

BiHom-Associative Algebras, BiHom-Lie Algebras and BiHom-Bialgebras

Giacomo GRAZIANI^{†1}, Abdenacer MAKHLOUF^{†2}, Claudia MENINI^{†3} and Florin PANAITTE^{†4}

^{†1} *Université Joseph Fourier Grenoble I Institut Fourier,
100, Rue des Maths BP74 38402 Saint-Martin-d'Hères, France*
E-mail: Giacomo.Graziani@ujf-grenoble.fr

^{†2} *Université de Haute Alsace, Laboratoire de Mathématiques, Informatique et Applications,
4, Rue des frères Lumière, F-68093 Mulhouse, France*
E-mail: Abdenacer.Makhlouf@uha.fr

^{†3} *University of Ferrara, Department of Mathematics,
Via Machiavelli 30, Ferrara, I-44121, Italy*
E-mail: men@unife.it

^{†4} *Institute of Mathematics of the Romanian Academy,
PO-Box 1-764, RO-014700 Bucharest, Romania*
E-mail: florin.panaite@imar.ro

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Abstract. A BiHom-associative algebra is a (nonassociative) algebra A endowed with two commuting multiplicative linear maps $\alpha, \beta: A \rightarrow A$ such that $\alpha(a)(bc) = (ab)\beta(c)$, for all $a, b, c \in A$. This concept arose in the study of algebras in so-called group Hom-categories. In this paper, we introduce as well BiHom-Lie algebras (also by using the categorical approach) and BiHom-bialgebras. We discuss these new structures by presenting some basic properties and constructions (representations, twisted tensor products, smash products etc).

Key words: BiHom-associative algebra; BiHom-Lie algebra; BiHom-bialgebra; representation; twisting; smash product

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1 Introduction

The origin of Hom-structures may be found in the physics literature around 1990, concerning q -deformations of algebras of vector fields, especially Witt and Virasoro algebras, see for instance [1, 10, 12, 19]. Hartwig, Larsson and Silvestrov studied this kind of algebras in [15, 18] and called them Hom-Lie algebras because they involve a homomorphism in the defining identity. More precisely, a Hom-Lie algebra is a linear space L endowed with two linear maps $[-]: L \otimes L \rightarrow L$ and $\alpha: L \rightarrow L$ such that $[-]$ is skew-symmetric and α is an algebra endomorphism with respect to the bracket satisfying the so-called Hom-Jacobi identity

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in L.$$

Since any associative algebra becomes a Lie algebra by taking the commutator $[a, b] = ab - ba$, it was natural to look for a Hom-analogue of this property. This was accomplished in [24], where the concept of Hom-associative algebra was introduced, as being a linear space A endowed with a multiplication $\mu: A \otimes A \rightarrow A$, $\mu(a \otimes b) = ab$, and a linear map $\alpha: A \rightarrow A$ satisfying the

so-called Hom-associativity condition

$$\alpha(a)(bc) = (ab)\alpha(c), \quad \forall a, b, c \in A.$$

If A is Hom-associative then $(A, [a, b] = ab - ba, \alpha)$ becomes a Hom-Lie algebra, denoted by $L(A)$. Notice that Hom-Lie algebras, in this paper, were considered without the assumption of multiplicativity of α .

In subsequent literature (see for instance [30]) were studied subclasses of these classes of algebras where the linear maps α involved in the definition of a Hom-Lie algebra or Hom-associative algebra are required to be multiplicative, that is $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in L$, respectively $\alpha(ab) = \alpha(a)\alpha(b)$ for all $a, b \in A$, and these subclasses were called multiplicative Hom-Lie algebras, respectively multiplicative Hom-associative algebras. Since we will always assume multiplicativity of the maps α and to simplify terminology, we will call Hom-Lie or Hom-associative algebras what was called above multiplicative Hom-Lie or Hom-associative algebras.

The Hom-analogues of coalgebras, bialgebras and Hopf algebras have been introduced in [25, 26]. The original definition of a Hom-bialgebra involved two linear maps, one twisting the associativity condition and the other one the coassociativity condition. Later, two directions of study on Hom-bialgebras were developed, one in which the two maps coincide (these are still called Hom-bialgebras) and another one, started in [8], where the two maps are assumed to be inverse to each other (these are called monoidal Hom-bialgebras).

In the last years, many concepts and properties from classical algebraic theories have been extended to the framework of Hom-structures, see for instance [2, 3, 7, 8, 11, 16, 20, 22, 25, 26, 27, 31, 30].

The main tool for constructing examples of Hom-type algebras is the so-called “twisting principle” introduced by D. Yau for Hom-associative algebras and extended afterwards to other types of Hom-algebras. For instance, if A is an associative algebra and $\alpha: A \rightarrow A$ is an algebra map, then A with the new multiplication defined by $a * b = \alpha(a)\alpha(b)$ is a Hom-associative algebra, called the *Yau twist* of A .

A categorical interpretation of Hom-associative algebras has been given by Caenepeel and Goyvaerts in [8]. First, to any monoidal category \mathcal{C} they associate a new monoidal category $\tilde{\mathcal{H}}(\mathcal{C})$, called a Hom-category, whose objects are pairs consisting of an object of \mathcal{C} and an automorphism of this object ($\tilde{\mathcal{H}}(\mathcal{C})$ has nontrivial associativity constraint even if the one of \mathcal{C} is trivial). By taking \mathcal{C} to be ${}_{\mathbb{k}}\mathcal{M}$, the category of linear spaces over a base field \mathbb{k} , it turns out that an algebra in the (symmetric) monoidal category $\tilde{\mathcal{H}}({}_{\mathbb{k}}\mathcal{M})$ is the same thing as a Hom-associative algebra (A, μ, α) with bijective α . The bialgebras in $\tilde{\mathcal{H}}({}_{\mathbb{k}}\mathcal{M})$ are the monoidal Hom-bialgebras we mentioned before.

In [14], the first author extended the construction of the Hom-category $\tilde{\mathcal{H}}(\mathcal{C})$ to include the action of a given group \mathcal{G} . Namely, given a monoidal category \mathcal{C} , a group \mathcal{G} , two elements $c, d \in Z(\mathcal{G})$ and ν an automorphism of the unit object of \mathcal{C} , the group Hom-category $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ has as objects pairs (A, f_A) , where A is an object in \mathcal{C} and $f_A: \mathcal{G} \rightarrow \text{Aut}_{\mathcal{C}}(A)$ is a group homomorphism. The associativity constraint of $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ is naturally defined by means of c, d, ν (see Claim 2.3 and Theorem 2.4) and it is, in general, non trivial. A braided structure is also defined on $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ (see Claim 2.7 and Theorem 2.8) turning it into a braided category which is symmetric whenever \mathcal{C} is. When $\mathcal{G} = \mathbb{Z}$, $c = d = 1_{\mathbb{Z}}$ and $\nu = \text{id}_{\mathbf{1}}$ one gets the category $\mathcal{H}(\mathcal{C})$ from [8], while for $c = 1_{\mathbb{Z}}$, $d = -1_{\mathbb{Z}}$ and $\nu = \text{id}_{\mathbf{1}}$ one gets the category $\tilde{\mathcal{H}}(\mathcal{C})$.

We first look at the case when $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}$, $c = (1, 0)$, $d = (0, 1)$, $\nu = \text{id}_{\mathbf{1}}$ and $\mathcal{C} = {}_{\mathbb{k}}\mathcal{M}$.

If $M \in {}_{\mathbb{k}}\mathcal{M}$, a group homomorphism $f_M: \mathbb{Z} \times \mathbb{Z} \rightarrow \text{Aut}_{\mathbb{k}}(M)$ is completely determined by

$$f_M((1, 0)) = \alpha_M \quad \text{and} \quad f_M((0, 1)) = \beta_M^{-1}.$$

Thus, an object in $\mathcal{H}(\mathbb{Z} \times \mathbb{Z}, {}_{\mathbb{k}}\mathcal{M})$ identifies with a triple (M, α_M, β_M) , where $\alpha_M, \beta_M \in \text{Aut}_{\mathbb{k}}(M)$ and $\alpha_M \circ \beta_M = \beta_M \circ \alpha_M$. For (X, α_X, β_X) , (Y, α_Y, β_Y) , (Z, α_Z, β_Z) objects in the category

$\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$, the associativity constraint in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ is given by

$$(\bar{a}^{c,d,\nu})_{(X,\alpha_X,\beta_X),(Y,\alpha_Y,\beta_Y),(Z,\alpha_Z,\beta_Z)} = a_{X,Y,Z} \circ [(\alpha_X \otimes Y) \otimes \beta_Z^{-1}],$$

and the braiding is

$$\bar{\gamma}_{(X,\alpha_X,\beta_X),(Y,\alpha_Y,\beta_Y)}^{c,d,\nu} = \tau[(\alpha_X \beta_X^{-1}) \otimes (\alpha_Y^{-1} \beta_Y)],$$

where $\tau: X \otimes Y \rightarrow Y \otimes X$ denotes the usual flip in the category of linear spaces. Note that $\bar{\gamma}$ is a symmetric braiding. Being $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ an additive braided monoidal category, all the concepts of algebra, Lie algebra and so on, can be introduced in this case.

By writing down the axioms for an algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ and discarding the invertibility of α and β if not needed, we arrived at the following concept. A BiHom-associative algebra over \mathbb{k} is a linear space A endowed with a multiplication $\mu: A \otimes A \rightarrow A$, $\mu(a \otimes b) = ab$, and two commuting multiplicative linear maps $\alpha, \beta: A \rightarrow A$ satisfying what we call the BiHom-associativity condition

$$\alpha(a)(bc) = (ab)\beta(c), \quad \forall a, b, c \in A.$$

Thus, a BiHom-associative algebra with *bijective* structure maps is exactly an algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$.

Obviously, a BiHom-associative algebra for which $\alpha = \beta$ is just a Hom-associative algebra.

The remarkable fact is that the twisting principle may be also applied: if A is an associative algebra and $\alpha, \beta: A \rightarrow A$ are two commuting algebra maps, then A with the new multiplication defined by $a * b = \alpha(a)\beta(b)$ is a BiHom-associative algebra, called the *Yau twist* of A . As a matter of fact, although we arrived at the concept of BiHom-associative algebra via the categorical machinery presented above, it is the possibility of twisting the multiplication of an associative algebra by *two* commuting algebra endomorphisms that led us to believe that BiHom-associative algebras are interesting objects in their own. One can think of this as follows. Take again an associative algebra A and $\alpha, \beta: A \rightarrow A$ two commuting algebra endomorphisms; define a new multiplication on A by $a * b = \alpha(a)\beta(b)$. Then it is natural to ask the following question: what kind of structure is $(A, *)$? Example 3.9 in this paper shows that, in general, $(A, *)$ is *not* a Hom-associative algebra, so the theory of Hom-associative algebras is *not* general enough to cover this natural operation of twisting the multiplication of an associative algebra by *two* maps; but this operation fits in the framework of BiHom-associative algebras. The Yau twisting of an associative algebra by two maps should thus be considered as the “natural” example of a BiHom-associative algebra. We would like to emphasize that for this operation the two maps are *not* assumed to be bijective, so the resulting BiHom-associative algebra has possibly *non bijective* structure maps and as such it *cannot* be regarded, to our knowledge, as an algebra in a monoidal category.

Take now the group \mathcal{G} to be arbitrary. It is natural to describe how an algebra in the monoidal category $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ looks like. By writing down the axioms, it turns out (see Claim 3.1 and Remark 3.5) that an algebra in such a category is a BiHom-associative algebra with bijective structure maps (the associativity of the algebra in the category is equivalent to the BiHom-associativity condition) having some extra structure (like an action of the group on the algebra). So, morally, the group $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}$ leads to BiHom-associative algebras but any other group would not lead to something like a “higher” structure than BiHom-associative algebras (for instance, one cannot have something like TriHom-associative algebras).

We initiate in this paper the study of what we will call BiHom-structures. The next structure we introduce is that of a BiHom-Lie algebra; for this, we use also a categorical approach. Unlike the Hom case, to obtain a BiHom-Lie algebra from a BiHom-associative algebra we need the structure maps α and β to be bijective; the commutator is defined by the formula

$[a, b] = ab - \alpha^{-1}\beta(b)\alpha\beta^{-1}(a)$. Nevertheless, just as in the Hom-case, the Yau twist works: if $(L, [-])$ is a Lie algebra over a field \mathbb{k} and $\alpha, \beta: L \rightarrow L$ are two commuting multiplicative linear maps and we define the linear map $\{-\}: L \otimes L \rightarrow L$, $\{a, b\} = [\alpha(a), \beta(b)]$, for all $a, b \in L$, then $L_{(\alpha, \beta)} := (L, \{-\}, \alpha, \beta)$ is a BiHom-Lie algebra, called the *Yau twist* of $(L, [-])$.

We define representations of BiHom-associative algebras and BiHom-Lie algebras and find some of their basic properties. Then we introduce BiHom-coassociative coalgebras and BiHom-bialgebras together with some of the usual ingredients (comodules, duality, convolution product, primitive elements, module and comodule algebras). We define antipodes for a certain class of BiHom-bialgebras, called monoidal BiHom-bialgebras, leading thus to the concept of monoidal BiHom-Hopf algebras. We define smash products, as particular cases of twisted tensor products, introduced in turn as a particular case of twisting a BiHom-associative algebra by what we call a BiHom-pseudotwistor. We write down explicitly such a smash product, obtained from an action of a Yau twist of the quantum group $U_q(\mathfrak{sl}_2)$ on a Yau twist of the quantum plane $\mathbb{A}_q^{2|0}$.

As a final remark, let us note that one could introduce a less restrictive concept of BiHom-associative algebra by dropping the assumptions that α and β are multiplicative and/or that they commute (note that all the examples of q -deformations of Witt or Virasoro algebras are not multiplicative). Unfortunately, by dropping any of these assumptions, one loses the main class of examples, the Yau twists, in the sense that if A is an associative algebra and $\alpha, \beta: A \rightarrow A$ are two arbitrary linear maps, and we define as before $a * b = \alpha(a)\beta(b)$, then $(A, *)$ in general is not a BiHom-associative algebra even in this more general sense.

2 The category $\mathcal{H}(\mathcal{G}, \mathcal{C})$

Our aim in this section is to introduce so-called group Hom-categories; proofs of the results in this section may be found in [14].

Definition 2.1. Let \mathcal{G} be a group and let \mathcal{C} be a category. The *group Hom-category* $\mathcal{H}(\mathcal{G}, \mathcal{C})$ associated to \mathcal{G} and \mathcal{C} is the category having as objects pairs (A, f_A) , where $A \in \mathcal{C}$ and f_A is a group homomorphism $\mathcal{G} \rightarrow \text{Aut}_{\mathcal{C}}(A)$. A morphism $\xi: (A, f_A) \rightarrow (B, f_B)$ in $\mathcal{H}(\mathcal{G}, \mathcal{C})$ is a morphism $\xi: A \rightarrow B$ in \mathcal{C} such that $f_B(g) \circ \xi = \xi \circ f_A(g)$, for all $g \in \mathcal{G}$.

Definition 2.2. A *monoidal category* (see [17, Chapter XI]) is a category \mathcal{C} endowed with an object $\mathbf{1} \in \mathcal{C}$ (called *unit*), a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (called *tensor product*) and functorial isomorphisms $a_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $l_X: \mathbf{1} \otimes X \rightarrow X$, $r_X: X \otimes \mathbf{1} \rightarrow X$, for every X, Y, Z in \mathcal{C} . The functorial isomorphisms a are called the *associativity constraints* and satisfy the pentagon axiom, that is

$$(U \otimes a_{V,W,X}) \circ a_{U,V \otimes W, X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W \otimes X} \circ a_{U \otimes V, W, X}$$

holds true, for every U, V, W, X in \mathcal{C} . The isomorphisms l and r are called the *unit constraints* and they obey the Triangle Axiom, that is

$$(V \otimes l_W) \circ a_{V, \mathbf{1}, W} = r_V \otimes W, \quad \text{for every } V, W \text{ in } \mathcal{C}.$$

A *monoidal functor* $(F, \phi_2, \phi_0): (\mathcal{C}, \otimes, \mathbf{1}, a, l, r) \rightarrow (\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ between two monoidal categories consists of a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$, an isomorphism $\phi_2(U, V): F(U) \otimes' F(V) \rightarrow F(U \otimes V)$, natural in $U, V \in \mathcal{C}$, and an isomorphism $\phi_0: \mathbf{1}' \rightarrow F(\mathbf{1})$ such that the diagram

$$\begin{array}{ccccc} (F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{\phi_2(U,V) \otimes' F(W)} & F(U \otimes V) \otimes' F(W) & \xrightarrow{\phi_2(U \otimes V, W)} & F((U \otimes V) \otimes W) \\ \downarrow a'_{F(U), F(V), F(W)} & & & & \downarrow F(a_{U,V,W}) \\ F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{F(U) \otimes' \phi_2(V,W)} & F(U) \otimes' F(V \otimes W) & \xrightarrow{\phi_2(U, V \otimes W)} & F(U \otimes (V \otimes W)) \end{array}$$

is commutative, and the following conditions are satisfied

$$F(l_U) \circ \phi_2(\mathbf{1}, U) \circ (\phi_0 \otimes' F(U)) = l'_{F(U)}, \quad F(r_U) \circ \phi_2(U, \mathbf{1}) \circ (F(U) \otimes' \phi_0) = r'_{F(U)}.$$

Claim 2.3. Let \mathcal{G} be a group and let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. Given any pair of objects $(A, f_A), (B, f_B) \in \mathcal{H}(\mathcal{G}, \mathcal{C})$, consider the map $f_A \otimes f_B: \mathcal{G} \rightarrow \text{Aut}_{\mathcal{C}}(A \otimes B)$ defined by setting

$$(f_A \otimes f_B)(g) = f_A(g) \otimes f_B(g),$$

for all $g \in \mathcal{G}$. Then $f_A \otimes f_B$ is a group homomorphism and hence

$$(A \otimes B, f_A \otimes f_B) \in \mathcal{H}(\mathcal{G}, \mathcal{C}).$$

Moreover, if $\phi: (A, f_A) \rightarrow (\tilde{A}, f_{\tilde{A}})$ and $\xi: (B, f_B) \rightarrow (\tilde{B}, f_{\tilde{B}})$ are morphisms in $\mathcal{H}(\mathcal{G}, \mathcal{C})$, then

$$\phi \otimes \xi: (A \otimes B, f_A \otimes f_B) \rightarrow (\tilde{A} \otimes \tilde{B}, f_{\tilde{A}} \otimes f_{\tilde{B}})$$

is a morphism in $\mathcal{H}(\mathcal{G}, \mathcal{C})$.

Let $Z(\mathcal{G})$ be the center of \mathcal{G} and let $c \in Z(\mathcal{G})$. Then we can consider the functorial isomorphism $\varphi(c): \text{Id}_{\mathcal{H}(\mathcal{G}, \mathcal{C})} \rightarrow \text{Id}_{\mathcal{H}(\mathcal{G}, \mathcal{C})}$ defined by setting

$$\varphi(c)(A, f_A) = f_A(c), \quad \text{for every } (A, f_A) \text{ in } \mathcal{H}(\mathcal{G}, \mathcal{C}).$$

Also, let $\widehat{\text{Id}}_{\mathbf{1}}: \mathcal{G} \rightarrow \text{Aut}_{\mathcal{C}}(\mathbf{1})$ denote the constant map equal to $\text{Id}_{\mathbf{1}}$.

Let $c, d \in Z(\mathcal{G})$ and let $\nu \in \text{Aut}_{\mathcal{C}}(\mathbf{1})$. We set

$$\begin{aligned} \bar{a}^{c,d,\nu} &= a \circ [(\varphi(c) \otimes \text{Id}_{\mathcal{H}(\mathcal{G}, \mathcal{C})}) \otimes \varphi(d)], & \bar{l}^{c,d,\nu} &= \varphi(d^{-1}) \circ l \circ (\nu \otimes \text{Id}_{\mathcal{H}(\mathcal{G}, \mathcal{C})}), \\ \bar{r}^{c,d,\nu} &= \varphi(c) \circ r \circ (\text{Id}_{\mathcal{H}(\mathcal{G}, \mathcal{C})} \otimes \nu). \end{aligned}$$

Theorem 2.4. *In the setting of Claim 2.3, the category*

$$\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C}) = (\mathcal{H}(\mathcal{G}, \mathcal{C}), \otimes, (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}), \bar{a}^{c,d,\nu}, \bar{l}^{c,d,\nu}, \bar{r}^{c,d,\nu})$$

is monoidal.

