

# The Use of Lagrange Multiplier Formalism to Solve the Problem of Transverse Vibrations of the Lower-Limb Exoskeleton

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In the paper, the Lagrange multiplier formalism has been used to find a solution to the problem of free transverse vibrations of the lower-limb exoskeleton. Parts of the exoskeleton were replaced with continuous (beams) and discrete elements. The beams have been circumscribed according to the Bernoulli–Euler theory. The physical model takes into account properties that characterize the interaction between exoskeleton elements, contact with the ground and system load. On the basis of the presented mathematical model, an algorithm and a computational program were created. It permitted the research on the impact of selected model parameters on the transverse vibrations of the analyzed system. The presented model, coupled with the longitudinal vibration model, may constitute a complete description of the vibrations of the lower-limb exoskeleton.

topics: discrete-continuous system, exoskeleton, free vibration, Lagrange multiplier formalism

## 1. Introduction

Mechanical vibrations are usually an undesirable phenomenon. They are often considered in the case of rotating elements (such as shafts or rotors), however, they just as frequently occur in structural elements (cases, support beams). Vibration analysis is extremely important when mechanisms cooperate with the human body because of two major reasons. In the event of resonance due to vibrations, the mechanism may become unstable and may be damaged. It may cause permanent damage to the health of the user [1, 2]. Many standards related to health and safety at work specify the maximum allowable vibration values at workplaces [2, 3]. Exoskeletons must take both areas into account in order to minimize the risk of damage to the mechanism and maximize the comfort of using such mechanisms.

Exoskeletons must take into account both of these areas in order to minimize the risk of damage to the mechanism and maximize the comfort of using such mechanisms. Therefore, the article analyzes the transverse vibrations of the lower limb exoskele-

ton in the bending plane of the knee and in the plane perpendicular to it, and is a continuation of the research from the paper [4], in which the longitudinal vibrations of the above-mentioned device were analyzed. Both of these works can be a coupled model of lower limb exoskeleton vibrations. The method of Lagrange's multipliers formalism [5–7] was used to formulate and solve the problem, as before. The real object is represented by a discrete-continuous model that takes into account the transferred mass and the influence of the ground/foot parameters on the structure vibrations.

## 2. Discrete-continuous model of the analyzed system

The analyzed discrete-continuous model of the exoskeleton's lower limb and its mechanical implementation are shown in Fig. 1.

In the computational model, the main members of the exoskeleton are replaced by Bernoulli–Euler beams of lengths  $L_1$  (calf) and  $L_2$  (thigh). On the

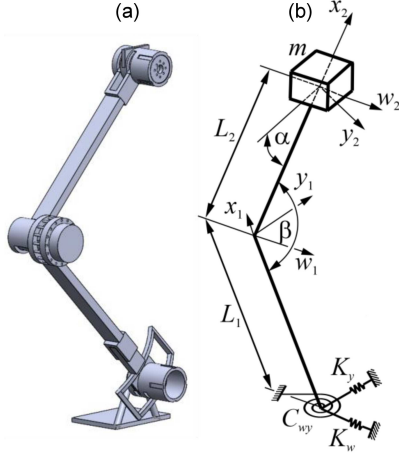


Fig. 1. The exoskeleton lower limb in (a) the 3D model and (b) the discrete-continuous model.

top of the model, a mass  $m$  is joined, which represents the structural components, batteries, controllers of drives, and the on-board computer, and takes into account additional elements such as the carried load. In the place where the  $L_1$  member is

attached to the foot, translational springs with the constants  $K_y$  and  $K_w$  as well as a rotary spring with the constant  $C_{wy}$  are fastened. Such a system of springs replaces the actual conditions, and the constants of these springs may take into account the foot and soil influence on the tested system.

It should also be noted that the transverse vibrations will be considered independently in two planes, i.e., the plane of leg bending in the knee ( $xy$ ) and the plane perpendicular to it ( $xw$ ).

During the duty cycle (the same at work [4]) the values of the angles  $\alpha$  and  $\beta$  are changed. These angles are directly bound up with the angular displacement in the joints and are included in the control system of the device.

