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Semilinear p-evolution equations in Sobolev spaces

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Abstract

We prove local in time well-posedness in Sobolev spaces of the Cauchy problem for semi-linear p-evolution equations of the first order with real principal part, but complex valued coefficients for the lower order terms, assuming decay conditions on the imaginary parts as $|x| \to \infty$.

MSC: primary 35G25; secondary 46A61

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1. Introduction and main result

Given an integer $p \geq 2$, we consider the Cauchy problem

$$\begin{cases} P_u(D)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1.1}$$

for the semi-linear p-evolution operator of the first order

$$P_u(D)u = P(t,x,u(t,x),D_t,D_x)u := D_t u + a_p(t)D_x^p u + \sum_{j=0}^{p-1} a_j(t,x,u)D_x^j u, \tag{1.2}$$

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where $D = \frac{1}{i}\partial$, $a_p \in C([0,T];\mathbb{R})$, and $a_j \in C([0,T]; C^\infty(\mathbb{R}\times\mathbb{C}))$ with $x \mapsto a_j(t,x,w) \in \mathcal{B}^\infty(\mathbb{R})$, for $0 \le j \le p-1$ ($\mathcal{B}^\infty(\mathbb{R})$ is the space of complex valued functions which are bounded on $\mathbb{R}$ together with all their derivatives).

The condition that a_p is real valued means that the principal symbol (in the sense of Petrowski) of P has the real characteristic $\tau = -a_p(t)\xi^p$; by the Lax–Mizohata theorem, this is a necessary condition to have a unique solution, in Sobolev spaces, of the Cauchy problem (1.1) in a neighborhood of $t = 0$, for any $p \ge 1$ (cf. [26]).

The condition that $a_j(t,x,w) \in \mathbb{C}$ for $0 \le j \le p-1$ implies some decay conditions on the coefficients as $|x| \to \infty$, because of the well-known necessary condition of Ichinose (cf. [21]) for 2-evolution linear equations, and of the very recent necessary condition of [9] for p-evolution linear equations with arbitrary $p \ge 2$.

We give the main result of this paper, Theorem 1.3, for the operator P in (1.2), i.e. under the condition that a_p does not depend on the monodimensional space variable $x \in \mathbb{R}$, only for simplicity's sake. Indeed, we can generalize Theorem 1.3 to the case $a_p = a_p(t,x)$, $x \in \mathbb{R}$: see Theorem 4.1. Moreover, Theorems 1.3 and 4.1 can be generalized to the case $x \in \mathbb{R}^n$, up to some technical complications, following the ideas of [23,16] and [14]: see Remark 4.2.

The monodimensional problem for either operator (1.2) or (4.3) is of interest by itself; for instance, in the case $p = 3$ our model recovers equations of Korteweg–de Vries type, widely used to describe the propagation of monodimensional waves of small amplitudes in waters of constant depth.

The original Korteweg–de Vries equation is

$$\partial_t u = \frac{3}{2}\sqrt{\frac{g}{h}}\,\partial_x\left(\frac{1}{2}u^2 + \frac{2}{3}\alpha u + \frac{1}{3}\sigma\partial_x^2 u\right), \tag{1.3}$$

where u represents the wave elevation with respect to the water's surface, g is the gravity constant, h the (constant) level of water, α a fixed small constant and

$$\sigma = \frac{h^3}{3} - \frac{Th}{\rho g},$$

with T the surface tension, ρ the density of the fluid. The operator in (1.3) is of the form (1.2) with constant coefficients $a_p = a_3$ and $a_2 = 0$, and with $a_1 = a_1(u)$.

Assuming that the water's level h depends on x, we are led to an operator of the form (4.3). Our model can also be applied to study the evolution of the wave when the seabed is variable.

The n-dimensional problem can find applications in the study of Schrödinger type equations ($p = 2$), and of higher order equations that can be factorized into the product of Schrödinger type equations, as for instance the Euler–Bernoulli equation of a vibrating plate

$$\partial_t^2 u + a^2(t)\Delta_x^2 u + \sum_{|\alpha|\le 3} b_\alpha(t,x)\partial_x^\alpha u = 0$$

(vibrating beam in the case $n = 1$). The above operator factorizes into the product $(\partial_t - ia(t)\Delta_x - b(t,x,\partial_x))(\partial_t + ia(t)\Delta_x + b(t,x,\partial_x))$ of two (pseudo-differential) Schrödinger type operators, modulo terms of order zero.

The aim of this paper is to give sufficient decay conditions on the coefficients of $P_u(D)$ in order that the Cauchy problem (1.1) be locally in time well-posed in H^∞.

Results of H^∞ well-posedness of the Cauchy problem for linear p-evolution equations of the first order

$$D_t u + a_p(t) D_x^p u + \sum_{j=0}^{p-1} a_j(t,x) D_x^j u = f(t,x), \tag{1.4}$$

or for linear p-evolution equations of higher order, have already been obtained, first for real valued (or complex valued, with imaginary part not depending on x) coefficients (see for instance [1,11] and the references therein), then for complex valued coefficients depending on the space variable x under suitable decay conditions on the coefficients as $|x| \to +\infty$ (see [22,23,13, 7,4] for equations of the form (1.4), [5,12,15,16,27] for higher order equations, and [10] for equation (1.4) in a different framework). Among all these results, in the present paper we shall need an extension of the following theorem of [7] (see Theorem 2.1 below):

Theorem 1.1. *Let us consider the operator (1.4) with $a_p \in C([0,T];\mathbb{R})$, $a_p(t) \geq 0$ $\forall t \in [0,T]$ and $a_j \in C([0,T];\mathcal{B}^\infty)$ for $0 \leq j \leq p-1$. Suppose that there exists $C > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$:*

$$|\operatorname{Re} D_x^\beta a_j(t,x)| \leq C a_p(t), \qquad 0 \leq \beta \leq j-1,\ 3 \leq j \leq p-1, \tag{1.5}$$

$$|\operatorname{Im} D_x^\beta a_j(t,x)| \leq \frac{C a_p(t)}{\langle x \rangle^{\frac{j-[\beta/2]}{p-1}}}, \qquad 0 \leq \left[\frac{\beta}{2}\right] \leq j-1,\ 3 \leq j \leq p-1, \tag{1.6}$$

$$|\operatorname{Im} a_2(t,x)| \leq \frac{C a_p(t)}{\langle x \rangle^{\frac{2}{p-1}}}, \tag{1.7}$$

$$|\operatorname{Im} a_1(t,x)| + |\operatorname{Im} D_x a_2(t,x)| \leq \frac{C a_p(t)}{\langle x \rangle^{\frac{1}{p-1}}}, \tag{1.8}$$

where $[\beta/2]$ denotes the integer part of $\beta/2$ and $\langle x \rangle := \sqrt{1+x^2}$.

Then, the Cauchy problem associated to equation (1.4) with $u(0,x) = u_0(x)$, $x \in \mathbb{R}$, is well-posed in H^∞ (with loss of derivatives). More precisely, there exists a positive constant σ such that for all $f \in C([0,T];H^s(\mathbb{R}))$ and $u_0 \in H^s(\mathbb{R})$ there is a unique solution $u \in C([0,T];H^{s-\sigma}(\mathbb{R}))$ which satisfies the following energy estimate:

$$\|u(t,\cdot)\|_{s-\sigma}^2 \leq C_s \left(\|u_0\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 \, d\tau \right) \qquad \forall t \in [0,T], \tag{1.9}$$

for some $C_s > 0$, with $\|\cdot\|_s = \|\cdot\|_{H^s}$.

Remark 1.2. Condition (1.6) with $j = p-1$, $\beta = 0$, i.e.

$$|\operatorname{Im} a_{p-1}(t,x)| \leq C a_p(t) \langle x \rangle^{-1},$$

is strictly consistent with the necessary condition

$$\exists M, N > 0: \sup_{x\in\mathbb{R}} \min_{0\le\tau\le t\le T} \int_{-\varrho}^{\varrho} \operatorname{Im} a_{p-1}(t, x + pa_p(\tau)\theta) d\theta \le M \log(1+\varrho) + N, \quad \forall \varrho > 0$$

for H^∞ well-posedness of the Cauchy problem associated to equation (1.4), proved in [9].

As far as we know, semi-linear equations $P_u(D)u = f$ of the form (1.2), or of higher order, have been considered in the case of complex valued coefficients with imaginary part not depending on x (see, for instance, [2]), or in the hyperbolic case (see, for instance, [3,17]). Recently, we considered in [8] semi-linear 3-evolution equations of the first order and in [6] a 2-evolution equation of order 2 that generalizes the Boussinesq equation. We gave sufficient decay conditions on the coefficients for H^∞ well-posedness of the Cauchy problem (see the comments after Theorem 1.3).

Here we consider the general case ($p \ge 2$) of non-linear p-evolution equations of the first order, proving the following:

Theorem 1.3. *Let us assume that there exist constants $C_p > 0$ and $C > 0$ and a function $\gamma : \mathbb{C} \to \mathbb{R}^+$ of class C^7 such that the coefficients of the semi-linear equation (1.2) satisfy for all $(t, x, w) \in [0, T] \times \mathbb{R} \times \mathbb{C}$:*

$$a_p(t) \ge C_p, \tag{1.10}$$

$$|\operatorname{Im}(D_x^\beta a_j)(t, x, w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}}, \quad 0 \le \left[\frac{\beta}{2}\right] \le j-1,\ 3 \le j \le p-1, \tag{1.11}$$

$$|\operatorname{Re}(D_x^\beta a_j)(t, x, w)| \le C\gamma(w) \qquad 0 \le \beta \le j-1,\ 3 \le j \le p-1, \tag{1.12}$$

$$|(D_w^\alpha D_x^\beta a_j)(t, x, w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{j-[(\alpha+\beta)/2]}{p-1}}}, \tag{1.13}$$

$$\alpha \ge 1, \beta \ge 0, \left[\frac{\alpha+\beta}{2}\right] \le j-1,\ 3 \le j \le p-1,$$

$$|\operatorname{Re} a_2(t, x, w)| \le C\gamma(w), \tag{1.14}$$

$$|\operatorname{Im} a_2(t, x, w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{2}{p-1}}}, \tag{1.15}$$

$$|\operatorname{Im} a_1(t, x, w)| + |\operatorname{Im} D_x a_2(t, x, w)| + |D_w a_2(t, x, w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{1}{p-1}}}. \tag{1.16}$$

Then the Cauchy problem (1.1) is locally in time well-posed in H^∞: for all $f \in C([0, T]; H^\infty(\mathbb{R}))$ and $u_0 \in H^\infty(\mathbb{R})$, there exist $0 < T^* \le T$ and a unique solution $u \in C([0, T^*]; H^\infty(\mathbb{R}))$ of (1.1).

Notice that conditions (1.10)–(1.16) correspond exactly to (1.5)–(1.8) for linear equations. In [8] we had to strengthen the leading condition (1.11) for $\beta = 0$, $j = p-1$, $p = 3$, requiring a decay of order $1 + \varepsilon$ instead of order 1 on the sub-leading coefficient:

$$|\operatorname{Im} a_2(t, x, w)| \le \frac{C\gamma(w)}{\langle x\rangle^{1+\varepsilon}}, \quad \epsilon > 0.$$

Also in [6] such a faster decay is required. This is because, both in [8] and [6], we used a fixed point argument to show the existence of a solution to the semi-linear equation. To avoid such a stronger condition, here we make use of the Nash–Moser theorem. Inspired by [18] (see also [17]), the idea is to linearize the equation, fixing $u \in C([0,T], H^\infty(\mathbb{R}))$ and solving the linear Cauchy problem in the unknown $v(t,x)$

$$\begin{cases} P_u(D)v(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}, \\ v(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.17}$$

and then apply the Nash–Moser theorem.

The paper is organized as follows. In Section 2 we briefly retrace the proof of Theorem 1.1 for the linear problem (1.17) taking care of the dependence on u of the constants in the energy estimate (1.9). Then, in Section 3, we prove Theorem 1.3. Section 4 is devoted to generalizations of the main result to the cases $a_p = a_p(t,x)$ and/or $x \in \mathbb{R}^n$. Finally, in Appendix A we collect the main notions about tame spaces and the Nash–Moser Theorem A.11, according to [18,20]; in Appendix B we recall the main tools used to write an energy estimate for the solution v of the linear Cauchy problem (1.17), i.e. sharp-Gårding theorem (with remainder in explicit form, see [24, Ch. 3, Thm. 4.2]) and Fefferman–Phong inequality, [19].

2. The linearized problem

In this section we consider, for a fixed function $u \in C([0,T]; H^\infty(\mathbb{R}))$, the linear Cauchy problem (1.17) in the unknown v. By Theorem 1.1, we know that this Cauchy problem is well-posed with loss of derivatives in Sobolev spaces, and the solution v satisfies the energy estimate (1.9), with a positive constant $C_s = C_s(u)$. The aim of this section is to retrace as briefly as possible the proof of Theorem 1.1 taking care of the dependence of the operator $P_u(D)$ on u, to compute precisely $C_s(u)$: this will be needed in the proof of Theorem 1.3. The computation of $C_s(u)$ clarifies that to write an energy estimate of the form (1.9) it is enough to fix $u \in C([0,T]; H^{4p-3}(\mathbb{R}))$.

The present section is devoted to prove the following:

Theorem 2.1. *Under the assumptions of Theorem 1.3, there exists $\sigma > 0$ such that for every $u \in C([0,T]; H^{4p-3}(\mathbb{R}))$, $f \in C([0,T]; H^s(\mathbb{R}))$ and $u_0 \in H^s(\mathbb{R})$, there exists a unique solution $v \in C([0,T]; H^{s-\sigma}(\mathbb{R}))$ of the Cauchy problem (1.17) and the following energy estimate is satisfied:*

$$\|v(t,\cdot)\|^2_{s-\sigma} \le C_{s,\gamma} e^{(1+\|u\|^{4p-3}_{4p-3})t} \left(\|u_0\|^2_s + \int_0^t \|f(\tau,\cdot)\|^2_s \, d\tau \right) \quad \forall t \in [0,T], \tag{2.1}$$

for some $C_{s,\gamma} > 0$.

The proof of Theorem 2.1 is based on the energy method, after an appropriate change of variable of the form

$$v(t,x) = e^{\Lambda(x,D)} w(t,x), \tag{2.2}$$

and makes use of the sharp-Gårding theorem and the Fefferman–Phong inequality, see Appendix B.

The operator Λ is suitably constructed as a real valued, invertible on L^2 operator with symbol $e^{\Lambda(x,\xi)} \in S^\delta$, $\delta > 0$. The change of variable (2.2) transforms the Cauchy problem (1.17) into the equivalent problem

$$\begin{cases} P_\Lambda(t,x,u(t,x),D_t,D_x)w(t,x) = f_\Lambda(t,x) & (t,x) \in [0,T]\times\mathbb{R} \\ w(0,x) = u_{0,\Lambda}(x) & x \in \mathbb{R} \end{cases} \tag{2.3}$$

for

$$P_\Lambda := (e^\Lambda)^{-1} P e^\Lambda, \quad f_\Lambda := (e^\Lambda)^{-1} f, \quad u_{0,\Lambda} := (e^\Lambda)^{-1} u_0.$$

We are so reduced to show the well-posedness of (2.3) in Sobolev spaces, which is equivalent, since e^Λ has order $\delta > 0$, to the desired well-posedness with loss of $\sigma = 2\delta$ derivatives for the Cauchy problem (1.17).

The operator $\Lambda(x, D_x)$ is a pseudo-differential operator having symbol

$$\Lambda(x,\xi) := \lambda_{p-1}(x,\xi) + \lambda_{p-2}(x,\xi) + \ldots + \lambda_1(x,\xi), \tag{2.4}$$

with

$$\lambda_{p-k}(x,\xi) := M_{p-k}\,\omega\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\,\langle\xi\rangle_h^{-k+1}, \quad 1 \le k \le p-1, \tag{2.5}$$

where $\langle\xi\rangle_h := \sqrt{h^2+\xi^2}$ for $h \ge 1$, $\langle y\rangle := \langle y\rangle_1$, $M_{p-k} > 0$ will be chosen large enough throughout the proof, ω is a $C^\infty(\mathbb{R})$ function such that $\omega(y) = 0$ for $|y| \le 1$ and $\omega(y) = |y|^{p-1}/y^{p-1}$ for $|y| \ge 2$, $\psi \in C_0^\infty(\mathbb{R})$ is such that $0 \le \psi(y) \le 1$ for all $y \in \mathbb{R}$, $\psi(y) = 1$ for $|y| \le \frac{1}{2}$, and $\psi(y) = 0$ for $|y| \ge 1$.

In the following lemma we list the main properties of the symbols $\lambda_k(x,\xi)$ and $\Lambda(x,\xi)$, referring to [7] for all proofs.

