solution.

Proposition 4.5. *The SD-prenucleolus satisfies the NPO property on the class of balanced games.*

Proof. Consider a balanced game (N, v) where $i \in N$ is a null player. Let $x \in C(N, v)$ be a core allocation. To prove the NPO property of the SD-prenucleolus it is sufficient to show that for every $S \subset N \setminus \{i\}$

$$F(S, x) = F(S \cup \{i\}, x).$$

It is immediately apparent that $x_i = 0$ and coalition $\{i\}$ has the minimal satisfaction, which is 0. Therefore for coalition $P = \arg\min_{S \subset N \setminus \{i\}} \frac{x(S) - v(S)}{|S|}$ it holds that $F(P, x) = F(P \cup \{i\}, x)$.

Consider that for coalitions that obtain their satisfaction before step k it holds that $F(S,x) = F(S \cup \{i\}, x)$. We will prove that for the step k of the algorithm and any coalition $S \in \mathcal{H}_k$, $S \subset N \setminus \{i\}$ it also holds that

$$F_{\mathcal{H}_k}(S,x) = F_{\mathcal{H}_k}(S \cup \{i\}, x). \tag{4.1}$$

Note that

$$\sigma_{\mathcal{H}_k}(S \cup \{i\}) = S \cup \{i\} \Leftrightarrow \sigma_{\mathcal{H}_k}(S) = S$$

Consider two cases (relevant and non relevant coalitions):

1. $\sigma_{\mathcal{H}_k}(S \cup \{i\}) \neq S \cup \{i\}$. Then

$$F_{\mathcal{H}_k}(S \cup \{i\}) = \frac{x(S \cup \{i\}) - v(S \cup \{i\}) - \sum\limits_{j \in \sigma_{\mathcal{H}_k}(S \cup \{i\})} f_{\mathcal{H}_k,F}(j,S)}{|S| + 1 - |\sigma_{\mathcal{H}_k}(S \cup \{i\})|} =$$

$$= \frac{x(S) - v(S) - \sum\limits_{j \in \sigma_{\mathcal{H}_k}(S)} f_{\mathcal{H}_k,F}(j,S)}{|S| + 1 - |\sigma_{\mathcal{H}_k}(S)| - 1} = F_{\mathcal{H}_k}(S)$$

2. $\sigma_{\mathcal{H}_k}(S \cup \{i\}) = S \cup \{i\}$. Then

$$F_{\mathcal{H}_k}(S \cup \{i\}) =$$

$$= x(S \cup \{i\}) - v(S \cup \{i\}) - \sum_{j \in S \cup \{i\}} f_{\mathcal{H}_k,F}(j,S) + \max_{j \in S \cup \{i\}} f_{\mathcal{H}_k,F}(j, S \cup \{i\}) =$$

$$= x(S) - v(S) - \sum_{j \in S} f_{\mathcal{H}_k,F}(j,S) + \max_{j \in S} f_{\mathcal{H}_k,F}(j, S \cup \{i\})$$

To show that $F_{\mathcal{H}_k}(S \cup \{i\}) = F_{\mathcal{H}_k}(S)$ it suffices to check that

$$\max_{j \in S} f_{\mathcal{H}_k,F}(j, S \cup \{i\}) = \max_{j \in S} f_{\mathcal{H}_k,F}(j,S).$$

But

$$f_{\mathcal{H}_k,F}(j, S \cup \{i\}) = \min_{T \in \mathcal{H}_k, j \in T \subset S \cup \{i\}} F_{\mathcal{H}_k}(T)$$

From the fact (4.1) for $k' < k$ it can be concluded that for every $T \in \mathcal{H}_k$ it holds that $F_{\mathcal{H}_k}(T) = F_{\mathcal{H}_k}(T \cup \{i\})$. Therefore

$$\min_{T \in \mathcal{H}_k, j \in T \subset S \cup \{i\}} F_{\mathcal{H}_k}(T) = \min_{T \in \mathcal{H}_k, j \in T \subset S} F_{\mathcal{H}_k}(T) \Rightarrow$$

$$\Rightarrow f_{\mathcal{H}_k,F}(j, S \cup \{i\}) = f_{\mathcal{H}_k,F}(j, S)$$

and the proposition has been proved. ■

In the class of balanced games the SD-prenucleolus is the only known single-valued core selector that satisfies the NPO property and strong aggregate monotonicity. The per capita prenucleolus violates the NPO property as the following example shows.

