Classifying BPS bosonic Wilson loops in 3d $\mathcal{N} = 4$ Chern-Simons-matter theories

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ABSTRACT: We study the possible BPS Wilson loops in three-dimensional $\mathcal{N} = 4$ Chern-Simons-matter theory which involve only the gauge field and bilinears of the scalars. Previously known examples are the analogues of the Gaiotto-Yin loops preserving four supercharges and "latitude" loops preserving two. We carry out a careful classification and find, in addition, loops preserving three supercharges, further inequivalent classes of loops preserving two supercharges and loops preserving a single supercharge. For each of the classes of loops, we present a representative example and analyse their full orbit under the broken symmetries.

Contents

1 Introduction and conclusions

Over the past few years, more and more examples of supersymmetry preserving (BPS) line operators have been found in Chern-Simons-matter theories, starting from the ABJM case [1] for which the bosonic loops were first found [2–4], then the vortex loops [5] and the 1/2 BPS fermionic loop [6]. This field of study was expanded with new 1/6 BPS fermionic loops discovered in [7, 8] and then extended to less supersymmetric settings, such as $\mathcal{N} = 4$ quiver theories, with the 1/2 BPS loops in [9, 10] and then with other examples in [11–15]. For a relatively recent introduction to the topic, see [16]. While many papers discuss the bosonic 1/6 BPS loop and fermionic 1/2 BPS loop, there are in fact many more Wilson loops including rich moduli spaces of 1/6 BPS loops with fermionic fields and Wilson loops preserving fewer supercharges.

Following work on $\mathcal{N} = 2$ theories [13], the recent papers [14, 15] started to methodically treat the space of BPS Wilson loops in the context of theories with $\mathcal{N} = 4$ supersymmetry, called there "hyperloops". In the course of writing the last paper in the series, we realised that even the case of Wilson loops with a single gauge field and no fermi fields is very rich and includes previously unnoticed operators preserving 1, 2 and 3 supercharges. As the classification of those was beyond the scope of that paper and requires many different tools, it is the topic of this paper.

A generic $\mathcal{N} = 4$ Chern-Simons-matter theory has vector multiplets, hypermultiplets as well as twisted hypers which can be organised graphically in either a circular or linear quiver diagram [17–20]. Restricting to bosonic fields, those are the vector fields A_μ and the bi-fundamental scalars in the hypermultiplet, q^a, $\bar{q}_a$ and in the twisted hypermultiplets, $\tilde{q}_{\dot{a}}$, $\bar{\tilde{q}}^{\dot{a}}$.

The scalar fields in the hypermultiplet have undotted indices and are doublets of the $\mathrm{SU}(2)_L$ R-symmetry. The fermions $\psi_{\dot{b}}$, $\bar{\psi}^{\dot{a}}$ with dotted indices are charged instead under $\mathrm{SU}(2)_R$. This is reversed for the twisted hypermultiplets. We use the usual epsilon symbols to raise and lower indices: $v^a = \epsilon^{ab} v_b$ and $v_a = \epsilon_{ab} v^b$ with $\epsilon^{12} = \epsilon_{21} = 1$, and likewise for the dotted indices.

We consider the theory on S^3 with the loops supported along the equator of this sphere with coordinate φ. The theories are conformal and this setup allows for conformal line operators preserving $\mathrm{SO}(2,1) \times \mathrm{SO}(2) \subset \mathrm{SO}(4,1)$, but the loops we study are not necessarily conformal, as is discussed below. Still, we restrict the connection $\mathcal{L}$ to have canonical dimension one, so it does not have dimensionful couplings. With this restriction, $\mathcal{L}$ is comprised of the gauge field and bilinear of the scalars. This leads to the natural ansatz

$$W = \mathrm{Tr}\,\mathcal{P} \exp \oint i\mathcal{L}\,d\varphi\,, \qquad \mathcal{L} = A_\varphi + \frac{i}{k} q^a M_a{}^b \bar{q}_b + \frac{i}{k} \bar{\tilde{q}}^{\dot{a}} \widetilde{M}_{\dot{a}}{}^{\dot{b}} \tilde{q}_{\dot{b}}\,. \qquad (1.1)$$

M and $\widetilde{M}$ — the couplings of the scalar bilinears — are the main protagonists of this paper. We allow for them to have explicit φ-dependence, which breaks rotational symmetry and therefore also the conformal symmetry along the circle.