Lemma 2.2. *Let χ_E be the characteristic function of the set $E = \{\langle y\rangle \le \langle\xi\rangle_h^{p-1}\} \subset \mathbb{R}^2$. We have that*

$$|\lambda_{p-1}(x,\xi)| \le M_{p-1}\log 2 + M_{p-1}(p-1)\log\langle\xi\rangle_h,$$
$$|\lambda_{p-k}(x,\xi)| \le M_{p-k}\frac{p-1}{k-1}\langle x\rangle^{\frac{k-1}{p-1}}\langle\xi\rangle_h^{-k+1}\chi_E(x), \quad 2 \le k \le p-1;$$

moreover, for every $\alpha \ne 0$, $\beta \ne 0$, there exist $C_{\alpha,\beta} > 0$ such that:

$$|\partial_x^\beta \lambda_{p-k}(x,\xi)| \le C_{0,\beta}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-k+1}\chi_E(x), \quad 1 \le k \le p-1,$$
$$|\partial_\xi^\alpha \lambda_{p-1}(x,\xi)| \le C_{\alpha,0}M_{p-1}\langle\xi\rangle_h^{-\alpha}(1+\log\langle\xi\rangle_h\chi_{\{|\xi|<2h\}}),$$
$$|\partial_\xi^\alpha \lambda_{p-k}(x,\xi)| \le C_{\alpha,0}M_{p-k}\langle x\rangle^{\frac{k-1}{p-1}}\langle\xi\rangle_h^{-\alpha-k+1}\chi_E(x), \quad 2 \le k \le p-1,$$
$$|\partial_\xi^\alpha\partial_x^\beta \lambda_{p-k}(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-\alpha-k+1}\chi_E(x), \quad 1 \le k \le p-1. \tag{2.6}$$

Again, there exist positive constants C, δ independent of h and, for every $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{N}$, positive constants $\delta_{\alpha,\beta}$, independent of h, such that:

$$|\Lambda(x,\xi)| \le C + \delta \log\langle\xi\rangle_h, \tag{2.7}$$

$$|\partial_\xi^\alpha D_x^\beta \Lambda(x,\xi)| \le \delta_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}. \tag{2.8}$$

By (2.7) and (2.8), it can be easily shown that the operator $e^{\Lambda(x,D_x)}$ with symbol $e^{\Lambda(x,\xi)} \in S^\delta$ is invertible for h large enough, say $h \ge h_0$, with inverse

$$(e^\Lambda)^{-1} = e^{-\Lambda}(I+R) \tag{2.9}$$

where I is the identity operator and R is the operator defined by means of the convergent Neumann series $R = \sum_{n=1}^{+\infty} r^n$, with principal symbol

$$r_{-1}(x,\xi) = \partial_\xi \Lambda(x,\xi) D_x \Lambda(x,\xi) \in S^{-1}.$$

Moreover, for $h \ge h_0$ we have

$$|\partial_\xi^\alpha e^{\pm\Lambda(x,\xi)}| \le C_\alpha \langle\xi\rangle_h^{-\alpha} e^{\pm\Lambda(x,\xi)} \qquad \forall \alpha \in \mathbb{N}_0, \tag{2.10}$$

$$|D_x^\beta e^{\pm\Lambda(x,\xi)}| \le C_\beta \langle x\rangle^{-\beta} e^{\pm\Lambda(x,\xi)} \qquad \forall \beta \in \mathbb{N}_0. \tag{2.11}$$

Finally, denoting $iP = \partial_t + A(t,x,u(t,x),D_x)$, with

$$A(t,x,u(t,x),\xi) = ia_p(t)\xi^p + \sum_{j=0}^{p-1} ia_j(t,x,u(t,x))\xi^j,$$

we have that

$$iP_\Lambda = \partial_t + (e^{\Lambda(x,D_x)})^{-1} A(t,x,u,D_x) e^{\Lambda(x,D_x)} \tag{2.12}$$

with

$$\begin{aligned}(e^{\Lambda(x,D_x)})^{-1} A(t,x,u,D_x) e^{\Lambda(x,D_x)} &= e^{-\Lambda(x,D_x)} A(t,x,u,D_x) e^{\Lambda(x,D_x)} \\ &\quad + \sum_{m=0}^{p-2} \frac{1}{m!} \sum_{n=1}^{p-1-m} e^{-\Lambda(x,D_x)} A^{n,m}(t,x,u,D_x) e^{\Lambda(x,D_x)} \\ &\quad + A_0(t,x,D_x),\end{aligned} \tag{2.13}$$

where $A_0(t,x,\xi) \in S^0$ and $A^{n,m}(t,x,u,\xi) = \partial_\xi^m r^n(x,\xi) D_x^m(A(t,x,u,\xi)) \in S^{p-m-n}$ (see [7, Lemma 2.6]). The lower order terms in (2.13) have the same structure as the principal term, so the structure of the operator $(e^\Lambda)^{-1} A e^\Lambda$ is the same as that of $e^{-\Lambda} A e^\Lambda$.

In the proof of Theorem 2.1, the following lemma will be crucial:

Lemma 2.3. *Under the assumptions (1.11), (1.12), (1.13) there exists a positive constant C' such that for every fixed $u \in C([0,T]; H^{4p-3}(\mathbb{R}))$ the coefficients $a_j(t,x,u(t,x))$ of the operator $P_u(D)$ satisfy for all $(t,x) \in [0,T] \times \mathbb{R}$:*

$$|\operatorname{Re} D_x^\beta(a_j(t,x,u(t,x)))| \le C'\gamma(u)(1+\|u\|^\beta_{1+\beta}), \quad 1 \le \beta \le j-1,\ 3 \le j \le p-1 \tag{2.14}$$

$$|\operatorname{Im} D_x^\beta(a_j(t,x,u(t,x)))| \le \frac{C'\gamma(u)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}}(1+\|u\|^\beta_{1+\beta}), \quad \beta \ge 1,\ \left[\frac{\beta}{2}\right] \le j-1,\ 3 \le j \le p-1. \tag{2.15}$$

Proof. Let us compute for $3 \le j \le p-1$ and $\beta \ge 1$ the derivative

$$D_x^\beta(a_j(t,x,u)) = (D_x^\beta a_j)(t,x,u) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \ge 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \ge 1}} c_{q,r}\partial_w^q(D_x^{\beta_1}a_j)(t,x,u)(D_x^{r_1}u)\cdots(D_x^{r_q}u)$$

for some $c_\beta, c_{q,r} > 0$.

From conditions (1.12) and (1.13), using the relationship between geometric and arithmetic mean value and Sobolev's inequality (see [24, Ch. 3, Lemma 2.5]) for the fixed function $u(t) \in H^{4p-3}(\mathbb{R})$, we immediately get that for every $3 \le j \le p-1$, $\beta \ge 1$ and $n \ge 1$

$$\begin{aligned}
|\operatorname{Re} D_x^\beta(a_j(t,x,u))| &\le |\operatorname{Re}(D_x^\beta a_j)(t,x,u)| \\
&\quad + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \ge 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \ge 1}} c_{q,r}|(\partial_w^q D_x^{\beta_1}a_j)(t,x,u)| \cdot |D_x^{r_1}u|\cdots|D_x^{r_q}u| \\
&\le C\gamma(u)\left(1 + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \ge 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \ge 1}} c_{q,r}|D_x^{r_1}u|\cdots|D_x^{r_q}u|\right) \\
&\le C\gamma(u)\left(1 + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \ge 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \ge 1}} c_{q,r}\left(\frac{|D_x^{r_1}u|+\cdots+|D_x^{r_q}u|}{q}\right)^q\right) \\
&\le C\gamma(u)\left(1 + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \ge 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \ge 1}} c_{q,r}\|u\|^q_{n+\beta_2}\right) \\
&\le C'\gamma(u)(1+\|u\|^\beta_{n+\beta});
\end{aligned} \tag{2.16}$$

taking $n = 1$ we get (2.14). On the other hand, from (1.11) and (1.13) we have, with similar computations, that for every $\beta \geq 1$ such that $[\beta/2] \leq j-1$, $3 \leq j \leq p-1$ and $n \geq 1$

$$\begin{aligned} |\operatorname{Im}(D_x^\beta(a_j(t,x,u)))| &\leq |\operatorname{Im}((D_x^\beta a_j)(t,x,u))| \\ &\quad + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} c_{q,r}|(\partial_w^q D_x^{\beta_1} a_j)(t,x,u)| \cdot |D_x^{r_1}u| \cdots |D_x^{r_q}u| \\ &\leq \frac{C\gamma(u)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}} \left(1 + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} c_{q,r}|D_x^{r_1}u| \cdots |D_x^{r_q}u|\right) \\ &\leq \frac{C'\gamma(u)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}}(1 + \|u\|_{n+\beta}^\beta); \end{aligned} \tag{2.17}$$

taking $n = 1$ we have (2.15). □

Proof of Theorem 2.1. We divide the proof of Theorem 2.1 into the following steps:

Step 1. Compute the symbol of the operator $e^{-\Lambda}Ae^{\Lambda}$ and show that its terms of order $p-k$, $1 \leq k \leq p-1$, denoted by $(e^{-\Lambda}Ae^{\Lambda})\big|_{\operatorname{ord}(p-k)}$, satisfy

$$\left|\operatorname{Re}(e^{-\Lambda}Ae^{\Lambda})\big|_{\operatorname{ord}(p-k)}(t,x,u,\xi)\right| \leq C_{(M_{p-1},\ldots,M_{p-k})}(u)\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{p-k} \tag{2.18}$$

for a positive constant

$$C_{(M_{p-1},\ldots,M_{p-k})}(u) = C_{M_{p-1},\ldots,M_{p-k}}(1+\gamma(u))(1+\|u\|_{1+(k-1)}^{k-1})$$

depending on u, $M_{p-1}, \ldots, M_{p-k}$ and not on $M_{p-k-1}, \ldots, M_1$. This allows the constants $M_{p-1}, \ldots, M_1$ to be chosen recursively, since at each step (say "step $p-k$") we have something which depends only on the already chosen $M_{p-1}, \ldots, M_{p-k+1}$ and on the new M_{p-k} that is going to be chosen, and not on the constants $M_{p-k-1}, \ldots, M_1$ which will be chosen in the next steps.

Step 2. We choose $M_{p-1} > 0$ such that

$$\operatorname{Re}\,(e^{-\Lambda}Ae^{\Lambda})\big|_{\operatorname{ord}(p-1)} + \tilde{C}(u) \geq 0$$

for some positive constant $\tilde{C}(u) > 0$ depending on u, and apply the sharp-Gårding Theorem B.1 to $(e^{-\Lambda}Ae^{\Lambda})\big|_{\operatorname{ord}(p-1)} + \tilde{C}(u)$ to get

$$\sigma(e^{-\Lambda}Ae^{\Lambda}) = ia_p\xi^p + Q_{p-1} + \sum_{k=2}^{p-1}(e^{-\Lambda}Ae^{\Lambda})\big|_{\operatorname{ord}(p-k)} + R_{p-1} + A_0,$$

where Q_{p-1} is a positive operator of order $p-1$, $A_0 \in S^0$ and $R_{p-1}(t, x, u, D_x)$ is a remainder of order $p-2$, consisting in the sum of terms satisfying (2.18) with a new constant which is similar to $C_{(M_{p-1},\dots,M_{p-k})}(u)$ but depends on a higher number of derivatives of the fixed function u.

Step 3. To iterate this process, finding positive constants $M_{p-2}, \dots, M_1$ such that

$$\mathrm{Re}\,(e^{-\Lambda} A e^{\Lambda})\big|_{\mathrm{ord}(p-k)} + \tilde{C}(u) \geq 0$$

for some $\tilde{C}(u) > 0$ depending on u, we need to investigate the action of the sharp-Gårding Theorem B.1 to each term of the form

$$(e^{-\Lambda} A e^{\Lambda})\big|_{\mathrm{ord}(p-k)} + S_{p-k},$$

where S_{p-k} denotes terms of order $p-k$ coming from remainders of previous applications of the sharp-Gårding Theorem, for $p-k \geq 3$.

We show at this step that remainders are sums of terms with "the right decay at the right level", in the sense that they satisfy (2.18) with a new constant which is similar to $C_{(M_{p-1},\dots,M_{p-k})}(u)$ but depends on a higher number of derivatives of the fixed function u. Then we apply the sharp-Gårding theorem to terms of order $p-k$, up to order $p-k=3$.

Step 4. In this step we apply the Fefferman–Phong inequality (B.5) to terms of order $p-k=2$ and the sharp-Gårding inequality (B.4) to terms of order $p-k=1$, finally obtaining that

$$\sigma(e^{-\Lambda} A e^{\Lambda}) = i a_p \xi^p + \sum_{s=1}^{p} Q_{p-s}$$

with

$$\begin{aligned}
&\mathrm{Re}\langle Q_{p-s} w, w\rangle \geq 0 \qquad \forall w(t,\cdot) \in H^{p-s}, \quad s = 1, \dots, p-3 \\
&\mathrm{Re}\langle Q_{p-s} w, w\rangle \geq -\bar{C}(u)\|w\|_0^2 \qquad \forall w(t,\cdot) \in H^{p-s}, \quad s = p-2, p-1 \\
&Q_0 \in S^0,
\end{aligned}$$

where $\bar{C}(u)$ is a positive constant which depends on $\gamma(u)$ and on a finite number of derivatives of the fixed function u.

Step 5. We finally look at the full operator in (2.13) and prove that $e^{-\Lambda} A^{n,m} e^{\Lambda}$ satisfies the same estimates (2.18) as $e^{-\Lambda} A e^{\Lambda}$, with suitable constants still depending on γ and on a finite number of derivatives of u. Thus, the results of Step 4 hold for the full operator $(e^{\Lambda})^{-1} A e^{\Lambda}$ and not only for $e^{-\Lambda} A e^{\Lambda}$, i.e. there exists a constant $\bar{C}(u) > 0$, still depending on a (higher) finite number of derivatives of u, such that

$$\mathrm{Re}\langle (e^{\Lambda})^{-1} A e^{\Lambda} w, w\rangle \geq -\bar{C}(u)\|w\|_0^2 \qquad \forall w(t,\cdot) \in H^{p-1}.$$

From this, the thesis follows by the energy method.

These steps have already been followed in the proof of Theorem 1.1 in [7]. Here we briefly retrace the proof of the five steps, outlining what is new with respect to [7] and referring to it for all the other computations.

Step 1. By developing asymptotically the symbols of the products of pseudo-differential operators, we have, as in [7], that:

$$\begin{aligned}\sigma(e^{-\Lambda}Ae^{\Lambda}) &= \sum_{m=0}^{p-1}\sum_{j=m+1}^{p}\binom{j}{m}(ia_j)(e^{-\Lambda}D_x^m e^{\Lambda})\xi^{j-m} \\ &\quad+\sum_{m=0}^{p-2}\sum_{j=m+2}^{p}\sum_{\alpha=1}^{j-m-1}\sum_{\beta=0}^{\alpha}\frac{1}{\alpha!}\binom{j}{m}\binom{\alpha}{\beta} \\ &\quad\times(\partial_\xi^\alpha e^{-\Lambda})(iD_x^\beta a_j)(D_x^{m+\alpha-\beta}e^{\Lambda})\xi^{j-m}+A_0 \\ &=: A_I+A_{II}+A_0, \end{aligned} \tag{2.19}$$

for some $A_0 \in S^0$; the difference is that here $a_j = a_j(t, x, u(t, x))$ depend on x both in the second and in the third variable, so in A_{II} we have to make use of Lemma 2.3 to estimate

$$\mathrm{Re}(iD_x^\beta a_j) = \mathrm{Im}(D_x^\beta a_j) \text{ and } |iD_x^\beta a_j|, \quad \beta \neq 0.$$

In [7] we have shown that the terms in A_{II} with $m+\alpha-\beta \geq 1$ satisfy, for some $c > 0$,

$$\begin{aligned}&|(\partial_\xi^\alpha e^{-\Lambda})(iD_x^\beta a_j)(D_x^{m+\alpha-\beta}e^{\Lambda})\xi^{j-m}| \\ &\leq c|D_x^\beta a_j| \sum_{\substack{\alpha_1+\ldots+\alpha_{p-1}=\alpha \\ \gamma_1+\ldots+\gamma_{p-1}=m+\alpha-\beta \\ r_1+\ldots+r_{q_k}=\alpha_k;\, r_i,\alpha_k\geq 1 \\ s_1+\ldots+s_{p_{k'}}=\gamma_{k'};\, s_i,\gamma_{k'}\geq 1}} \prod_{k,k'=1}^{p-1} M_{p-k}^{q_k}\frac{\langle x\rangle^{\frac{k-1}{p-1}q_k}}{\langle\xi\rangle_h^{\alpha_k+q_k(k-1)}}\cdot M_{p-k'}^{p_{k'}}\frac{\langle x\rangle^{\frac{k'-1}{p-1}p_{k'}-\gamma_{k'}}}{\langle\xi\rangle_h^{p_{k'}(k'-1)}}\langle\xi\rangle_h^{j-m}. \end{aligned} \tag{2.20}$$

In (2.20) the term $-\partial_\xi\Lambda(ia_p\xi^p)D_x\Lambda = -r_{-1}(x,\xi)(ia_p\xi^p)$ appears. This term will cancel, in Step 5, with $r_{-1}(x,\xi)(ia_p\xi^p)$, coming from $A^{1,0}$ of (2.13); thus we shall omit this term in the following.