Consider the games (N, v_1) and $(N \backslash \{4\}, v_2)$ where $N = \{1, 2, 3, 4\}$ and

$$v_1(S) = \begin{cases} 7 & \text{if } S \in \{\{1,2,3\}, N\} \\ 4 & \text{if } S \in \{\{1,2\}, \{1,2,4\}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$v_2(S) = \begin{cases} 7 & \text{if } S = N \backslash \{4\} = \{1,2,3\} \\ 4 & \text{if } S = \{1,2\} \\ 0 & \text{otherwise.} \end{cases}$$

In game (N, v_1) player 4 is a null player and game $(N \backslash \{4\}, v_2)$ results after eliminating player 4 from game (N, v_1). The per capita prenucleolus of game (N, v_1) is $(2.25, 2.25, 1.5, 0)$ and the per capita prenucleolus of game $(N \backslash \{4\}, v_2)$ is $(3, 3, 1)$.

4.3. Kohlberg's characterization

We provide the equivalent of Kohlberg's theorem for the SD-prenucleolus. For this purpose we introduce the following notation. Given an allocation x and a real number α we define the following set of coalitions

$$\mathcal{B}_\alpha = \{S \subsetneq N : F(S, x) \leq \alpha\}.$$

The theorem is useful for checking whether an allocation is the SD-prenucleolus of a game or not. In fact, it is used to prove the main result of Section 6.

Theorem 4.6. *Let (N, v) be a TU game and x be an allocation. Then $x = SD(N, v)$ if and only if the collection of sets $\mathcal{B}_\alpha$ is empty or balanced*[4] *for every α.*

[4] See Peleg and Sudholter (2007) for the definition of a balanced collection of sets.

Proof. Assume that $x = SD(N, v)$ and that the theorem is not true. Let us choose the minimal α for which the collection of sets $\mathcal{B}_\alpha$ is nonempty and not balanced. It is immediate that the collection $\mathcal{B}_\alpha$ coincides with the collection $\mathcal{H}_k$ for some k.

The assumption of the minimality of the value α implies that for every $m < k$ the collection of sets $\mathcal{H}_m$ is balanced. The non-balancedness of the collection $\mathcal{H}_k$ implies that there exists a vector y such that

1. $\sum\limits_{i \in N} y_i = 0$
2. $\sum\limits_{i \in S} y_i \geq 0$ for each $S \in \mathcal{H}_k$
3. There is $S \in \mathcal{H}_k$ such that $\sum\limits_{i \in S} y_i > 0$

Moreover, by using the fact that the collection $\mathcal{H}_{k-1}$ is balanced we can conclude that $\sum\limits_{i \in T} y_i = 0$ for every $T \in \mathcal{H}_{k-1}$.

Let us consider the vector $x + \varepsilon y$ for "‘small"’ positive value ε. It holds that

1. for every $T \in \mathcal{H}_{k-1}$ the satisfaction with respect to vector $x + \varepsilon y$ is equal to the satisfaction with respect to vector x
2. for every $T \in \mathcal{H}_k \setminus \mathcal{H}_{k-1}$ the satisfaction with respect to vector $x + \varepsilon y$ is higher than or equal to the satisfaction with respect to vector x and there exists the coalition U such that this inequality is strong.

It is also immediately apparent that a value ε be chosen that is so small that the following collections $\mathcal{H}_m$ for $m > k$ will be not important. Therefore the vector $x + \varepsilon y$ dominates the vector x.

Therefore if a collection $\mathcal{H}_k$ is not balanced then the allocation x is not the SD-prenucleolus of the game. And if the allocation x is not the SD-prenucleolus of the game then there exists some collection $\mathcal{H}_k$ that is not balanced. ■

Using this theorem it can be asserted that the allocation $(5, 1, 0)$ is not the SD-prenucleolus of the second TU game in Section 3.

In general, the computation of the new solution is not an easy task. Like the prenucleolus, the calculation of the SD-prenucleolus of a game is an open challenge. In this sense, the characterization above is a first step that allows it to be checked whether an allocation is the SD-prenucleolus of the game. In Section

6 we introduce a formula for computing the SD-prenucleolus of veto balanced games.