The scalar fields may be charged under a flavour group or in fact other gauge groups. The ansatz above assumes that they form singlets of those groups, since the supersymmetry variation (2.2) below requires a cancellation between the variation of the gauge fields that are not charged under these other groups and that of the scalar bilinears.

More general Wilson loops in these theories include also fermi fields and a connection that is naturally extended to a supermatrix with multiple gauge fields, fermions and scalar bilinears. The restriction to bosonic structures still allows the scalar bilinears. Those appearing in (1.1) are in the adjoint of the gauge group (plus the singlet), but in a quiver theory one could also construct bilinears of scalars from different multiplets $q^a \tilde{q}_{\dot{a}}$ which transform in the bifundamental of next to nearest neighbours in the quiver. Wilson loops without fermions but with these couplings are BPS only when the supercharge annihilates these bilinears. As this is a rather trivial additional constraint on our general analysis of (1.1), we do not discuss this possibility further.

With the increasingly rich and intricate structure of BPS loops in 3d theories, it is worth mentioning some of the possible applications of such operators. First, BPS protected quantities serve as a rich laboratory for refining the tools of quantum field theory, for example Seiberg-Witten theory [21, 22] or AdS/CFT [23]. The circular BPS Wilson loop, in particular, is amenable to exact calculations [24–30] and the rich spectrum of BPS Wilson loops in 4d [26, 31–35] allows for further exact results in quantities such as the bremsstrahlung function [36, 37] and its 3d generalisations [38–44]. Some of these loops or close analogues of them arise in our analysis and we hope that the new examples we uncover here will play a similar role in future work.

With this analysis of the bosonic loops complete, it is evident that the story of Wilson loops involving fermi fields and more than one gauge field is still richer than all those already identified in [7–15]. This will be explored in the forthcoming paper [45].

2 General analysis

We start by looking at the variation of (1.1) under a generic supercharge. This leads to a set of conditions on the supercharges that can preserve such a loop. These are then used in the subsequent section to reconstruct the loop operator invariant under the possible supercharges.

$\mathcal{N} = 4$ Chern-Simons matter theories were constructed in [17–20] and the supersymmetry transformations in flat space were presented there. Those were adapted to S^3 in [14], relying also on the decomposition to $\mathcal{N} = 2$ theories and the transformation rules in [20, 46]. Suppressing spinor indices, the variations of the bosonic fields are

$$
\begin{aligned}
\mathcal{Q}A_\mu &= \frac{i}{k}\xi_{a\dot{b}}\gamma_\mu(q^a\bar{\psi}^{\dot{b}} - \epsilon^{ac}\epsilon^{\dot{b}\dot{c}}\psi_{\dot{c}}\bar{q}_c - \bar{\tilde{q}}^{\dot{b}}\tilde{\psi}^a + \epsilon^{\dot{b}\dot{c}}\epsilon^{ac}\bar{\tilde{\psi}}_c\tilde{q}_{\dot{c}}) \,, \\
\mathcal{Q}q^a &= \xi^{a\dot{b}}\psi_{\dot{b}} \,, \qquad\qquad \mathcal{Q}\bar{q}_a = \xi_{a\dot{b}}\bar{\psi}^{\dot{b}} \,, \\
\mathcal{Q}\tilde{q}_{\dot{b}} &= -\xi_{a\dot{b}}\tilde{\psi}^a \,, \qquad\quad \mathcal{Q}\bar{\tilde{q}}^{\dot{b}} = -\xi^{a\dot{b}}\bar{\tilde{\psi}}_a \,,
\end{aligned}
\tag{2.1}
$$

This is all in Euclidean signature and in the frame outlined below $\gamma_\varphi = \sigma_3$.