From now on, when $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a monoidal category, \mathcal{G} is a group, $c, d \in Z(\mathcal{G})$ and $\nu \in \text{Aut}_{\mathcal{C}}(\mathbf{1})$, we will indicate the monoidal category defined in Theorem 2.4 by $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$. In the case when $c = d = \mathbf{1}_{\mathcal{G}}$ and $\nu = \text{Id}_{\mathbf{1}}$, we will simply write $\mathcal{H}(\mathcal{G}, \mathcal{C})$.

Theorem 2.5. *Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category and \mathcal{G} a group. Then the identity functor $\mathcal{I}: \mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C}) \rightarrow \mathcal{H}(\mathcal{G}, \mathcal{C})$ is a monoidal isomorphism via*

$$\phi_0 = \nu^{-1}: (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}) \rightarrow (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}) \quad \text{and} \quad \phi_2((A, f_A), (B, f_B)) = f_A(c^{-1}) \otimes f_B(d),$$

for every $(A, f_A), (B, f_B) \in \mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$.

Definition 2.6 (see [17]). A *braided monoidal category* $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, \gamma)$ is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ equipped with a *braiding* γ , that is, an isomorphism $\gamma_{U,V}: U \otimes V \rightarrow V \otimes U$, natural in $U, V \in \mathcal{C}$, satisfying, for all $U, V, W \in \mathcal{C}$, the hexagon axioms

$$\begin{aligned} a_{V,W,U} \circ \gamma_{U,V \otimes W} \circ a_{U,V,W} &= (V \otimes \gamma_{U,W}) \circ a_{V,U,W} \circ (\gamma_{U,V} \otimes W), \\ a_{W,U,V}^{-1} \circ \gamma_{U \otimes V,W} \circ a_{U,V,W}^{-1} &= (\gamma_{U,W} \otimes V) \circ a_{U,W,V}^{-1} \circ (U \otimes \gamma_{V,W}). \end{aligned}$$

A braided monoidal category is called *symmetric* if we further have $\gamma_{V,U} \circ \gamma_{U,V} = \text{Id}_{U \otimes V}$ for every $U, V \in \mathcal{C}$. A *braided monoidal functor* is a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ such that

$$F(\gamma_{U,V}) \circ \phi_2(U, V) = \phi_2(V, U) \circ \gamma'_{F(U), F(V)}, \quad \text{for every } U, V \in \mathcal{C}.$$

Claim 2.7. Let \mathcal{G} be a group and let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, \gamma)$ be a braided monoidal category. Let $c, d \in Z(\mathcal{G})$ and let $\nu \in \text{Aut}_{\mathcal{C}}(\mathbf{1})$. We will introduce a braided structure on the monoidal category $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ by setting, for every (A, f_A) and (B, f_B) in $\mathcal{H}(\mathcal{G}, \mathcal{C})$,

$$\overline{\gamma}_{(A,f_A),(B,f_B)}^{c,d,\nu} = \gamma_{A,B} \circ (f_A(cd) \otimes f_B(c^{-1}d^{-1})).$$

Theorem 2.8. *In the setting of Claim 2.7, the category*

$$(\mathcal{H}(\mathcal{G}, \mathcal{C}), \otimes, (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}), \overline{a}^{c,d,\nu}, \overline{l}^{c,d,\nu}, \overline{r}^{c,d,\nu}, \overline{\gamma}^{c,d,\nu})$$

is a braided monoidal category.

From now on, when $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, \gamma)$ is a braided monoidal category and \mathcal{G} is a group, we will still denote the braided monoidal structure defined in Theorem 2.8 with $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$. In the case when $c = d = \mathbf{1}_{\mathcal{G}}$ and $\nu = \text{id}_{\mathbf{1}}$, we will simply write respectively $\mathcal{H}(\mathcal{G}, \mathcal{C})$ instead of $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ and $\gamma_{(A,f_A),(B,f_B)}$ instead of $\overline{\gamma}_{(A,f_A),(B,f_B)}^{c,d,\nu}$.

Theorem 2.9. *Let \mathcal{G} be a group and let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, \gamma)$ be a braided monoidal category. Then the identity functor $\mathcal{I}: \mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C}) \rightarrow \mathcal{H}(\mathcal{G}, \mathcal{C})$ is a braided monoidal isomorphism via*

$$\phi_0 = \nu^{-1}: (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}) \rightarrow (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}) \quad \text{and} \quad \phi_2((A, f_A), (B, f_B)) = f_A(c^{-1}) \otimes f_B(d),$$

for every $(A, f_A), (B, f_B) \in \mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$.

Remark 2.10. Let \mathcal{G} be a torsion-free abelian group. Corollary 4 in [4] states that, up to a braided monoidal category isomorphism, there is a unique braided monoidal structure (actually symmetric) on the category of representations over the group algebra $\mathbb{k}[\mathcal{G}]$, considered monoidal via a structure induced by that of vector spaces over the field \mathbb{k} . Thus Theorem 2.9 can be deduced from this result whenever \mathcal{G} is a torsion-free abelian group. We should remark that this result in [4] stems from the fact that the third Harrison cohomology group $H_{\text{Harr}}^3(\mathcal{G}, \mathbb{k}, \mathbb{G}_m)$ has, in this case, just one element. If \mathcal{G} is not a torsion-free abelian group then this might not happen. As one of the referees pointed out, in the case when $\mathbb{k} = \mathbb{C}$ and $\mathcal{G} = C_2$ then $H_{\text{Harr}}^3(\mathcal{G}, \mathbb{k}, \mathbb{G}_m)$ has exactly two elements and so in this case there are two distinct equivalence classes of braided monoidal structures on the category of representations over the group algebra $\mathbb{k}[\mathcal{G}]$, considered monoidal via a structure induced by that of vector spaces over the field \mathbb{k} . This does not contradict our Theorem 2.9. In fact, there might exist braided monoidal structures different from the ones considered in the statement of Theorem 2.9.

Claim 2.11. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category and \mathcal{G} a group, let $c, d \in Z(\mathcal{G})$ and $\nu \in \text{Aut}_{\mathcal{C}}(\mathbf{1})$. A *unital algebra* in $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathcal{C})$ is a triple $((A, f_A), \mu, u)$ where

- 1) $(A, f_A) \in \mathcal{H}(\mathcal{G}, \mathcal{C})$;
- 2) $\mu: (A \otimes A, f_A \otimes f_A) \rightarrow (A, f_A)$ is a morphism in $\mathcal{H}(\mathcal{G}, \mathcal{C})$;
- 3) $u: (\mathbf{1}, \widehat{\text{Id}}_{\mathbf{1}}) \rightarrow (A, f_A)$ is a morphism in $\mathcal{H}(\mathcal{G}, \mathcal{C})$;
- 4) $\mu \circ (\mu \otimes A) = \mu \circ (A \otimes \mu) \circ \overline{a}_{A,A,A}^{c,d,\nu}$;
- 5) $\mu \circ (u \otimes A) \circ (\overline{l}_A^{c,d,\nu})^{-1} = \text{Id}_A$;
- 6) $\text{Id}_A = \mu \circ (A \otimes u) \circ (\overline{r}_A^{c,d,\nu})^{-1}$.

Definition 2.12. Given a monoidal category \mathcal{M} , a quadruple (A, μ, u, c) is called a *braided unital algebra* in \mathcal{M} if (for simplicity, we will omit to write the associators):

- (A, μ, u) is a unital algebra in \mathcal{M} ;
- (A, c) is a braided object in \mathcal{M} , i.e., $c: A \otimes A \rightarrow A \otimes A$ is invertible and satisfies the Yang–Baxter equation

$$(c \otimes A)(A \otimes c)(c \otimes A) = (A \otimes c)(c \otimes A)(A \otimes c);$$

- the following conditions hold:

$$\begin{aligned} c(\mu \otimes A) &= (A \otimes \mu)(c \otimes A)(A \otimes c), & c(A \otimes \mu) &= (\mu \otimes A)(A \otimes c)(c \otimes A), \\ c(u \otimes A)l_A^{-1} &= (A \otimes u)r_A^{-1}, & c(A \otimes u)r_A^{-1} &= (u \otimes A)l_A^{-1}. \end{aligned}$$

A braided unital algebra is called *symmetric* whenever $c^2 = \text{Id}_A$.

Definition 2.13. Given an additive monoidal category \mathcal{M} , a *braided Lie algebra* in \mathcal{M} consists of a triple $(L, c, [-]: L \otimes L \rightarrow L)$ where (L, c) is a braided object and the following equalities hold true:

$$\begin{aligned} [-] &= -[-] \circ c \quad (\text{skew-symmetry}); \\ [-] \circ (L \otimes [-]) \circ [\text{Id}_{L \otimes (L \otimes L)} + (L \otimes c)a_{L,L,L}(c \otimes L)a_{L,L,L}^{-1} \\ &\quad + a_{L,L,L}(c \otimes L)a_{L,L,L}^{-1}(L \otimes c)] = 0 \quad (\text{Jacobi condition}); \\ c \circ (L \otimes [-])a_{L,L,L} &= ([-] \otimes L)a_{L,L,L}^{-1}(L \otimes c)a_{L,L,L}(c \otimes L); \end{aligned} \tag{2.1}$$

$$c \circ ([-] \otimes L)a_{L,L,L}^{-1} = (L \otimes [-])a_{L,L,L}(c \otimes L)a_{L,L,L}^{-1}(L \otimes c). \tag{2.2}$$

Let \mathcal{M} be an additive braided monoidal category. A *Lie algebra* in \mathcal{M} consists of a pair $(L, [-]: L \otimes L \rightarrow L)$ such that $(L, c_{L,L}, [-])$ is a braided Lie algebra in the additive monoidal category \mathcal{M} , where $c_{L,L}$ is the braiding c of \mathcal{M} evaluated on L (note that in this case the conditions (2.1) and (2.2) are automatically satisfied).

Claim 2.14. Given a symmetric algebra (A, μ, u, c) , one has that $[-] := \mu \circ (\text{Id}_{A \otimes A} - c)$ defines a braided Lie algebra structure on A (see [13, Construction 2.16]).

In a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$, it is well known that any unital algebra (A, μ, u) gives rise to a braided unital algebra $(A, \mu, u, c_{A,A})$.

3 Generalized Hom-structures

Let \mathbb{k} be a field and let ${}_{\mathbb{k}}\mathcal{M}$ be the category of linear spaces regarded as a braided monoidal category in the usual way. Then, for every group \mathcal{G} , the category $\mathcal{H}(\mathcal{G}, {}_{\mathbb{k}}\mathcal{M})$ identifies with the category $\mathbb{k}[\mathcal{G}]\text{-Mod}$ of left modules over the group algebra $\mathbb{k}[\mathcal{G}]$.

Let $c, d \in Z(\mathcal{G})$ and ν an automorphism of \mathbb{k} regarded as linear space over \mathbb{k} , that is ν is the multiplication by an element of $\mathbb{k} \setminus \{0\}$ that we will also denote by ν . Note that, given $X, Y, Z \in \mathbb{k}[\mathcal{G}]\text{-Mod}$, we have

$$\begin{aligned} \bar{a}_{X,Y,Z}^{c,d,\nu}((x \otimes y) \otimes z) &= c \cdot x \otimes (y \otimes d \cdot z), & \text{for every } x \in X, y \in Y, z \in Z, \\ \bar{l}_X^{c,d,\nu}(t \otimes x) &= d^{-1} \cdot (\nu t x) & \text{and} & \quad \bar{r}_X^{c,d,\nu}(x \otimes t) = c \cdot (\nu t x), \\ & \text{for every } t \in \mathbb{k} \text{ and } x \in X, \end{aligned}$$

so that

$$\begin{aligned} (\bar{l}_A^{c,d,\nu})^{-1}(x) &= (\nu^{-1} \otimes d \cdot x) & \text{and} & \quad (\bar{r}_A^{c,d,\nu})^{-1}(x) = (c^{-1} \cdot x \otimes \nu^{-1}), \\ & \text{for every } x \in X. \end{aligned}$$

The unit object of $\mathcal{H}^{c,d,\nu}(\mathcal{G}, {}_{\mathbb{k}}\mathcal{M})$ is $\{1_{\mathbb{k}}\}$ regarded as a left $\mathbb{k}[\mathcal{G}]$ -module in the trivial way.

Claim 3.1. In view of 2.11, a unital algebra in $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ is a triple $((A, f_A), \mu, u)$, where

- 1) $A \in \mathbb{k}[\mathcal{G}]\text{-Mod}$;
- 2) $\mu: A \otimes A \rightarrow A$ is a morphism in $\mathbb{k}[\mathcal{G}]\text{-Mod}$, i.e., $g \cdot (ab) = (g \cdot a)(g \cdot b)$, for every $g \in \mathcal{G}$, $a, b \in A$;
- 3) $u: \{1_{\mathbb{k}}\} \rightarrow A$ is a morphism in $\mathbb{k}[\mathcal{G}]\text{-Mod}$, i.e., $g \cdot u(1_{\mathbb{k}}) = u(1_{\mathbb{k}})$, for every $g \in \mathcal{G}$;
- 4) $(x \cdot y) \cdot z = (c \cdot x) \cdot [y \cdot (d \cdot z)]$, for every $x, y, z \in A$, which is equivalent to
$$(c \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (d^{-1} \cdot z), \quad \forall x, y, z \in A;$$
- 5) $u(\nu^{-1}) \cdot (d \cdot x) = x$, for every $x \in A$;
- 6) $(c^{-1} \cdot x) \cdot u(\nu^{-1}) = x$, for every $x \in A$.

Note that when $c = d = 1_{\mathcal{G}}$ and $\nu = 1_{\mathbb{k}}$, it turns out that A is simply a $\mathbb{k}[\mathcal{G}]\text{-module algebra}$.

Example 3.2. Let M be a \mathbb{k} -linear space and $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}$. Then a group morphism $f_M: \mathbb{Z} \times \mathbb{Z} \rightarrow \text{Aut}_{\mathbb{k}}(M)$ is completely determined by

$$f_M((1, 0)) = \alpha_M \quad \text{and} \quad f_M((0, 1)) = \beta_M^{-1}.$$

Thus an object in $\mathcal{H}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ identifies with a triple (M, α_M, β_M) , where $\alpha_M, \beta_M \in \text{Aut}_{\mathbb{k}}(M)$ and $\alpha_M \circ \beta_M = \beta_M \circ \alpha_M$. Also, a morphism $f: (M, \alpha_M, \beta_M) \rightarrow (N, \alpha_N, \beta_N)$ is just a linear map $f: M \rightarrow N$ such that $f \circ \alpha_M = \alpha_N \circ f$ and $f \circ \beta_M = \beta_N \circ f$. Moreover, the tensor product, in the category, of the objects (M, α_M, β_M) and (N, α_N, β_N) is the object $(M \otimes N, \alpha_M \otimes \alpha_N, \beta_M \otimes \beta_N)$.

We set $c = (1, 0)$, $d = (0, 1)$ and $\nu = 1_{\mathbb{k}}$. For (X, α_X, β_X) , (Y, α_Y, β_Y) , (Z, α_Z, β_Z) objects in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$, the associativity constraints in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ are given by

$$\begin{aligned} & (\bar{a}^{c,d,\nu})_{(X,\alpha_X,\beta_X),(Y,\alpha_Y,\beta_Y),(Z,\alpha_Z,\beta_Z)} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z), \\ & (\bar{a}^{c,d,\nu})_{(X,\alpha_X,\beta_X),(Y,\alpha_Y,\beta_Y),(Z,\alpha_Z,\beta_Z)} = a_{X,Y,Z} \circ [(\alpha_X \otimes Y) \otimes \beta_Z^{-1}], \end{aligned}$$

and the braiding is

$$\bar{\gamma}_{(X,\alpha_X,\beta_X),(Y,\alpha_Y,\beta_Y)}^{c,d,\nu} = \tau [(\alpha_X \beta_X^{-1}) \otimes (\alpha_Y \beta_Y^{-1})^{-1}] = \tau [(\alpha_X \beta_X^{-1}) \otimes (\alpha_Y^{-1} \beta_Y)],$$

where $\tau: X \otimes Y \rightarrow Y \otimes X$ denotes the usual flip in the category of linear spaces. Note that $\bar{\gamma}$ is a symmetric braiding.

Then, in view of 3.1, an algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ is a triple $((A, \alpha, \beta), \mu, u)$, where

- 1) $\alpha, \beta \in \text{Aut}_{\mathbb{k}}(A)$ and $\alpha \circ \beta = \beta \circ \alpha$;
- 2) $\mu: (A \otimes A, \alpha \otimes \alpha, \beta \otimes \beta) \rightarrow (A, \alpha, \beta)$ is a morphism in $\mathbb{k}[\mathbb{Z} \times \mathbb{Z}]\text{-Mod}$, i.e., $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$ and $\beta(a \cdot b) = \beta(a) \cdot \beta(b)$ for every $a, b \in A$;
- 3) $u: \{1_{\mathbb{k}}\} \rightarrow (A, \alpha, \beta)$ is a morphism in $\mathbb{k}[\mathbb{Z} \times \mathbb{Z}]\text{-Mod}$, i.e., $\alpha(u(1_{\mathbb{k}})) = u(1_{\mathbb{k}})$ and $\beta(u(1_{\mathbb{k}})) = u(1_{\mathbb{k}})$;
- 4) $\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z)$, for every $x, y, z \in A$;
- 5) $u(1_{\mathbb{k}}) \cdot (\beta^{-1}(x)) = x$, for every $x \in A$, which is equivalent to $u(1_{\mathbb{k}}) \cdot x = \beta(x)$, for every $x \in A$;
- 6) $(\alpha^{-1}(x)) \cdot u(1_{\mathbb{k}}) = x$, for every $x \in A$, which is equivalent to $x \cdot u(1_{\mathbb{k}}) = \alpha(x)$, for every $x \in A$.

Inspired by Example 3.2, we introduce the following concept.

Definition 3.3. Let \mathbb{k} be a field. A *BiHom-associative algebra* over \mathbb{k} is a 4-tuple (A, μ, α, β) , where A is a \mathbb{k} -linear space, $\alpha: A \rightarrow A$, $\beta: A \rightarrow A$ and $\mu: A \otimes A \rightarrow A$ are linear maps, with notation $\mu(a \otimes a') = aa'$, satisfying the following conditions, for all $a, a', a'' \in A$:

$$\begin{aligned} \alpha \circ \beta &= \beta \circ \alpha, \\ \alpha(aa') &= \alpha(a)\alpha(a') \quad \text{and} \quad \beta(aa') = \beta(a)\beta(a') \quad (\text{multiplicativity}), \\ \alpha(a)(a'a'') &= (aa')\beta(a'') \quad (\text{BiHom-associativity}). \end{aligned}$$

We call α and β (in this order) the *structure maps* of A .

A morphism $f: (A, \mu_A, \alpha_A, \beta_A) \rightarrow (B, \mu_B, \alpha_B, \beta_B)$ of BiHom-associative algebras is a linear map $f: A \rightarrow B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

A BiHom-associative algebra (A, μ, α, β) is called *unital* if there exists an element $1_A \in A$ (called a *unit*) such that $\alpha(1_A) = 1_A$, $\beta(1_A) = 1_A$ and

$$a1_A = \alpha(a) \quad \text{and} \quad 1_A a = \beta(a), \quad \forall a \in A.$$

A morphism of unital BiHom-associative algebras $f: A \rightarrow B$ is called *unital* if $f(1_A) = 1_B$.

Remark 3.4. A Hom-associative algebra (A, μ, α) can be regarded as the BiHom-associative algebra (A, μ, α, α) .

Remark 3.5. A BiHom-associative algebra with *bijective* structure maps is exactly an algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$. On the other hand, in the setting of Claim 3.1, if we define the maps $\alpha, \beta: A \rightarrow A$ by $\alpha(a) = c \cdot a$ and $\beta(a) = d^{-1} \cdot a$, for all $a \in A$, the axiom 2) in Claim 3.1 implies that α and β are multiplicative and then the axiom 4) in Claim 3.1 says that (A, μ, α, β) is a BiHom-associative algebra.