5. Axiomatizations

5.1. The SD-Reduced Game Property (SD-RGP)

Similarly to the axiomatization of the prenucleolus by Sobolev (1975), we axiomatize the SD-prenucleolus with an almost identical set of axioms where the Davis-Maschler reduced game property is replaced by another reduced game property: the SD-reduced game property.

We introduce a lemma that is used to prove that the SD-RGP is well-defined.

Lemma 5.1. *Let N be a finite set of players, vector $x \in \mathbb{R}^N$, number V and a vector $f \in \mathbb{R}^{2^N \setminus \{N\}}$. Then there is a unique TU game (N, v) such that*

1. *$v(N) = V$*
2. *for every $S \subsetneq N$ it holds that $F(S, x) = f_S$.*

Proof. Consider subsets of N starting from subsets with minimal satisfaction. The calculation of $v(S)$ for current coalition S requires only coalitions with lower satisfactions and these coalitions have been considered before. ■

Definition 5.2. *Let (N, v) be a TU game, $S \subset N$ and $x \in X(N)$. A game (S, v_S^x) is the SD-reduced game with respect to D and x if*

1. *$v_S^x(S) = v(N) - x(N \setminus S)$*
2. *for every $T \subsetneq S$*

$$F^{(S,v_S^x)}(T, x_S) = \min_{U \in N \setminus S} F^{(N,v)}(U \cup T, x).$$

From Lemma 5.1 we conclude that for any game (N, v) and any allocation x the SD-reduced game exists and is unique.

In the characterization of the SD-prenucleolus,this new reduced game property plays the role played by the DM-reduced game property in the characterization of the prenucleolus (Sobolev, 1975.)

5.2. The SD-prenucleolus

We provide the equivalent of Sobolev's theorem for the SD-prenucleolus (by using SD-RGP instead of the Davis-Maschler reduced game property). Kleppe (2010) axiomatizes the per capita prenucleolus by replacing the DM-RGP with another reduced game property.

Let $\mathcal{G}$ be the class of all TU games for players from some universal infinite player set $\mathcal{N}$. Let $X(N, v)$ be a set of efficient allocations for a game (N, v). Given a game (N, v) and an allocation $x \in X(N, v)$ we define game $(N, \phi_x(v))$ as follows:

1. $\phi_x(v)(N) = v(N)$
2. for every $T \subsetneq N$

$$F^{(N,v)}(T, x) = x(T) - \phi_x(v)(T).$$

With this transformation we obtain that a satisfaction of coalition S in game (N, v) with respect to vector x is equal to the negative excess of S in game $(N, \phi_x(v))$. Therefore if the vector x coincides with the SD-prenucleolus in the game (N, v) then it coincides with the prenucleolus in the game $(N, \phi_x(v))$.

This transformation is well-defined for an arbitrary game (N, v). Also it is simple to see that the reverse transformation ϕ_x^{-1} is also well-defined.

The TU game (N, v) is said to be *transitive* if the group of symmetries of this game $(SYM(N, v))$ is transitive. We now prove several auxiliary facts.

Lemma 5.3. *1. If a TU game (N, v) is transitive and $x = SD(N, v)$ then the game $(N, \phi_x(v))$ is also transitive.*

2. If a TU game (N, v) is transitive and $x = PN(N, v)$ then the game $(N, \phi_x^{-1}(v))$ is also transitive.

Proof. It is sufficient to note that by anonymity of the SD-prenucleolus (or the prenucleolus) it holds that $x_i = x_j$ for every $i, j \in N$. ■

Proposition 5.4. *If a TU game (N, v) is transitive then there is a game (N, w) such that $(N, v) = (N, \phi_x(w))$ where $x = SD(N, w)$.*

Proof. Define vector y as follows: $y_i = \frac{v(N)}{|N|}$ for every $i \in N$. It is true that $y = PN(N, v)$. Consider the game $(N, \phi_y^{-1}(v))$. From the previous Lemma this game is transitive and therefore $y = SD(N, v)$.

Therefore we find the transitive game (N, w) (it is $(N, \phi_y^{-1}(v))$) such that $(N, v) = (N, \phi_y(w))$ where $y = SD(N, w)$. ■

Proposition 5.5. *Let N be a finite set and $S \subset N$. Let (N, v) and (S, w) be two TU games and $x \in X(N, v)$. Consider games $(N, \phi_x(v))$ and $(S, \phi_{x|_S}(w))$. Then the following two conditions are equivalent:*

1. (S, w) is the SD-reduced game of (N, v) with respect to S and x

2. $(S, \phi_{x|_S}(w))$ is the Davis-Maschler reduced game of $(N, \phi_x(v))$ with respect to S and x.