Each other term of (2.20) has order

$$j-m-\alpha-\sum_{k=1}^{p-1}q_k(k-1)-\sum_{k'=1}^{p-1}p_{k'}(k'-1)\leq\min\{p-k-1,\, p-k'-1\}$$

and, in view of (2.14), decay in x of the form

$$\langle x\rangle^{\frac{\sum_{k=1}^{p-1}q_k(k-1)+\sum_{k'=1}^{p-1}p_{k'}(k'-1)}{p-1}-m-\alpha+\beta}\leq\langle x\rangle^{-\frac{j-m-\alpha-\sum_{k=1}^{p-1}q_k(k-1)-\sum_{k'=1}^{p-1}p_{k'}(k'-1)}{p-1}}$$

since $-(p-1)(m+\alpha-\beta) \leq -j+m+\alpha$ for $m+\alpha-\beta \geq 1$. Thus, whenever M_{p-k} or $M_{p-k'}$ appear in (2.20), then the order is at most $p-k-1$ and $p-k'-1$ respectively.

On the other hand, for $m+\alpha-\beta=0$, from [7] we have that

$$
\begin{aligned}
&|\operatorname{Re}[(\partial_\xi^\alpha e^{-\Lambda})(iD_x^\beta a_j)e^\Lambda \xi^{j-m}]| \\
&\leq c|\operatorname{Re}(iD_x^\beta a_j)| \sum_{\substack{\alpha_1+\ldots+\alpha_{p-1}\\=\alpha}} \prod_{k=1}^{p-1} \sum_{\substack{r_1+\ldots+r_{q_k}=\alpha_k\\ r_i,\alpha_k\geq 1}} M_{p-k}^{q_k} \langle x\rangle^{\frac{k-1}{p-1}q_k} \langle\xi\rangle_h^{-\alpha_k-q_k(k-1)} \langle\xi\rangle_h^{j-m},
\end{aligned}
\tag{2.21}
$$

for some $c>0$.

Inserting (2.15) in (2.21), and reminding that $D_x^\beta a_p(t)=0$ for $\beta\neq 0$, we see that each term of (2.21) is a symbol of order

$$
j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1) \leq p-k-1 \tag{2.22}
$$

with decay in x of the form

$$
\langle x\rangle^{\frac{\sum_{k=1}^{p-1} q_k(k-1)-j+[\beta/2]}{p-1}} \leq \langle x\rangle^{-\frac{j-m-\alpha-\sum_{k=1}^{p-1} q_k(k-1)}{p-1}}
$$

since $[\beta/2] \leq \beta \leq \alpha+m$; hence M_{p-k} appears in (2.21) only when the order is at most $p-k-1$. Moreover, the terms of order $p-k$ of A_{II}, denoted by $A_{II}|_{\operatorname{ord}(p-k)}$, all have the "right decay for the right level" in the sense that they satisfy (2.18). To compute the positive constant $C_{(M_{p-1},\ldots,M_{p-k})}$ we notice that for every $1\leq k\leq p-1$ we have that $A_{II}|_{\operatorname{ord}(p-k)}$ contains derivatives of the fixed function u. Let us compute the maximum number of derivatives of u in $A_{II}|_{\operatorname{ord}(p-k)}$: the general term $(\partial_\xi^\alpha e^{-\Lambda})(iD_x^\beta a_j)(D_x^{m+\alpha-\beta}e^\Lambda)\xi^{j-m}$, with $\beta\leq\alpha$, is at level $j-m-\alpha$ because of (2.10) and hence at a fixed level $j-m-\alpha=p-k$ the maximum number of β-derivatives on u appears when $\alpha\geq 1$ is maximum and hence $m=0$ and j maximum, i.e. $j=p-1$ ($j=p$ is not considered because $D_x^\beta a_p(t)=0$); in this case $p-k=j-m-\alpha=p-1-\alpha$ and the maximum number of β-derivatives on u at level $p-k$ is given by $\beta=\alpha=k-1$. On the other hand, the minimum number of derivatives on u at level $p-k$ is zero (think, for instance, at terms of the form $D_x^0 a_j$) and there are also terms which do not depend neither on γ neither on u (think at $D_x^0 a_p(t)$). Summing up, formulas (2.14), (2.15), (2.20), and (2.21) give that

$$
\left|\operatorname{Re} A_{II}|_{\operatorname{ord}(p-k)}\right| \leq \frac{C_k(1+\gamma(u))(1+\|u\|_{1+(k-1)}^{k-1})}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k} \tag{2.23}
$$

for some $C_k>0$, and moreover, $\operatorname{Re} A_{II}|_{\operatorname{ord}(p-k)}$ depends only on $M_{p-1},\ldots,M_{p-k+1}$ and not on $M_{p-k},\ldots,M_1$.

As it concerns A_I, following [7] it is straightforward, by means of Lemma 2.2, to show that it can be written as

$$A_I = ia_p\xi^p + \sum_{k=1}^{p-1}(A^0_{p-k} + A^1_{p-k}) + \tilde{B}_0, \tag{2.24}$$

where

$$A^0_{p-k} := ia_{p-k}\xi^{p-k} + ipa_p D_x\lambda_{p-k}\xi^{p-1} \in S^{p-k}$$

is the sum of a term $ia_{p-k}(t,x,u)\xi^{p-k}$ depending on u and a term $ipa_p(t)D_x\lambda_{p-k}(x,\xi)\xi^{p-1}$ not depending on u, while $A^1_{p-k} \in S^{p-k}$ depends on the coefficients $a_j(t,x,u)$ (but not on their derivatives) and on the symbols $\lambda_{p-1}, \lambda_{p-2}, \dots, \lambda_{p-1-(k-2)}$ (cf. [7]), $\tilde{B}_0 \in S^0$, and from (1.11), (1.12) with $\beta = 0$ and (1.15), (1.16) we get that

$$|\operatorname{Re} A^0_{p-k}(t,x,u)| + |A^1_{p-k}(t,x,u)| \le \frac{C_k(1+\gamma(u))\langle\xi\rangle_h^{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}} \tag{2.25}$$

possibly enlarging the constant C_k of (2.23), $1 \le k \le p-1$; moreover, A^0_{p-k} depends only on M_{p-k} and A^1_{p-k} depends only on $M_{p-1}, \dots, M_{p-k+1}$. Step 1 is so completed.

Step 2. We now look at the real part of the terms A_{p-k} of order $p-k$ in (2.19):

$$\begin{aligned} A_{p-k} &:= A_I|_{\operatorname{ord}(p-k)} + A_{II}|_{\operatorname{ord}(p-k)} \\ &= A^0_{p-k} + A^1_{p-k} + A_{II}|_{\operatorname{ord}(p-k)}, \qquad k = 1, \dots, p-1. \end{aligned} \tag{2.26}$$

From the definition (2.5) of λ_{p-k}, conditions (1.10) and (1.11) with $\beta = 0$, estimates (2.23) and (2.25), for $|\xi| \ge 2h$ we have:

$$\begin{aligned} \operatorname{Re} A_{p-k} &= \operatorname{Re}(ipa_p D_x\lambda_{p-k}\xi^{p-1} + ia_{p-k}\xi^{p-k}) + \operatorname{Re}(A^1_{p-k}) + \operatorname{Re}(A_{II}|_{\operatorname{ord}(p-k)}) \\ &= pa_p\xi^{p-1}M_{p-k}\frac{|\xi|^{p-1}}{\xi^{p-1}}\langle x\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\langle\xi\rangle_h^{-k+1} - \operatorname{Im} a_{p-k}\cdot\xi^{p-k} \\ &\quad + \operatorname{Re}(A^1_{p-k}) + \operatorname{Re}(A_{II}|_{\operatorname{ord}(p-k)}) \\ &\ge \left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_p\frac{M_{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k}\psi - \frac{C\gamma(u)}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k}\psi - \frac{C\gamma(u)}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k}(1-\psi) \\ &\quad - C_k(1+1+\|u\|^{k-1}_{1+(k-1)})(1+\gamma(u))\frac{\langle\xi\rangle_h^{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}} \\ &\ge \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\cdot\left[\left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_pM_{p-k}\right. \\ &\quad \left. - \left(C + C_k(2+\|u\|^{k-1}_{1+(k-1)})\right)(1+\gamma(u))\right]\frac{\langle\xi\rangle_h^{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}} \\ &\quad - \tilde{C}_k(1+\gamma(u))(1+\|u\|^{k-1}_{1+(k-1)}) \end{aligned} \tag{2.27}$$

for some $\tilde{C}_k > 0$ since $|\xi| \geq \frac{2}{\sqrt{5}}\langle\xi\rangle_h$ and $\langle\xi\rangle_h^{p-1}/\langle x\rangle$ is bounded on the support of $(1-\psi)$. Notice that the constants C_k, $\tilde{C}_k$ depend only on $M_{p-1}, \dots, M_{p-k+1}$ and not on M_{p-k}, and that with a new constant $C'_k > 0$ we can write

$$\begin{aligned}\operatorname{Re} A_{p-k} \geq \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left[\left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_pM_{p-k} - C'_k(1+\|u\|_{1+(k-1)}^{k-1})(1+\gamma(u))\right]\frac{\langle\xi\rangle_h^{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}}\\ - \tilde{C}_k(1+\gamma(u))(1+\|u\|_{1+(k-1)}^{k-1}).\end{aligned} \tag{2.28}$$

In particular, for $k=1$,

$$\begin{aligned}\operatorname{Re} A_{p-1} \geq \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\cdot\left[\left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_pM_{p-1} - 2C'_1(1+\gamma(u))\right]\frac{\langle\xi\rangle_h^{p-1}}{\langle x\rangle}\\ - 2\tilde{C}_1(1+\gamma(u)),\end{aligned}$$

and we can choose $M_{p-1} > 0$ sufficiently large, i.e.

$$M_{p-1} = M_{p-1}(u) \geq \frac{C'_1(1+\gamma(u))}{2^{p-2}/\sqrt{5}^{p-1}pC_p}$$

so that

$$\operatorname{Re} A_{p-1}(t,x,u,\xi) \geq -2\tilde{C}_1(1+\gamma(u)) \qquad \forall (t,x,\xi) \in [0,T]\times\mathbb{R}^2.$$

Applying the sharp-Gårding Theorem B.1 to $A_{p-1} + 2\tilde{C}_1(1+\gamma(u))$ we can thus find pseudo-differential operators $Q_{p-1}(t,x,u,D_x)$ and $R_{p-1}(t,x,u,D_x)$ with symbols $Q_{p-1}(t,x,u,\xi) \in S^{p-1}$ and $R_{p-1}(t,x,u,\xi) \in S^{p-2}$ such that

$$A_{p-1}(t,x,u,D_x) = Q_{p-1}(t,x,u,D_x) + R_{p-1}(t,x,u,D_x) - 2\tilde{C}_1(1+\gamma(u)) \tag{2.29}$$

where

$$\operatorname{Re}\langle Q_{p-1}(t,x,u,D_x)w(t,x), w(t,x)\rangle \geq 0 \qquad \forall (t,x)\in[0,T]\times\mathbb{R},\ \forall w(t,\cdot)\in H^{p-1}(\mathbb{R}),$$

and R_{p-1} has the asymptotic development given in (B.2) for A_{p-1}. From (2.19), (2.24), (2.26) and (2.29) we get:

$$\begin{aligned}\sigma(e^{-\Lambda}Ae^{\Lambda}) &= ia_p\xi^p + A_{p-1} + \sum_{k=2}^{p-1} A_{p-k} + A'_0\\ &= ia_p\xi^p + Q_{p-1} + \sum_{k=2}^{p-1}(A_I|_{\mathrm{ord}(p-k)} + A_{II}|_{\mathrm{ord}(p-k)} + R_{p-1}|_{\mathrm{ord}(p-k)}) + A''_0\end{aligned} \tag{2.30}$$

for some $A_0', A_0'' \in S^0$, where $R_{p-1}\big|_{\mathrm{ord}(p-k)}$ denotes the terms of order $p-k$ of R_{p-1}, for $2 \le k \le p$. To complete step 2, we need to check that the terms $R_{p-1}\big|_{\mathrm{ord}(p-k)}$ satisfy (2.18) with a new constant $C_{(M_{p-1},\dots,M_{p-k})}(u)$. In [7] it has already been proved, using Lemma 2.2 and conditions (1.5), (1.6) instead of (1.11), (1.12), (1.13), that the terms of $R_{p-1}\big|_{\mathrm{ord}(p-k)}$ all have the right decay for the right level $p-k$; here we only need to find by means of Lemma 2.3 the precise constant $C_{M_{p-1}}(u)$ (note that the constant depends only on M_{p-1} since R_{p-1} depends only on A_{p-1}). The term A_{p-1} does not contain derivatives of u, since in order that a term of type $(\partial_\xi^\alpha e^{-\Lambda})(iD_x^\beta a_j)(D_x^{m+\alpha-\beta}e^\Lambda)\xi^{j-m}$ of (2.19) is at level $j-m-\alpha = p-1$ with $\alpha \ge 1$ we must have $j = p$. Therefore, looking at the asymptotic development of R_{p-1} given by Remark B.2 with $m = p-1$ and $\ell = k-1$, the maximum number of derivatives on u in $R_{p-1}\big|_{\mathrm{ord}(p-k)}$ appears in (B.3) when $\alpha = 0$ and $\beta = 2\ell+1 = 2k-1$.

By Lemma 2.3 we come so to the estimate

$$|R_{p-1}(t,x,u,\xi)|_{\mathrm{ord}(p-k)}| \le \frac{C_k(1+\gamma(u))(1+\|u\|^{2k-1}_{1+(2k-1)})}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k}, \tag{2.31}$$

possibly enlarging the constant C_k of (2.23), (2.25). Step 2 is completed.

Step 3. In order to reapply sharp-Gårding Theorem B.1 we now have to investigate the action of that theorem to each term of the form $A_I|_{\mathrm{ord}(p-k)} + A_{II}|_{\mathrm{ord}(p-k)} + S_{p-k}$, where S_{p-k} denotes terms of order $p-k$ coming from remainders of previous applications of the sharp-Gårding Theorem, for $p-k \ge 3$. In [7] we have computed and estimated the generic remainder

$$R(A_I|_{\mathrm{ord}(p-k)}) + R(A_{II}|_{\mathrm{ord}(p-k)}) + R(S_{p-k})$$

under the assumptions of Theorem 1.1, showing that it is sum of terms of order $p-j$, $k+1 \le j \le p$, each one of them with the right decay $(p-j)/(p-1)$ and the right constants $M_{p-1},\dots,M_{p-k+1}$ for the right level $p-j$. Here, we can argue with the same (quite long and technical) computations and make use of Lemma 2.3 instead of assumptions (1.5) and (1.6) to get that this generic remainder consists in a sum of terms of order $p-j$, $k+1 \le j \le p$, each one of them satisfying (2.18). It only remains to compute precisely the corresponding constant $C_{(M_{p-1},\dots,M_{p-k+1})}(u)$.

To this aim, we need to better understand the dependence of S_{p-k} on u; let us first focus on the second application of the sharp-Gårding Theorem B.1. From (2.30) with $R_{p-1} = R(A_{p-1})$ we have

$$\begin{aligned}\sigma(e^{-\Lambda}Ae^{\Lambda}) = {}& ia_p\xi^p + Q_{p-1} + A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)}\\ &+ \sum_{k=3}^{p-1}(A_{p-k} + R(A_{p-1})\big|_{\mathrm{ord}(p-k)}) + A_0''.\end{aligned}$$

Since from (2.31) $R(A_{p-1})\big|_{\mathrm{ord}(p-2)}$ has the same structure as A_{p-2}, depends on the same constant, and bears $2k-1=3$ derivatives of u (much more than A_{p-2}, see (2.23) and (2.25)), we can follow the computations in (2.27) to get, instead of (2.28),

$$
\begin{aligned}
&\mathrm{Re}\left(A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)}\right)(t,x,u,\xi) \geq \\
&\quad \geq \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left[\left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_pM_{p-2} - C_2'\left(1+\|u\|_4^3\right)(1+\gamma(u))\right]\frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}} \\
&\qquad - \tilde{C}_2(1+\gamma(u))(1+\|u\|_4^3)
\end{aligned}
$$

for some $C_2', \tilde{C}_2 > 0$.