According to the Lagrange multiplier formalism [5], the solution of the free vibration problem of the above system can be reduced to the following set of equations in the matrix form

$$C \bar{A} = 0, \quad (1)$$

where the amplitudes  $\bar{A}_n$  ( $n = 1, 2, 3, 4, 5$ ) of Lagrange multipliers create the vector

$$\bar{A} = [\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5]^T \quad (2)$$

and the square matrix  $C$  has the form

$$C = \begin{bmatrix} C_{111} + \varepsilon_1 & C_{112} & C_{113} \cos(\gamma) & C_{114} \cos(\gamma) & 0 \\ C_{121} & C_{122} + \varepsilon_2 & C_{123} \cos(\gamma) & C_{124} \cos(\gamma) & 0 \\ C_{131} \cos(\gamma) & C_{132} \cos(\gamma) & C_{133} \cos^2(\gamma) + C_{233} \cos^2(\alpha) & C_{134} \cos^2(\gamma) + C_{234} \cos^2(\alpha) & C_{235} \cos(\alpha) \\ C_{141} \cos(\gamma) & C_{142} \cos(\gamma) & C_{143} \cos^2(\gamma) + C_{243} \cos^2(\alpha) & C_{144} \cos^2(\gamma) + C_{244} \cos^2(\alpha) & C_{245} \cos(\alpha) \\ 0 & 0 & C_{253} \cos(\alpha) & C_{254} \cos(\alpha) & C_{255} + \varepsilon_3 \end{bmatrix}, \quad (3)$$

where  $\gamma = \beta - \alpha$ .

The following denotations have been introduced concerning the matrices and the quantities in (3), respectively,

$$C_{nkr} = \sum_{i=0}^{N_n} \frac{b_{nik} b_{nir}}{K_{n_i} - \omega^2 M_{n_i}} \quad (4)$$

and

$$\begin{aligned} \varepsilon_1^{(y)} &= 1/K_y \quad \text{or} \quad \varepsilon_1^{(w)} = 1/K_w, \\ \varepsilon_2 &= 1/C_{wy}, \quad \text{and} \quad \varepsilon_3 = -1/(m\omega^2). \end{aligned} \quad (5)$$

The coefficients  $C_{nkr}$  (4) characterize dynamic properties of the beams representing the main members of the exoskeleton.

At the same time (in the  $xy$  plane),

$$\begin{aligned} b_{1_{ir}} &= Y_{1_i}(x_{1,r}), \quad \text{for} \quad r = 1, 3, \quad i = 0, 1, \dots, N_1, \\ b_{1_{i2}} &= Y'_{1_i}(x_{1,r}), \quad \text{for} \quad r = 2, 4, \quad i = 0, 1, \dots, N_1, \\ b_{2_{ir}} &= Y_{2_i}(x_{2,r}), \quad \text{for} \quad r = 3, 5, \quad i = 0, 1, \dots, N_2, \\ b_{2_{i4}} &= Y'_{2_i}(x_{2,4}), \quad \text{for} \quad i = 0, 1, \dots, N_2, \end{aligned} \quad (6)$$

or (in the  $xw$  plane):

$$\begin{aligned} b_{1_{ir}} &= W_{1_i}(x_{1,r}), \quad \text{for} \quad r = 1, 3, \quad i = 0, 1, \dots, N_1, \\ b_{1_{i2}} &= W'_{1_i}(x_{1,r}), \quad \text{for} \quad r = 2, 4, \quad i = 0, 1, \dots, N_1, \\ b_{2_{ir}} &= W_{2_i}(x_{2,r}), \quad \text{for} \quad r = 3, 5, \quad i = 0, 1, \dots, N_2, \\ b_{2_{i4}} &= W'_{2_i}(x_{2,4}), \quad \text{for} \quad i = 0, 1, \dots, N_2 \end{aligned} \quad (7)$$

represent the mode shapes of the free-free beams calculated for each element without any influence from the other elements. The coefficients  $x_{n,r}$  in (6) and (7) describe the place where the discrete elements are joined to the beams, and so

$$\begin{aligned} x_{1,1} = x_{1,2} &= 0, \quad x_{1,3} = x_{1,4} = L_1, \\ x_{2,3} = x_{2,4} &= 0, \quad x_{2,5} = L_2. \end{aligned} \quad (8)$$

The quantities (5) define the influence of discrete elements on the system vibration and represent the elasticity of the foot and soil ( $\varepsilon_1, \varepsilon_2$ ) and the system load ( $\varepsilon_3$ ).