Example 3.6. We give now two families of examples of 2-dimensional unital BiHom-associative algebras, that are obtained by a computer algebra system. Let $\{e_1, e_2\}$ be a basis; for $i = 1, 2$ the maps α_i, β_i and the multiplication μ_i are defined by

$$\begin{aligned} \alpha_1(e_1) &= e_1, & \alpha_1(e_2) &= \frac{2a}{b-1}e_1 - e_2, \\ \beta_1(e_1) &= e_1, & \beta_1(e_2) &= -ae_1 + be_2, \\ \mu_1(e_1, e_1) &= e_1, & \mu_1(e_1, e_2) &= -ae_1 + be_2, \\ \mu_1(e_2, e_1) &= \frac{2a}{b-1}e_1 - e_2, & \mu_1(e_2, e_2) &= -\frac{a^2(b-2)}{(b-1)^2}e_1 + ae_2, \end{aligned}$$

and

$$\begin{aligned} \alpha_2(e_1) &= e_1, & \alpha_2(e_2) &= \frac{b(1-a)}{a}e_1 + ae_2, \\ \beta_2(e_1) &= e_1, & \beta_2(e_2) &= be_1 + (1-a)e_2, \\ \mu_2(e_1, e_1) &= e_1, & \mu_2(e_1, e_2) &= be_1 + (1-a)e_2, \\ \mu_2(e_2, e_1) &= \frac{b(1-a)}{a}e_1 + ae_2, & \mu_2(e_2, e_2) &= \frac{b}{a}e_2, \end{aligned}$$

where a, b are parameters in \mathbb{k} , with $b \neq 1$ in the first case and $a \neq 0$ in the second. In both cases, the unit is e_1 .

Claim 3.7. In view of Theorem 2.5, if (A, μ, α, β) is a BiHom-associative algebra, and α and β are invertible, then $(A, \mu \circ (\alpha^{-1} \otimes \beta^{-1}), \text{Id}_A, \text{Id}_A)$ is a BiHom-associative algebra, i.e., the multiplication $\mu \circ (\alpha^{-1} \otimes \beta^{-1})$ is associative in the usual sense.

On the other hand, if $(A, \mu: A \otimes A \rightarrow A)$ is an associative algebra and $\alpha, \beta: A \rightarrow A$ are commuting algebra endomorphisms, then one can easily check that $(A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-associative algebra, denoted by $A_{(\alpha, \beta)}$ and called the *Yau twist* of (A, μ) .

In view of Claim 3.7, a BiHom-associative algebra with bijective structure maps is a Yau twist of an associative algebra.

The Yau twisting procedure for BiHom-associative algebras admits a more general form, which we state in the next result (the proof is straightforward and left to the reader).

Proposition 3.8. *Let $(D, \mu, \tilde{\alpha}, \tilde{\beta})$ be a BiHom-associative algebra and $\alpha, \beta: D \rightarrow D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Then $(D, \mu \circ (\alpha \otimes \beta), \tilde{\alpha} \circ \alpha, \tilde{\beta} \circ \beta)$ is also a BiHom-associative algebra, denoted by $D_{(\alpha, \beta)}$.*

Example 3.9. We present an example of a BiHom-associative algebra that cannot be expressed as a Hom-associative algebra. Let \mathbb{k} be a field and $A = \mathbb{k}[X]$. Let $\alpha: A \rightarrow A$ be the algebra map defined by setting $\alpha(X) = X^2$ and let $\beta = \text{Id}_{\mathbb{k}[X]}$. Then we can consider the BiHom-associative algebra $A_{(\alpha, \beta)} = (A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$, where $\mu: A \otimes A \rightarrow A$ is the usual multiplication. For every $a, a' \in A$ set

$$a * a' = \mu \circ (\alpha \otimes \beta)(a \otimes a') = \alpha(a)a'.$$

Let us assume that there exists $\theta \in \text{End}(\mathbb{k}[X])$ such that $(A, \mu \circ (\alpha \otimes \beta), \theta)$ is a Hom-associative algebra. Then we should have that

$$\theta(X) * (X * X) = (X * X) * \theta(X). \quad (3.1)$$

Write

$$\theta(X) = \sum_{i=0}^n a_i X^i, \quad \text{where } a_i \in \mathbb{k} \text{ for every } i = 0, 1, \dots, n \text{ and } a_n \neq 0.$$

Since

$$X * X = \alpha(X)X = X^3,$$

(3.1) rewrites as

$$\sum_{i=0}^n a_i X^i * X^3 = X^3 * \sum_{i=0}^n a_i X^i,$$

and hence as

$$\sum_{i=0}^n a_i \alpha(X)^i X^3 = \alpha(X)^3 \sum_{i=0}^n a_i X^i,$$

i.e.,

$$\sum_{i=0}^n a_i X^{2i+3} = \sum_{i=0}^n a_i X^{6+i},$$

which implies that

$$2n + 3 = 6 + n, \quad \text{i.e., } n = 3, \quad \text{and hence}$$

$$a_0 X^3 + a_1 X^5 + a_2 X^7 + a_3 X^9 = a_0 X^6 + a_1 X^7 + a_2 X^8 + a_3 X^9,$$

so that

$$\theta(X) = a_3 X^3.$$

Let us set $c = a_3$ and let us check the equality

$$\theta(X^2) * (X * X) = (X^2 * X) * \theta(X).$$

The left-hand side is

$$\theta(X^2) * (X * X) = c^2 X^6 * X^3 = \alpha(c^2 X^6) X^3 = c^2 X^{15}.$$

The right-hand side is

$$(X^2 * X) * \theta(X) = (\alpha(X^2) X) * \theta(X) = X^5 * \theta(X) = c X^{10} X^3 = c X^{13}.$$

Thus the equality does not hold.

Remark 3.10. Given two algebras $(A, \mu_A, 1_A)$ and $(B, \mu_B, 1_B)$ in a braided monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$, it is well known that $A \otimes B$ becomes also an algebra in the category, with multiplication $\mu_{A \otimes B}$ defined by

$$\begin{aligned} \mu_{A \otimes B} &= (\mu_A \otimes \mu_B) \circ a_{A,A,B \otimes B}^{-1} \circ (A \otimes a_{A,B,B}) \\ &\quad \circ (A \otimes (c_{B,A} \otimes B)) \circ (A \otimes a_{B,A,B}^{-1}) \circ a_{A,B,A \otimes B}. \end{aligned}$$

In the case of our category $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$, we have, for every $x, y \in A, x', y' \in B$:

$$\begin{aligned} \mu_{A \otimes B}((x \otimes y) \otimes (x' \otimes y')) &= ((\mu_A \otimes \mu_B) \circ a_{A,A,B \otimes B}^{-1} \circ (A \otimes a_{A,B,B}) \\ &\quad \circ (A \otimes (c_{B,A} \otimes B)) \circ (A \otimes a_{B,A,B}^{-1}) \circ a_{A,B,A \otimes B})((x \otimes y) \otimes (x' \otimes y')) \\ &= ((\mu_A \otimes \mu_B) \circ a_{A,A,B \otimes B}^{-1} \circ (A \otimes a_{A,B,B}) \circ (A \otimes (c_{B,A} \otimes B)) \\ &\quad \circ (A \otimes a_{B,A,B}^{-1})) (cx \otimes (y \otimes (dx' \otimes dy'))) \\ &= ((\mu_A \otimes \mu_B) \circ a_{A,A,B \otimes B}^{-1} \circ (A \otimes a_{A,B,B})) (cx \otimes ((c^{-1}x' \otimes dy) \otimes y')) \\ &= ((\mu_A \otimes \mu_B) \circ a_{A,A,B \otimes B}^{-1}) (cx \otimes (x' \otimes (dy \otimes dy'))) \\ &= (\mu_A \otimes \mu_B)((x \otimes x') \otimes (y \otimes y')) = (x \cdot x') \otimes (y \cdot y'). \end{aligned}$$

In particular, if (A, α_A, β_A) and (B, α_B, β_B) are two algebras in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$, their braided tensor product $A \underline{\otimes} B$ in the category is the algebra $(A \otimes B, \alpha_A \otimes \alpha_B, \beta_A \otimes \beta_B)$, whose multiplication is given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, for all $a, a' \in A$ and $b, b' \in B$.

Remark 3.11. If $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ are two BiHom-associative algebras over a field \mathbb{k} , then $(A \otimes B, \mu_{A \otimes B}, \alpha_A \otimes \alpha_B, \beta_A \otimes \beta_B)$ is a BiHom-associative algebra (called the tensor product of A and B), where $\mu_{A \otimes B}$ is the usual multiplication: $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. If A and B are unital with units 1_A and respectively 1_B then $A \otimes B$ is also unital with unit $1_A \otimes 1_B$. This is consistent with Remark 3.10.

Example 3.12. In view of Definition 2.13, a Lie algebra in $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ is a pair $((L, f_L), [-])$, where

- 1) $(L, f_L) \in \mathbb{k}[\mathcal{G}]\text{-Mod}$;
- 2) $[-]: L \otimes L \rightarrow L$ is a morphism in $\mathbb{k}[\mathcal{G}]\text{-Mod}$;
- 3) $[-] = -[-] \circ \bar{\gamma}_{L,L}$;

4)

$$\begin{aligned} & [-] \circ (L \otimes [-]) + [-] \circ (L \otimes [-]) \circ (L \otimes \bar{\gamma}_{L,L}) \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1} \\ & + [-] \circ (L \otimes [-]) \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1} (L \otimes \bar{\gamma}_{L,L}) = 0, \end{aligned}$$

where $\bar{\gamma}_{L,L} = \tau \circ (f_L(cd) \otimes f_L(c^{-1}d^{-1}))$ and τ is the usual flip.

We will write down 4) explicitly. We have

$$\begin{aligned} & ((L \otimes \bar{\gamma}_{L,L}) \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1})(x \otimes (y \otimes z)) \\ & = (L \otimes \bar{\gamma}_{L,L}) \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) ((c^{-1}x \otimes y) \otimes d^{-1}z) \\ & = (L \otimes \bar{\gamma}_{L,L}) \bar{a}_{L,L,L} ((c^{-1}d^{-1}y \otimes cdc^{-1}x) \otimes d^{-1}z) \\ & = (L \otimes \bar{\gamma}_{L,L}) (cc^{-1}d^{-1}y \otimes (cdc^{-1}x \otimes dd^{-1}z)) \\ & = cc^{-1}d^{-1}y \otimes (c^{-1}d^{-1}dd^{-1}z \otimes cdc^{-1}x) \\ & = d^{-1}y \otimes (c^{-1}d^{-1}z \otimes dcdx), \end{aligned}$$

therefore

$$\begin{aligned} & [-] \circ (L \otimes [-]) ((L \otimes \bar{\gamma}_{L,L}) \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1})(x \otimes (y \otimes z)) \\ & = [d^{-1}y, [c^{-1}d^{-1}z, cd^2x]], \end{aligned}$$

and

$$\begin{aligned} & (\bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1} (L \otimes \bar{\gamma}_{L,L}))(x \otimes (y \otimes z)) \\ & = \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1} (x \otimes (c^{-1}d^{-1}z \otimes cdy)) \\ & = \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) (c^{-1}x \otimes (c^{-1}d^{-1}z \otimes d^{-1}cdy)) \\ & = \bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) ((c^{-1}x \otimes c^{-1}d^{-1}z) \otimes cy) \\ & = \bar{a}_{L,L,L} ((c^{-2}d^{-2}z \otimes cdc^{-1}x) \otimes cy) \\ & = ((c^{-1}d^{-2}z \otimes dx) \otimes cdy), \end{aligned}$$

hence

$$[-] \circ (L \otimes [-]) (\bar{a}_{L,L,L} (\bar{\gamma}_{L,L} \otimes L) \bar{a}_{L,L,L}^{-1} (L \otimes \bar{\gamma}_{L,L}))(x \otimes (y \otimes z)) = [c^{-1}d^{-2}z, [dx, cdy]].$$

Thus 4) is equivalent to

$$[x, [y, z]] + [d^{-1}y, [c^{-1}d^{-1}z, cd^2x]] + [c^{-1}d^{-2}z, [dx, cdy]] = 0, \quad \text{for every } x, y, z \in L,$$

which is equivalent to

$$[d^{-2}x, [d^{-1}y, cz]] + [d^{-2}y, [d^{-1}z, cx]] + [d^{-2}z, [d^{-1}x, cy]] = 0, \quad \text{for every } x, y, z \in L.$$

Thus, a Lie algebra in $\mathcal{H}^{c,d,\nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ is a pair $(L, [-])$, where

- 1) $L \in \mathbb{k}[\mathcal{G}]\text{-Mod}$;
- 2) $g[x, y] = [gx, gy]$, for every $x, y \in L$;
- 3) $[x, y] = -[c^{-1}d^{-1}y, cdx]$, for every $x, y \in L$, i.e., $[x, cdy] = -[y, cdx]$, for every $x, y \in L$ (skew-symmetry);

- 4) $[d^{-2}x, [d^{-1}y, cz]] + [d^{-2}y, [d^{-1}z, cx]] + [d^{-2}z, [d^{-1}x, cy]] = 0$, for every $x, y, z \in L$ (Jacobi condition).

In particular, a Lie algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ is a pair $((L, \alpha, \beta), [-])$, where

- 1) $\alpha, \beta \in \text{Aut}_{\mathbb{k}}(L)$ and $\alpha \circ \beta = \beta \circ \alpha$;
- 2) $[-]: (L \otimes L, \alpha \otimes \alpha, \beta \otimes \beta) \rightarrow (L, \alpha, \beta)$ is a morphism in $\mathbb{k}[\mathbb{Z} \times \mathbb{Z}]\text{-Mod}$, i.e., $\alpha[a, b] = [\alpha(a), \alpha(b)]$ and $\beta[a, b] = [\beta(a), \beta(b)]$, for every $a, b \in L$;
- 3) $[a, \alpha\beta^{-1}(b)] = -[b, \alpha\beta^{-1}(a)]$, for every $a, b \in L$, which is equivalent to $[\beta(a), \alpha(b)] = -[\beta(b), \alpha(a)]$, for every $a, b \in L$;
- 4) $[\beta^2x, [\beta y, \alpha z]] + [\beta^2y, [\beta z, \alpha x]] + [\beta^2z, [\beta x, \alpha y]] = 0$, for every $x, y, z \in L$.

Inspired by Example 3.12, we introduce the following concept.

Definition 3.13. A *BiHom-Lie algebra* over a field \mathbb{k} is a 4-tuple $(L, [-], \alpha, \beta)$, where L is a \mathbb{k} -linear space, $\alpha: L \rightarrow L$, $\beta: L \rightarrow L$ and $[-]: L \otimes L \rightarrow L$ are linear maps, with notation $[-](a \otimes a') = [a, a']$, satisfying the following conditions, for all $a, a', a'' \in L$:

$$\begin{aligned} \alpha \circ \beta &= \beta \circ \alpha, \\ \alpha([a', a'']) &= [\alpha(a'), \alpha(a'')] \quad \text{and} \quad \beta([a', a'']) = [\beta(a'), \beta(a'')], \\ [\beta(a), \alpha(a')] &= -[\beta(a'), \alpha(a)] \quad (\text{skew-symmetry}), \\ [\beta^2(a), [\beta(a'), \alpha(a'')]] &+ [\beta^2(a'), [\beta(a''), \alpha(a)]] + [\beta^2(a''), [\beta(a), \alpha(a')]] = 0 \\ &(\text{BiHom-Jacobi condition}). \end{aligned}$$

We call α and β (in this order) the *structure maps* of L . A morphism $f: (L, [-], \alpha, \beta) \rightarrow (L', [-]', \alpha', \beta')$ of BiHom-Lie algebras is a linear map $f: L \rightarrow L'$ such that $\alpha' \circ f = f \circ \alpha$, $\beta' \circ f = f \circ \beta$ and $f([x, y]) = [f(x), f(y)]'$, for all $x, y \in L$.

Thus, a Lie algebra in $\mathcal{H}^{(1,0),(0,1),1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ is exactly a BiHom-Lie algebra with *bijective* structure maps.

Remark 3.14. Obviously, a Hom-Lie algebra $(L, [-], \alpha)$ is a particular case of a BiHom-Lie algebra, namely $(L, [-], \alpha, \alpha)$. Conversely, a BiHom-Lie algebra $(L, [-], \alpha, \alpha)$ with bijective α is the Hom-Lie algebra $(L, [-], \alpha)$.

In view of Claim 2.14, we have:

Proposition 3.15. *If (A, μ, α, β) is a BiHom-associative algebra with bijective α and β , then, for every $a, a' \in A$, we can set*

$$[a, a'] = aa' - (\alpha^{-1}\beta(a'))(\alpha\beta^{-1}(a)).$$

Then $(A, [-], \alpha, \beta)$ is a BiHom-Lie algebra, denoted by $L(A)$.

The proofs of the following three results are straightforward and left to the reader.

Proposition 3.16. *Let $(L, [-])$ be an ordinary Lie algebra over a field \mathbb{k} and let $\alpha, \beta: L \rightarrow L$ two commuting linear maps such that $\alpha([a, a']) = [\alpha(a), \alpha(a')]$ and $\beta([a, a']) = [\beta(a), \beta(a')]$, for all $a, a' \in L$. Define the linear map $\{-\}: L \otimes L \rightarrow L$,*

$$\{a, b\} = [\alpha(a), \beta(b)], \quad \text{for all } a, b \in L.$$

*Then $L_{(\alpha, \beta)} := (L, \{-\}, \alpha, \beta)$ is a BiHom-Lie algebra, called the *Yau twist* of $(L, [-])$.*

Claim 3.17. More generally, let $(L, [-], \alpha, \beta)$ be a BiHom-Lie algebra and $\alpha', \beta': L \rightarrow L$ linear maps such that $\alpha'([a, b]) = [\alpha'(a), \alpha'(b)]$ and $\beta'([a, b]) = [\beta'(a), \beta'(b)]$ for all $a, b \in L$, and any two of the maps $\alpha, \beta, \alpha', \beta'$ commute. Then $(L, [-]_{(\alpha', \beta')}) := [-] \circ (\alpha' \otimes \beta'), \alpha \circ \alpha', \beta \circ \beta'$ is a BiHom-Lie algebra.

Proposition 3.18. Let (A, μ) be an associative algebra and $\alpha, \beta: A \rightarrow A$ two commuting algebra isomorphisms. Then $L(A_{(\alpha, \beta)}) = L(A)_{(\alpha, \beta)}$, as BiHom-Lie algebras.

Remark 3.19. Let \mathcal{G} be a group and $c, d \in Z(\mathcal{G}), \nu \in \text{Aut}_{\mathcal{C}}(\mathbf{1})$. It is straightforward to prove that the category $\mathcal{H}^{c, d, \nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ fulfills the assumption of [5, Theorem 6.4]. Hence, for any Lie algebra $(L, [-])$ in $\mathcal{H}^{c, d, \nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$, we can consider the universal enveloping bialgebra $\bar{\mathcal{U}}((L, [-]))$ as introduced in [5]. By [5, Remark 6.5], $\bar{\mathcal{U}}((L, [-]))$ as a bialgebra is a quotient of the tensor bialgebra \bar{TL} . The morphism giving the projection is induced by the canonical projection $p: TL \rightarrow \mathcal{U}(L, [-])$ defining the universal enveloping algebra. At algebra level we have

$$\begin{aligned} \mathcal{U}(L, [-]) &= \frac{TL}{([x, y] - x \otimes y + \bar{\gamma}_{L, L}(x \otimes y) \mid x, y \in L)} \\ &= \frac{TL}{([x, y] - x \otimes y + (f_L(c^{-1}d^{-1})(y)) \otimes f_L(cd)(x) \mid x, y \in L)}. \end{aligned}$$

By Theorem 2.9, the identity functor $\mathcal{I}: \mathcal{H}^{c, d, \nu}(\mathcal{G}, \mathbb{k}\mathcal{M}) \rightarrow \mathcal{H}(\mathcal{G}, \mathbb{k}\mathcal{M})$ is a braided monoidal isomorphism. Let $F: \mathcal{H}(\mathcal{G}, \mathbb{k}\mathcal{M}) \rightarrow \mathbb{k}\mathcal{M}$ be the forgetful functor. Then $F \circ \mathcal{I}$ is a monoidal functor $\mathcal{H}^{c, d, \nu}(\mathcal{G}, \mathbb{k}\mathcal{M}) \rightarrow \mathbb{k}\mathcal{M}$ to which we can apply [5, Theorem 8.5] to get that $\mathcal{H}^{c, d, \nu}(\mathcal{G}, \mathbb{k}\mathcal{M})$ is what is called in [5] a Milnor–Moore category. This implies that, by [5, Theorem 7.2], we have an isomorphism $(L, [-]) \rightarrow \mathcal{P}\bar{\mathcal{U}}((L, [-]))$, where $\mathcal{P}\bar{\mathcal{U}}((L, [-]))$ denotes the primitive part of $\bar{\mathcal{U}}((L, [-]))$. That is, half of the Milnor–Moore theorem holds.