Proof. The proof is evident because the satisfactions are transformed to usual negative excesses under function ϕ. ■

Proposition 5.6. *Let (N, v) be a TU game such that $PN_i(N, v) = 0$ for each $i \in N$. Then there is a finite set $M \supset N$ and a transitive game (M, w) with $w(M) = 0$ such that the game (N, v) is the Davis-Maschler reduced game of (M, w) with respect to the prenucleolus (because of transitivity $PN_i(M, w) = 0$ for all $i \in M$).*

Proof. This fact was proved by Sobolev in his characterization of the prenucleolus. See Sobolev (1975). ■

The main theorem of this section can now be proven:

Theorem 5.7. *On the class of all TU games for universal infinite set of players the SD-prenucleolus is the unique single-valued solution that satisfies covariance, anonymity and SD-RGP.*

Proof. It has been shown that the SD-prenucleolus satisfies all these properties. Assume that A is other solution satisfying the properties. Consider an arbitrary game (N, v). We will show that $A_i(N, v) = SD_i(N, v)$ for all $i \in N$.

Consider game (N, v_0), defined as follows:

$$v_0(S) = v(S) - \sum_{i \in S} SD_i(N, v) \text{ for all } S \subset N$$

Consider game $(N, \phi_x(v_0))$ for $x = SD(N, v_0)$. In this game (by Kohlberg´s theorem) the vector x coincides with the prenucleolus and Proposition 5.6 can therefore be used..

Construct a transitive game (M, w) such that $PN_i(M, w) = 0$ for each $i \in M$ and that $(N, \phi_x(v_0))$ is the Davis-Maschler reduced game of (M, w) with respect to the prenucleolus.

Now consider the game $(M, \phi_y^{-1}(w))$ where $y_i = 0$ for all i (therefore y coincides with the prenucleolus of the game (M, w)). By Lemma 5.3 this game is also transitive and therefore (by anonymity and single-valuedness of A) it holds that $A_i(M, \phi_y^{-1}(w)) = 0$ for each $i \in M$.

From Proposition 5.5 it holds that the game (N, v_0) is the SD-reduced game of $(M, \phi_y^{-1}(w))$ with respect to vector y. By SD-RGP of A it can be concluded that for each $i \in N$ it holds that $A_i(N, v_0) = 0$. Therefore in the initial game (N, v) for every $i \in N$, $A_i(N, v) = SD_i(N, v)$ is obtained. ■

5.3. The SD-prekernel

The SD-RGP is also one of the main axioms in the characterization of the SD-prekernel. This solution arises naturally whenever the satisfaction vector is defined. With this tool we can define the complaint of a player i against a player j as the minimal satisfaction obtained with coalitions that contain player i but not player j. The similarities with the classic prekernel are immediately apparent and the axiomatization of the new SD-prekernel is almost identical to the axiomatization of the classic prekernel (see Peleg, 1986). Obviously, the only change is to replace the DM-reduced game property by the SD-reduced game property. Given a TU game (N, v) and an allocation $x \in X(N, v)$ the complaint of player i against player j is defined as follows:

$$s_{ij}(x) = \min_{S: i \in S, j \notin S} F(S, x).$$

The SD-prekernel of a TU game (N, v) is:

$$SDPK(N, v) = \{x \in X(N, v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\}$$

It is immediately apparent that the SD-prekernel satisfies ETP, covariance and SD-RGP. We now show that it is the maximal solution with all these properties.

Theorem 5.8. *In the class of all TU games the SD-prekernel is the maximal solution that satisfies ETP, covariance and the SD-reduced game property.*

The proof is similar to that for the characterization of the prekernel provided by Peleg.

An immediate corollary of the theorem is that the SD-prenucleolus of a game is an element of the SD-prekernel of the game. In some cases, such as glove games, this inclusion is strict.

Consider a 4-person glove game (N, v) defined as follows:

$$v(S) = \begin{cases} 4 & \text{if } S = N \\ 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1, 2\}, \{3, 4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

It is immediately apparent that $(2, 2, 0, 0)$ is an element of the SD-prekernel of the game and that the SD-prenucleolus of (N, v) is $(1, 1, 1, 1)$.