The equation set (1) yields the eigenvalue problem

$$\det [C] = 0, \quad (9)$$

TABLE I

Transverse vibration frequencies of the discrete-continuous model of the exoskeleton's lower limb in the  $xw$  plane (structure made of AISI 1020).

	$\alpha = 90^\circ, \beta = 180^\circ$			$\alpha = 45^\circ, \beta = 90^\circ$		
$m$ [kg]	50	60	70	50	60	70
$K_w = K_y = C_{wy} = 1 \times 10^8$ N/m						
$\omega_1$ [Hz]	1.09	0.98	0.89	1.95	1.77	1.63
$\omega_2$ [Hz]	48.32	48.16	48.03	21.41	21.31	21.22
$\omega_3$ [Hz]	181.2	180.6	180.1	218.4	215.4	213.2
$K_w = K_y = C_{wy} = 1 \times 10^{15}$ N/m						
$\omega_1$ [Hz]	2.48	2.26	2.09	3.48	3.17	2.93
$\omega_2$ [Hz]	62.37	62.20	62.06	40.29	40.11	39.94
$\omega_3$ [Hz]	199.2	198.5	197.7	234.4	232.3	230.6

TABLE II

Transverse vibration frequencies of the discrete-continuous model of the exoskeleton's lower limb in the  $xy$  plane (structure made of AISI 1020).

	$\alpha = 90^\circ, \beta = 180^\circ$			$\alpha = 45^\circ, \beta = 90^\circ$		
$m$ [kg]	50	60	70	50	60	70
$K_w = K_y = C_{wy} = 1 \times 10^8$ N/m						
$\omega_1$ [Hz]	2.71	2.02	1.86	3.16	2.89	2.67
$\omega_2$ [Hz]	95.21	94.81	94.46	17.65	16.18	15.03
$\omega_3$ [Hz]	345.9	342.6	339.4	366.6	361.7	356.8
$K_w = K_y = C_{wy} = 1 \times 10^{15}$ N/m						
$\omega_1$ [Hz]	5.04	4.61	4.27	6.64	6.07	5.63
$\omega_2$ [Hz]	123.1	122.6	122.1	21.96	20.15	18.72
$\omega_3$ [Hz]	381.3	377.4	373.6	396.9	390.5	384.2

which enables one to calculate the free vibration frequencies independently in the  $xw$  and  $xy$  plane, and then on the basis of (1), the values of the Lagrange multipliers  $\bar{\Lambda}_r$ .

### 3. Sample numerical calculations

On the basis of presented mathematical model, the algorithms and computer programs have been worked out and numerical calculations have been carried out. The sample computations have been conducted for the following parameters:  $L_1 = 0.5$  m,  $L_2 = 0.5$  m, beams with a rectangular section:  $0.02$  m (in the  $w$  direction)  $\times$   $0.045$  m (in the  $y$  direction),  $N_1 = 10$ ,  $N_2 = 10$ . The influence of:

- configuration ( $\alpha = 90^\circ, \beta = 180^\circ$  and  $\alpha = 45^\circ, \beta = 90^\circ$ ),
- carried mass ( $m = 50$  kg,  $m = 60$  kg,  $m = 70$  kg),
- beams material (AISI 1020:  $E = 2 \times 10^{11}$  Pa,  $\rho = 7700$  kg/m<sup>3</sup>; 1060 Alloy:  $E = 6.9 \times 10^{10}$  Pa,  $\rho = 2700$  kg/m<sup>3</sup>),
- stiffness of foot and soil ( $K_w = K_y = C_{wy} = 1 \times 10^8$  N/m,  $K_w = K_y = C_{wy} = 1 \times 10^{15}$  N/m),

TABLE III

Transverse vibration frequencies of the discrete-continuous model of the exoskeleton's lower limb in the  $xw$  plane (structure made of 1060 Alloy).