The case $\mathcal{G} = \mathbb{Z}$ can be found in [5, Remark 9.10].

In the particular case of a Lie algebra $((L, \alpha, \beta), [-])$ in $\mathcal{H}^{(1,0), (0,1), 1}(\mathbb{Z} \times \mathbb{Z}, \mathbb{k}\mathcal{M})$ we have that

$$\mathcal{U}(L, [-]) = \frac{TL}{([x, y] - x \otimes y + (\alpha^{-1}\beta)(y) \otimes (\alpha\beta^{-1})(x) \mid x, y \in L)}.$$

Enveloping algebras of Hom-Lie algebras were introduced in [29] (see also [8, Section 8]).

4 Representations

From now on, we will always work over a base field \mathbb{k} . All algebras, linear spaces etc. will be over \mathbb{k} ; unadorned \otimes means $\otimes_{\mathbb{k}}$. For a comultiplication $\Delta: C \rightarrow C \otimes C$ on a linear space C , we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are *not* supposed to be (co)unital, and a multiplication $\mu: V \otimes V \rightarrow V$ on a linear space V is denoted by juxtaposition: $\mu(v \otimes v') = vv'$. For the composition of two maps f and g , we will write either $g \circ f$ or simply gf . For the identity map on a linear space V we will use the notation id_V .

Definition 4.1. Let $(A, \mu_A, \alpha_A, \beta_A)$ be a BiHom-associative algebra. A *left A -module* is a triple (M, α_M, β_M) , where M is a linear space, $\alpha_M, \beta_M: M \rightarrow M$ are linear maps and we have a linear map $A \otimes M \rightarrow M$, $a \otimes m \mapsto a \cdot m$, such that, for all $a, a' \in A$, $m \in M$, we have

$$\alpha_M \circ \beta_M = \beta_M \circ \alpha_M, \tag{4.1}$$

$$\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \tag{4.2}$$

$$\beta_M(a \cdot m) = \beta_A(a) \cdot \beta_M(m), \tag{4.3}$$

$$\alpha_A(a) \cdot (a' \cdot m) = (aa') \cdot \beta_M(m). \quad (4.4)$$

If (M, α_M, β_M) and (N, α_N, β_N) are left A -modules (both A -actions denoted by \cdot), a morphism of left A -modules $f: M \rightarrow N$ is a linear map satisfying the conditions $\alpha_N \circ f = f \circ \alpha_M$, $\beta_N \circ f = f \circ \beta_M$ and $f(a \cdot m) = a \cdot f(m)$, for all $a \in A$ and $m \in M$.

If $(A, \mu_A, \alpha_A, \beta_A, 1_A)$ is a unital BiHom-associative algebra and (M, α_M, β_M) is a left A -module, then M is called *unital* if $1_A \cdot m = \beta_M(m)$, for all $m \in M$.

Remark 4.2. If (A, μ, α, β) is a BiHom-associative algebra, then (A, α, β) is a left A -module with action defined by $a \cdot b = ab$, for all $a, b \in A$.

Lemma 4.3. *Let $(E, \mu, 1_E)$ be an associative unital algebra and $u, v \in E$ two invertible elements such that $uv = vu$. Define the linear maps $\tilde{\alpha}, \tilde{\beta}: E \rightarrow E$, $\tilde{\alpha}(a) = uau^{-1}$, $\tilde{\beta}(a) = vav^{-1}$, for all $a \in E$, and the linear map $\tilde{\mu}: E \otimes E \rightarrow E$, $\tilde{\mu}(a \otimes b) := a * b = uau^{-1}bv^{-1}$, for all $a, b \in E$. Then $(E, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta})$ is a unital BiHom-associative algebra with unit v , denoted by $E(u, v)$.*

Proof. Obviously $\tilde{\alpha} \circ \tilde{\beta} = \tilde{\beta} \circ \tilde{\alpha}$ because $uv = vu$. Then, for all $a, b, c \in E$:

$$\begin{aligned} \tilde{\alpha}(a) * \tilde{\alpha}(b) &= (uau^{-1}) * (ubu^{-1}) = uuau^{-1}u^{-1}ubu^{-1}v^{-1} \\ &= uuau^{-1}bu^{-1}v^{-1} = \tilde{\alpha}(uau^{-1}bv^{-1}) = \tilde{\alpha}(a * b), \\ \tilde{\beta}(a) * \tilde{\beta}(b) &= (vav^{-1}) * (v bv^{-1}) = uvav^{-1}u^{-1}v bv^{-1}v^{-1} \\ &= uvau^{-1}bv^{-1}v^{-1} = \tilde{\beta}(uau^{-1}bv^{-1}) = \tilde{\beta}(a * b), \\ \tilde{\alpha}(a) * (b * c) &= (uau^{-1}) * (ubu^{-1}cv^{-1}) = uuau^{-1}u^{-1}ubu^{-1}cv^{-1}v^{-1} \\ &= uuau^{-1}bv^{-1}u^{-1}vcv^{-1}v^{-1} = (uau^{-1}bv^{-1}) * (vcv^{-1}) = (a * b) * \tilde{\beta}(c), \end{aligned}$$

so $(E, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta})$ is indeed a BiHom-associative algebra. To prove that v is the unit, we compute

$$\begin{aligned} \tilde{\alpha}(v) &= uvu^{-1} = v, & \tilde{\beta}(v) &= vvv^{-1} = v, \\ a * v &= uau^{-1}vv^{-1} = uau^{-1} = \tilde{\alpha}(a), & v * a &= uvu^{-1}av^{-1} = vav^{-1} = \tilde{\beta}(a), \end{aligned}$$

finishing the proof. ■

Proposition 4.4. *Let $(A, \mu_A, \alpha_A, \beta_A)$ be a BiHom-associative algebra, M a linear space and $\alpha_M, \beta_M: M \rightarrow M$ two commuting linear isomorphisms. Consider the associative unital algebra $E = \text{End}(M)$ with its usual structure, denote $u := \alpha_M$, $v := \beta_M$, and construct the BiHom-associative algebra $(E, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta}) = \text{End}(M)(\alpha_M, \beta_M)$ as in Lemma 4.3. Then setting a structure of a left A -module on (M, α_M, β_M) is equivalent to giving a morphism of BiHom-associative algebras $\varphi: (A, \mu_A, \alpha_A, \beta_A) \rightarrow (E, \tilde{\mu}, \tilde{\alpha}, \tilde{\beta})$. If A is moreover unital with unit 1_A , then the module (M, α_M, β_M) is unital if and only if the morphism φ is unital.*

Proof. The correspondence is given as follows: the module structure $A \otimes M \rightarrow M$ is defined by setting $a \otimes m \mapsto a \cdot m$ if and only if $a \cdot m = \varphi(a)(m)$, for all $a \in A$, $m \in M$. It is easy to see that conditions (4.2) and (4.3) are equivalent to $\tilde{\alpha} \circ \varphi = \varphi \circ \alpha_A$ and respectively $\tilde{\beta} \circ \varphi = \varphi \circ \beta_A$. We prove that, assuming (4.2) and (4.3), we have that (4.4) is equivalent to $\varphi \circ \mu_A = \tilde{\mu} \circ (\varphi \otimes \varphi)$. Note first that (4.2) may be written as $\alpha_M \circ \varphi(a) = \varphi(\alpha_A(a)) \circ \alpha_M$, for all $a \in A$, or equivalently $\alpha_M \circ \varphi(a) \circ \alpha_M^{-1} = \varphi(\alpha_A(a))$, for all $a \in A$. Thus, for all $a, b \in A$, we have

$$\tilde{\mu} \circ (\varphi \otimes \varphi)(a \otimes b) = \varphi(a) * \varphi(b) = \alpha_M \circ \varphi(a) \circ \alpha_M^{-1} \circ \varphi(b) \circ \beta_M^{-1} = \varphi(\alpha_A(a)) \circ \varphi(b) \circ \beta_M^{-1}.$$

Hence, we have

$$\varphi \circ \mu_A = \tilde{\mu} \circ (\varphi \otimes \varphi) \iff \varphi(ab) = \varphi(a) * \varphi(b), \quad \forall a, b \in A,$$

$$\begin{aligned}
&\iff \varphi(ab)(n) = (\varphi(a) * \varphi(b))(n), \quad \forall a, b \in A, n \in M, \\
&\iff (ab) \cdot n = (\varphi(\alpha_A(a)) \circ \varphi(b) \circ \beta_M^{-1})(n), \quad \forall a, b \in A, n \in M, \\
&\iff (ab) \cdot \beta_M(m) = (\varphi(\alpha_A(a)) \circ \varphi(b))(m), \quad \forall a, b \in A, m \in M, \\
&\iff (ab) \cdot \beta_M(m) = \alpha_A(a) \cdot (b \cdot m), \quad \forall a, b \in A, m \in M,
\end{aligned}$$

which is exactly (4.4).

Assume that A is unital with unit 1_A . The fact that φ is unital is equivalent to $\varphi(1_A) = \beta_M$, which is equivalent to $1_A \cdot m = \beta_M(m)$, for all $m \in M$, which is equivalent to saying that the module M is unital. \blacksquare

We recall the following concept from [27] (see also [7] on this subject).

Definition 4.5 ([27]). Let $(L, [-], \alpha)$ be a Hom-Lie algebra. A *representation* of L is a triple (M, ρ, A) , where M is a linear space, $A: M \rightarrow M$ and $\rho: L \rightarrow \text{End}(M)$ are linear maps such that, for all $x, y \in L$, the following conditions are satisfied:

$$\rho(\alpha(x)) \circ A = A \circ \rho(x), \quad \rho([x, y]) \circ A = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x).$$

Remark 4.6. Let $(L, [-], \alpha)$ be a Hom-Lie algebra, M a linear space, $A: M \rightarrow M$ and $\rho: L \rightarrow \text{End}(M)$ linear maps such that A is bijective. We can consider the Hom-associative algebra $\text{End}(M)(A, A)$ as in Lemma 4.3, and then the Hom-Lie algebra $L(\text{End}(M)(A, A))$. Then one can check that (M, ρ, A) is a representation of L if and only if ρ is a morphism of Hom-Lie algebras from L to $L(\text{End}(M)(A, A))$.

Inspired by this remark, we can introduce now the following concept:

Definition 4.7. Let $(L, [-], \alpha, \beta)$ be a BiHom-Lie algebra. A *representation* of L is a 4-tuple $(M, \rho, \alpha_M, \beta_M)$, where M is a linear space, $\alpha_M, \beta_M: M \rightarrow M$ are two commuting linear maps and $\rho: L \rightarrow \text{End}(M)$ is a linear map such that, for all $x, y \in L$, we have

$$\rho(\alpha(x)) \circ \alpha_M = \alpha_M \circ \rho(x), \tag{4.5}$$

$$\rho(\beta(x)) \circ \beta_M = \beta_M \circ \rho(x), \tag{4.6}$$

$$\rho([\beta(x), y]) \circ \beta_M = \rho(\alpha\beta(x)) \circ \rho(y) - \rho(\beta(y)) \circ \rho(\alpha(x)). \tag{4.7}$$

A first indication that this is indeed the appropriate concept of representation for BiHom-Lie algebras is provided by the following result (extending the corresponding one for Hom-associative algebras in [6]), whose proof is straightforward and left to the reader.

Proposition 4.8. *Let $(A, \mu_A, \alpha_A, \beta_A)$ be a BiHom-associative algebra with bijective structure maps and (M, α_M, β_M) a left A -module, with action $A \otimes M \rightarrow M$, $a \otimes m \mapsto a \cdot m$. Then we have a representation $(M, \rho, \alpha_M, \beta_M)$ of the BiHom-Lie algebra $L(A)$, where $\rho: L(A) \rightarrow \text{End}(M)$ is the linear map defined by $\rho(a)(m) = a \cdot m$, for all $a \in A$, $m \in M$.*

A second indication is provided by the fact that, under certain circumstances, we can construct the semidirect product (the Hom-case is done in [27]).

Proposition 4.9. *Let $(L, [-], \alpha, \beta)$ be a BiHom-Lie algebra and $(M, \rho, \alpha_M, \beta_M)$ a representation of L , with notation $\rho(x)(a) = x \cdot a$, for all $x \in L$, $a \in M$. Assume that the maps α and β_M are bijective. Then $L \ltimes M := (L \oplus M, [-], \alpha \oplus \alpha_M, \beta \oplus \beta_M)$ is a BiHom-Lie algebra (called the semidirect product), where $\alpha \oplus \alpha_M, \beta \oplus \beta_M: L \oplus M \rightarrow L \oplus M$ are defined by $(\alpha \oplus \alpha_M)(x, a) = (\alpha(x), \alpha_M(a))$ and $(\beta \oplus \beta_M)(x, a) = (\beta(x), \beta_M(a))$, and, for all $x, y \in L$ and $a, b \in M$, the bracket $[-]$ is defined by*

$$[(x, a), (y, b)] = ([x, y], x \cdot b - \alpha^{-1}\beta(y) \cdot \alpha_M\beta_M^{-1}(a)).$$

Proof. Follows by a direct computation that is left to the reader. \blacksquare

Proposition 4.10. *Let $(L, [-], \alpha, \beta)$ be a BiHom-Lie algebra such that the map β is surjective, M a linear space, $\alpha_M, \beta_M: M \rightarrow M$ two commuting linear isomorphisms and $\rho: L \rightarrow \text{End}(M)$ a linear map. Then $(M, \rho, \alpha_M, \beta_M)$ is a representation of L if and only if ρ is a morphism of BiHom-Lie algebras from L to $L(\text{End}(M)(\alpha_M, \beta_M))$.*

Proof. Obviously, (4.5) and (4.6) are respectively equivalent to $\tilde{\alpha} \circ \rho = \rho \circ \alpha$ and $\tilde{\beta} \circ \rho = \rho \circ \beta$, so we only need to prove that, assuming (4.5) and (4.6), (4.7) is equivalent to $\rho([x, y]) = [\rho(x), \rho(y)]$ for all $x, y \in L$. First we write down explicitly the bracket of $L(\text{End}(M)(\alpha_M, \beta_M))$. In view of Proposition 3.15, this bracket looks as follows, for $f, g \in \text{End}(M)$:

$$\begin{aligned} [f, g] &= f * g - (\tilde{\alpha}^{-1} \tilde{\beta}(g)) * (\tilde{\alpha} \tilde{\beta}^{-1}(f)) \\ &= f * g - (\tilde{\alpha}^{-1}(\beta_M \circ g \circ \beta_M^{-1})) * (\tilde{\alpha}(\beta_M^{-1} \circ f \circ \beta_M)) \\ &= f * g - (\alpha_M^{-1} \circ \beta_M \circ g \circ \beta_M^{-1} \circ \alpha_M) * (\alpha_M \circ \beta_M^{-1} \circ f \circ \beta_M \circ \alpha_M^{-1}) \\ &= \alpha_M \circ f \circ \alpha_M^{-1} \circ g \circ \beta_M^{-1} \\ &\quad - \alpha_M \circ \alpha_M^{-1} \circ \beta_M \circ g \circ \beta_M^{-1} \circ \alpha_M \circ \alpha_M^{-1} \circ \alpha_M \circ \beta_M^{-1} \circ f \circ \beta_M \circ \alpha_M^{-1} \circ \beta_M^{-1} \\ &= \alpha_M \circ f \circ \alpha_M^{-1} \circ g \circ \beta_M^{-1} - \beta_M \circ g \circ \beta_M^{-1} \circ \alpha_M \circ \beta_M^{-1} \circ f \circ \beta_M \circ \alpha_M^{-1} \circ \beta_M^{-1}. \end{aligned}$$

Let $x, y \in L$; we take $f = \rho(\beta(x))$, $g = \rho(y)$. We obtain

$$\begin{aligned} [\rho(\beta(x)), \rho(y)] \circ \beta_M &= \alpha_M \circ \rho(\beta(x)) \circ \alpha_M^{-1} \circ \rho(y) \\ &\quad - \beta_M \circ \rho(y) \circ \beta_M^{-1} \circ \alpha_M \circ \beta_M^{-1} \circ \rho(\beta(x)) \circ \beta_M \circ \alpha_M^{-1} \\ &\stackrel{(4.5), (4.6)}{=} \rho(\alpha\beta(x)) \circ \rho(y) - \rho(\beta(y)) \circ \alpha_M \circ \rho(x) \circ \alpha_M^{-1} \\ &\stackrel{(4.5)}{=} \rho(\alpha\beta(x)) \circ \rho(y) - \rho(\beta(y)) \circ \rho(\alpha(x)), \end{aligned}$$

which is the right-hand side of (4.7). So, we have that (4.7) holds if and only if $\rho([\beta(x), y]) = [\rho(\beta(x)), \rho(y)]$ for all $x, y \in L$, which is equivalent to $\rho([a, b]) = [\rho(a), \rho(b)]$, for all $a, b \in L$, because β is surjective. \blacksquare

Proposition 4.11. *Let $(L, [-], \alpha, \beta)$ be a BiHom-Lie algebra and define the linear map $\text{ad}: L \rightarrow \text{End}(L)$, $\text{ad}(x)(y) = [x, y]$, for all $x, y \in L$. If the maps α and β are bijective, then $(L, \text{ad}, \alpha, \beta)$ is a representation of L .*

Proof. The conditions (4.5) and (4.6) are equivalent to $\alpha([a, b]) = [\alpha(a), \alpha(b)]$ and $\beta([a, b]) = [\beta(a), \beta(b)]$ for all $a, b \in L$, so we only need to prove (4.7). Note first that the skew-symmetry condition implies

$$\text{ad}(x)(y) = -[\alpha^{-1}\beta(y), \alpha\beta^{-1}(x)], \quad \forall x, y \in L.$$

We compute the left-hand side of (4.7) applied to $z \in L$:

$$\begin{aligned} (\text{ad}([\beta(x), y]) \circ \beta)(z) &= \text{ad}([\beta(x), y])(\beta(z)) = -[\alpha^{-1}\beta^2(z), \alpha\beta^{-1}([\beta(x), y])] \\ &= -[\beta^2(\alpha^{-1}(z)), [\alpha(x), \alpha\beta^{-1}(y)]] \\ &= -[\beta^2(\alpha^{-1}(z)), [\beta(\alpha\beta^{-1}(x)), \alpha(\beta^{-1}(y))]]. \end{aligned}$$

We compute the right-hand side of (4.7) applied to $z \in L$:

$$\begin{aligned} (\text{ad}(\alpha\beta(x)) \circ \text{ad}(y))(z) - (\text{ad}(\beta(y)) \circ \text{ad}(\alpha(x)))(z) \\ = \text{ad}(\alpha\beta(x))(-[\alpha^{-1}\beta(z), \alpha\beta^{-1}(y)]) - \text{ad}(\beta(y))(-[\alpha^{-1}\beta(z), \alpha^2\beta^{-1}(x)]) \end{aligned}$$

$$\begin{aligned}
&= [\alpha^{-1}\beta([\alpha^{-1}\beta(z), \alpha\beta^{-1}(y)]), \alpha\beta^{-1}\alpha\beta(x)] \\
&\quad - [\alpha^{-1}\beta([\alpha^{-1}\beta(z), \alpha^2\beta^{-1}(x)]), \alpha\beta^{-1}\beta(y)] \\
&= [\beta([\alpha^{-2}\beta(z), \beta^{-1}(y)]), \alpha^2(x)] - [\beta([\alpha^{-2}\beta(z), \alpha\beta^{-1}(x)]), \alpha(y)] \\
&\stackrel{\text{skew-symmetry}}{=} -[\beta\alpha(x), [\alpha^{-1}\beta(z), \alpha\beta^{-1}(y)]] + [\beta(y), [\alpha^{-1}\beta(z), \alpha^2\beta^{-1}(x)]] \\
&= [\beta\alpha(x), [y, z]] + [\beta(y), [\beta\alpha^{-1}(z), \alpha^2\beta^{-1}(x)]] \\
&= [\beta^2(\alpha\beta^{-1}(x)), [\beta(\beta^{-1}(y)), \alpha(\alpha^{-1}(z))]] + [\beta^2(\beta^{-1}(y)), [\beta(\alpha^{-1}(z)), \alpha(\alpha\beta^{-1}(x))]],
\end{aligned}$$

and (4.7) holds because of the BiHom-Jacobi identity applied to the elements $a = \alpha\beta^{-1}(x)$, $a' = \beta^{-1}(y)$ and $a'' = \alpha^{-1}(z)$. \blacksquare

5 BiHom-coassociative coalgebras and BiHom-bialgebras

We introduce now the dual concept to the one of BiHom-associative algebra.