Alternatively, in this characterization the maximality can be replaced by converse SD-RGP[5].

6. Games with Veto Players

The class of games with veto players has been widely used to model economic situations where the presence of special players is needed in order to achieve some positive outcome. The list of papers that consider TU games with veto players is

[5] We omit the formal definition of this property. We only want to emphasize axiomatic similarities between the classic prekernel and the SD-prekernel.

long. Our main purpose is to provide an easy way to compute the SD-prenucleolus of games with veto players.

Arin and Feltkamp introduce the Serial Rule for the class of veto balanced games. Let (N, v) be a game with veto players and let player 1 be a veto player. Define for each player i a value d_i as follows:

$$d_i = \max_{S \subseteq N \setminus \{i\}} v(S).$$

Then $d_1 = 0$. Let $d_{n+1} = v(N)$ and rename players according to the nondecreasing order of those values. That is, player 2 is the player with the lowest value besides player 1 and so on. The solution SR associates to each game with veto players, (N, v), the following payoff vector:

$$SR_l(N, v) = \sum_{i=l}^{n} \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1, ..., n\}.$$

Note that since $d_1 = 0$ the solution is efficient. If there is no veto player the solution is not efficient.

The example in Section 3 illustrates how the solution behaves. The 4-person game has a veto player, player 1. Recall the characteristic function of the game:

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,4\}, \{1,2,4\}\} \\ 4 & \text{if } S = \{1,2,3\} \\ 8 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the vector of d-values we get:

$$(d_1, d_2, d_3, d_4, d_5) = (0, 1, 1, 4, 8).$$

Applying the formula

$$\begin{aligned} SR_1 &= \tfrac{d_2-d_1}{1} + \tfrac{d_3-d_2}{2} + \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 3 \\ SR_2 &= \tfrac{d_3-d_2}{2} + \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 2 \\ SR_3 &= \tfrac{d_4-d_3}{3} + \tfrac{d_5-d_4}{4} = 2 \\ SR_4 &= \tfrac{d_5-d_4}{4} = 1. \end{aligned}$$

We prove that for monotonic[6] games with veto players the Serial Rule and the SD-prenucleolus coincide.

We present several lemmas that are used in the proof of the main theorem.

Lemma 6.1. *Let (N, v) be a monotonic veto game and let $x = SR(N, v)$. Let l be a non veto player and let S be a coalition such that $l \in S$ and $F(S, x) > x_l$. Then $f_{\mathcal{H},F}(l, S) = x_l$.*

Proof. It is immediately apparent that the lemma is true for player n (the player with highest d-value) since $x_n = \min\limits_{S \subset N} \frac{x(S)-v(S)}{|S|} = \frac{x(N\setminus\{n\})\_d_n}{n-1}$. The lemma also must be true for player n-1 since

$$x_{n-1} = \min_{S \subset N, S \notin \{N\setminus\{n\},\{n\}\}} F(S, x) = F(N\setminus\{n-1\}, x) = \frac{\sum\limits_{l=1}^{n-2} x_l - d_{n-2}}{n-2}.$$

Following similar arguments, it is not difficult to check that if the lemma holds for player k it must hold for player k-1. ■

Lemma 6.2. *Let (N, v) be a monotonic veto game. Let $x = SR(N, v)$ and let i be a non veto player. Then $F(N\setminus\{i\}, x) = x_i$.*

Proof. Let $T = \{l \in N\setminus\{i\} : x_i < x_l\}$ and let $P = \{l \in N\setminus\{i\} : x_i \geq x_l\}$. Note that since the game is monotonic $v(N\setminus\{i\}) = d_i$. Then by lemma 6.1

$$F(N\setminus\{i\}, x) = \frac{\sum\limits_{l \in P} SR_l - d_i}{|P|} = SR_i(N, v).$$

This last equality is a consequence of the fact that for any k

$$\sum_{l=1}^{k-1} (SR_l - SR_k) = d_k.$$

■

[6] If a game with veto players is monotonic then is balanced since the allocation where a veto player receives $v(N)$ and the rest receive 0 is a core allocation.

Lemma 6.3. *Let* (N, v) *be a monotonic veto game and let* $x = SR(N, v)$. *Let* S *be a coalition without the veto player. Then* $F(S, x) = \max_{i \in S} x_i$.