We can so choose $M_{p-2} > 0$ sufficiently large, i.e.

$$
M_{p-2} \geq \frac{C_2'(1+\|u\|_4^3)(1+\gamma(u))}{2^{p-1}/\sqrt{5}^{p-1} pC_p}
$$

so that

$$
\begin{aligned}
&\mathrm{Re}\left(A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)}\right)(t,x,u,\xi) \geq -\tilde{C}_2(1+\gamma(u))(1+\|u\|_4^3), \\
&\forall (t,x,\xi) \in [0,T]\times\mathbb{R}^2
\end{aligned}
$$

and we can apply the sharp-Gårding Theorem B.1 to

$$
A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)} + \tilde{C}_2(1+\gamma(u))(1+\|u\|_4^3)
$$

obtaining that there exist pseudo-differential operators Q_{p-2} and R_{p-2}, with symbols in S^{p-2} and S^{p-3} respectively, such that

$$
\begin{aligned}
&\mathrm{Re}\langle Q_{p-2}w, w\rangle \geq 0 \qquad \forall w(t,\cdot) \in H^{p-2} \\
&A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)} = Q_{p-2} + R_{p-2} - \tilde{C}_2(1+\gamma(u))(1+\|u\|_4^3),
\end{aligned}
$$

with

$$
R_{p-2} = R(A_{p-2} + R(A_{p-1})\big|_{\mathrm{ord}(p-2)}) = R(A_{p-2}) + R(R(A_{p-1})\big|_{\mathrm{ord}(p-2)}),
$$

so that with the notation $R^2(A_{p-1}) = R(R(A_{p-1}))$

$$
\begin{aligned}
\sigma(e^{-\Lambda}Ae^{\Lambda}) = {} & ia_p\xi^p + Q_{p-1} + Q_{p-2} \\
& + A_{p-3} + R(A_{p-1})\big|_{\mathrm{ord}(p-3)} + R(A_{p-2})\big|_{\mathrm{ord}(p-3)} + R^2(A_{p-1})\big|_{\mathrm{ord}(p-3)} \\
& + \sum_{k=4}^{p-1}\Big(A_{p-k} + R(A_{p-1})\big|_{\mathrm{ord}(p-k)} + R(A_{p-2})_{\mathrm{ord}(p-k)} \\
& + R^2(A_{p-1})\big|_{\mathrm{ord}(p-k)}\Big) + A_0'.
\end{aligned}
$$

At the second application of sharp-Gårding Theorem B.1 the term $R^2(A_{p-1})$ appears at (maximum) level $p-3$. By (2.31) we know that $R(A_{p-1})\big|_{\mathrm{ord}(p-2)}$ contains $2k-1=3$ derivatives on u; so, its remainder, given by (B.2), has the structure of Remark B.2 and by (B.3) with $m=p-2$ and $\ell=k-2$ we see that the maximum number of derivatives with respect to u in $R(R(A_{p-1})\big|_{\mathrm{ord}(p-2)})\big|_{\mathrm{ord}(p-k)}$ appears when $\beta=2\ell+1=2k-3$ and is given by $3+(2k-3)=2k$.

Analogously, by formula (B.2) and Remark B.2, we have that $R(A_{p-2})$ consists of terms $R(A_{p-2})\big|_{\mathrm{ord}(p-k)}$ with $3\leq k\leq p$; formula (B.3) with $m=p-2$ and $\ell=k-2$, together with (2.23) and (2.25), give that $R(A_{p-2})\big|_{\mathrm{ord}(p-k)}$ contains at most $1+(2k-3)=2k-2$ derivatives on u.

Summing up, the maximum number of derivatives of u appears in $R^2(A_{p-1})$ and we get:

$$\left|\left(R(A_{p-1})\big|_{\mathrm{ord}(p-k)}+R(A_{p-2})_{\mathrm{ord}(p-k)}\ +R^2(A_{p-1})\big|_{\mathrm{ord}(p-k)}\right)(t,x,u,\xi)\right|\leq$$
$$\leq\frac{C_k(1+\gamma(u))(1+\|u\|^{2k}_{1+2k})}{\langle x\rangle^{\frac{p-k}{p-1}}}\langle\xi\rangle_h^{p-k},\ 3\leq k\leq p. \tag{2.32}$$

Now, let us come back to the general case.

At the j-th application of sharp-Gårding theorem we find, at level $p-j$, the terms

$$A_{p-j}+\sum_{\substack{1\leq k\leq j-1\\ 1\leq s\leq j-k}}R^s(A_{p-k})|_{p-j}.$$

These terms depend on u and its derivatives; reminding that

$$A_{p-j}=A_I|_{\mathrm{ord}(p-j)}+A_{II}|_{\mathrm{ord}(p-j)}=ia_{p-j}\xi^{p-j}+ipa_pD_x\lambda_{p-j}\xi^{p-1}+A^1_{p-k}+A_{II}|_{\mathrm{ord}(p-j)},$$

see also (2.23) and (2.25), we see that the maximum number of derivatives of u is in the term $R^{j-1}(A_{p-1})|_{\mathrm{ord}(p-j)}$, i.e. in the principal part of $R^{j-1}(A_{p-1})$. To compute this number, we work by induction. For $j=2$, by (2.31) with $k=2$ we know that $R^{j-1}(A_{p-1})|_{\mathrm{ord}(p-j)}$ contains $2k-1=3$ derivatives of u; for $j=3$, by (2.32) with $k=3$ we know that $R^{j-1}(A_{p-1})|_{\mathrm{ord}(p-j)}$ contains $2k=6$ derivatives of u; let us now suppose that for all $2\leq s\leq j-1$ we have that $R^{s-1}(A_{p-1})|_{\mathrm{ord}(p-s)}$ contains $3(s-1)$ derivatives of u, and prove that $R^{j-1}(A_{p-1})|_{\mathrm{ord}(p-j)}$ contains $3(j-1)$ derivatives of u. Arguing as for $R(A_{p-1})$ and $R^2(A_{p-1})$ we obtain that the remainder $R^{j-1}(A_{p-1})=R(R^{j-2}(A_{p-1}))$ is the sharp-Gårding remainder of the operator $R^{j-2}(A_{p-1})$ with symbol of order $p-1-(j-2)=p-j+1$; the principal part of $R(R^{j-2}(A_{p-1}))$ consists so in a term of order $p-j$, depending on u and its derivatives, and given by (B.3) with $m=p-j+1$ and $\ell=1$. The maximum number of derivatives on u appears so when $\alpha=0$ and $\beta=3$ in (B.3) and is given, by the inductive hypothesis, by $3(j-2)+3=3(j-1)$.

It follows that

$$|R^{j-1}(A_{p-1})|_{\mathrm{ord}(p-j)}(t,x,u,\xi)|\leq\frac{C_j(1+\gamma(u))(1+\|u\|^{3(j-1)}_{1+3(j-1)})}{\langle x\rangle^{\frac{p-j}{p-1}}}\langle\xi\rangle_h^{p-j},\ 2\leq j\leq p.$$

Thus, at each level $p-j$, $2 \le j \le p$, we have

$$\left| A^1_{p-j} + A_{II}|_{\mathrm{ord}(p-j)} + \sum_{\substack{1\le k\le j-1\\ 1\le s\le j-k}} R^s(A_{p-k})|_{p-j} \right| (t,x,u,\xi)$$
$$\le \frac{C_j(1+\gamma(u))(1+\|u\|^{3(j-1)}_{1+3(j-1)})}{\langle x\rangle^{\frac{p-j}{p-1}}} \langle\xi\rangle_h^{p-j},$$

with C_j depending on $M_{p-1}, \dots, M_{p-j+1}$ and not on $M_{p-j}, \dots, M_1$.

Thanks to the estimates given here above, we can apply again and again the sharp-Gårding Theorem B.1 to find pseudo-differential operators $Q_{p-1}, Q_{p-2}, \dots, Q_3$ of order $p-1, p-2, \dots, 3$ respectively and all positive definite, such that

$$\sigma(e^{-\Lambda}Ae^{\Lambda}) = ia_p\xi^p + Q_{p-1} + Q_{p-2} + \dots + Q_3 + (A_2+S_2) + (A_1+S_1) + (A_0+S_0),$$

with S_j, $j=0,1,2$ coming from remainders of the sharp-Gårding theorem, and such that

$$|(A_j+S_j)(t,x,u,\xi)| \le C_{p-j}(1+\gamma(u))(1+\|u\|^{3(p-j-1)}_{1+3(p-j-1)})\langle x\rangle^{-\frac{j}{p-1}}\langle\xi\rangle_h^j, \ j=0,1,2, \tag{2.33}$$

with $C_{p-j} > 0$ depending on $M_{p-1}, \dots, M_{j+1}$ and not on $M_j, \dots, M_1$.

Step 4. Let us split the term of order 2 into $\mathrm{Re}(A_2+S_2) + i\,\mathrm{Im}(A_2+S_2)$; by (2.28) and the discussion of step 3 we have that

$$\mathrm{Re}(A_2+S_2) \ge \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \cdot \Bigg[\left(\frac{2}{\sqrt{5}}\right)^{p-1} pC_pM_2$$
$$- C''_{p-2}(1+\|u\|^{3(p-3)}_{1+3(p-3)})(1+\gamma(u))\Bigg]\langle x\rangle^{-\frac{2}{p-1}}\langle\xi\rangle_h^2$$
$$- \bar{C}_{p-2}(1+\|u\|^{3(p-3)}_{1+3(p-3)})(1+\gamma(u)),$$

for some $C''_{p-2}, \bar{C}_{p-2} > 0$ and we can choose M_2 large enough so that

$$\mathrm{Re}(A_2+S_2) \ge -\bar{C}_{p-2}(1+\gamma(u))(1+\|u\|^{3(p-3)}_{1+3(p-3)}).$$

Then, the Fefferman–Phong inequality (B.5) applied to

$$\tilde{A}_2(t,x,u(t,x),\gamma(u(t,x)),D_x) := \mathrm{Re}(A_2+S_2) + \bar{C}_{p-2}(1+\gamma(u))(1+\|u\|^{3(p-3)}_{1+3(p-3)})$$

gives

$$\mathrm{Re}\langle(\mathrm{Re}(A_2+S_2))w, w\rangle \ge -c(u)(1+\gamma(u))(1+\|u\|^{3(p-3)}_{1+3(p-3)})\|w\|_0^2$$

without any remainder, for a new constant $c(u) > 0$ depending on the derivatives $\partial_\xi^\alpha \partial_x^\beta$ of the symbol of $\tilde{A}_2$ with $|\alpha| + |\beta| \le 7$, by Remark B.5. Being the function γ of class C^7 by assumption we can then find a constant $C_\gamma > 0$, depending only on γ, such that

$$\begin{aligned}\mathrm{Re}\langle(\mathrm{Re}(A_2 + S_2))w, w\rangle &\ge -C_\gamma(1 + \|u\|_{1+3(p-3)+7}^{1+3(p-3)+7})\|w\|_0^2 \\ &= -C_\gamma(1 + \|u\|_{3p-1}^{3p-1})\|w\|_0^2. \end{aligned} \tag{2.34}$$

On the other hand, we split $i\,\mathrm{Im}(A_2 + S_2)$ into its hermitian and antihermitian part:

$$\frac{i\,\mathrm{Im}(A_2 + S_2) + (i\,\mathrm{Im}(A_2 + S_2))^*}{2} + \frac{i\,\mathrm{Im}(A_2 + S_2) - (i\,\mathrm{Im}(A_2 + S_2))^*}{2},$$

and we have that $\mathrm{Re}\langle \frac{i\,\mathrm{Im}(A_2+S_2)-(i\,\mathrm{Im}(A_2+S_2))^*}{2} w, w\rangle = 0$, while $i\,\mathrm{Im}(A_2 + S_2) + (i\,\mathrm{Im}(A_2 + S_2))^*$ can be put together with $A_1 + S_1$ since it has a real principal part of order 1, does not depend on M_1, and has the "right decay" for level 1. Therefore we can choose $M_1 > 0$ sufficiently large so that, by (2.33),

$$\mathrm{Re}\left(A_1 + S_1 + \frac{i\,\mathrm{Im}(A_2 + S_2) + (i\,\mathrm{Im}(A_2 + S_2))^*}{2}\right) \ge -\bar{C}_{p-1}(1 + \gamma(u))(1 + \|u\|_{1+3(p-2)}^{3(p-2)}),$$

for some $\bar{C}_{p-1} > 0$ and hence, by the sharp-Gårding inequality (B.4) we get

$$\begin{aligned}&\mathrm{Re}\langle\big(A_1 + S_1 + \frac{i\,\mathrm{Im}(A_2 + S_2) + (i\,\mathrm{Im}(A_2 + S_2))^*}{2}\big)w, w\rangle \ge \\ &\ge -C'_\gamma(1 + \|u\|_{1+3(p-2)}^{1+3(p-2)})\|w\|_0^2, \end{aligned} \tag{2.35}$$

for a new constant $C'_\gamma > 0$.

Summing up, throughout steps 1–4 we have obtained

$$\sigma(e^{-\Lambda} A e^{\Lambda}) = i a_p \xi^p + \sum_{j=1}^{p-3} Q_{p-j} + (A_2 + S_2) + (A_1 + S_1) + (A_0 + S_0)$$

with

$$\begin{aligned}&\mathrm{Re}\langle Q_{p-j} w, w\rangle \ge 0 \qquad \forall w(t,\cdot) \in H^{p-j},\ 1 \le j \le p-3 \\ &\mathrm{Re}\langle(A_2 + S_2 + A_1 + S_1)w, w\rangle \ge -\tilde{C}_\gamma(1 + \|u\|_{3p-1}^{3p-1})\|w\|_0^2 \qquad \forall w(t,\cdot) \in H^2,\end{aligned}$$

for a positive constant $\tilde{C}_\gamma$, because of (2.34) and (2.35), since $3p - 1 > 1 + 3(p - 2)$.

Step 5. Now, we come back to (2.13), and remark that $A^{n,m}$ is of the same kind of A with $D_x^m a_j$ instead of a_j and $(\partial_\xi^m r^n)\xi^j$ instead of ξ^j, with $0 \le m \le p-2$ and $1 \le n \le p-1-m$. This implies that we have m more x-derivatives on a_j, but the level in ξ decreases of $-n-m < -m$, so we still have the right decay for the right level and the right dependence on the constants M_{p-j}, $j \le k$, at each level $p-k$. As far as the derivatives of the fixed function u are concerned, the maximum number of derivatives of u appears in the term $\partial_\xi^m r^n D_x^m(ia_j(t,x,u)\xi^j) \in S^{j-m-n}$ with $j = p-1$, $n = 1$, $m = p-2$, so that we argue as for $\sigma(e^{-\Lambda}Ae^{\Lambda})$ and find that also

$$\sigma(e^{-\Lambda}A^{n,m}e^{\Lambda}) = \sum_{s=0}^{p} Q_{p-s}^{n,m}$$

with $Q_0^{n,m} \in S^0$ and

$$\begin{aligned}\mathrm{Re}\langle Q_{p-s}^{n,m}w, w\rangle &\geq -C_\gamma^{n,m}(1 + \|u\|_{3p-1+p-2}^{3p-1+p-2})\|w\|_0^2 \\ &= -C_\gamma^{n,m}(1 + \|u\|_{4p-3}^{4p-3})\|w\|_0^2 \qquad \forall w(t,\cdot) \in H^{p-s},\ 1 \leq s \leq p-1\end{aligned}$$

for some $C_\gamma^{n,m} > 0$.

Summing up, we have proved that

$$\mathrm{Re}\langle (e^{\Lambda})^{-1}Ae^{\Lambda}w, w\rangle \geq -c_\gamma(1 + \|u\|_{4p-3}^{4p-3})\|w\|_0^2 \qquad \forall w(t,\cdot) \in H^p \tag{2.36}$$

for some $c > 0$. From (2.12) and (2.36) it follows that every $w \in C([0,T]; H^p) \cap C^1([0,T]; H^0)$ satisfies:

$$\begin{aligned}\frac{d}{dt}\|w\|_0^2 &= 2\,\mathrm{Re}\langle \partial_t w, w\rangle = 2\,\mathrm{Re}\langle iP_\Lambda w, w\rangle - 2\,\mathrm{Re}\langle (e^{\Lambda})^{-1}Ae^{\Lambda}w, w\rangle \\ &\leq \|P_\Lambda w\|_0^2 + \|w\|_0^2 - 2\,\mathrm{Re}\langle (e^{\Lambda})^{-1}Ae^{\Lambda}w, w\rangle \\ &\leq c'_\gamma(1 + \|u\|_{4p-3}^{4p-3})(\|P_\Lambda w\|_0^2 + \|w\|_0^2)\end{aligned}$$

for some $c'_\gamma > 0$. Applying Gronwall's inequality, we deduce that for all $w \in C([0,T]; H^p) \cap C^1([0,T]; H^0)$, the estimate

$$\|w(t,\cdot)\|_0^2 \leq e^{c'_\gamma(1+\|u\|_{4p-3}^{4p-3})t}\left(\|w(0,\cdot)\|_0^2 + \int_0^t \|P_\Lambda w(\tau,\cdot)\|_0^2 d\tau\right),\ \forall t \in [0,T]$$

holds. Since $\langle D_x\rangle^s P_\Lambda \langle D_x\rangle^{-s}$ satisfies, for every $s \in \mathbb{R}$, the same hypotheses as P_Λ, we immediately get that for every $s \in \mathbb{R}$, $w \in C([0,T]; H^{s+p}) \cap C^1([0,T]; H^s)$ we have

$$\|w(t,\cdot)\|_s^2 \leq e^{C_{s,\gamma}(1+\|u\|_{4p-3}^{4p-3})t}\left(\|w(0,\cdot)\|_s^2 + \int_0^t \|P_\Lambda w(\tau,\cdot)\|_s^2 d\tau\right),\ \forall t \in [0,T] \tag{2.37}$$

for a positive constant $C_{s,\gamma}$.