Definition 5.1. A *BiHom-coassociative coalgebra* is a 4-tuple $(C, \Delta, \psi, \omega)$, in which C is a linear space, $\psi, \omega: C \rightarrow C$ and $\Delta: C \rightarrow C \otimes C$ are linear maps, such that

$$\begin{aligned}
\psi \circ \omega &= \omega \circ \psi, & (\psi \otimes \psi) \circ \Delta &= \Delta \circ \psi, & (\omega \otimes \omega) \circ \Delta &= \Delta \circ \omega, \\
(\Delta \otimes \psi) \circ \Delta &= (\omega \otimes \Delta) \circ \Delta.
\end{aligned}$$

We call ψ and ω (in this order) the *structure maps* of C .

A morphism $g: (C, \Delta_C, \psi_C, \omega_C) \rightarrow (D, \Delta_D, \psi_D, \omega_D)$ of BiHom-coassociative coalgebras is a linear map $g: C \rightarrow D$ such that $\psi_D \circ g = g \circ \psi_C$, $\omega_D \circ g = g \circ \omega_C$ and $(g \otimes g) \circ \Delta_C = \Delta_D \circ g$.

A BiHom-coassociative coalgebra $(C, \Delta, \psi, \omega)$ is called *counital* if there exists a linear map $\varepsilon: C \rightarrow \mathbb{k}$ (called a *counit*) such that

$$\varepsilon \circ \psi = \varepsilon, \quad \varepsilon \circ \omega = \varepsilon, \quad (\text{id}_C \otimes \varepsilon) \circ \Delta = \omega \quad \text{and} \quad (\varepsilon \otimes \text{id}_C) \circ \Delta = \psi.$$

A morphism of counital BiHom-coassociative coalgebras $g: C \rightarrow D$ is called *counital* if $\varepsilon_D \circ g = \varepsilon_C$, where ε_C and ε_D are the counits of C and D , respectively.

Remark 5.2. If $(C, \Delta_C, \psi_C, \omega_C)$ and $(D, \Delta_D, \psi_D, \omega_D)$ are two BiHom-coassociative coalgebras, then $(C \otimes D, \Delta_{C \otimes D}, \psi_C \otimes \psi_D, \omega_C \otimes \omega_D)$ is also a BiHom-coassociative coalgebra (called the tensor product of C and D), where $\Delta_{C \otimes D}: C \otimes D \rightarrow C \otimes D \otimes C \otimes D$ is defined by $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$, for all $c \in C$, $d \in D$. If C and D are counital with counits ε_C and respectively ε_D , then $C \otimes D$ is also counital with counit $\varepsilon_C \otimes \varepsilon_D$.

Definition 5.3. Let $(C, \Delta_C, \psi_C, \omega_C)$ be a BiHom-coassociative coalgebra. A *right C -comodule* is a triple (M, ψ_M, ω_M) , where M is a linear space, $\psi_M, \omega_M: M \rightarrow M$ are linear maps and we have a linear map (called a coaction) $\rho: M \rightarrow M \otimes C$, with notation $\rho(m) = m_{(0)} \otimes m_{(1)}$, for all $m \in M$, such that the following conditions are satisfied

$$\begin{aligned}
\psi_M \circ \omega_M &= \omega_M \circ \psi_M, & (\psi_M \otimes \psi_C) \circ \rho &= \rho \circ \psi_M, & (\omega_M \otimes \omega_C) \circ \rho &= \rho \circ \omega_M, \\
(\omega_M \otimes \Delta_C) \circ \rho &= (\rho \otimes \psi_C) \circ \rho.
\end{aligned}$$

If (M, ψ_M, ω_M) and (N, ψ_N, ω_N) are right C -comodules with coactions ρ_M and respectively ρ_N , a morphism of right C -comodules $f: M \rightarrow N$ is a linear map satisfying the conditions $\psi_N \circ f = f \circ \psi_M$, $\omega_N \circ f = f \circ \omega_M$ and $\rho_N \circ f = (f \otimes \text{id}_C) \circ \rho_M$.

If $(C, \Delta_C, \psi_C, \omega_C, \varepsilon_C)$ is a counital BiHom-coassociative coalgebra and (M, ψ_M, ω_M) is a right C -comodule with coaction ρ , then M is called *counital* if $(\text{id}_M \otimes \varepsilon_C) \circ \rho = \omega_M$.

Remark 5.4. If $(C, \Delta, \psi, \omega)$ is a BiHom-coassociative coalgebra, then (C, ψ, ω) is a right C -comodule, with coaction $\rho = \Delta$.

We discuss now the duality between BiHom-associative and BiHom-coassociative structures.

Theorem 5.5. *Let $(C, \Delta, \psi, \omega)$ be a BiHom-coassociative coalgebra. Then its dual linear space is provided with a structure of BiHom-associative algebra $(C^*, \Delta^*, \omega^*, \psi^*)$, where $\Delta^*, \psi^*, \omega^*$ are the transpose maps. Moreover, the BiHom-associative algebra C^* is unital whenever the BiHom-coassociative coalgebra C is counital.*

Proof. The product $\mu = \Delta^*$ is defined from $C^* \otimes C^*$ to C^* by

$$(fg)(x) = \Delta^*(f, g)(x) = \langle \Delta(x), f \otimes g \rangle = (f \otimes g)(\Delta(x)) = f(x_1)g(x_2), \quad \forall x \in C,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between the linear space $C \otimes C$ and its dual linear space. For $f, g, h \in C^*$ and $x \in C$, we have

$$\begin{aligned} (fg)\psi^*(h)(x) &= \langle (\Delta \otimes \psi) \circ \Delta(x), f \otimes g \otimes h \rangle, \\ \omega^*(f)(gh)(x) &= \langle (\omega \otimes \Delta) \circ \Delta(x), f \otimes g \otimes h \rangle. \end{aligned}$$

Therefore, the BiHom-associativity condition $\mu \circ (\mu \otimes \psi^* - \omega^* \otimes \mu) = 0$ follows from the BiHom-coassociativity condition $(\Delta \otimes \psi - \omega \otimes \Delta) \circ \Delta = 0$.

Moreover, if C has a counit ε then for $f \in C^*$ and $x \in C$ we have

$$\begin{aligned} (\varepsilon f)(x) &= \varepsilon(x_1)f(x_2) = f(\varepsilon(x_1)x_2) = f(\psi(x)) = \psi^*(f)(x), \\ (f\varepsilon)(x) &= f(x_1)\varepsilon(x_2) = f(x_1\varepsilon(x_2)) = f(\omega(x)) = \omega^*(f)(x), \end{aligned}$$

which shows that ε is the unit of C^* . ■

The dual of a BiHom-associative algebra (A, μ, α, β) is not always a BiHom-coassociative coalgebra, because $(A \otimes A)^* \supsetneq A^* \otimes A^*$. Nevertheless, it is the case if the BiHom-associative algebra is finite-dimensional, since $(A \otimes A)^* = A^* \otimes A^*$ in this case.

More generally, we can define the finite dual of A by

$$A^\circ = \{f \in A^* / f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where a cofinite ideal I is an ideal $I \subset A$ such that A/I is finite-dimensional and where we say that I is an ideal of A if for $x \in I$ and $y \in A$ we have $xy \in I$, $yx \in I$ and $\alpha(x) \in I$, $\beta(x) \in I$.

A° is a subspace of A^* since it is closed under multiplication by scalars and the sum of two elements of A° is again in A° because the intersection of two cofinite ideals is again a cofinite ideal. If A is finite-dimensional, of course $A^\circ = A^*$. As in the classical case, one can show that if A and B are two BiHom-associative algebras and $f: A \rightarrow B$ is a morphism of BiHom-associative algebras, then the dual map $f^*: B^* \rightarrow A^*$ satisfies $f^*(B^\circ) \subset A^\circ$.

Therefore, a similar proof to the one of the previous theorem leads to:

Theorem 5.6. *Let (A, μ, α, β) be a BiHom-associative algebra. Then its finite dual is provided with a structure of BiHom-coassociative coalgebra $(A^\circ, \Delta, \beta^\circ, \alpha^\circ)$, where $\Delta = \mu^\circ = \mu^*|_{A^\circ}$ and $\beta^\circ, \alpha^\circ$ are the transpose maps on A° . Moreover, the BiHom-coassociative coalgebra is counital whenever A is unital, with counit $\varepsilon: A^\circ \rightarrow \mathbb{k}$ defined by $\varepsilon(f) = f(1_A)$.*

We can now define the notion of BiHom-bialgebra.

Definition 5.7. A *BiHom-bialgebra* is a 7-tuple $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$, with the property that (H, μ, α, β) is a BiHom-associative algebra, $(H, \Delta, \psi, \omega)$ is a BiHom-coassociative coalgebra and moreover the following relations are satisfied, for all $h, h' \in H$:

$$\begin{aligned} \Delta(hh') &= h_1 h'_1 \otimes h_2 h'_2, & (5.1) \\ \alpha \circ \psi &= \psi \circ \alpha, & \alpha \circ \omega &= \omega \circ \alpha, & \beta \circ \psi &= \psi \circ \beta, & \beta \circ \omega &= \omega \circ \beta, \\ (\alpha \otimes \alpha) \circ \Delta &= \Delta \circ \alpha, & (\beta \otimes \beta) \circ \Delta &= \Delta \circ \beta, \\ \psi(hh') &= \psi(h)\psi(h'), & \omega(hh') &= \omega(h)\omega(h'). \end{aligned}$$

We say that H is a unital and counital BiHom-bialgebra if, in addition, it admits a unit 1_H and a counit ε_H such that

$$\begin{aligned} \Delta(1_H) &= 1_H \otimes 1_H, & \varepsilon_H(1_H) &= 1, & \psi(1_H) &= 1_H, & \omega(1_H) &= 1_H, \\ \varepsilon_H \circ \alpha &= \varepsilon_H, & \varepsilon_H \circ \beta &= \varepsilon_H, & \varepsilon_H(hh') &= \varepsilon_H(h)\varepsilon_H(h'), & \forall h, h' \in H. \end{aligned}$$

Let us record the formula expressing the BiHom-coassociativity of Δ :

$$\Delta(h_1) \otimes \psi(h_2) = \omega(h_1) \otimes \Delta(h_2), \quad \forall h \in H. \quad (5.2)$$

Remark 5.8. Obviously, a BiHom-bialgebra $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ with $\alpha = \beta = \psi = \omega$ reduces to a Hom-bialgebra, as used for instance in [22, 23], while a BiHom-bialgebra for which $\psi = \omega = \alpha^{-1} = \beta^{-1}$ reduces to a monoidal Hom-bialgebra, in the terminology of [8].

We see now that analogues of Yau's twisting principle hold for the BiHom-structures we defined (proofs are straightforward and left to the reader):

Proposition 5.9.

- (i) Let (A, μ) be an associative algebra and $\alpha, \beta: A \rightarrow A$ two commuting algebra endomorphisms. Define a new multiplication $\mu_{(\alpha, \beta)}: A \otimes A \rightarrow A$, by $\mu_{(\alpha, \beta)} := \mu \circ (\alpha \otimes \beta)$. Then $(A, \mu_{(\alpha, \beta)}, \alpha, \beta)$ is a BiHom-associative algebra, denoted by $A_{(\alpha, \beta)}$. If A is unital with unit 1_A , then $A_{(\alpha, \beta)}$ is also unital with unit 1_A .
- (ii) Let (C, Δ) be a coassociative coalgebra and $\psi, \omega: C \rightarrow C$ two commuting coalgebra endomorphisms. Define a new comultiplication $\Delta_{(\psi, \omega)}: C \rightarrow C \otimes C$, by $\Delta_{(\psi, \omega)} := (\omega \otimes \psi) \circ \Delta$. Then $(C, \Delta_{(\psi, \omega)}, \psi, \omega)$ is a BiHom-coassociative coalgebra, denoted by $C_{(\psi, \omega)}$. If C is counital with counit ε_C , then $C_{(\psi, \omega)}$ is also counital with counit ε_C .
- (iii) Let (H, μ, Δ) be a bialgebra and $\alpha, \beta, \psi, \omega: H \rightarrow H$ bialgebra endomorphisms such that any two of them commute. If we define $\mu_{(\alpha, \beta)}$ and $\Delta_{(\psi, \omega)}$ as in (i) and (ii), then $H_{(\alpha, \beta, \psi, \omega)} := (H, \mu_{(\alpha, \beta)}, \Delta_{(\psi, \omega)}, \alpha, \beta, \psi, \omega)$ is a BiHom-bialgebra.

More generally, a BiHom-bialgebra $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ and multiplicative and comultiplicative linear maps $\alpha', \beta', \psi', \omega'$ such that any two of the maps $\alpha, \beta, \psi, \omega, \alpha', \beta', \psi', \omega'$ commute, give rise to a new BiHom-bialgebra $(H, \mu \circ (\alpha' \otimes \beta'), (\omega' \otimes \psi') \circ \Delta, \alpha \circ \alpha', \beta \circ \beta', \psi \circ \psi', \omega \circ \omega')$. Hence, if the maps $\alpha, \beta, \psi, \omega$ are invertible, one can untwist the BiHom-bialgebra and get a bialgebra by taking $\alpha' = \alpha^{-1}, \beta' = \beta^{-1}, \psi' = \psi^{-1}, \omega' = \omega^{-1}$.

Proposition 5.10. Let (A, μ_A) be an associative algebra and $\alpha_A, \beta_A: A \rightarrow A$ two commuting algebra endomorphisms. Assume that M is a left A -module, with action $A \otimes M \rightarrow M$, $a \otimes m \mapsto a \cdot m$. Let $\alpha_M, \beta_M: M \rightarrow M$ be two commuting linear maps such that $\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)$ and $\beta_M(a \cdot m) = \beta_A(a) \cdot \beta_M(m)$, for all $a \in A, m \in M$. Then (M, α_M, β_M) becomes a left module over $A_{(\alpha_A, \beta_A)}$, with action $A_{(\alpha_A, \beta_A)} \otimes M \rightarrow M$, $a \otimes m \mapsto a \triangleright m := \alpha_A(a) \cdot \beta_M(m)$.

Proposition 5.11. *Let (C, Δ_C) be a coassociative coalgebra and $\psi_C, \omega_C: C \rightarrow C$ two commuting coalgebra endomorphisms. Assume that M is a right C -comodule, with coaction $\rho: M \rightarrow M \otimes C$, $\rho(m) = m_{(0)} \otimes m_{(1)}$, for all $m \in M$. Let $\psi_M, \omega_M: M \rightarrow M$ be two commuting linear maps such that $(\psi_M \otimes \psi_C) \circ \rho = \rho \circ \psi_M$ and $(\omega_M \otimes \omega_C) \circ \rho = \rho \circ \omega_M$. Then (M, ψ_M, ω_M) becomes a right comodule over the BiHom-coassociative coalgebra $C_{(\psi_C, \omega_C)}$, with coaction*

$$M \rightarrow M \otimes C_{(\psi_C, \omega_C)}, \quad m \mapsto m_{(0)} \otimes m_{(1)} := \omega_M(m_{(0)}) \otimes \psi_C(m_{(1)}).$$

We describe in what follows primitive elements of a BiHom-bialgebra.

Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ be a unital and counital BiHom-bialgebra with a unit $1 = \eta(1)$ and a counit ε . We assume that α and β are bijective.

An element $x \in H$ is called *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$.

Lemma 5.12. *Let x be a primitive element in H . Then $\varepsilon(x)1 = \omega(x) - x = \psi(x) - x$, and therefore $\omega(x) = \psi(x)$. Moreover, $\alpha^p \beta^q(x)$ is also a primitive element for any $p, q \in \mathbb{Z}$.*

Proof. By the counit property, we have $\omega(x) = (\text{id}_H \otimes \varepsilon)(1 \otimes x + x \otimes 1) = \varepsilon(x)1 + \varepsilon(1)x = \varepsilon(x)1 + x$, and similarly $\psi(x) = \varepsilon(x)1 + x$.

Since α and β are comultiplicative maps and $\alpha^p \beta^q(1) = 1$, it follows that $\alpha^p \beta^q(x)$ is a primitive element whenever x is a primitive element. \blacksquare

Proposition 5.13. *Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ be a unital and counital BiHom-bialgebra, with unit $1 = \eta(1)$ and counit ε . Assume that α and β are bijective. If x and y are two primitive elements in H , then the commutator $[x, y] = xy - \alpha^{-1}\beta(y)\alpha\beta^{-1}(x)$ is also a primitive element.*

Consequently, the set of all primitive elements of H , denoted by $\text{Prim}(H)$, has a structure of BiHom-Lie algebra.

Proof. We compute

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= 1 \otimes xy + \beta(y) \otimes \alpha(x) + \alpha(x) \otimes \beta(y) + xy \otimes 1, \\ \Delta(\alpha^{-1}\beta(y)\alpha\beta^{-1}(x)) &= \Delta(\alpha^{-1}\beta(y))\Delta(\alpha\beta^{-1}(x)) \\ &= (1 \otimes \alpha^{-1}\beta(y) + \alpha^{-1}\beta(y) \otimes 1)(1 \otimes \alpha\beta^{-1}(x) + \alpha\beta^{-1}(x) \otimes 1) \\ &= 1 \otimes \alpha^{-1}\beta(y)\alpha\beta^{-1}(x) + \beta(\alpha\beta^{-1}(x)) \otimes \alpha(\alpha^{-1}\beta(y)) \\ &\quad + \alpha(\alpha^{-1}\beta(y)) \otimes \beta(\alpha\beta^{-1}(x)) + \alpha^{-1}\beta(y)\alpha\beta^{-1}(x) \otimes 1 \\ &= 1 \otimes \alpha^{-1}\beta(y)\alpha\beta^{-1}(x) + \alpha(x) \otimes \beta(y) \\ &\quad + \beta(y) \otimes \alpha(x) + \alpha^{-1}\beta(y)\alpha\beta^{-1}(x) \otimes 1. \end{aligned}$$

Therefore, we have

$$\Delta([x, y]) = \Delta(xy) - \Delta(\alpha^{-1}\beta(y)\alpha\beta^{-1}(x)) = 1 \otimes [x, y] + [x, y] \otimes 1,$$

which means that $\text{Prim}(H)$ is closed under the bracket multiplication $[\cdot, \cdot]$. Hence, $\text{Prim}(H)$ is a BiHom-Lie algebra by Proposition 3.15. \blacksquare

Now, we introduce the notion of H -module BiHom-algebra, where H is a BiHom-bialgebra.

Definition 5.14. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra for which the maps $\alpha_H, \beta_H, \psi_H, \omega_H$ are bijective. A BiHom-associative algebra $(A, \mu_A, \alpha_A, \beta_A)$ is called a *left H -module BiHom-algebra* if (A, α_A, β_A) is a left H -module, with action denoted by $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \cdot a$, such that the following condition is satisfied

$$h \cdot (aa') = [\alpha_H^{-1}(\omega_H^{-1}(h_1)) \cdot a][\beta_H^{-1}(\psi_H^{-1}(h_2)) \cdot a'], \quad \forall h \in H, \quad a, a' \in A. \quad (5.3)$$

Remark 5.15. This concept contains as particular cases the concepts of module algebras over a Hom-bialgebra, respectively monoidal Hom-bialgebra, introduced in [31], respectively [11].