	$\alpha = 90^\circ, \beta = 180^\circ$			$\alpha = 45^\circ, \beta = 90^\circ$		
$m$ [kg]	50	60	70	50	60	70
$K_w = K_y = C_{wy} = 1 \times 10^8$ N/m						
$\omega_1$ [Hz]	0.88	0.78	0.70	1.48	1.33	1.21
$\omega_2$ [Hz]	50.75	50.52	50.29	26.39	26.18	25.97
$\omega_3$ [Hz]	180.2	178.7	177.2	209.2	206.4	203.6
$K_w = K_y = C_{wy} = 1 \times 10^{15}$ N/m						
$\omega_1$ [Hz]	1.41	1.27	1.16	1.98	1.79	1.64
$\omega_2$ [Hz]	61.58	61.31	61.05	38.69	38.22	37.75
$\omega_3$ [Hz]	194.4	192.7	191.0	221.6	218.3	215.2

TABLE IV

Transverse vibration frequencies of the discrete-continuous model of the exoskeleton's lower limb in the  $xy$  plane (structure made of 1060 Alloy).

	$\alpha = 90^\circ, \beta = 180^\circ$			$\alpha = 45^\circ, \beta = 90^\circ$		
$m$ [kg]	50	60	70	50	60	70
$K_w = K_y = C_{wy} = 1 \times 10^8$ N/m						
$\omega_1$ [Hz]	1.89	1.71	1.57	2.61	2.38	2.19
$\omega_2$ [Hz]	98.56	97.77	97.04	11.10	10.14	9.40
$\omega_3$ [Hz]	322.6	313.5	304.9	328.5	314.7	302.0
$K_w = K_y = C_{wy} = 1 \times 10^{15}$ N/m						
$\omega_1$ [Hz]	2.96	2.70	2.50	3.91	3.57	3.30
$\omega_2$ [Hz]	119.6	118.5	117.4	13.16	12.04	11.15
$\omega_3$ [Hz]	346.5	336.2	326.7	339.7	324.2	310.1

on the values of the first three free vibration frequencies of the exoskeleton were analyzed.

The obtained results are presented in Tables I–IV. Tables I and II refer to the exoskeleton in which the beams are made of steel (AISI 1020), and the results in Tables III and IV are for aluminum (1060 Alloy) beams.

By analyzing the results in Tables I–IV, it can be concluded that:

- the configuration of the exoskeleton's operation ( $\alpha$  and  $\beta$  angles) has the greatest impact on the vibration frequencies, which is especially visible for the second vibration frequency ( $\omega_2$ ). At the first ( $\omega_1$ ) and the third ( $\omega_3$ ) vibration frequency, the bent structure vibrates with a higher frequency than the straightened exoskeleton;
- the ground on which the exoskeleton can move has a significant effect (the greater the stiffness of the soil and the foot, the higher the vibration frequency values);
- the vibration frequencies in the plane ( $xw$ ) perpendicular to the plane of knee bending are lower than in the plane where the knee

bends ( $xy$ ). It results directly from the shape of the beam sections, and thus their moments of inertia;

- steel structure has higher vibration frequencies than aluminum structure;
- an increase in the mass loading the system causes a reduction in the vibration frequency, but the influence of the additional mass is not significant. This is due to the fact that the mass loads the system, first of all, perpendicular to the considered planes of vibration.

#### 4. Conclusions

The developed model enables the analysis of free transverse vibrations of the lower-limb exoskeleton and is a continuation of the research from the paper [4], in which the longitudinal vibrations of the mechanism were analyzed. Both of these works create a coupled model of exoskeleton vibrations.

In the paper [4] it was proved that the drive systems must be selected so that they do not generate vibrations in the range from 120 to 420 Hz, which corresponds to the first frequency of longitudinal vibrations of the system in various work configurations. In the case of transverse vibrations, there are additional forbidden ranges, and they are much more critical due to the low frequencies of the system vibrations.

The results obtained in this paper, together with the results in [4], will allow to design the exoskeleton in such a way (select the material from which it will be made, configurations and work ranges, introduce possible damping elements when moving on different grounds) so that it is both safe for the user and the risk of damaging the device is minimal.

At the same time, it should be emphasized that the presented discrete-continuous model significantly simplifies the real object (e.g. the mass of motors and gears, which may have a meaningful impact on the vibration frequency values, have not been taken into account) and the final results will be obtained only after experimental verification.

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