The a priori estimate (2.37) implies, by standard arguments from the energy method, that for every $u_{0,\Lambda} \in H^s$ and $f_\Lambda \in C([0,T], H^s)$ the Cauchy problem (2.3) has a unique solution $w \in C([0,T]; H^s)$ satisfying

$$\|w(t,\cdot)\|_s^2 \leq e^{C_{s,\gamma}(1+\|u\|_{4p-3}^{4p-3})t}\left(\|u_{0,\Lambda}\|_s^2 + \int_0^t \|f_\Lambda(\tau,\cdot)\|_s^2 d\tau\right) \qquad \forall t \in [0,T]. \tag{2.38}$$

Since $e^{\Lambda} \in S^{\delta}$, for $v = e^{\Lambda} w$ we finally have, from (2.38) with $s - \delta$ instead of s:

$$\begin{aligned}\|v\|_{s-2\delta}^2 &\le c_1 \|w\|_{s-\delta}^2 \le c_2 e^{C_{s,\gamma}(1+\|u\|_{4p-3}^{4p-3})t} \left(\|u_{0,\Lambda}\|_{s-\delta}^2 + \int_0^t \|f_\Lambda\|_{s-\delta}^2 d\tau \right) \\ &\le C'_{s,\gamma} e^{(1+\|u\|_{4p-3}^{4p-3})t} \left(\|u_0\|_s^2 + \int_0^t \|f\|_s^2 d\tau \right)\end{aligned}$$

for some $c_1, c_2, C'_{s,\gamma} > 0$, that is (2.1). This proves Theorem 2.1. In particular, we have that for every $f \in C([0, T]; H^\infty)$ and $u_0 \in H^\infty$ there exists a solution $v \in C([0, T]; H^\infty(\mathbb{R}))$ of (1.17) which satisfies an energy estimate of the form (1.9) with constant

$$C_s = C_s(u) = C'_{s,\gamma} e^{(1+\|u\|_{4p-3}^{4p-3})t}. \qquad \square$$

3. The semilinear problem

In this section we consider the semilinear Cauchy problem (1.1) and give the proof of the main result of this paper, Theorem 1.3.

We set $X := C^1([0, T]; H^\infty(\mathbb{R}))$ and consider the map

$$\begin{aligned} J : X &\longrightarrow X \\ u &\longmapsto J(u) \end{aligned}$$

defined by

$$\begin{aligned} J(u) := u(t,x) - u_0(x) + i \int_0^t a_p(s) D_x^p u(s,x) ds \\ + i \sum_{j=0}^{p-1} \int_0^t a_j(s,x,u(s,x)) D_x^j u(s,x) ds - i \int_0^t f(s,x) ds. \end{aligned} \tag{3.1}$$

Remark 3.1. The existence of a local solution $u \in C^1([0, T^*]; H^\infty(\mathbb{R}))$ of the Cauchy problem (1.1) is equivalent to the existence of a solution $u \in C^1([0, T^*]; H^\infty(\mathbb{R}))$ of

$$J(u) \equiv 0 \quad \text{in } [0, T^*] \times \mathbb{R}. \tag{3.2}$$

Indeed, if $J(u) = 0$ then

$$\begin{aligned} u(t,x) = u_0(x) - i \int_0^t a_p(s) D_x^p u(s,x) ds \\ - i \sum_{j=0}^{p-1} \int_0^t a_j(s,x,u(s,x)) D_x^j u(s,x) ds + i \int_0^t f(s,x) ds \end{aligned} \tag{3.3}$$

and hence $u(0, x) = u_0(x)$ and

$$D_t u(t,x) = -i\partial_t u(t,x) = -a_p(t)D_x^p u(t,x) - \sum_{j=0}^{p-1} a_j(t,x,u(t,x))D_x^j u(t,x) + f(t,x) \tag{3.4}$$

so that u solves (1.1). Vice versa, if $u \in C^1([0,T^*]; H^\infty(\mathbb{R}))$ is a solution of the Cauchy problem (1.1), then integrating (3.4) with respect to time we get (3.3), i.e. $J(u) = 0$.

We are so reduced to prove the existence of $0 < T^* \le T$ and of a unique solution $u \in C^1([0,T^*]; H^\infty(\mathbb{R}))$ of (3.2).

To this aim we shall use the Nash–Moser Theorem A.11. Note that $X = C^1([0,T]; H^\infty(\mathbb{R}))$, with the family of semi-norms

$$|||g|||_n = \sup_{[0,T]} (\|g(t,\cdot)\|_n + \|D_t g(t,\cdot)\|_n), \qquad n \in \mathbb{N}_0,$$

is a tame space, see Example A.6 in Appendix A.

The map J is smooth tame, since it is a composition of linear and nonlinear operators and of integrations, which are all smooth tame by Remark A.10, and since it does not contain time derivatives (this is important since ∂_t does not operate from X to X, so it cannot be a tame map).

In order to apply the Nash–Moser Theorem A.11, denoting by $DJ(u)v$ the Fréchet derivative of J at u in the direction v, we shall prove that the equation $DJ(u)v = h$ has a unique solution $v := S(u,h)$ for all $u, h \in X$ and that $S: X \times X \to X$ is smooth tame. This is going to be done in the following lemmas.

Lemma 3.2. *For every $u, h \in X$, the equation $DJ(u)v = h$ admits a unique solution $v \in X$, and the solution satisfies for every $n \in \mathbb{N}_0$ the following estimate:*

$$\|v(t,\cdot)\|_n^2 \le C_n(u)\left(\|h(0,\cdot)\|_{n+r}^2 + \int_0^t \|D_t h(\tau,\cdot)\|_{n+r}^2 d\tau\right) \qquad \forall t \in [0,T], \tag{3.5}$$

for any $r \ge \sigma$, with $C_n(u) := C_{n+\sigma,\gamma} \exp\left\{\left(1 + \|u\|_{4p-3}^{4p-3}\right)T\right\}$ as in (2.1).

Proof. Let us compute by the definition (3.1) of the map J, the Fréchet derivative of J, for $u, v \in X$:

$$\begin{aligned} DJ(u)v &= \lim_{\varepsilon\to 0} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \Big\{ v + i\int_0^t a_p(s)D_x^p v(s)ds + i\sum_{j=0}^{p-1}\int_0^t \frac{a_j(s,x,u+\varepsilon v) - a_j(s,x,u)}{\varepsilon} D_x^j u(s)ds \\ &\quad + i\sum_{j=0}^{p-1}\int_0^t a_j(s,x,u+\varepsilon v)D_x^j v(s)ds \Big\} \\ &= v + i\int_0^t a_p(s)D_x^p v(s)ds + i\sum_{j=0}^{p-1}\int_0^t \partial_w a_j(s,x,u)v(s)D_x^j u(s)ds \end{aligned}$$

$$
\begin{aligned}
&+i\sum_{j=0}^{p-1}\int_0^t a_j(s,x,u)D_x^j v(s)ds\\
&= v+i\int_0^t a_p(s)D_x^p v(s)ds+i\sum_{j=1}^{p-1}\int_0^t a_j(s,x,u)D_x^j v(s)ds\\
&\quad +i\int_0^t\Big(a_0(s,x,u)+\sum_{j=0}^{p-1}\partial_w a_j(s,x,u)D_x^j u\Big)v(s)ds\\
&= v-0+i\int_0^t a_p(s)D_x^p v(s)ds+i\sum_{j=0}^{p-1}\int_0^t \tilde a_j(s,x,u)D_x^j v(s)ds-0\\
&=: J_{0,u,0}(v),
\end{aligned}
$$

where

$$
\tilde a_j(s,x,u)=\begin{cases} a_j(s,x,u), & 1\le j\le p-1\\ a_0(s,x,u)+\sum_{h=0}^{p-1}\partial_w a_h(s,x,u)D_x^h u, & j=0\end{cases}
$$

and for every $u, u_0, f\in X$ the map $J_{u_0,u,f}: X\to X$ is defined by

$$
\begin{aligned}
J_{u_0,u,f}v &:= v(t,x)-u_0(x)+i\int_0^t a_p(s)D_x^p v(s,x)ds\\
&\quad +i\sum_{j=0}^{p-1}\int_0^t \tilde a_j(s,x,u(s,x))D_x^j v(s,x)ds-i\int_0^t f(s,x)ds.
\end{aligned}
$$

As in Remark 3.1, we notice that v is a solution of $J_{u_0,u,f}(v)\equiv 0$ if and only if it is a solution of the linearized Cauchy problem

$$
\begin{cases} \tilde P_u(D)v(t,x)=f(t,x)\\ u(0,x)=u_0(x),\end{cases}
$$

where $\tilde P_u(D)$ is obtained from $P_u(D)$ substituting a_j with $\tilde a_j$.

Therefore v is a solution of $DJ(u)v=h$ if and only if $J_{0,u,0}(v)=h$; writing

$$
J_{0,u,0}(v)-h=J_{0,u,0}(v)-h_0-i\int_0^t D_t h(s,x)ds=J_{h_0,u,D_th}(v)
$$

with $h_0:=h(0,x)$, we have that v is a solution of $DJ(u)v=h$ if and only if it is a solution of $J_{h_0,u,D_th}(v)=0$, i.e. it is a solution of the linearized Cauchy problem

$$
\begin{cases} \tilde P_u(D)v(t,x)=D_th(t,x)\\ v(0,x)=h_0(x).\end{cases} \tag{3.6}
$$

We now want to apply Theorem 2.1 with $\tilde{P}_u(D)$ instead of $P_u(D)$. Note that conditions (1.10)–(1.16) are the same for $\tilde{P}_u(D)$ and $P_u(D)$, since there are no conditions for $j=0$. Applying Theorem 2.1 we have that, for any $u, h \in X$ there is a unique solution $v \in X$ of (3.6) satisfying the energy estimate

$$\|v(t,\cdot)\|_n^2 \leq C_{n+\sigma,\gamma} e^{(1+\|u\|_{4p-3}^{4p-3})T} \left(\|h_0\|_{n+r}^2 + \int_0^t \|D_t h(\tau,\cdot)\|_{n+r}^2 d\tau \right) \qquad \forall t \in [0,T],$$

for any $r \geq \sigma$, which is exactly (3.5). This completes the proof of the lemma. □

We can therefore define the map

$$\begin{aligned} S:\ X \times X &\longrightarrow X \\ (u,h) &\longmapsto v, \end{aligned} \tag{3.7}$$

where v is the unique solution of the Cauchy problem (3.6), i.e. of $DJ(u)v = h$, and satisfies the energy estimate (3.5).

Lemma 3.3. *The map S defined in (3.7) is smooth tame.*

Proof. To prove that S is smooth tame, we work by induction. The proof is divided into 4 steps. In steps 1, 2, 3 we prove, respectively, that S, DS, D^2S are tame maps; step 4 is the inductive step.

Step 1. Let us show that S is a tame map. To this aim we first remark that, for fixed $(u_0, h_0) \in X \times X$ and (u,h) in a neighborhood of (u_0, h_0) we have that $C_n(u)$ is bounded and hence, from the energy estimate (3.5),

$$\|v(t,\cdot)\|_n^2 \leq C'_n \|\|h\|\|_{n+r}^2 \qquad \forall t \in [0,T] \tag{3.8}$$

for some $C'_n > 0$. A similar estimate also holds for $D_t v$ since

$$\begin{aligned} \|D_t v(t,\cdot)\|_n &= \left\| -a_p(t) D_x^p v(t,\cdot) - \sum_{j=0}^{p-1} \tilde{a}_j(t,\cdot,u) D_x^j v(t,\cdot) \right\|_n + \|D_t h(t,\cdot)\|_n \\ &\leq C(\|v(t,\cdot)\|_{n+p} + \|\|h\|\|_n) \end{aligned}$$

for some $C > 0$.

Therefore

$$\|\|S(u,h)\|\|_n = \sup_{t\in[0,T]} (\|v(t,\cdot)\|_n + \|D_t v(t,\cdot)\|_n) \leq C_n \|\|h\|\|_{n+r'} \leq C_n \|\|(u,h)\|\|_{n+r'} \tag{3.9}$$

for some $C_n > 0$ and $r' \in \mathbb{N}$, $r' \geq \sigma + p$, and S is tame.

Step 2. We start by computing the Fréchet derivative of S, for $(u, h), (u_1, h_1) \in X \times X$:

$$DS(u,h)(u_1,h_1) = \lim_{\varepsilon\to 0} \frac{S(u+\varepsilon u_1, h+\varepsilon h_1) - S(u,h)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{v_\varepsilon - v}{\varepsilon} = \lim_{\varepsilon\to 0} w_1^\varepsilon, \tag{3.10}$$

where v is the solution of the Cauchy problem (3.6) and v_ε is the solution of the Cauchy problem

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1}(D)v_\varepsilon = D_t(h+\varepsilon h_1) \\ v_\varepsilon(0,x) = h(0,x) + \varepsilon h_1(0,x). \end{cases}$$

Therefore

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1}(D)v_\varepsilon - \tilde{P}_u(D)v = \varepsilon D_t h_1 \\ v_\varepsilon(0,x) - v(0,x) = \varepsilon h_1(0,x) \end{cases} \tag{3.11}$$

and, writing explicitly the equation in (3.11) we come to the equivalent equation

$$\begin{aligned} D_t(v_\varepsilon - v) + a_p(t)D_x^p(v_\varepsilon - v) \quad &+ \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u+\varepsilon u_1) D_x^j(v_\varepsilon - v) \\ &+ \sum_{j=0}^{p-1} \big(\tilde{a}_j(t,x,u+\varepsilon u_1) - \tilde{a}_j(t,x,u)\big) D_x^j v = \varepsilon D_t h_1. \end{aligned}$$

This means that w_1^ε in (3.10) satisfies

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1} w_1^\varepsilon = D_t h_1 - \displaystyle\sum_{j=0}^{p-1} \frac{\tilde{a}_j(t,x,u+\varepsilon u_1) - \tilde{a}_j(t,x,u)}{\varepsilon} D_x^j v =: f_1^\varepsilon \\ w_1^\varepsilon(0,x) = h_1(0,x). \end{cases}$$

If we prove that $\{w_1^\varepsilon\}_\varepsilon$ is a Cauchy sequence in X, there exists then $w_1 \in X$ such that $w_1^\varepsilon \to w_1$ in X, so that $DS(u,h)(u_1,h_1) = w_1$, and w_1 solves the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_1 = f_1 \\ w_1(0,x) = h_1(0,x) \end{cases} \tag{3.12}$$

for

$$f_1 := \lim_{\varepsilon\to 0} f_1^\varepsilon = D_t h_1 - \sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u) u_1 D_x^j v. \tag{3.13}$$

Then, by Theorem 2.1 the solution w_1 of the Cauchy problem (3.12) would satisfy the energy estimate

$$\|w_1(t,\cdot)\|_n^2 \le C_n(u)\Big(\|h_1(0,\cdot)\|_{n+r}^2 + \int_0^t \|f_1(\tau,\cdot)\|_{n+r}^2 d\tau\Big)$$

for $C_n(u)$ defined in (3.5), and

$$\begin{aligned}\|w_1(t,\cdot)\|_n &\le C_n'(u,u_1) \sup_{t\in[0,T]} \big(\|h_1(t,\cdot)\|_{n+r} + \|D_t h_1(t,\cdot)\|_{n+r} + \|v(t,\cdot)\|_{n+r+p-1}\big)\\ &\le C_n''\left(|||h_1|||_{n+r'} + |||h|||_{n+r'}\right) \end{aligned} \tag{3.14}$$

by (3.8), for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1) in a neighborhood of some fixed $(\bar{u}_1,\bar{h}_1) \in X \times X$, and for some $C_n'(u,u_1)$, $C_n'' > 0$ and $r' \ge 2r + p - 1$. Also

$$D_t w_1 = -a_p(t) D_x^p w_1 - \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u) D_x^j w_1 + f_1$$

would satisfy a similar estimate, so that the first derivative DS would be tame. Summing up, to gain that DS is a tame map, it only remains to show that $\{w_1^\varepsilon\}_\varepsilon$ is a Cauchy sequence in X. To this aim, let us take $w_1^{\varepsilon_1}$ and $w_1^{\varepsilon_2}$ solutions, respectively, of the Cauchy problems

$$\begin{cases} \tilde{P}_{u+\varepsilon_1 u_1}(D) w_1^{\varepsilon_1} = f_1^{\varepsilon_1}, \\ w_1^{\varepsilon_1}(0,x) = h_1(0,x), \end{cases} \qquad \begin{cases} \tilde{P}_{u+\varepsilon_2 u_1}(D) w_1^{\varepsilon_2} = f_1{}^{\varepsilon_2}, \\ w_1^{\varepsilon_2}(0,x) = h_1(0,x), \end{cases} \tag{3.15}$$

then $w_1^{\varepsilon_1} - w_1^{\varepsilon_2}$ is solution of

$$\begin{aligned} D_t w_1^{\varepsilon_1} &+ a_p(t) D_x^p w_1^{\varepsilon_1} + \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u+\varepsilon_1 u_1) D_x^j w_1^{\varepsilon_1} - D_t w_1^{\varepsilon_2} - a_p(t) D_x^p w_1^{\varepsilon_2} \\ &- \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u+\varepsilon_2 u_1) D_x^j w_1^{\varepsilon_2} + \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u+\varepsilon_1 u_1) D_x^j w_1^{\varepsilon_2} \\ &- \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u+\varepsilon_1 u_1) D_x^j w_1^{\varepsilon_2} \\ =\, & f_1^{\varepsilon_1} - f_1^{\varepsilon_2} \end{aligned}$$

with initial condition $(w_1^{\varepsilon_1} - w_1^{\varepsilon_2})(0,x) = 0$, i.e.