The choice of (5.3) is motivated by the following result, whose proof is also left to the reader:

Proposition 5.16. *Let (H, μ_H, Δ_H) be a bialgebra and (A, μ_A) a left H -module algebra in the usual sense, with action denoted by $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \cdot a$. Let $\alpha_H, \beta_H, \psi_H, \omega_H: H \rightarrow H$ be bialgebra endomorphisms of H such that any two of them commute; let $\alpha_A, \beta_A: A \rightarrow A$ be two commuting algebra endomorphisms such that, for all $h \in H$ and $a \in A$, we have*

$$\alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a) \quad \text{and} \quad \beta_A(h \cdot a) = \beta_H(h) \cdot \beta_A(a).$$

If we consider the BiHom-bialgebra $H_{(\alpha_H, \beta_H, \psi_H, \omega_H)}$ and the BiHom-associative algebra $A_{(\alpha_A, \beta_A)}$ as defined before, then $A_{(\alpha_A, \beta_A)}$ is a left $H_{(\alpha_H, \beta_H, \psi_H, \omega_H)}$ -module BiHom-algebra in the above sense, with action

$$H_{(\alpha_H, \beta_H, \psi_H, \omega_H)} \otimes A_{(\alpha_A, \beta_A)} \rightarrow A_{(\alpha_A, \beta_A)}, \quad h \otimes a \mapsto h \triangleright a := \alpha_H(h) \cdot \beta_A(a).$$

6 Monoidal BiHom-Hopf algebras and BiHom-Hopf algebras

In this section, we introduce the concept of monoidal BiHom-Hopf algebra and discuss a possible generalization of Hom-Hopf algebras to BiHom-Hopf algebras.

We begin with a lemma whose proof is obvious.

Lemma 6.1. *Let (A, μ, α, β) be a BiHom-associative algebra. Define $\underline{A} := \{a \in A / \alpha(a) = \beta(a) = a\}$. Then (\underline{A}, μ) is an associative algebra. If A is unital with unit 1_A , then 1_A is also the unit of \underline{A} (in particular, it follows that the unit of a BiHom-associative algebra, if it exists, is unique).*

Proposition 6.2. *Let (A, μ, α, β) be a BiHom-associative algebra and $(C, \Delta, \psi, \omega)$ a BiHom-coassociative coalgebra. Set, for $f, g \in \text{Hom}(C, A)$, $f \star g = \mu \circ (f \otimes g) \circ \Delta$. Define the linear maps $\phi, \gamma: \text{Hom}(C, A) \rightarrow \text{Hom}(C, A)$ by $\phi(f) = \alpha \circ f \circ \omega$ and $\gamma(f) = \beta \circ f \circ \psi$, for all $f \in \text{Hom}(C, A)$. Then $(\text{Hom}(C, A), \star, \phi, \gamma)$ is a BiHom-associative algebra.*

Moreover, if A is unital with unit 1_A and C is counital with counit ε , then $\text{Hom}(C, A)$ is a unital BiHom-associative algebra with unit $\eta \circ \varepsilon$, where we denote by η the linear map $\eta: \mathbb{k} \rightarrow A$, $\eta(1) = 1_A$.

In particular, if we denote by $\underline{\text{Hom}}(C, A)$ the linear subspace of $\text{Hom}(C, A)$ consisting of the linear maps $f: C \rightarrow A$ such that $\alpha \circ f \circ \omega = f$ and $\beta \circ f \circ \psi = f$, then $(\underline{\text{Hom}}(C, A), \star, \eta \circ \varepsilon)$ is an associative unital algebra.

Proof. Let $f, g, h \in \text{Hom}(C, A)$. We have

$$\begin{aligned} \phi(f) \star (g \star h) &= \mu \circ (\phi(f) \otimes (g \star h)) \Delta = \mu \circ (\phi(f) \otimes (\mu \circ (g \otimes h) \circ \Delta)) \Delta \\ &= \mu \circ ((\alpha \otimes \mu) \circ (f \otimes g \otimes h) \circ (\omega \otimes \Delta)) \Delta. \end{aligned}$$

Similarly,

$$(f \star g) \star \gamma(h) = \mu \circ ((\mu \otimes \beta) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \psi)) \Delta.$$

The BiHom-associativity of μ and the BiHom-coassociativity of Δ lead to the BiHom-associativity of the convolution product \star .

The map $\eta \circ \varepsilon$ is the unit for the convolution product. Indeed, for $f \in \text{Hom}(C, A)$ and $x \in C$, we have

$$\begin{aligned} (f \star (\eta \circ \varepsilon))(x) &= \mu \circ (f \otimes \eta \circ \varepsilon) \circ \Delta(x) = \mu(f(x_1) \otimes \eta \circ \varepsilon(x_2)) = \varepsilon(x_2)\mu(f(x_1) \otimes \eta(1)) \\ &= \varepsilon(x_2)(\alpha \circ f)(x_1) = (\alpha \circ f)(x_1\varepsilon(x_2)) = \alpha \circ f \circ \omega(x). \end{aligned}$$

A similar calculation shows that $(\eta \circ \varepsilon) \star f = \beta \circ f \circ \psi$.

The last statement follows from Lemma 6.1. ■

Definition 6.3. Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ be a unital and counital BiHom-bialgebra. We say that H is a *monoidal BiHom-bialgebra* if $\alpha, \beta, \psi, \omega$ are bijective and $\omega = \alpha^{-1}$ and $\psi = \beta^{-1}$. We will refer to a monoidal BiHom-bialgebra as the 5-tuple $(H, \mu, \Delta, \alpha, \beta)$.

If $(H, \mu, \Delta, \alpha, \beta)$ is a monoidal BiHom-bialgebra, we can consider the associative unital algebra $\underline{\text{Hom}}(H, H)$, and since $\omega = \alpha^{-1}$ and $\psi = \beta^{-1}$, it follows that $\text{id}_H \in \underline{\text{Hom}}(H, H)$.

Definition 6.4. Let $(H, \mu, \Delta, \alpha, \beta)$ be a monoidal BiHom-bialgebra with a unit 1_H and a counit ε_H . A linear map $S: H \rightarrow H$ is called an *antipode* if $\alpha \circ S = S \circ \alpha$ and $\beta \circ S = S \circ \beta$ (i.e., $S \in \underline{\text{Hom}}(H, H)$) and S is the convolution inverse of id_H in $\underline{\text{Hom}}(H, H)$, that is

$$S(h_1)h_2 = \varepsilon_H(h)1_H = h_1S(h_2), \quad \forall h \in H.$$

A *monoidal BiHom-Hopf algebra* is a monoidal BiHom-bialgebra endowed with an antipode.

Obviously, if the antipode exists, it is unique; we will refer to the monoidal BiHom-Hopf algebra as the 8-tuple $(H, \mu, \Delta, \alpha, \beta, 1_H, \varepsilon_H, S)$.

Proposition 6.5. Let $(H, \mu, \Delta, 1_H, \varepsilon_H)$ be a Hopf algebra (in the usual sense) with antipode S . Let $\alpha, \beta: H \rightarrow H$ be two unital and counital commuting bialgebra automorphisms. Then $(H, \mu \circ (\alpha \otimes \beta), (\alpha^{-1} \otimes \beta^{-1}) \circ \Delta, \alpha, \beta, 1_H, \varepsilon_H, S)$ is a monoidal BiHom-bialgebra.

Proof. A straightforward computation. Let us only note that α, β being bialgebra maps, they automatically commute with S . ■

We state now the basic properties of the antipode.

Proposition 6.6. Let $(H, \mu, \Delta, \alpha, \beta, 1_H, \varepsilon_H, S)$ be a monoidal BiHom-Hopf algebra. Then

- (i) $S(1_H) = 1_H$ and $\varepsilon_H \circ S = \varepsilon_H$;
- (ii) $S(\beta(a)\alpha(b)) = S(\beta(b))S(\alpha(a))$, for all $a, b \in H$;
- (iii) $\alpha(S(h)_1) \otimes \beta(S(h)_2) = \beta(S(h_2)) \otimes \alpha(S(h_1))$, for all $h \in H$.

Proof. (i) By $\Delta(1_H) = 1_H \otimes 1_H$ we obtain $S(1_H)1_H = \varepsilon_H(1_H)1_H$, so $\alpha(S(1_H)) = 1_H$, and since $\alpha \circ S = S \circ \alpha$ and $\alpha(1_H) = 1_H$ we obtain $S(1_H) = 1_H$. Then, if $h \in H$, we apply ε_H to the equality $h_1S(h_2) = \varepsilon_H(h)1_H$, and we obtain $\varepsilon_H(h_1)\varepsilon_H(S(h_2)) = \varepsilon_H(h)$, so $\varepsilon_H(S(\varepsilon_H(h_1)h_2)) = \varepsilon_H(h)$, hence $\varepsilon_H(S(\beta^{-1}(h))) = \varepsilon_H(h)$, and since $S \circ \beta = \beta \circ S$ and $\varepsilon_H \circ \beta = \varepsilon_H$ we obtain $\varepsilon_H \circ S = \varepsilon_H$.

(ii) We define the linear maps $R, L, m: H \otimes H \rightarrow H$ by the formulae (for all $a, b \in H$):

$$R(a \otimes b) = S(\beta(b))S(\alpha(a)), \quad L(a \otimes b) = S(\beta(a)\alpha(b)), \quad m(a \otimes b) = \beta(a)\alpha(b).$$

One can easily check that $R, L, m \in \underline{\text{Hom}}(H \otimes H, H)$ (where $H \otimes H$ is the tensor product BiHom-coassociative coalgebra). Thus, to prove that $R = L$, it is enough to prove that L (respectively R) is a left (respectively right) convolution inverse of m in $\underline{\text{Hom}}(H \otimes H, H)$. We compute

$$(L \star m)(a \otimes b) = L(a_1 \otimes b_1)m(a_2 \otimes b_2) = S(\beta(a_1)\alpha(b_1))(\beta(a_2)\alpha(b_2))$$

$$\begin{aligned}
&= S(\beta(a)_1\alpha(b)_1)(\beta(a)_2\alpha(b)_2) = S((\beta(a)\alpha(b))_1)(\beta(a)\alpha(b))_2 \\
&= \varepsilon_H(\beta(a)\alpha(b))1_H = \varepsilon_H(a)\varepsilon_H(b)1_H, \\
(m \star R)(a \otimes b) &= m(a_1 \otimes b_1)R(a_2 \otimes b_2) = (\beta(a_1)\alpha(b_1))(S(\beta(b_2))S(\alpha(a_2))) \\
&= \alpha(\alpha^{-1}\beta(a_1)b_1)(\beta(S(b_2))\alpha(S(a_2))) = ((\alpha^{-1}\beta(a_1)b_1)\beta(S(b_2)))\alpha\beta(S(a_2)) \\
&= (\beta(a_1)(b_1S(b_2)))\alpha\beta(S(a_2)) = (\beta(a_1)\varepsilon_H(b)1_H)\alpha\beta(S(a_2)) \\
&= \varepsilon_H(b)\alpha\beta(a_1)\alpha\beta(S(a_2)) = \varepsilon_H(b)\alpha\beta(a_1S(a_2)) \\
&= \varepsilon_H(b)\alpha\beta(\varepsilon_H(a)1_H) = \varepsilon_H(a)\varepsilon_H(b)1_H,
\end{aligned}$$

finishing the proof.

(iii) similar to the proof of (ii), by defining the linear maps $\mathcal{L}, \mathcal{R}, \delta: H \rightarrow H \otimes H$,

$$\mathcal{L}(h) = \alpha(S(h)_1) \otimes \beta(S(h)_2), \quad \mathcal{R}(h) = \beta(S(h_2)) \otimes \alpha(S(h_1)), \quad \delta(h) = \alpha(h_1) \otimes \beta(h_2),$$

for all $h \in H$, and proving that \mathcal{L} (respectively \mathcal{R}) is a left (respectively right) convolution inverse of δ in $\underline{\text{Hom}}(H, H \otimes H)$. \blacksquare

Remark 6.7. We had to restrict the definition of the antipode to the class of monoidal BiHom-bialgebras because, if H is a Hopf algebra with antipode S and we make an arbitrary Yau twist of H , then in general S will not satisfy the defining property of an antipode for the Yau twist, as the next example shows.

Example 6.8. Let \mathbb{k} be a field and let $H = \mathbb{k}[X]$, regarded as a Hopf algebra in the usual way. Let $\alpha: H \rightarrow H$ be the algebra map defined by setting $\alpha(X) = X^2$ and let $\beta = \omega = \psi = \text{Id}_H$. Then we can consider the BiHom-bialgebra $H_{(\alpha, \beta, \psi, \omega)} := (H, \mu_{(\alpha, \beta)}, \Delta_{(\psi, \omega)}, \alpha, \beta, \psi, \omega)$, where $\mu: H \otimes H \rightarrow H$ is the usual multiplication and $\Delta: H \rightarrow H \otimes H$ is the usual comultiplication. Moreover $H_{(\alpha, \beta, \psi, \omega)}$ has unit $1_H = \eta_H(1_{\mathbb{k}})$ and counit ε_H that coincide with the ones of H .

Assume that there exists a linear map $S: H \rightarrow H$ such that $S \star \text{Id} = \text{Id} \star S = \eta_H \circ \varepsilon_H$, i.e.,

$$\mu_{(\alpha, \beta)} \circ (S \otimes \text{Id}) \circ \Delta_{(\psi, \omega)} = \mu_{(\alpha, \beta)} \circ (\text{Id} \otimes S) \circ \Delta_{(\psi, \omega)} = \eta_H \circ \varepsilon_H. \quad (6.1)$$

Then we compute

$$\begin{aligned}
(\mu_{(\alpha, \beta)} \circ (\text{Id} \otimes S) \circ \Delta_{(\psi, \omega)})(X) &= \alpha(X)S(1) + \alpha(1)S(X) = X^2S(1) + S(X), \\
(\mu_{(\alpha, \beta)} \circ (S \otimes \text{Id}) \circ \Delta_{(\psi, \omega)})(X) &= \alpha(S(X))1 + \alpha(S(1))X,
\end{aligned}$$

and

$$(\eta_H \circ \varepsilon_H)(X) = 01_H,$$

so that from (6.1) we get

$$S(X) = -X^2S(1) \quad (6.2)$$

and

$$\alpha(S(X)) = -\alpha(S(1))X, \quad (6.3)$$

and hence

$$-\alpha(S(1))X \stackrel{(6.3)}{=} \alpha(S(X)) \stackrel{(6.2)}{=} \alpha(-X^2S(1)) \stackrel{\text{def. } \alpha}{=} -\alpha(X^2)\alpha(S(1)) \stackrel{\text{def. } \alpha}{=} -X^4\alpha(S(1)),$$

so that we get $\alpha(S(1))X = X^4\alpha(S(1))$, which implies that $\alpha(S(1)) = 0$.

On the other hand, we have

$$1 = (\eta_H \circ \varepsilon_H)(1) \stackrel{(6.1)}{=} (\mu_{(\alpha, \beta)} \circ (S \otimes \text{Id}) \circ \Delta_{(\psi, \omega)})(1) = \alpha(S(1))1 = 0,$$

and this is a contradiction.

In view of all the above, we propose the following definition for what might be a BiHom-Hopf algebra, that is moreover invariant under Yau twisting:

Definition 6.9. Let $(H, \mu, \Delta, \alpha, \beta, \psi, \omega)$ be a unital and counital BiHom-bialgebra with a unit 1_H and a counit ε_H . A linear map $S: H \rightarrow H$ is called an *antipode* if it commutes with all the maps $\alpha, \beta, \psi, \omega$ and it satisfies the following relation:

$$\beta\psi(S(h_1))\alpha\omega(h_2) = \varepsilon_H(h)1_H = \beta\psi(h_1)\alpha\omega(S(h_2)), \quad \forall h \in H.$$

A *BiHom-Hopf algebra* is a unital and counital BiHom-bialgebra with an antipode.

We hope to make a more detailed analysis of these structures in a forthcoming paper.

7 BiHom-pseudotwistors and BiHom-twisted tensor products

Inspired by Proposition 3.8, by the concept of pseudotwistor for associative algebras introduced in [21] and its generalization for Hom-associative algebras introduced in [23], we arrive at the following concept and result:

Theorem 7.1. *Let $(D, \mu, \tilde{\alpha}, \tilde{\beta})$ be a BiHom-associative algebra and $\alpha, \beta: D \rightarrow D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Let $T: D \otimes D \rightarrow D \otimes D$ a linear map and assume that there exist two linear maps $\tilde{T}_1, \tilde{T}_2: D \otimes D \otimes D \rightarrow D \otimes D \otimes D$ such that the following relations hold:*

$$(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha), \tag{7.1}$$

$$(\beta \otimes \beta) \circ T = T \circ (\beta \otimes \beta), \tag{7.2}$$

$$(\tilde{\alpha} \otimes \tilde{\alpha}) \circ T = T \circ (\tilde{\alpha} \otimes \tilde{\alpha}), \tag{7.3}$$

$$(\tilde{\beta} \otimes \tilde{\beta}) \circ T = T \circ (\tilde{\beta} \otimes \tilde{\beta}), \tag{7.4}$$

$$T \circ (\tilde{\alpha} \otimes \mu) = (\tilde{\alpha} \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes \text{id}_D), \tag{7.5}$$

$$T \circ (\mu \otimes \tilde{\beta}) = (\mu \otimes \tilde{\beta}) \circ \tilde{T}_2 \circ (\text{id}_D \otimes T), \tag{7.6}$$

$$\tilde{T}_1 \circ (T \otimes \text{id}_D) \circ (\alpha \otimes T) = \tilde{T}_2 \circ (\text{id}_D \otimes T) \circ (T \otimes \beta). \tag{7.7}$$

Then $D_{\alpha, \beta}^T := (D, \mu \circ T, \tilde{\alpha} \circ \alpha, \tilde{\beta} \circ \beta)$ is also a BiHom-associative algebra. The map T is called an (α, β) -BiHom-pseudotwistor and the two maps \tilde{T}_1, \tilde{T}_2 are called the companions of T . In the particular case $\alpha = \beta = \text{id}_D$, we simply call T a BiHom-pseudotwistor and we denote $D_{\alpha, \beta}^T$ by D^T .

Proof. The fact that $\tilde{\alpha} \circ \alpha$ and $\tilde{\beta} \circ \beta$ are multiplicative with respect to $\mu \circ T$ follows immediately from (7.1)–(7.4) and the fact that $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ are multiplicative with respect to μ . Now we prove the BiHom-associativity of $\mu \circ T$:

$$\begin{aligned} (\mu \circ T) \circ ((\mu \circ T) \otimes (\tilde{\beta} \circ \beta)) &= \mu \circ T \circ (\mu \otimes \tilde{\beta}) \circ (T \otimes \beta) \\ &\stackrel{(7.6)}{=} \mu \circ (\mu \otimes \tilde{\beta}) \circ \tilde{T}_2 \circ (\text{id}_D \otimes T) \circ (T \otimes \beta) \\ &\stackrel{(7.7)}{=} \mu \circ (\mu \otimes \tilde{\beta}) \circ \tilde{T}_1 \circ (T \otimes \text{id}_D) \circ (\alpha \otimes T) = \mu \circ (\tilde{\alpha} \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes \text{id}_D) \circ (\alpha \otimes T) \\ &\stackrel{(7.5)}{=} \mu \circ T \circ (\tilde{\alpha} \otimes \mu) \circ (\alpha \otimes T) = (\mu \circ T) \circ ((\tilde{\alpha} \circ \alpha) \otimes (\mu \circ T)), \end{aligned}$$

finishing the proof. ■

Obviously, if (D, μ) is an associative algebra and $\tilde{\alpha} = \tilde{\beta} = \alpha = \beta = \text{id}_D$, an (α, β) -BiHom-pseudotwistor reduces to a pseudotwistor (as defined in [21]) and the BiHom-associative algebra $D_{\alpha, \beta}^T$ is actually associative.

We show now that Proposition 3.8 is a particular case of Theorem 7.1.

Proposition 7.2. *Let $(D, \mu, \tilde{\alpha}, \tilde{\beta})$ be a BiHom-associative algebra and $\alpha, \beta: D \rightarrow D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Define the maps*

$$\begin{aligned} T: D \otimes D &\rightarrow D \otimes D, & T &= \alpha \otimes \beta, \\ \tilde{T}_1: D \otimes D \otimes D &\rightarrow D \otimes D \otimes D, & \tilde{T}_1 &= \text{id}_D \otimes \text{id}_D \otimes \beta, \\ \tilde{T}_2: D \otimes D \otimes D &\rightarrow D \otimes D \otimes D, & \tilde{T}_2 &= \alpha \otimes \text{id}_D \otimes \text{id}_D. \end{aligned}$$

Then T is an (α, β) -BiHom-pseudotwistor with companions \tilde{T}_1, \tilde{T}_2 and the BiHom-associative algebras $D_{\alpha, \beta}^T$ and $D_{(\alpha, \beta)}$ coincide.