Proof. Let $p = \max_{i \in S} x_i$. Let $T = \{i \in S : x_i = p\}$ and let $P = \{i \in S : x_i < p\}$. Then for $l \in P$ and applying lemma 6.1 it holds that

$$x_l = F(l, x) < F(S, x) = \frac{x(T)}{|T|} = p.$$

■

Lemma 6.4. *Let* (N, v) *be a monotonic veto game. Let* $x = SR(N, v)$ *and let* l *a non veto player. Let* S *be a coalition containing the veto players such that* $l \notin S$ *and* $x_l = \max_{i \notin S} x_i$. *Then* $F(\{l\}, x) = x_l \leq F(S, x)$.

Proof. Assume on the contrary that $x_l > F(S, x)$.

Let l be a non veto player such that $l \notin S$ and $x_l = \max_{i \notin S} x_i$. Let $T = \{i \in S : x_i \geq x_l\}$ and let $P = \{i \in S : x_i < x_l\}$. It is immediate that for $i \in P$ it holds that $F(\{i\}, x) < F(S; x)$. Therefore

$$F(S, x) = \frac{x(T) - v(S)}{|T|} \geq \frac{x(T) - v(N \setminus \{l\})}{|T|} = x_l.$$

The first equality results from applying lemma 6.1. The last inequality holds because of the monotonicity of (N, v) and the last equality is a consequence of lemma 6.2. ■

The main theorem of this section establishes the coincidence of the Serial Rule and the SD-prenucleolus on the class of veto monotonic games.

Theorem 6.5. *Let* (N, v) *be a monotonic veto game. Then* $SR(N, v) = SD(N, v)$.

Proof. The proof is based in the above lemmas. Consider the collection of coalitions S for which $F(S, SR(N, v) \leq k$. From lemma 6.3 if this collection contains a coalition without a veto player all players of this coalition appear also in the collection as singletons. By lemma 6.4 and 6.2 if there is a coalition S containing veto players and without non veto player l then coalitions $N\setminus\{l\}$ and $\{l\}$ are also present. Therefore for a non veto player i one of the two statements is true: either coalition $\{l\}$ is in the collection or all coalitions containing the veto players also contain player i. It is clear that such collection is always balanced. ■

Similarly to the case of the prenucleolus (see Arin and Feltkamp (1997)) for the SD-prenucleolus it holds that the complaint of non veto players against the veto player uses the singletons as a coalition[7]. This can be used to prove that the SD-prenucleolus is the only element of the SD-prekernel of a monotonic veto game.

It is clear that the solution denoted by SR satisfies monotonicity. That is, on the class of monotonic veto games the SD-prenucleolus satisfies coalitional monotonicity. (See Arin and Feltkamp (2011) to check that this result is not true for the prenucleolus and the per capita prenucleolus.)

The result of Theorem 6.5 is not necessarily true if the game is not monotonic. Consider the following 3-person balanced game. Let $N = \{1,2,3\}$ and $v(\{1\}) = v(\{1,3\}) = v(\{1,2\}) = -3$ and $v(S) = 0$ otherwise. Then $SR(N,v) = (0,0,0) \neq SD(N,v) = (-2,1,1)$.

7. Conclusions

We introduce a new solution concept for TU games: a solution that is a lexicographic value and can therefore be seen as a member of a family of solutions that includes the prenucleolus and the per capita prenucleolus. The new solution is not a weighted prenucleolus and incorporates into its definition the idea that the surplus obtained by a coalition is divided among its members in a coherent way. This interpretation links the solution with the per capita prenucleolus and both solutions can be seen as members of a family of solutions that provides this distribution of the surplus[8]. Apart from the different way of interpreting the clas-

[7] This is not the case for the per capita prenucleolus, so a different argument is needed to find an algorithm for computing the per capita prenucleolus of a veto monotonic game.

[8] The definition of the prenucleolus does not allow such interpretation.

sic concept of excess/satisfaction, the attractiveness of the new solution relies on two interesting facts: The SD-prenucleolus is the only known solution that satisfies core stability, strong aggregate monotonicity and NPO property in the class of balanced games. The SD-prenucleolus is the only known solution defined in the class of all TU games that satisfies core stability, continuity and coalitional monotonicity in the class of veto balanced games.

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