$$\begin{cases} \tilde{P}_{u+\varepsilon_1 u_1}(D)(w_1^{\varepsilon_1} - w_1^{\varepsilon_2}) = f_1^{\varepsilon_1} - f_1^{\varepsilon_2} + \displaystyle\sum_{j=0}^{p-1} \big(\tilde{a}_j(t,x,u+\varepsilon_2 u_1) - \tilde{a}_j(t,x,u+\varepsilon_1 u_1)\big) D_x^j w_1^{\varepsilon_2} \\ (w_1^{\varepsilon_1} - w_1^{\varepsilon_2})(0,x) = 0. \end{cases}$$

By the energy estimate (2.1) and the Lagrange theorem, there exists $u_{1,2}$ between $u + \varepsilon_1 u_1$ and $u + \varepsilon_2 u_1$ such that, for all $t \in [0,T]$,

$$\|(w_1^{\varepsilon_1} - w_1^{\varepsilon_2})(t,\cdot)\|_n \le C_n(u+\varepsilon_1 u_1)\Big(\sup_{t\in[0,T]} \|f_1^{\varepsilon_1}(t,\cdot) - f_1^{\varepsilon_2}(t,\cdot)\|_{n+r} + \sum_{j=0}^{p-1} \sup_{t\in[0,T]} \|\partial_w a_j(t,x,u_{1,2})(\varepsilon_1-\varepsilon_2)u_1 D_x^j w_1^{\varepsilon_2}\|_{n+r}\Big)$$

for some $C_n(u+\varepsilon_1 u_1) > 0$. This goes to 0 as $\varepsilon_1 \to \varepsilon_2 \to 0$ because $f_1^{\varepsilon_1} - f_1^{\varepsilon_2} \to 0$ and, being $H^s(\mathbb{R})$ an algebra and satisfying Sobolev inequality for $s > 1/2$,

$$\|\partial_w a_j(t,x,u_{1,2})(\varepsilon_1-\varepsilon_2)u_1 D_x^j w_1^{\varepsilon_2}\|_{n+r} \le \|\partial_w a_j(t,x,u_{1,2})\|_{n+r}(\varepsilon_1-\varepsilon_2)\|u_1\|_{n+r}\|w_1^{\varepsilon_2}\|_{n+r+j}$$

is bounded for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1) in a neighborhood of some fixed $(\bar u_1,\bar h_1) \in X\times X$, since $u_{1,2}$ is between $u+\varepsilon_1 u_1$ and $u+\varepsilon_2 u_1$ and $\|w_1^{\varepsilon_2}\|_{n+r+j}$ is bounded by the energy estimate

$$\|w_1^{\varepsilon_2}\|_{n+r+j}^2 \le C_{n+r+j}(u+\varepsilon_2 u_1)\left(\|h_1(0,\cdot)\|_{n+2r+j}^2 + \int_0^t \|f_1^{\varepsilon_2}(\tau,\cdot)\|_{n+2r+j}^2 d\tau\right).$$

Then $\{w_1^\varepsilon\}_\varepsilon$ is a Cauchy sequence in X and the Fréchet derivative

$$\begin{aligned} DS:\ (X\times X)^2 &\longrightarrow X \\ \big((u,h),(u_1,h_1)\big) &\longmapsto w_1, \end{aligned}$$

with w_1 solution of (3.12), is tame by the above considerations.

Step 3. Let us now consider the second derivative of S:

$$\begin{aligned} D^2S:\ (X\times X)^3 &\longrightarrow X \\ \big((u,h),(u_1,h_1),(u_2,h_2)\big) &\longmapsto D^2S(u,h)(u_1,h_1)(u_2,h_2) \end{aligned}$$

defined by

$$\begin{aligned} D^2S(u,h)(u_1,h_1)(u_2,h_2) &= \lim_{\varepsilon\to 0} \frac{DS(u+\varepsilon u_2, h+\varepsilon h_2)(u_1,h_1) - DS(u,h)(u_1,h_1)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\bar w_1^\varepsilon - w_1}{\varepsilon} =: \lim_{\varepsilon\to 0} w_2^\varepsilon \end{aligned}$$

where w_1 is the solution of the Cauchy problem (3.12) and $\bar w_1^\varepsilon$ is the solution of the Cauchy problem

$$\begin{cases} \tilde P_{u+\varepsilon u_2} \bar w_1^\varepsilon = \bar f_1^\varepsilon := D_t h_1 - \displaystyle\sum_{j=0}^{p-1} \partial_w \tilde a_j(t,x,u+\varepsilon u_2) u_1 D_x^j v \\ \bar w_1^\varepsilon(0,x) = h_1(0,x). \end{cases} \tag{3.16}$$

Writing

$$\bar{f}_1^\varepsilon - f_1 = \tilde{P}_{u+\varepsilon u_2}(D)\bar{w}_1^\varepsilon - \tilde{P}_u(D)w_1$$
$$= \tilde{P}_{u+\varepsilon u_2}(D)(\bar{w}_1^\varepsilon - w_1) + \sum_{j=0}^{p-1}\big(\tilde{a}_j(t,x,u+\varepsilon u_2) - \tilde{a}_j(t,x,u)\big)D_x^j w_1$$

we have from (3.13) and (3.16) that

$$\tilde{P}_{u+\varepsilon u_2}(D)w_2^\varepsilon = \frac{f_1^\varepsilon - f_1}{\varepsilon} - \sum_{j=0}^{p-1}\frac{\tilde{a}_j(t,x,u+\varepsilon u_2) - \tilde{a}_j(t,x,u)}{\varepsilon}D_x^j w_1$$
$$= -\sum_{j=0}^{p-1}\frac{\tilde{a}_j(t,x,u+\varepsilon u_2) - \tilde{a}_j(t,x,u)}{\varepsilon}D_x^j w_1$$
$$- \sum_{j=0}^{p-1}\frac{\partial_w\tilde{a}_j(t,x,u+\varepsilon u_2) - \partial_w\tilde{a}_j(t,x,u)}{\varepsilon}u_1 D_x^j v$$
$$=: f_2^\varepsilon$$

i.e.

$$\begin{cases} \tilde{P}_{u+\varepsilon u_2}(D)w_2^\varepsilon = f_2^\varepsilon \\ w_2^\varepsilon(0,x) = 0. \end{cases}$$

Arguing as for $\{w_1^\varepsilon\}_\varepsilon$, we can prove that $\{w_2^\varepsilon\}_\varepsilon$ is a Cauchy sequence and hence $w_2^\varepsilon \to w_2 \in X$, where w_2 is the solution of the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_2 = -\sum_{j=0}^{p-1}\partial_w\tilde{a}_j(t,x,u)u_2 D_x^j w_1 - \sum_{j=0}^{p-1}\partial_w^2\tilde{a}_j(t,x,u)u_1u_2 D_x^j v =: f_2 \\ w_2(0,x) = 0 \end{cases}$$

and satisfies the following energy estimate for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1), (u_2,h_2) in a neighborhood of some fixed $(\bar{u}_1,\bar{h}_1)$, $(\bar{u}_2,\bar{h}_2) \in X \times X$:

$$\|w_2\|_n^2 \le C_n(u)\int_0^t \|f_2(\tau,\cdot)\|_{n+r}^2 d\tau$$

which gives

$$\|w_2\|_n \le C'_n(u,u_1,u_2)\sum_{j=0}^{p-1}\Big(\|D_x^j w_1\|_{n+r} + \|D_x^j v\|_{n+r}\Big)$$
$$\le C'_n(u,u_1,u_2)\big(\|w_1\|_{n+r+p-1} + \|v\|_{n+r+p-1}\big) \le C''_n\big(|||h_1|||_{n+r''} + |||h|||_{n+r''}\big)$$

for some $C'_n(u, u_1, u_2), C''_n > 0$ and $r'' \geq r + p - 1 + r'$, by (3.14) and (3.8). Then also $D^2 S$ is tame.

Step 4. We prove by induction on $m \in \mathbb{N}$ that, for all $m \geq 2$,

$$D^m S(u,h)(u_1,h_1)\cdots(u_m,h_m) = w_m \tag{3.17}$$

is the solution of the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_m = f_m \\ w_m(0,x) = 0, \end{cases} \tag{3.18}$$

with

$$\begin{aligned} f_m := &-\sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u) u_m D_x^j w_{m-1} - \sum_{j=0}^{p-1} \partial_w^2 \tilde{a}_j(t,x,u) u_{m-1} u_m D_x^j w_{m-2} \\ &- \cdots - \sum_{j=0}^{p-1} \partial_w^m \tilde{a}_j(t,x,u) u_1 \cdots u_{m-1} u_m D_x^j w_0 \end{aligned} \tag{3.19}$$

and $w_0 := v$, and satisfies, in a neighborhood of (u,h), $(u_1,h_1), \ldots (u_m,h_m)$, the estimate

$$|||w_m|||_n \leq C_n \sum_{j=0}^{m-1} |||h_j|||_{n+r(m)} \tag{3.20}$$

for some $C_n > 0$ and $r(m) \in \mathbb{N}$, where $h_0 := h$.

Let us assume (3.17)–(3.20) to be true for all $j \leq m$ and let us prove them for $j = m + 1$:

$$\begin{aligned} &D^{m+1} S(u,h)(u_1,h_1)\cdots(u_{m+1},h_{m+1}) \\ &\quad = \lim_{\varepsilon\to 0} \frac{D^m S(u+\varepsilon u_{m+1}, h+\varepsilon u_{m+1})(u_1,h_1)\cdots(u_m,h_m) - D^m S(u,h)(u_1,h_1)\cdots(u_m,h_m)}{\varepsilon} \\ &\quad = \lim_{\varepsilon\to 0} \frac{\bar{w}_m^\varepsilon - w_m}{\varepsilon} =: \lim_{\varepsilon\to 0} w_{m+1}^\varepsilon, \end{aligned} \tag{3.21}$$

where w_m is the solution of (3.18) and $\bar{w}_m^\varepsilon$ is the solution of

$$\begin{cases} \tilde{P}_{u+\varepsilon u_{m+1}}(D)\bar{w}_m^\varepsilon = f_m^\varepsilon \\ \bar{w}_m^\varepsilon(0,x) = 0 \end{cases}$$

with

$$\begin{aligned} f_m^\varepsilon := &-\sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u+\varepsilon u_{m+1}) u_m D_x^j w_{m-1} - \sum_{j=0}^{p-1} \partial_w^2 \tilde{a}_j(t,x,u+\varepsilon u_{m+1}) u_{m-1} u_m D_x^j w_{m-2} \\ &- \cdots - \sum_{j=0}^{p-1} \partial_w^m \tilde{a}_j(t,x,u+\varepsilon u_{m+1}) u_1 \cdots u_m D_x^j w_0. \end{aligned} \tag{3.22}$$

Then

$$\begin{aligned}
f_m^\varepsilon - f_m &= \tilde{P}_{u+\varepsilon u_{m+1}}(D)\bar{w}_m^\varepsilon - \tilde{P}_u(D)w_m \\
&= D_t(\bar{w}_m^\varepsilon - w_m) + a_p(t)D_x^p(\bar{w}_m^\varepsilon - w_m) + \sum_{j=0}^{p-1}\tilde{a}_j(t,x,u+\varepsilon u_{m+1})D_x^j(\bar{w}_m^\varepsilon - w_m) \\
&\quad + \sum_{j=0}^{p-1}\big(\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \tilde{a}_j(t,x,u)\big)D_x^j w_m \\
&= \tilde{P}_{u+\varepsilon u_{m+1}}(D)(\bar{w}_m^\varepsilon - w_m) + \sum_{j=0}^{p-1}\big(\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \tilde{a}_j(t,x,u)\big)D_x^j w_m \qquad (3.23)
\end{aligned}$$

and hence, by (3.21), (3.23), (3.22) and (3.19), w_{m+1}^ε is solution of the Cauchy problem

$$\begin{cases} \tilde{P}_{u+\varepsilon u_{m+1}}(D)w_{m+1}^\varepsilon = f_{m+1}^\varepsilon \\ w_{m+1}^\varepsilon(0) = 0 \end{cases}$$

where

$$\begin{aligned}
f_{m+1}^\varepsilon := &-\sum_{j=0}^{p-1}\frac{\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \tilde{a}_j(t,x,u)}{\varepsilon}D_x^j w_m + \frac{f_m^\varepsilon - f_m}{\varepsilon} \\
= \quad &-\sum_{j=0}^{p-1}\frac{\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \tilde{a}_j(t,x,u)}{\varepsilon}D_x^j w_m \\
&-\sum_{j=0}^{p-1}\frac{\partial_w\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \partial_w\tilde{a}_j(t,x,u)}{\varepsilon}u_m D_x^j w_{m-1} \\
&-\sum_{j=0}^{p-1}\frac{\partial_w^2\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \partial_w^2\tilde{a}_j(t,x,u)}{\varepsilon}u_{m-1}u_m D_x^j w_{m-2} \\
&\vdots \\
&-\sum_{j=0}^{p-1}\frac{\partial_w^m\tilde{a}_j(t,x,u+\varepsilon u_{m+1}) - \partial_w^m\tilde{a}_j(t,x,u)}{\varepsilon}u_1\cdots u_m D_x^j w_0.
\end{aligned}$$

Arguing as for $\{w_1^\varepsilon\}_\varepsilon$, we can prove that $\{w_{m+1}^\varepsilon\}_\varepsilon$ is a Cauchy sequence and therefore $w_{m+1}^\varepsilon \to w_{m+1} \in X$, where w_{m+1} is the solution of the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_{m+1} = f_{m+1} \\ w_{m+1}(0,x) = 0, \end{cases}$$

with

$$f_{m+1} := -\sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u) u_{m+1} D_x^j w_m - \sum_{j=0}^{p-1} \partial_w^2 \tilde{a}_j(t,x,u) u_m u_{m+1} D_x^j w_{m-1}$$
$$- \sum_{j=0}^{p-1} \partial_w^3 \tilde{a}_j(t,x,u) u_{m-1} u_m u_{m+1} D_x^j w_{m-2} \cdots$$
$$- \sum_{j=0}^{p-1} \partial_w^{m+1} \tilde{a}_j(t,x,u) u_1 \cdots u_{m+1} D_x^j w_0$$

and (3.18) is proved for $j = m+1$. Moreover, by the energy estimate (2.1) and the inductive assumption (3.20)

$$\|w_{m+1}\|_n^2 \leq C_n(u) \int_0^t \|f_{m+1}(\tau,\cdot)\|_{n+r}^2 d\tau$$

and so, for (u,h) in a neighborhood of (u_0,h_0) and $(u_1,h_1),\dots,(u_m,h_m)$ in a neighborhood of some fixed $(\bar{u}_1,\bar{h}_1),\dots(\bar{u}_m,\bar{h}_m) \in X \times X$,

$$\|w_{m+1}\|_n \leq C'_n(u) \sum_{s=0}^{m} \sum_{j=0}^{p-1} \|u_{m-s+1} \cdots u_m \cdot u_{m+1} D_x^j w_{m-s}\|_{n+r}$$
$$\leq C_n(u,u_1,\dots,u_m) \sum_{j=0}^{p-1} \sum_{s=0}^{m} \|w_{m-s}\|_{n+r+j} \leq C'_n \sum_{j=0}^{p-1} \sum_{s=0}^{m} \sum_{i=0}^{m-s-1} |||h_i|||_{n+r+j+r(m-s)}$$
$$\leq C''_n \sum_{i=0}^{m-1} |||h_i|||_{n+r'(m)}$$

for some $C'_n(u)$, $C_n(u,u_1,\dots,u_m)$, $C'_n, C''_n > 0$, $r'(m) \in \mathbb{N}$.