Proof. The conditions (7.1)–(7.4) are obviously satisfied. We check (7.5), for $a, b, c \in D$:

$$\begin{aligned} ((\tilde{\alpha} \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes \text{id}_D))(a \otimes b \otimes c) &= ((\tilde{\alpha} \otimes \mu) \circ \tilde{T}_1)(\alpha(a) \otimes \beta(b) \otimes c) \\ &= (\tilde{\alpha} \otimes \mu)(\alpha(a) \otimes \beta(b) \otimes \beta(c)) = (\tilde{\alpha} \circ \alpha)(a) \otimes \beta(bc) \\ &= T(\tilde{\alpha}(a) \otimes bc) = (T \circ (\tilde{\alpha} \otimes \mu))(a \otimes b \otimes c). \end{aligned}$$

The condition (7.6) is similar, so we check (7.7):

$$\begin{aligned} (\tilde{T}_1 \circ (T \otimes \text{id}_D) \circ (\alpha \otimes T))(a \otimes b \otimes c) &= (\tilde{T}_1 \circ (T \otimes \text{id}_D))(\alpha(a) \otimes \alpha(b) \otimes \beta(c)) \\ &= \tilde{T}_1(\alpha^2(a) \otimes \beta\alpha(b) \otimes \beta(c)) = \alpha^2(a) \otimes \beta\alpha(b) \otimes \beta^2(c) = \tilde{T}_2(\alpha(a) \otimes \alpha\beta(b) \otimes \beta^2(c)) \\ &= (\tilde{T}_2 \circ (\text{id}_D \otimes T))(\alpha(a) \otimes \beta(b) \otimes \beta(c)) = (\tilde{T}_2 \circ (\text{id}_D \otimes T) \circ (T \otimes \beta))(a \otimes b \otimes c). \end{aligned}$$

It is obvious that $D_{\alpha, \beta}^T$ and $D_{(\alpha, \beta)}$ coincide. ■

Example 7.3. We consider the 2-dimensional BiHom-associative algebra $(D, \mu, \tilde{\alpha}, \tilde{\beta})$ defined with respect to a basis $\mathcal{B} = \{e_1, e_2\}$ by

$$\begin{aligned} \mu(e_1, e_1) &= \mu(e_1, e_2) = e_1, & \mu(e_2, e_1) &= \mu(e_2, e_2) = e_2, \\ \tilde{\alpha}(e_1) &= e_1, & \tilde{\alpha}(e_2) &= e_2, & \tilde{\beta}(e_1) &= e_1, & \tilde{\beta}(e_2) &= e_1. \end{aligned}$$

We have the following multiplicative linear maps α, β defined with respect to the basis \mathcal{B} by

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = ae_1 + (1-a)e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = be_1 + (1-b)e_2,$$

where a, b are parameters in \mathbb{k} . One can easily see that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. By the previous proposition, we can construct the BiHom-associative algebra $D_{(\alpha, \beta)} = (D, \mu_T = \mu \circ (\alpha \otimes \beta), \alpha_T = \tilde{\alpha} \circ \alpha, \beta_T = \tilde{\beta} \circ \beta)$ defined on the basis \mathcal{B} by

$$\begin{aligned} \mu_T(e_1, e_1) &= e_1, & \mu_T(e_1, e_2) &= e_1, & \mu_T(e_2, e_1) &= ae_1 + (1-a)e_2, \\ \mu_T(e_2, e_2) &= ae_1 + (1-a)e_2, & \alpha_T(e_1) &= e_1, & \alpha_T(e_2) &= ae_1 + (1-a)e_2, \\ \beta_T(e_1) &= e_1, & \beta_T(e_2) &= e_1. \end{aligned}$$

Definition 7.4 ([9, 28]). Let $(A, \mu_A), (B, \mu_B)$ be two associative algebras. A *twisting map* between A and B is a linear map $R: B \otimes A \rightarrow A \otimes B$ satisfying the conditions

$$\begin{aligned} R \circ (\text{id}_B \otimes \mu_A) &= (\mu_A \otimes \text{id}_B) \circ (\text{id}_A \otimes R) \circ (R \otimes \text{id}_A), \\ R \circ (\mu_B \otimes \text{id}_A) &= (\text{id}_A \otimes \mu_B) \circ (R \otimes \text{id}_B) \circ (\text{id}_B \otimes R). \end{aligned} \tag{7.8}$$

If this is the case, the map $\mu_R = (\mu_A \otimes \mu_B) \circ (\text{id}_A \otimes R \otimes \text{id}_B)$ is an associative product on $A \otimes B$; the associative algebra $(A \otimes B, \mu_R)$ is denoted by $A \otimes_R B$ and called the *twisted tensor product* of A and B afforded by R .

We introduce now twisted tensor products of BiHom-associative algebras.

Definition 7.5. Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ be two BiHom-associative algebras such that the maps $\alpha_A, \beta_A, \alpha_B, \beta_B$ are bijective. A linear map $R: B \otimes A \rightarrow A \otimes B$ is called a *BiHom-twisting map* between A and B if the following conditions are satisfied

$$(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A), \quad (7.9)$$

$$(\beta_A \otimes \beta_B) \circ R = R \circ (\beta_B \otimes \beta_A), \quad (7.10)$$

$$R \circ (\alpha_B \otimes \mu_A) = (\mu_A \otimes \beta_B) \circ (\text{id}_A \otimes R) \circ (\text{id}_A \otimes \alpha_B \beta_B^{-1} \otimes \text{id}_A) \circ (R \otimes \text{id}_A), \quad (7.11)$$

$$R \circ (\mu_B \otimes \beta_A) = (\alpha_A \otimes \mu_B) \circ (R \otimes \text{id}_B) \circ (\text{id}_B \otimes \alpha_A^{-1} \beta_A \otimes \text{id}_B) \circ (\text{id}_B \otimes R). \quad (7.12)$$

If we use the standard Sweedler-type notation $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$, for $a \in A, b \in B$, then the above conditions may be rewritten (for all $a, a' \in A$ and $b, b' \in B$) as follows

$$\alpha_A(a_R) \otimes \alpha_B(b_R) = \alpha_A(a)_R \otimes \alpha_B(b)_R, \quad (7.13)$$

$$\beta_A(a_R) \otimes \beta_B(b_R) = \beta_A(a)_R \otimes \beta_B(b)_R, \quad (7.14)$$

$$(aa')_R \otimes \alpha_B(b)_R = a_R a'_r \otimes \beta_B([\alpha_B \beta_B^{-1}(b_R)]_r), \quad (7.15)$$

$$\beta_A(a)_R \otimes (bb')_R = \alpha_A([\alpha_A^{-1} \beta_A(a_R)]_r) \otimes b_r b'_R. \quad (7.16)$$

Proposition 7.6. Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ be two BiHom-associative algebras with bijective structure maps, $R: B \otimes A \rightarrow A \otimes B$ a BiHom-twisting map. Define the linear map

$$T: (A \otimes B) \otimes (A \otimes B) \rightarrow (A \otimes B) \otimes (A \otimes B),$$

$$T((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_R) \otimes (a'_R \otimes b').$$

Then T is a BiHom-pseudotwistor for the tensor product $(A \otimes B, \mu_{A \otimes B}, \alpha_A \otimes \alpha_B, \beta_A \otimes \beta_B)$ of A and B , with companions

$$\tilde{T}_1 = (\text{id}_A \otimes \alpha_B^{-1} \beta_B \otimes \text{id}_A \otimes \text{id}_B \otimes \text{id}_A \otimes \text{id}_B) \circ T_{13}$$

$$\circ (\text{id}_A \otimes \alpha_B \beta_B^{-1} \otimes \text{id}_A \otimes \text{id}_B \otimes \text{id}_A \otimes \text{id}_B),$$

$$\tilde{T}_2 = (\text{id}_A \otimes \text{id}_B \otimes \text{id}_A \otimes \text{id}_B \otimes \alpha_A \beta_A^{-1} \otimes \text{id}_B) \circ T_{13}$$

$$\circ (\text{id}_A \otimes \text{id}_B \otimes \text{id}_A \otimes \text{id}_B \otimes \alpha_A^{-1} \beta_A \otimes \text{id}_B),$$

where we use the standard notation for T_{13} . The BiHom-associative algebra $(A \otimes B)^T$ is denoted by $A \otimes_R B$ and is called the *BiHom-twisted tensor product* of A and B ; its multiplication is defined by $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$, and the structure maps are $\alpha_A \otimes \alpha_B$ and $\beta_A \otimes \beta_B$.

Proof. We begin by proving the following relation, for all $a \in A, b \in B$:

$$\alpha_B^{-1} \beta_B([\alpha_B \beta_B^{-1}(b)]_R) \otimes a_R = b_R \otimes \alpha_A \beta_A^{-1}([\alpha_A^{-1} \beta_A(a)]_R). \quad (7.17)$$

This relation is equivalent to

$$\beta_B([\alpha_B \beta_B^{-1}(b)]_R) \otimes \beta_A(a_R) = \alpha_B(b_R) \otimes \alpha_A([\alpha_A^{-1} \beta_A(a)]_R),$$

which, by using (7.13) and (7.14), is equivalent to

$$\alpha_B(b)_R \otimes \beta_A(a)_R = \alpha_B(b)_R \otimes \beta_A(a)_R,$$

which is obviously true.

We need to prove the relations (7.1)–(7.7) (with $\tilde{\alpha} = \alpha_A \otimes \alpha_B$, $\tilde{\beta} = \beta_A \otimes \beta_B$, $\alpha = \beta = \text{id}_A \otimes \text{id}_B$). We will prove only (7.7), while (7.1)–(7.6) are very easy and left to the reader. We compute (r and \mathcal{R} are two more copies of R)

$$\begin{aligned} \tilde{T}_1 \circ (T \otimes \text{id}) \circ (\text{id} \otimes T)(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'') &= \tilde{T}_1(a \otimes b_r \otimes a'_r \otimes b'_R \otimes a''_R \otimes b'') \\ &= a \otimes \alpha_B^{-1} \beta_B([\alpha_B \beta_B^{-1}(b_r)]_{\mathcal{R}}) \otimes a'_r \otimes b'_R \otimes (a''_R)_{\mathcal{R}} \otimes b'', \\ \tilde{T}_2 \circ (\text{id} \otimes T) \circ (T \otimes \text{id})(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'') &= \tilde{T}_2(a \otimes b_r \otimes a'_r \otimes b'_R \otimes a''_R \otimes b'') \\ &= a \otimes (b_r)_{\mathcal{R}} \otimes a'_r \otimes b'_R \otimes \alpha_A \beta_A^{-1}([\alpha_A^{-1} \beta_A(a''_R)]_{\mathcal{R}}) \otimes b'', \end{aligned}$$

and the two terms are equal because of the relation (7.17). \blacksquare

Remark 7.7. Let $(A, \mu_A, \alpha_A, \beta_A)$ and $(B, \mu_B, \alpha_B, \beta_B)$ be two BiHom-associative algebras with bijective structure maps. Then obviously the linear map $R: B \otimes A \rightarrow A \otimes B$, $R(b \otimes a) = a \otimes b$, is a BiHom-twisting map and the BiHom-twisted tensor product $A \otimes_R B$ coincides with the ordinary tensor product $A \otimes B$.

Proposition 7.8. Let (A, μ_A) and (B, μ_B) be two associative algebras, $\alpha_A, \beta_A: A \rightarrow A$ two commuting algebra isomorphisms of A and $\alpha_B, \beta_B: B \rightarrow B$ two commuting algebra isomorphisms of B . Let $P: B \otimes A \rightarrow A \otimes B$ be a twisting map satisfying the conditions

$$(\alpha_A \otimes \alpha_B) \circ P = P \circ (\alpha_B \otimes \alpha_A), \quad (7.18)$$

$$(\beta_A \otimes \beta_B) \circ P = P \circ (\beta_B \otimes \beta_A). \quad (7.19)$$

Define the linear map

$$U: B \otimes A \rightarrow A \otimes B, \quad U(b \otimes a) = \beta_A^{-1}(\beta_A(a)_P) \otimes \alpha_B^{-1}(\alpha_B(b)_P).$$

Then U is a BiHom-twisting map between the BiHom-associative algebras $A_{(\alpha_A, \beta_A)}$ and $B_{(\alpha_B, \beta_B)}$ and the BiHom-associative algebras $A_{(\alpha_A, \beta_A)} \otimes_U B_{(\alpha_B, \beta_B)}$ and $(A \otimes_P B)_{(\alpha_A \otimes \alpha_B, \beta_A \otimes \beta_B)}$ coincide.

Proof. We only prove (7.15) for U and leave the rest to the reader. We compute (by denoting by p another copy of P and by u another copy of U)

$$\begin{aligned} (aa')_U \otimes \alpha_B(b)_U &= [\alpha_A(a)\beta_A(a')]_U \otimes \alpha_B(b)_U = \beta_A^{-1}([\beta_A \alpha_A(a)\beta_A^2(a')]_P) \otimes \alpha_B^{-1}(\alpha_B^2(b)_P) \\ &\stackrel{(7.8)}{=} \beta_A^{-1}(\beta_A \alpha_A(a)_P) \beta_A^{-1}(\beta_A^2(a')_P) \otimes \alpha_B^{-1}(\alpha_B^2(b)_P) \\ &= \beta_A^{-1}(\alpha_A(\beta_A(a)_P)) \beta_A^{-1}(\beta_A^2(a')_P) \otimes \alpha_B^{-1}([\alpha_B(\alpha_B(b)_P)]_P) \\ &\stackrel{(7.18)}{=} \beta_A^{-1} \alpha_A(\beta_A(a)_P) \beta_A^{-1}(\beta_A^2(a')_P) \otimes \alpha_B^{-1}(\alpha_B(\alpha_B(b)_P)_P), \\ aUa'_u \otimes \beta_B([\alpha_B \beta_B^{-1}(bU)]_u) &= \alpha_A(aU)\beta_A(a'_u) \otimes \beta_B([\alpha_B \beta_B^{-1}(bU)]_u) \\ &= \alpha_A \beta_A^{-1}(\beta_A(a)_P) \beta_A(a'_u) \otimes \beta_B([\beta_B^{-1}(\alpha_B(b)_P)]_u) \\ &= \alpha_A \beta_A^{-1}(\beta_A(a)_P) \beta_A(a')_P \otimes \alpha_B^{-1} \beta_B([\alpha_B \beta_B^{-1}(\alpha_B(b)_P)]_P) \\ &= \alpha_A \beta_A^{-1}(\beta_A(a)_P) \beta_A(a')_P \otimes \alpha_B^{-1} \beta_B(\beta_B^{-1}(\alpha_B(\alpha_B(b)_P))_P) \\ &= \alpha_A \beta_A^{-1}(\beta_A(a)_P) \beta_A^{-1}(\beta_A^2(a')_P) \otimes \alpha_B^{-1} \beta_B(\beta_B^{-1}(\alpha_B(\alpha_B(b)_P))_P) \\ &\stackrel{(7.19)}{=} \alpha_A \beta_A^{-1}(\beta_A(a)_P) \beta_A^{-1}(\beta_A^2(a')_P) \otimes \alpha_B^{-1}(\alpha_B(\alpha_B(b)_P)_P), \end{aligned}$$

finishing the proof. \blacksquare

8 BiHom-smash products

We construct first a large family of BiHom-twisting maps.

Theorem 8.1. *Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra, $(A, \mu_A, \alpha_A, \beta_A)$ a left H -module BiHom-algebra, with action denoted by $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \cdot a$, and assume that all structure maps $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ are bijective. Let $m, n, p \in \mathbb{Z}$. Define the linear map*

$$R_{m,n,p}: H \otimes A \rightarrow A \otimes H, \quad R_{m,n,p}(h \otimes a) = \alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(a) \otimes \psi_H^{-1}(h_2).$$

Then $R_{m,n,p}$ is a BiHom-twisting map between A and H .

Proof. The relations (7.9) and (7.10) are very easy to prove and left to the reader.

Proof of (7.11):

$$\begin{aligned} & (\mu_A \otimes \beta_H) \circ (\text{id}_A \otimes R_{m,n,p}) \circ (\text{id}_A \otimes \alpha_H \beta_H^{-1} \otimes \text{id}_A) \circ (R_{m,n,p} \otimes \text{id}_A)(h \otimes a \otimes a') \\ &= (\mu_A \otimes \beta_H) \circ (\text{id}_A \otimes R_{m,n,p})(\alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(a) \otimes \alpha_H \beta_H^{-1} \psi_H^{-1}(h_2) \otimes a') \\ &= (\mu_A \otimes \beta_H)(\alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(a) \otimes \alpha_H^m \beta_H^n \omega_H^p([\alpha_H \beta_H^{-1} \psi_H^{-1}(h_2)]_1) \cdot \beta_A^{-1}(a') \\ &\quad \otimes \psi_H^{-1}([\alpha_H \beta_H^{-1} \psi_H^{-1}(h_2)]_2)) \\ &= (\mu_A \otimes \beta_H)(\alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(a) \otimes \alpha_H^{m+1} \beta_H^{n-1} \omega_H^p \psi_H^{-1}((h_2)_1) \cdot \beta_A^{-1}(a') \\ &\quad \otimes \alpha_H \beta_H^{-1} \psi_H^{-2}((h_2)_2)) \\ &= [\alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(a)] [\alpha_H^{m+1} \beta_H^{n-1} \psi_H^{-1} \omega_H^p((h_2)_1) \cdot \beta_A^{-1}(a')] \otimes \alpha_H \psi_H^{-2}((h_2)_2) \\ &\stackrel{(5.2)}{=} [\alpha_H^m \beta_H^n \omega_H^{p-1}((h_1)_1) \cdot \beta_A^{-1}(a)] [\alpha_H^{m+1} \beta_H^{n-1} \psi_H^{-1} \omega_H^p((h_1)_2) \cdot \beta_A^{-1}(a')] \otimes \alpha_H \psi_H^{-1}(h_2) \\ &= [\alpha_H^{-1} \omega_H^{-1}(\alpha_H^{m+1} \beta_H^n \omega_H^p((h_1)_1)) \cdot \beta_A^{-1}(a)] [\beta_H^{-1} \psi_H^{-1}(\alpha_H^{m+1} \beta_H^n \omega_H^p((h_1)_2)) \cdot \beta_A^{-1}(a')] \\ &\quad \otimes \alpha_H \psi_H^{-1}(h_2) \\ &= \{ \alpha_H^{-1} \omega_H^{-1}([\alpha_H^{m+1} \beta_H^n \omega_H^p(h_1)]_1) \cdot \beta_A^{-1}(a) \} \{ \beta_H^{-1} \psi_H^{-1}([\alpha_H^{m+1} \beta_H^n \omega_H^p(h_1)]_2) \cdot \beta_A^{-1}(a') \} \\ &\quad \otimes \alpha_H \psi_H^{-1}(h_2) \\ &\stackrel{(5.3)}{=} \alpha_H^{m+1} \beta_H^n \omega_H^p(h_1) \cdot \beta_A^{-1}(aa') \otimes \alpha_H \psi_H^{-1}(h_2) = (R_{m,n,p} \circ (\alpha_H \otimes \mu_A))(h \otimes a \otimes a'). \end{aligned}$$

Proof of (7.12):

$$\begin{aligned} & (\alpha_A \otimes \mu_H) \circ (R_{m,n,p} \otimes \text{id}_H) \circ (\text{id}_H \otimes \alpha_A^{-1} \beta_A \otimes \text{id}_H) \circ (\text{id}_H \otimes R_{m,n,p})(h \otimes h' \otimes a) \\ &= (\alpha_A \otimes \mu_H) \circ (R_{m,n,p} \otimes \text{id}_H)(h \otimes \alpha_A^{-1} \beta_A (\alpha_H^m \beta_H^n \omega_H^p(h'_1) \cdot \beta_A^{-1}(a)) \otimes \psi_H^{-1}(h'_2)) \\ &= (\alpha_A \otimes \mu_H) \circ (R_{m,n,p} \otimes \text{id}_H)(h \otimes \alpha_H^{m-1} \beta_H^{n+1} \omega_H^p(h'_1) \cdot \alpha_A^{-1}(a) \otimes \psi_H^{-1}(h'_2)) \\ &= (\alpha_A \otimes \mu_H)(\alpha_H^m \beta_H^n \omega_H^p(h_1) \cdot (\alpha_H^{m-1} \beta_H^{n+1} \omega_H^p(h'_1) \cdot \alpha_A^{-1}(a)) \otimes \psi_H^{-1}(h_2) \otimes \psi_H^{-1}(h'_2)) \\ &= \alpha_H^{m+1} \beta_H^n \omega_H^p(h_1) \cdot (\alpha_H^m \beta_H^n \omega_H^p(h'_1) \cdot \beta_A^{-1}(a)) \otimes \psi_H^{-1}(h_2 h'_2) \\ &\stackrel{(4.4)}{=} \{ [\alpha_H^m \beta_H^n \omega_H^p(h_1)] [\alpha_H^m \beta_H^n \omega_H^p(h'_1)] \} \cdot a \otimes \psi_H^{-1}(h_2 h'_2) \\ &= \alpha_H^m \beta_H^n \omega_H^p(h_1 h'_1) \cdot a \otimes \psi_H^{-1}(h_2 h'_2) \\ &\stackrel{(5.1)}{=} \alpha_H^m \beta_H^n \omega_H^p((hh')_1) \cdot a \otimes \psi_H^{-1}((hh')_2) = (R_{m,n,p} \circ (\mu_H \otimes \beta_A))(h \otimes h' \otimes a), \end{aligned}$$

finishing the proof. ■

Definition 8.2. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra and $(A, \mu_A, \alpha_A, \beta_A)$ a left H -module BiHom-algebra, with left H -module structure $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \cdot a$, such

that all structure maps $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ are bijective. Consider the BiHom-twisting map

$$R = R_{0,-1,-1}: H \otimes A \rightarrow A \otimes H, \quad R(h \otimes a) = \beta_H^{-1} \omega_H^{-1}(h_1) \cdot \beta_A^{-1}(a) \otimes \psi_H^{-1}(h_2). \quad (8.1)$$

We denote the BiHom-associative algebra $A \otimes_R H$ by $A \# H$ (we denote $a \otimes h := a \# h$, for $a \in A, h \in H$) and call it the *BiHom-smash product* of A and H . Its structure maps are $\alpha_A \otimes \alpha_H$ and $\beta_A \otimes \beta_H$, and its multiplication is

$$(a \# h)(a' \# h') = a(\beta_H^{-1} \omega_H^{-1}(h_1) \cdot \beta_A^{-1}(a')) \# \psi_H^{-1}(h_2) h'.$$

Remark 8.3. If H is a Hom-bialgebra, i.e., $\alpha_H = \beta_H = \psi_H = \omega_H$, and A is a Hom-associative algebra, the multiplication of $A \# H$ becomes

$$(a \# h)(a' \# h') = a(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a')) \# \alpha_H^{-1}(h_2) h',$$

which is the formula introduced in [23]. If H is a monoidal Hom-bialgebra, i.e., $\psi_H = \omega_H = \alpha_H^{-1} = \beta_H^{-1}$, and A is a Hom-associative algebra, the multiplication of $A \# H$ becomes

$$(a \# h)(a' \# h') = a(h_1 \cdot \alpha_A^{-1}(a')) \# \alpha_H(h_2) h',$$

which is the formula introduced in [11], used also in [20] for defining the Radford biproduct for monoidal Hom-bialgebras.

Proposition 8.4. *In the same setting as in Proposition 5.16, and assuming moreover that the maps α_A and β_A are bijective, if we denote by $A \# H$ the usual smash product between A and H , then $\alpha_A \otimes \alpha_H$ and $\beta_A \otimes \beta_H$ are commuting algebra endomorphisms of $A \# H$ and the BiHom-associative algebras $(A \# H)_{(\alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)}$ and $A_{(\alpha_A, \beta_A)} \# H_{(\alpha_H, \beta_H, \psi_H, \omega_H)}$ coincide.*

Proof. We will apply Proposition 7.8. In our situation, we have the twisting map $P: H \otimes A \rightarrow A \otimes H$, $P(h \otimes a) = h_1 \cdot a \otimes h_2$, for which $A \# H = A \otimes_P H$. Obviously P satisfies the conditions (7.18) and (7.19), so, by Proposition 7.8, we obtain the map

$$U: H \otimes A \rightarrow A \otimes H, \quad U(h \otimes a) = \beta_A^{-1}(\beta_A(a)_P) \otimes \alpha_H^{-1}(\alpha_H(h)_P),$$

which is a BiHom-twisting map between $A_{(\alpha_A, \beta_A)}$ and $H_{(\alpha_H, \beta_H)}$ and we have

$$(A \# H)_{(\alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)} = A_{(\alpha_A, \beta_A)} \otimes_U H_{(\alpha_H, \beta_H)}.$$

Thus, the proof will be finished if we prove that the map U coincides with the map R affording the BiHom-smash product $A_{(\alpha_A, \beta_A)} \# H_{(\alpha_H, \beta_H, \psi_H, \omega_H)}$. We compute

$$\begin{aligned} U(h \otimes a) &= \beta_A^{-1}(\alpha_H(h)_1 \cdot \beta_A(a)) \otimes \alpha_H^{-1}(\alpha_H(h)_2) \\ &= \beta_A^{-1}(\alpha_H(h_1) \cdot \beta_A(a)) \otimes \alpha_H^{-1}(\alpha_H(h_2)) = \alpha_H \beta_H^{-1}(h_1) \cdot a \otimes h_2, \\ R(h \otimes a) &= \beta_H^{-1} \omega_H^{-1}(\omega_H(h_1)) \triangleright \beta_A^{-1}(a) \otimes \psi_H^{-1}(\psi_H(h_2)) \\ &= \beta_H^{-1}(h_1) \triangleright \beta_A^{-1}(a) \otimes h_2 = \alpha_H \beta_H^{-1}(h_1) \cdot a \otimes h_2, \end{aligned}$$

finishing the proof. ■

Example 8.5. We construct a class of examples of $U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)}$ -module BiHom-algebra structures on $\mathbb{A}_{q, \alpha, \beta}^{2|0}$, generalizing examples of $U_q(\mathfrak{sl}_2)_\alpha$ -module Hom-algebra structures on $\mathbb{A}_{q, \gamma}^{2|0}$ given in [32, Example 5.7] (here we take the base field $\mathbb{k} = \mathbb{C}$). The quantum group $U_q(\mathfrak{sl}_2)$ is generated as a unital associative algebra by 4 generators $\{E, F, K, K^{-1}\}$ with relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

where $q \in \mathbb{C}$ with $q \neq 0$, $q \neq \pm 1$. The comultiplication is defined by

$$\begin{aligned}\Delta(E) &= 1 \otimes E + E \otimes K, & \Delta(F) &= K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}.\end{aligned}$$

We fix $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$ some nonzero elements. The BiHom-bialgebra $U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)} = (U_q(\mathfrak{sl}_2), \mu_{(\alpha, \beta)}, \Delta_{(\psi, \omega)}, \alpha, \beta, \psi, \omega)$ is defined (as in Proposition 5.9(iii)) by $\mu_{(\alpha, \beta)} = \mu \circ (\alpha \otimes \beta)$ and $\Delta_{(\psi, \omega)} = (\omega \otimes \psi) \circ \Delta$, where μ and Δ are respectively the multiplication and comultiplication of $U_q(\mathfrak{sl}_2)$ and $\alpha, \beta, \psi, \omega: U_q(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2)$ are bialgebra morphisms such that

$$\begin{aligned}\alpha(E) &= \lambda_1 E, & \alpha(F) &= \lambda_1^{-1} F, & \alpha(K) &= K, & \alpha(K^{-1}) &= K^{-1}, \\ \beta(E) &= \lambda_2 E, & \beta(F) &= \lambda_2^{-1} F, & \beta(K) &= K, & \beta(K^{-1}) &= K^{-1}, \\ \psi(E) &= \lambda_3 E, & \psi(F) &= \lambda_3^{-1} F, & \psi(K) &= K, & \psi(K^{-1}) &= K^{-1}, \\ \omega(E) &= \lambda_4 E, & \omega(F) &= \lambda_4^{-1} F, & \omega(K) &= K, & \omega(K^{-1}) &= K^{-1}.\end{aligned}$$

Note that any two of the maps $\alpha, \beta, \psi, \omega$ commute.

Let $\mathbb{A}_q^{2|0} = k\langle x, y \rangle / (yx - qxy)$ be the quantum plane. We fix also some $\xi \in \mathbb{C}$, $\xi \neq 0$. The BiHom-quantum plane $\mathbb{A}_{q, \alpha, \beta}^{2|0} = (\mathbb{A}_q^{2|0}, \mu_{\mathbb{A}, \alpha_{\mathbb{A}}, \beta_{\mathbb{A}}}, \alpha_{\mathbb{A}}, \beta_{\mathbb{A}})$ is the BiHom-associative algebra defined (as in Proposition 5.9(i)) by $\mu_{\mathbb{A}, \alpha_{\mathbb{A}}, \beta_{\mathbb{A}}} = \mu_{\mathbb{A}} \circ (\alpha_{\mathbb{A}} \otimes \beta_{\mathbb{A}})$, where $\mu_{\mathbb{A}}$ is the multiplication of $\mathbb{A}_q^{2|0}$ and $\alpha_{\mathbb{A}}, \beta_{\mathbb{A}}: \mathbb{A}_q^{2|0} \rightarrow \mathbb{A}_q^{2|0}$ are the (commuting) algebra morphisms such that

$$\alpha_{\mathbb{A}}(x) = \xi x, \quad \alpha_{\mathbb{A}}(y) = \xi \lambda_1^{-1} y \quad \text{and} \quad \beta_{\mathbb{A}}(x) = \xi x, \quad \beta_{\mathbb{A}}(y) = \xi \lambda_2^{-1} y.$$

We consider $\mathbb{A}_q^{2|0}$ as a left $U_q(\mathfrak{sl}_2)$ -module algebra as in [32, Example 5.7] (we denote by $h \otimes a \mapsto h \cdot a$ the $U_q(\mathfrak{sl}_2)$ -action on $\mathbb{A}_q^{2|0}$). By the computations performed in [32, Example 5.7] we know that $\alpha_{\mathbb{A}}(h \cdot a) = \alpha(h) \cdot \alpha_{\mathbb{A}}(a)$ and $\beta_{\mathbb{A}}(h \cdot a) = \beta(h) \cdot \beta_{\mathbb{A}}(a)$, for all $h \in U_q(\mathfrak{sl}_2)$ and $a \in \mathbb{A}_q^{2|0}$. Then, according to Proposition 5.16, there exists a $U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)}$ -module BiHom-algebra structure on $\mathbb{A}_{q, \alpha, \beta}^{2|0}$ defined by

$$\rho: U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)} \otimes \mathbb{A}_{q, \alpha, \beta}^{2|0} \rightarrow \mathbb{A}_{q, \alpha, \beta}^{2|0}, \quad \rho(h \otimes a) = h \triangleright a = \alpha(h) \cdot \beta_{\mathbb{A}}(a).$$

By using also the computations performed in [32, Example 5.7] one can see that the map ρ is given on generators by

$$\begin{aligned}\rho(E \otimes x^m y^n) &= [n]_q \xi^{m+n} \lambda_1 \lambda_2^{-n} x^{m+1} y^{n-1}, \\ \rho(F \otimes x^m y^n) &= [m]_q \xi^{m+n} \lambda_1^{-1} \lambda_2^{-n} x^{m-1} y^{n+1}, \\ \rho(K^{\pm 1} \otimes P) &= P(q^{\pm 1} \xi x, q^{\mp 1} \xi \lambda_2^{-1} y),\end{aligned}$$

for any monomial $x^m y^n \in \mathbb{A}_q^{2|0}$, where $P = P(x, y) \in \mathbb{A}_q^{2|0}$ and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Since $\xi \neq 0$ and $\lambda_i \neq 0$ for all $i = 1, 2, 3, 4$, all the maps $\alpha, \beta, \psi, \omega, \alpha_{\mathbb{A}}, \beta_{\mathbb{A}}$ are bijective. According to Theorem 8.1, the map $R: U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)} \otimes \mathbb{A}_{q, \alpha, \beta}^{2|0} \rightarrow \mathbb{A}_{q, \alpha, \beta}^{2|0} \otimes U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)}$ defined by (8.1) leads to the smash product $\mathbb{A}_{q, \alpha, \beta}^{2|0} \# U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)}$ whose multiplication is defined by

$$(a \# h)(a' \# h') = a * (\beta^{-1} \omega^{-1}(h_{(1)}) \triangleright \beta_{\mathbb{A}}^{-1}(a')) \# \psi^{-1}(h_{(2)}) \bullet h',$$

where $h_{(1)} \otimes h_{(2)} = \Delta_{(\psi, \omega)}(h)$ and $*$ (respectively \bullet) is the multiplication of $\mathbb{A}_{q, \alpha, \beta}^{2|0}$ (respectively $U_q(\mathfrak{sl}_2)_{(\alpha, \beta, \psi, \omega)}$).

In particular, for any $G \in U_q(\mathfrak{sl}_2)$ and $m, n, r, s \in \mathbb{N}$ we have

$$\begin{aligned} (x^m y^n \# K^{\pm 1})(x^r y^s \# G) &= q^{\pm r \mp s + nr} \xi^{m+n+r+s} \lambda_1^{-n} \lambda_2^{-s} x^{m+r} y^{n+s} \# K^{\pm 1} \beta(G), \\ (x^m y^n \# E)(x^r y^s \# G) &= q^{nr} \xi^{m+n+r+s} \lambda_1^{-n+1} \lambda_2^{-s} x^{m+r} y^{n+s} \# E \beta(G) \\ &\quad + [s]_q q^{n(r+1)} \xi^{m+n+r+s} \lambda_1^{1-n} \lambda_2^{-s} x^{m+r+1} y^{n+s-1} \# K \beta(G), \\ (x^m y^n \# F)(x^r y^s \# G) &= q^{s-r+nr} \xi^{m+n+r+s} \lambda_1^{-n-1} \lambda_2^{-s} x^{m+r} y^{n+s} \# F \beta(G) \\ &\quad + [r]_q q^{n(r-1)} \xi^{m+n+r+s} \lambda_1^{-n-1} \lambda_2^{-s} x^{m+r-1} y^{n+s+1} \# \beta(G), \end{aligned}$$

where $K^{\pm 1} \beta(G)$, $E \beta(G)$ and $F \beta(G)$ are multiplications in $U_q(\mathfrak{sl}_2)$.

We introduce now the BiHom analogue of comodule Hom-algebras defined in [30].

Definition 8.6. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra. A *right H -comodule BiHom-algebra* is a 7-tuple $(D, \mu_D, \alpha_D, \beta_D, \psi_D, \omega_D, \rho_D)$, where $(D, \mu_D, \alpha_D, \beta_D)$ is a BiHom-associative algebra, (D, ψ_D, ω_D) is a right H -comodule via the coaction $\rho_D: D \rightarrow D \otimes H$ and moreover ρ_D is a morphism of BiHom-associative algebras.

Example 8.7. If $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ is a BiHom-bialgebra, then we have the right H -comodule BiHom-algebra $(H, \mu_H, \alpha_H, \beta_H, \psi_H, \omega_H, \Delta_H)$.

The next result generalizes Proposition 3.6 in [23].

Proposition 8.8. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra and $(A, \mu_A, \alpha_A, \beta_A)$ a left H -module BiHom-algebra, with notation $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \cdot a$, such that all structure maps $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ are bijective. Assume that there exist two more linear maps $\psi_A, \omega_A: A \rightarrow A$ such that any two of the maps $\alpha_A, \beta_A, \psi_A, \omega_A$ commute and moreover

$$\begin{aligned} \omega_A(aa') &= \omega_A(a)\omega_A(a'), \quad \forall a, a' \in A, \\ \omega_A(h \cdot a) &= \omega_H(h) \cdot \omega_A(a), \quad \forall a \in A, \quad h \in H. \end{aligned} \tag{8.2}$$

Define the linear map

$$\rho_{A\#H}: A\#H \rightarrow (A\#H) \otimes H, \quad \rho_{A\#H}(a\#h) = (\omega_A(a)\#h_1) \otimes h_2.$$

Then $(A\#H, \mu_{A\#H}, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H, \psi_A \otimes \psi_H, \omega_A \otimes \omega_H, \rho_{A\#H})$ is a right H -comodule BiHom-algebra.

Proof. We only prove that $\rho_{A\#H}$ is multiplicative and leave the other details to the reader:

$$\begin{aligned} \rho_{A\#H}((a\#h)(a'\#h')) &= \omega_A(a(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(a'))) \# (\psi_H^{-1}(h_2)h')_1 \otimes (\psi_H^{-1}(h_2)h')_2 \\ &= \omega_A(a)\omega_A(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(a')) \# \psi_H^{-1}((h_2)_1)h'_1 \otimes \psi_H^{-1}((h_2)_2)h'_2 \\ &\stackrel{(8.2)}{=} \omega_A(a)(\beta_H^{-1}(h_1) \cdot \omega_A\beta_A^{-1}(a')) \# \psi_H^{-1}((h_2)_1)h'_1 \otimes \psi_H^{-1}((h_2)_2)h'_2 \\ &\stackrel{(5.2)}{=} \omega_A(a)(\beta_H^{-1}\omega_H^{-1}((h_1)_1) \cdot \omega_A\beta_A^{-1}(a')) \# \psi_H^{-1}((h_1)_2)h'_1 \otimes h_2h'_2 \\ &= \omega_A(a)(\beta_H^{-1}\omega_H^{-1}((h_1)_1) \cdot \beta_A^{-1}\omega_A(a')) \# \psi_H^{-1}((h_1)_2)h'_1 \otimes h_2h'_2 \\ &= (\omega_A(a)\#h_1)(\omega_A(a')\#h'_1) \otimes h_2h'_2 = \rho_{A\#H}(a\#h)\rho_{A\#H}(a'\#h'), \end{aligned}$$

finishing the proof. ■

Example 8.9. Let $(H, \mu_H, \Delta_H, \alpha_H, \beta_H, \psi_H, \omega_H)$ be a BiHom-bialgebra such that all structure maps are bijective. Denote by A the linear space H^* . Then A becomes a BiHom-associative algebra with multiplication and structure maps defined by

$$\begin{aligned}(f \bullet g)(h) &= f(\alpha_H^{-1} \omega_H^{-1}(h_1))g(\beta_H^{-1} \psi_H^{-1}(h_2)), \\ \alpha_A: H^* &\rightarrow H^*, \quad \alpha_A(f)(h) = f(\alpha_H^{-1}(h)), \\ \beta_A: H^* &\rightarrow H^*, \quad \beta_A(f)(h) = f(\beta_H^{-1}(h)),\end{aligned}$$

for all $f, g \in H^*$ and $h \in H$. Moreover, A becomes a left H -module BiHom-algebra, with action

$$\dashv: H \otimes H^* \rightarrow H^*, \quad (h \dashv f)(h') = f(\alpha_H^{-1} \beta_H^{-1}(h')h),$$

for all $h, h' \in H$ and $f \in H^*$. Obviously, α_A and β_A are bijective maps. Define the linear map

$$\omega_A: H^* \rightarrow H^*, \quad \omega_A(f)(h) = f(\omega_H^{-1}(h)), \quad \forall f \in H^*, \quad h \in H,$$

and choose a linear map $\psi_A: H^* \rightarrow H^*$ that commutes with $\alpha_A, \beta_A, \omega_A$, for instance one can choose the map ψ_A defined by $\psi_A(f)(h) = f(\psi_H^{-1}(h))$, for all $f \in H^*$ and $h \in H$. Then one can check that the hypotheses of Proposition 8.8 are satisfied, and consequently $H^* \# H$ becomes a right H -comodule BiHom-algebra.

Note also that, if H is counital with counit ε_H such that $\varepsilon_H \circ \alpha_H = \varepsilon_H$ and $\varepsilon_H \circ \beta_H = \varepsilon_H$, then the BiHom-associative algebra $A = H^*$ is unital with unit ε_H .

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