Then also

$$\|D_t w_{m+1}\|_n = \| -a_p(t) D_x^p w_{m+1} - \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u) D_x^j w_{m+1} + f_{m+1}\|_n$$
$$\leq C_n(u) \left(\|w_{m+1}\|_{n+p} + \|f_{m+1}\|_n \right) \leq C'_n \sum_{i=0}^{m-1} |||h_i|||_{n+r''(m)}$$

for some $C'_n > 0$, $r''(m) = p + r'(m)$, and for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1), $\dots$, (u_m,h_m) in a neighborhood of some fixed $(\bar{u}_1,\bar{h}_1),\dots(\bar{u}_m,\bar{h}_m) \in X \times X$. Therefore (3.20) holds also for $m+1$.

We have thus proved (3.17)–(3.20). In particular, $D^m S$ is tame for every m and hence S is a smooth tame map. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Remark 3.1, our goal is to look for a local solution u of (1.1) as a local solution of (3.2), i.e. of

$$u(t,x)=u_0(x)-i\int_0^t a_p(s)D_x^p u(s,x)ds-i\sum_{j=0}^{p-1}\int_0^t a_j(s,x,u(s,x))D_x^j u(s,x)ds \\ +i\int_0^t f(s,x)ds, \tag{3.24}$$

by definition (3.1) of the map J. To this aim, let us notice that from (3.24) we have the Taylor expansion of the first order of u:

$$u(t,x) = u_0(x)-it\left(a_p(0)D_x^p u_0(x)+\sum_{j=0}^{p-1}a_j(0,x,u_0(x))D_x^j u_0(x)-f(0,x)\right)+o(t) \\ =: w(t,x)+o(t), \qquad \text{as } t\to 0. \tag{3.25}$$

The function $w\in X$ is in a neighborhood of the solution u we are looking for, if t is sufficiently small. The idea of the proof is to approximate Jw by a function ϕ_ε identically zero for $0\le t\le T_\varepsilon\le T$ and apply the Nash–Moser's Theorem A.11, in particular the fact that J is a bijection of a neighborhood U of w onto a neighborhood V of Jw. If $\phi_\varepsilon\in V$, then by the local invertibility of J there will be $u\in U$ such that $Ju=\phi_\varepsilon\equiv 0$ in $[0,T_\varepsilon]$ and hence the local (in time) solution of (3.2) will be found.

To construct ϕ_ε we compute first (see the definition (3.1) of J):

$$\partial_t(Jw(t,x))=\partial_t w+ia_p(t)D_x^p w+i\sum_{j=0}^{p-1}a_j(t,x,w)D_x^j w-if(t,x),$$

and using the definition (3.25) of w we get

$$\begin{aligned}\partial_t(Jw(t,x)) &= -ia_p(0)D_x^p u_0-i\sum_{j=0}^{p-1}a_j(0,x,u_0)D_x^j u_0+if(0,x)\\ &\quad +ia_p(t)D_x^p u_0+ta_p(t)D_x^p\Big(a_p(0)D_x^p u_0+\sum_{j=0}^{p-1}a_j(0,x,u_0)D_x^j u_0-f(0,x)\Big)\\ &\quad +i\sum_{j=0}^{p-1}a_j(t,x,w)D_x^j u_0+t\sum_{j=0}^{p-1}a_j(t,x,w)D_x^j\Big(a_p(0)D_x^p u_0\\ &\quad +\sum_{j=0}^{p-1}a_j(0,x,u_0)D_x^j u_0-f(0,x)\Big)-if(t,x)\\ &= i[a_p(t)-a_p(0)]D_x^p u_0+i\sum_{j=0}^{p-1}\big[a_j(t,x,w)-a_j(0,x,u_0)\big]D_x^j u_0\end{aligned}$$

$$+ a_p(t) t D_x^p \Big[a_p(0) D_x^p u_0 + \sum_{j=0}^{p-1} a_j(0,x,u_0) D_x^j u_0 - f(0,x) \Big]$$
$$+ \sum_{j=0}^{p-1} a_j(t,x,w) t D_x^j \Big[a_p(0) D_x^p u_0 + \sum_{s=0}^{p-1} a_s(0,x,u_0) D_x^s u_0 - f(0,x) \Big]$$
$$+ i\big(f(0,x) - f(t,x)\big).$$

Therefore

$$\|\partial_t Jw(t,\cdot)\|_n \le \sup_{t\in[0,T]} |a_p(t) - a_p(0)| \cdot \|u_0\|_{n+p} + \sum_{j=0}^{p-1} \Big\| \big[a_j(t,x,w) - a_j(0,x,u_0)\big] D_x^j u_0 \Big\|_n$$
$$+ t \sup_{t\in[0,T]} |a_p(t)| \cdot \Big\| a_p(0) D_x^p u_0 + \sum_{s=0}^{p-1} a_s(0,x,u_0) D_x^s u_0 - f(0,x) \Big\|_{n+p}$$
$$+ t \sum_{j=0}^{p-1} \Big\| a_j(t,x,w) D_x^j \Big[a_p(0) D_x^p u_0 + \sum_{s=0}^{p-1} a_s(0,x,u_0) D_x^s u_0 - f(0,x) \Big] \Big\|_n$$
$$+ \|f(0,x) - f(t,x)\|_n.$$

By Lagrange theorem and for t sufficiently small (so that w is in a sufficiently small neighborhood of u_0):

$$\begin{aligned} \|\partial_t Jw(t,\cdot)\|_n &\le C_0(a_p,u_0)t + \sum_{j=0}^{p-1} C(a_j,u_0)t + C_1(a_p,\dots,a_0,u_0,f)t + C_1(f)t \\ &\le C(a_p,\dots,a_0,u_0,f)t \end{aligned} \tag{3.26}$$

for some positive constants $C_0(a_p,u_0)$, $C(a_j,u_0)$, $C_1(a_p,\dots,a_0,u_0,f)$, $C_1(f)$, $C(a_p,\dots,a_0,u_0,f)$ depending only on the variables specified there.

Let us now choose $\rho \in C^\infty(\mathbb{R})$ with $0 \le \rho \le 1$ and

$$\rho(s) = \begin{cases} 0, & s \le 1 \\ 1, & s \ge 2. \end{cases}$$

Define then

$$\phi_\varepsilon(t,x) := \int_0^t \rho\left(\frac{s}{\varepsilon}\right)(\partial_t Jw)(s,x)\,ds,$$

and note that $\phi_\varepsilon \equiv 0$ for $0 \le t \le \varepsilon$. We are going to prove that, for every fixed neighborhood V of Jw in the topology of $X = C^1([0,T]; H^\infty(\mathbb{R}))$, we have $\phi_\varepsilon \in V$ if ε is sufficiently small. Indeed, by definition of ϕ_ε and using (3.26),

$$
\begin{aligned}
\|Jw-\phi_\varepsilon\|_n &= \left\| \int_0^t \left(1-\rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s,\cdot)ds \right\|_n \le \int_0^{2\varepsilon} \left\| \left(1-\rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s,\cdot) \right\|_n ds \\
&\le C(a_p,\dots,a_0,u_0,f)\int_0^{2\varepsilon} s\,ds = C(a_p,\dots,a_0,u_0,f)2\varepsilon^2. \qquad (3.27)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\|\partial_t(Jw-\phi_\varepsilon)\|_n &= \left\| \partial_t Jw(t,\cdot) - \rho\left(\frac{t}{\varepsilon}\right)(\partial_t Jw)(t,\cdot) \right\|_n \le \left(1-\rho\left(\frac{t}{\varepsilon}\right)\right)\|\partial_t Jw(t,\cdot)\|_n \\
&\le \left(1-\rho\left(\frac{t}{\varepsilon}\right)\right) C(a_p,\dots,a_0,u_0,f)t \le 2C(a_p,\dots,a_0,u_0,f)\varepsilon, \qquad (3.28)
\end{aligned}
$$

again by (3.26) and looking at the support of $1-\rho(t/\varepsilon)$.

From (3.27) and (3.28) we thus have, for $0<\varepsilon<1$, that

$$
|||Jw-\phi_\varepsilon|||_n \le 2C(a_p,\dots,a_0,u_0,f)\varepsilon
$$

and hence $\phi_\varepsilon \in V$ for ε sufficiently small, where V is the neighborhood of Jw such that $J: U\to V$ is invertible.

Then, there exists $u\in U\subset X$ such that $Ju=\phi_\varepsilon$ and hence, in particular,

$$
Ju \equiv 0 \qquad \text{for } 0\le t\le \varepsilon.
$$

This proves that $u\in C^1([0,\varepsilon];H^\infty(\mathbb{R}))$ is a local solution of the Cauchy problem (1.1).

Uniqueness follows by standard arguments. As a matter of fact, if u, v are two solutions of the Cauchy problem (1.1), we have

$$
\begin{aligned}
0 = P_u(D)u - P_v(D)v &= P_u(D)(u-v) + \sum_{j=0}^{p-1}\big(a_j(t,x,u)-a_j(t,x,v)\big)D_x^j v \\
&= P_u(D)(u-v) + \sum_{j=0}^{p-1}\int_v^u \partial_w a_j(t,x,s)ds\, D_x^j v \\
&= P_u(D)(u-v) + \sum_{j=0}^{p-1}\int_0^1 \partial_w a_j(t,x,v+t(u-v))(u-v)dt\, D_x^j v \\
&= \left(P_u(D) + \sum_{j=0}^{p-1}\int_0^1 \partial_w a_j(t,x,v+t(u-v))dt\, D_x^j v\right)(u-v) \\
&=: \tilde{\tilde{P}}(u-v).
\end{aligned}
$$

Therefore, for fixed $u, v\in X$, the function $w := u-v$ solves the linear Cauchy problem

$$
\begin{cases} \tilde{\tilde{P}}w = 0 \\ w(0,x) = 0, \end{cases} \qquad (3.29)
$$

and since $\tilde{\tilde{P}}$ is of the same form as $P_u(D)$ with

$$\tilde{\tilde{a}}_0(t,x,u) := a_0(t,x,u) + \sum_{j=0}^{p-1} \int_0^1 \partial_w a_j(t,x,v+t(u-v))dt\, D_x^j v$$

instead of $a_0(t,x,u)$, and has therefore the same kind of regularity on the coefficients. By the uniqueness of the linearized Cauchy problem (given by Theorem 1.1), we finally have that $w=0$. Therefore $u=v$ and uniqueness is proved. □

4. Further generalizations

In this section we focus on generalizations of Theorem 1.3; we first consider the Cauchy problem (1.1) with $a_p = a_p(t,x)$, $x \in \mathbb{R}$, and then give an idea on how the result can be extended to the case $x \in \mathbb{R}^n$.

The dependence of a_p on x means that in the explicit expression of the symbol $\sigma(e^{-\Lambda} A e^{\Lambda})$ in (2.19) some new terms containing $D_x^\beta a_p(t,x)$ appear for $\beta \neq 0$.

By assuming $a_p \in C([0,T], \mathcal{B}^\infty(\mathbb{R}))$ with $a_p(t,x) \in \mathbb{R}$ and

$$|\mathrm{Im}(D_x^\beta a_p)(t,x)| \le \frac{C}{\langle x \rangle^{\frac{p-[\beta/2]}{p-1}}}, \qquad 0 \le \left[\frac{\beta}{2}\right] \le p-1,\ \beta \neq 0 \tag{4.1}$$

$$|\mathrm{Re}(D_x^\beta a_p)(t,x)| \le C, \qquad 0 \le \beta \le p-1, \tag{4.2}$$

in analogy with (1.11)–(1.12), we shall retrace here below the proof of Theorem 1.3.

Notice that $a_p \in C([0,T]; \mathcal{B}^\infty(\mathbb{R}))$ implies that condition (4.2) is automatically satisfied, while a_p real valued implies that condition (4.1) reduces to

$$|D_x^\beta a_p(t,x)| \le \frac{C}{\langle x \rangle^{\frac{p-[\beta/2]}{p-1}}}, \qquad 0 \le \left[\frac{\beta}{2}\right] \le p-1,\ \beta \text{ odd}.$$

Therefore we can prove the following theorem:

Theorem 4.1. *Let $p \ge 2$ and consider the following p-evolution operator:*

$$P_u(D)u := D_t u + a_p(t,x) D_x^p u + \sum_{j=0}^{p-1} a_j(t,x,u) D_x^j u, \tag{4.3}$$

where $a_p \in C([0,T]; \mathcal{B}^\infty(\mathbb{R}))$ with $a_p(t,x) \in \mathbb{R}$, and $a_j \in C([0,T]; C^\infty(\mathbb{R} \times \mathbb{C}))$ with $x \mapsto a_j(t,x,w) \in \mathcal{B}^\infty(\mathbb{R})$, for $0 \le j \le p-1$.

Let us assume that there exist constants $C_p > 0$ and $C > 0$ and a function $\gamma: \mathbb{C} \to \mathbb{R}^+$ of class C^7 such that, for all $(t,x,w) \in [0,T] \times \mathbb{R} \times \mathbb{C}$:

$$a_p(t,x) \ge C_p,$$

$$|D_x^\beta a_p(t,x)| \le \frac{C}{\langle x \rangle^{\frac{p-[\beta/2]}{p-1}}}, \qquad 0 \le \left[\frac{\beta}{2}\right] \le p-1,\ \beta \text{ odd},$$

$$|\operatorname{Im}(D_x^\beta a_j)(t,x,w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{j-[\beta/2]}{p-1}}}, \quad 0 \le \left[\frac{\beta}{2}\right] \le j-1,\ 3 \le j \le p-1,$$

$$|\operatorname{Re}(D_x^\beta a_j)(t,x,w)| \le C\gamma(w) \qquad 0 \le \beta \le j-1,\ 3 \le j \le p-1,$$

$$|(D_w^\alpha D_x^\beta a_j)(t,x,w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{j-[(\alpha+\beta)/2]}{p-1}}}, \quad \alpha \ge 1, \beta \ge 0, \left[\frac{\alpha+\beta}{2}\right] \le j-1,\ 3 \le j \le p-1,$$

$$|\operatorname{Re} a_2(t,x,w)| \le C\gamma(w),$$

$$|\operatorname{Im} a_2(t,x,w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{2}{p-1}}},$$

$$|\operatorname{Im} a_1(t,x,w)| + |\operatorname{Im} D_x a_2(t,x,w)| + |D_w a_2(t,x,w)| \le \frac{C\gamma(w)}{\langle x\rangle^{\frac{1}{p-1}}}. \tag{4.4}$$

Then the Cauchy problem (1.1), for $P_u(D)$ defined as in (4.3), is locally in time well-posed in H^∞: for all $f \in C([0,T]; H^\infty(\mathbb{R}))$ and $u_0 \in H^\infty(\mathbb{R})$, there exists $0 < T^* \le T$ and a unique solution $u \in C([0,T^*]; H^\infty(\mathbb{R}))$ of (1.1).

Proof. We remark that from the assumptions (4.1)–(4.2) we can obtain (2.14)–(2.15) also for $j = p$ (indeed, inequalities (2.16)–(2.17) are valid also for $j = p$ with a fixed constant $\gamma(u) = \gamma$).

Now we follow the proof of Theorem 2.1 (see also [7]) outlining the needed changes. In formula (2.19) of step 1, on the one hand the symbol A_I remains unvaried if $a_p = a_p(t,x)$, so (2.25) is unvaried too; on the other hand, A_{II} has terms $D_x^\beta a_p(t,x)$ which are now different from zero also for $\beta \ne 0$. Deriving (2.23) from (2.20) and (2.21) we thus have to take into account these terms. The estimates of the order and decay of the terms in (2.20) are the same as in the case $a_p = a_p(t)$, while in the estimate of the order and decay of the terms in (2.21), the only term which works differently from the case $a_p = a_p(t)$ is $-(\partial_\xi \Lambda)(D_x i a_p)\xi^p$. This is sum of terms of the form $-(\partial_\xi \lambda_{p-k})(\partial_x a_p)\xi^p$: for $k = 1$ the term $-(\partial_\xi \lambda_{p-1})(\partial_x a_p)\xi^p$ is of order zero because $\partial_\xi \lambda_{p-1}$ has support in the set $\{\frac{1}{2}\langle\xi\rangle_h^{p-1} \le \langle x\rangle \le \langle\xi\rangle_h^{p-1}\}$; the other terms

$$-(\partial_\xi \lambda_{p-k})(\partial_x a_p)\xi^p, \qquad 2 \le k \le p-1, \tag{4.5}$$

have order $p-k$ and not $p-k-1$, as it was in (2.22).

Therefore all terms of $\operatorname{Re} A_{II}|_{\operatorname{ord}(p-k)}$ satisfy (2.23) except for the terms in (4.5), that we shall treat separately here below, following the same ideas as in the proof of the invertibility of e^Λ (cf. [7]).

By (2.6) and (4.1):

$$|\partial_\xi \lambda_{p-k} \partial_x a_p \xi^p| \le C_{1,0} M_{p-k} \langle x\rangle^{\frac{k-1}{p-1}} \langle\xi\rangle_h^{-k} \chi_E(x) \frac{C}{\langle x\rangle^{\frac{p}{p-1}}} \langle\xi\rangle_h^p$$

$$= C_{1,0} C \frac{M_{p-k}}{\langle\xi\rangle_h} \frac{\langle\xi\rangle_h^{p-k+1}}{\langle x\rangle^{\frac{p-k+1}{p-1}}} \chi_E(x)$$

$$\le C_{1,0}C\,\frac{M_{p-k}}{h}\,\frac{\langle\xi\rangle_h^{p-k+1}}{\langle x\rangle^{\frac{p-k+1}{p-1}}}$$

$$\le C_{1,0}C\,\frac{\langle\xi\rangle_h^{p-k+1}}{\langle x\rangle^{\frac{p-k+1}{p-1}}} \tag{4.6}$$

if $h \ge M_{p-k}$.

This means that we can insert $(\partial_\xi\lambda_{p-k})(\partial_x a_p)\xi^p$ at level $p-k+1$, since it has the "right decay" for the level $p-k+1$ and satisfies an estimate of the form (2.18), with a constant $C_{1,0}C$ that does not depend on any of the M_{p-k}, for $k \ge 2$.

Therefore we shall insert $(\partial_\xi\lambda_{p-k})(\partial_x a_p)\xi^p$ in $\mathrm{Re}\,A_{II}|_{\mathrm{ord}(p-k+1)}$ instead of $\mathrm{Re}\,A_{II}|_{\mathrm{ord}(p-k)}$, for $2 \le k \le p-1$, and act as if (2.23) holds as it was in Theorem 2.1.

All the other steps are based on the estimates (2.25) and (2.23), so that their proof follows as in Theorem 2.1, thanks to the added assumptions (4.1), (4.2), which ensure that the new terms still have the "right decay for the right level" and depend on the "right constants" M_{p-k}, if we choose $h \ge \max\{M_{p-2}, M_{p-3}, \dots, M_1\}$ and large enough to ensure the invertibility of the operator e^Λ as in (2.9).

Finally the Nash–Moser scheme of Section 3 does not involve the dependence of a_p on x.

Therefore Theorem 4.1 is proved. □

We conclude this paper with the following remark about the generalization of Theorems 1.3 and 4.1 to the case $x \in \mathbb{R}^n$, $n \ge 2$.

Remark 4.2. In the proof of Theorems 1.3 and 4.1, the symbol $\Lambda = \lambda_1 + \cdots + \lambda_{p-1}$ was constructed in (2.5), following [23], in a way such that in Steps 2 and 3 of the proof of Theorem 2.1, in order to apply the sharp-Gårding theorem, we got

$$\mathrm{Re}(ipa_pD_x\lambda_{p-k}\xi^{p-1} + i\tilde{a}_{p-k}) = pa_p\partial_x\lambda_{p-k}\xi^{p-1} - \mathrm{Im}\,\tilde{a}_{p-k} \ge -C(u), \quad k = 1, \dots, p-1, \tag{4.7}$$

where $\tilde{a}_{p-k}$ was given by the sum of $a_{p-k}\xi^{p-k}$ and (possible) other symbols of order $p-k$ with the "right decay for the right level" and dependence on the "right constants".

For the case of more space variables we have to choose $\Lambda = \lambda_1 + \cdots + \lambda_{p-1}$ in order that it satisfies a pseudo-differential inequality of the form:

$$\sum_{j=1}^{n} pa_p(t,x)\partial_{x_j}\lambda_{p-k}|\xi|^{p-2}\xi_j - \mathrm{Im}\,\tilde{\tilde{a}}_{p-k} \ge -C(u), \quad k = 1, \dots, p-1, \tag{4.8}$$

where $\tilde{\tilde{a}}_{p-k}$ is a symbol of order $p-k$ with the "right decay" and depending on the "right constants", i.e.

$$|\mathrm{Im}\,\tilde{\tilde{a}}_{p-k}| \le C(M_{p-1}, \dots, M_{p-k+1})\frac{\langle\xi\rangle_h^{p-k}}{\langle x\rangle^{\frac{p-k}{p-1}}} \tag{4.9}$$

for some $C(M_{p-1}, \dots, M_{p-k+1}) > 0$.

A solution λ_{p-k} to (4.8) can be constructed, following the ideas of [23], by solving the equation

$$\sum_{j=1}^{n} pC_p \partial_{x_j} \lambda_{p-k} |\xi|^{p-2} \xi_j = |\xi| g_k(x,\xi) \tag{4.10}$$

for C_p as in (4.4), and for some positive function $g_k(x,\xi)$ with a decay as in (4.9) and large enough so that (4.8) will therefore be satisfied.

But a solution, for large $|\xi|$, of an equation of the form

$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda(x,\xi) = |\xi| g(x,\xi)$$

is given by (cf. [23])

$$\lambda(x,\xi) = \int_0^{x\cdot\xi/|\xi|} g\left(x - \tau\frac{\xi}{|\xi|}, \xi\right) d\tau, \tag{4.11}$$

so that the functions λ_{p-k} can be constructed by means of functions λ of the form (4.11), as explained in [16].

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Appendix A. The Nash–Moser theorem

We recall here the basic notion of the Nash–Moser theory as in [18,20].

Definition A.1. A *graded* Fréchet space X is a Fréchet space whose topology is generated by a *grading*, i.e. an increasing sequence of semi-norms:

$$\|x\|_n \le \|x\|_{n+1}, \qquad \forall n \in \mathbb{N}_0,\ x \in X.$$

Definition A.2. For two graded Fréchet spaces X, Y, a linear map $L: X \to Y$ is said to be a *tame linear map* if there exist $r, n_0 \in \mathbb{N}$ such that for every integer $n \ge n_0$ there is a constant $C_n > 0$, depending only on n, s.t.

$$\|Lx\|_n \le C_n \|x\|_{n+r} \qquad \forall x \in X. \tag{A.1}$$

The number n_0 is called the *base* and r the *degree* of the *tame estimate* (A.1).

Definition A.3. Given a Banach space B, the *space of exponentially decreasing sequences* $\Sigma(B)$ is the graded space of all sequences $\{v_k\}_{k\in\mathbb{N}_0} \subset B$ such that

$$\|\{v_k\}\|_n := \sum_{k=0}^{+\infty} e^{nk} \|v_k\|_B < +\infty \qquad \forall n \in \mathbb{N}_0.$$

Definition A.4. A graded space X is said to be *tame* if there exist a Banach space B and two tame linear maps $L_1 : X \to \Sigma(B)$ and $L_2 : \Sigma(B) \to X$ such that $L_2 \circ L_1$ is the identity on X.

Remark A.5. The property of being tame is stable under the usual operations like direct sum, product, etc. (cf. [18,20]).

Example A.6. *The space $C^1([0,T]; H^\infty(\mathbb{R}^m))$, $m \geq 1$, endowed with the family of seminorms*

$$|||g|||_n := \sup_{[0,T]} (\|g(t,\cdot)\|_n + \|D_t g(t,\cdot)\|_n), \qquad n \in \mathbb{N}_0,$$

is a tame space.

Proof. Since $H^\infty(\mathbb{R}^m)$ is tame (see [17]), there exist a Banach space B and two tame linear maps

$$\begin{aligned} L_1 :\ H^\infty(\mathbb{R}^m) &\longrightarrow \Sigma(B) \\ g &\longmapsto \{g_k\} \\ L_2 :\ \Sigma(B) &\longrightarrow H^\infty(\mathbb{R}^m) \\ \{g_k\} &\longmapsto g \end{aligned}$$

such that

$$\begin{aligned} &\|L_1(g)\|_n = \|\{g_k\}\|_n := \sum_{k=0}^{+\infty} e^{nk}\|g_k\|_B \leq C_n\|g\|_{n+r} \qquad n \geq n_0 \\ &\|L_2(\{g_k\})\|_n = \|g\|_n \leq C'_n\|\{g_k\}\|_{n+r'} \qquad n \geq n_0 \end{aligned} \tag{A.2}$$

for some $C_n, C'_n, > 0$, $n_0, r, r' \in \mathbb{N}$, and

$$L_2 \circ L_1(g) = g \qquad \forall g \in H^\infty(\mathbb{R}^m). \tag{A.3}$$

We construct the linear map

$$\begin{aligned} \tilde{L}_1 :\ C^1([0,T]; H^\infty(\mathbb{R}^m)) &\longrightarrow C^1([0,T]; \Sigma(B)) \\ g(t,x) &\longmapsto \tilde{L}_1 g(t,x) \end{aligned}$$

defined, for every fixed $t \in [0,T]$, by $\tilde{L}_1 g = L_1 g(t,\cdot) = \{g_k(t,\cdot)\}$. Clearly $\{g_k(t,\cdot)\} \in \Sigma(B)$ for all $t \in [0,T]$ by construction. We now prove that $\tilde{L}_1$ is well defined. Let us first remark that, since $g \in C^1([0,T]; H^\infty(\mathbb{R}^m))$, there exists $g' \in C^0([0,T]; H^\infty(\mathbb{R}^m))$ such that

$$\lim_{h\to 0} \left\| \frac{g(t+h,\cdot) - g(t,\cdot)}{h} - g'(t,\cdot) \right\|_n = 0 \qquad \forall n \in \mathbb{N}_0. \tag{A.4}$$

In order to show that $\tilde{L}_1 g = \{g_k(t,\cdot)\} \in C^1([0,T];\Sigma(B))$ we shall prove that there exists, in $\Sigma(B)$, the ∂_t derivative of $\tilde{L}_1 g = \{g_k(t,\cdot)\}$ and this is given by $L_1(g') = \{(g')_k(t,\cdot)\}$. Indeed, for all $n \in \mathbb{N}$, from (A.2) and (A.4) we have that

$$\begin{aligned}\sum_{k=0}^{+\infty} e^{nk} &\left\| \frac{g_k(t+h,\cdot)-g_k(t,\cdot)}{h} - (g')_k(t,\cdot) \right\|_B \\ &= \left\| L_1\left(\frac{g(t+h,\cdot)-g(t,\cdot)}{h} - g'(t,\cdot) \right) \right\|_n \\ &\le C_n \left\| \frac{g(t+h,\cdot)-g(t,\cdot)}{h} - g'(t,\cdot) \right\|_{n+r} \longrightarrow 0 \quad \text{as } h \to 0.\end{aligned}$$

Therefore $(g_k)'(t,\cdot) = (g')_k(t,\cdot)$ and the linear map $\tilde{L}_1$ is a linear tame map because of (A.2):

$$\begin{aligned}|||\tilde{L}_1(g)|||_n &:= \sup_{t\in[0,T]} \left(\sum_{k=0}^{+\infty} e^{nk} \|g_k(t,\cdot)\|_n + \sum_{k=0}^{+\infty} e^{nk} \|(g_k)'(t,\cdot)\|_n \right) \\ &\le C_n \sup_{t\in[0,T]} \left(\|g(t,\cdot)\|_{n+r} + \|g'(t,\cdot)\|_{n+r} \right) \\ &= C_n |||g|||_{n+r}.\end{aligned}$$

Analogously we can construct a tame linear map

$$\begin{aligned}\tilde{L}_2 :\ C^1([0,T];\Sigma(B)) &\longrightarrow C^1([0,T];H^\infty(\mathbb{R}^m)) \\ \{g_k(t,x)\} &\longmapsto g(t,x)\end{aligned}$$

defined by $\tilde{L}_2(\{g_k\})(t,\cdot) = L_2(\{g_k(t,\cdot)\}) = g(t,\cdot)$ for all $t \in [0,T]$.

Moreover, $\tilde{L}_2 \circ \tilde{L}_1$ is the identity map on $C^1([0,T];H^\infty(\mathbb{R}^m))$ by construction.

The proof is complete. □

For nonlinear maps, the definition of tame map is given by:

Definition A.7. Let X, Y be two graded spaces, $U \subset X$ and $T: U \to Y$. We say that T satisfies a *tame estimate* of degree r and base n_0 if for every integer $n \ge n_0$ there exists a constant $C_n > 0$ such that

$$\|T(u)\|_n \le C_n(1 + \|u\|_{n+r}) \qquad \forall u \in U. \tag{A.5}$$

A map T defined on an open set U is said to be *tame* if it satisfies a tame estimate (A.5) in a neighborhood of each point $u \in U$ (with constants r, n_0 and C_n which may depend on the neighborhood).

Remark A.8. Let us remark that a linear map is tame if and only if it is a tame linear map. Moreover, the composition of tame maps is tame (cf. [18,20]).

Recalling the notion of *Fréchet* derivative $DT(u)v$ of a map $T : U \subset X \to Y$ at $u \in U$ in the direction $v \in X$, as

$$DT(u)v := \lim_{\epsilon \to 0} \frac{T(u + \epsilon v) - T(u)}{\epsilon}, \tag{A.6}$$

we say that T is $C^1(U)$ if it is differentiable, in the sense that the limit (A.6) exists, and if the derivative $DT : U \times X \to Y$ is continuous.

Recursively, we can define the successive derivatives $D^n T : U \times X^n \to Y$ and say that T is $C^\infty(U)$ if all the Fréchet derivatives of T exist and are continuous.

Definition A.9. Given two graded spaces X, Y and an open subset U of X, we say that a map $T : U \to Y$ is *smooth tame* if it is C^∞ and $D^n T$ is tame for all $n \in \mathbb{N}_0$.

Remark A.10. Sums and compositions of smooth tame maps are smooth tame. Moreover, linear and nonlinear partial differential operators and integration are smooth tame (cf. [18,20]).

We finally recall the theorem of Nash–Moser (see [20]):

Theorem A.11 *(Nash–Moser–Hamilton). Let X, Y be tame spaces, U an open subset of X and $T : U \to Y$ a smooth tame map. Assume that the equation $DT(u)v = h$ has a unique solution $v := S(u, h)$ for all $u \in U$ and $h \in Y$ and assume that $S : U \times Y \to X$ is smooth tame. Then T is locally invertible and each local inverse is smooth tame.*

Appendix B. Sharp-Gårding and Fefferman–Phong inequalities

Let $A(x, D_x)$ be a pseudo-differential operator of order m on $\mathbb{R}$ with symbol $A(x, \xi)$ in the standard class S^m defined by

$$|\partial_\xi^\alpha D_x^\beta A(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\alpha} \qquad \forall \alpha, \beta \in \mathbb{N}, \tag{B.1}$$

for some $C_{\alpha,\beta} > 0$.

The following theorem holds (cf. [24]):

Theorem B.1 *(Sharp-Gårding). Let $A(x, \xi) \in S^m$ and assume that $\operatorname{Re} A(x, \xi) \ge 0$. There exist then pseudo-differential operators $Q(x, D_x)$ and $R(x, D_x)$ with symbols, respectively, $Q(x, \xi) \in S^m$ and $R(x, \xi) \in S^{m-1}$, such that*

$$\begin{aligned} &A(x, D_x) = Q(x, D_x) + R(x, D_x) \\ &\operatorname{Re}\langle Q(x, D_x)u, u\rangle \ge 0 \qquad \forall u \in H^m \\ &R(x, \xi) \sim \psi_1(\xi) D_x A(x, \xi) + \sum_{\alpha+\beta \ge 2} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta A(x, \xi), \end{aligned} \tag{B.2}$$

with ψ_1, $\psi_{\alpha,\beta}$ real valued functions, $\psi_1 \in S^{-1}$ and $\psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}$.

Remark B.2. Terms of the form $\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A(x,\xi) \in S^{m-(\alpha+\beta)/2}$, $\alpha+\beta \geq 2$, of (B.2) can be rearranged so that we have

$$R(x,\xi) \sim \sum_{\ell\geq 1} R(x,\xi)\big|_{\mathrm{ord}(m-\ell)},$$

where

$$R(x,\xi)\big|_{\mathrm{ord}(m-\ell)} = \begin{cases} \psi_1(\xi)D_x A(x,\xi) + \displaystyle\sum_{2\leq\alpha+\beta\leq 3} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A(x,\xi), & \ell = 1, \\ \displaystyle\sum_{2\ell\leq\alpha+\beta\leq 2\ell+1} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A(x,\xi), & \ell \geq 2. \end{cases} \tag{B.3}$$

Remark B.3. Theorem B.1 implies the well-known sharp-Gårding inequality

$$\mathrm{Re}\langle A(x,D_x)u,u\rangle \geq -c\|u\|^2_{(m-1)/2} \tag{B.4}$$

for some fixed constant $c > 0$.

A well-known refinement of (B.4) is given by the following theorem (cf. [19]):

Theorem B.4 *(Fefferman–Phong inequality). Let $A(x,\xi) \in S^m$ with $A(x,\xi) \geq 0$. Then*

$$\mathrm{Re}\langle A(x,D_x)u,u\rangle \geq -c\|u\|^2_{(m-2)/2} \qquad \forall u \in H^m \tag{B.5}$$

for some $c > 0$ depending only on the constants $C_{\alpha,\beta}$ in (B.1).

Remark B.5. By [25] we know that for $m = 2$ the constant c in (B.5) depends only on $\max_{|\alpha|+|\beta|\leq 7} C_{\alpha,\beta}$, for $C_{\alpha,\beta}$ as in (B.1).

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