The supersymmetry variation of the connection (1.1) is then

$$
\begin{aligned}
\mathcal{Q}\mathcal{L} &= \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha + M_a{}^b\xi^\alpha_{b\dot{a}}\right)q^a\bar{\psi}^{\dot{a}}_\alpha - \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha - \xi^\alpha_{b\dot{a}}\epsilon^{bc}M_c{}^d\epsilon_{da}\right)\psi^{\dot{a}}_\alpha\bar{q}^a \\
&\quad - \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha + \widetilde{M}_{\dot{a}}{}^{\dot{b}}\xi^\alpha_{a\dot{b}}\right)\bar{\tilde{q}}^{\dot{a}}\tilde{\psi}^a_\alpha + \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha - \xi^\alpha_{a\dot{b}}\epsilon^{\dot{b}\dot{c}}\widetilde{M}_{\dot{c}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{a}}\right)\bar{\tilde{\psi}}^a_\alpha\tilde{q}^{\dot{a}} \,.
\end{aligned}
\tag{2.2}
$$

The supercharge $\mathcal{Q}$ is a linear combination of the 16 supercharges $Q_l^{a\dot{a}}$, $Q_r^{a\dot{a}}$, $Q_{\bar{l}}^{a\dot{a}}$ and $Q_{\bar{r}}^{a\dot{a}}$ given by ($\imath$ takes values l, r and likewise $\bar{\imath}$)

$$
\mathcal{Q} = \eta^\imath_{a\dot{a}}Q^{a\dot{a}}_\imath + \bar{\eta}^\imath_{a\dot{a}}(\sigma_1)_\imath{}^{\bar{\jmath}}Q^{a\dot{a}}_{\bar{\jmath}} \,.
\tag{2.3}
$$

The interpolating σ_1 guarantees that $\eta^\imath_{a\dot{a}}, \bar{\eta}^\imath_{a\dot{a}}$ transform in the same representation of the conformal group, see appendix A for details. The right-hand-side of (2.2) is expressed in terms of $\xi^\alpha_{a\dot{a}}$, which package together the η parameters and the four Killing spinors $\xi^\imath_\alpha$, $\xi^{\bar{\imath}}_\alpha$

$$
\xi^\alpha_{a\dot{a}} = \eta^\imath_{a\dot{a}}\xi^\alpha_\imath + \bar{\eta}^\imath_{a\dot{a}}(\sigma_1)_\imath{}^{\bar{\imath}}\xi^\alpha_{\bar{\imath}} \,.
\tag{2.4}
$$

Note that the Killing spinors appear here with raised spinor and lowered $\imath$, $\bar{\imath}$ indices compared to (2.7).

Let us recall some facts about the Killing spinors. They obey the equations

$$
\nabla_\mu\xi^{l,\bar{l}} = \frac{i}{2}\gamma_\mu\xi^{l,\bar{l}} \,, \qquad \nabla_\mu\xi^{r,\bar{r}} = -\frac{i}{2}\gamma_\mu\xi^{r,\bar{r}} \,.
\tag{2.5}
$$

Specifically, following [27, 47], we can use the Lie group structure of S^3 to construct a left-invariant dreibein e_i with spin connection $\omega_{ij} = \epsilon_{ijk}e^k$. Then the spin connection in the first equation cancels the right-hand-side and the solutions are simply constant spinors. For the great circle along the $\mu = \varphi$ direction, those can be chosen as eigenstates of σ_3. In this setup, the second equation reads

$$
\partial_\varphi\xi^{r,\bar{r}} = -i\sigma_3\xi^{r,\bar{r}} \,.
\tag{2.6}
$$

Clearly they can again be chosen along the circle to be eigenvectors of σ_3 such that we find

$$\xi_\alpha^l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_\alpha^{\bar{l}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \xi_\alpha^r = \begin{pmatrix} e^{-i\varphi} \\ 0 \end{pmatrix}, \qquad \xi_\alpha^{\bar{r}} = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix}. \qquad (2.7)$$

In particular note that the unbarred spinors have only $+$ components and the barred ones only $-$. This makes the indices redundant and allows us to eliminate some of them as already done in (2.3). With those expressions for the Killing spinors, (2.4) becomes

$$\xi_{a\dot{a}}^+ = \bar{\eta}_{a\dot{a}}^l - \bar{\eta}_{a\dot{a}}^r e^{i\varphi}, \qquad \xi_{a\dot{a}}^- = \eta_{a\dot{a}}^l e^{-i\varphi} - \eta_{a\dot{a}}^r. \qquad (2.8)$$

For the supersymmetry variation of the connection to vanish, all four terms in (2.2) must vanish. A Wilson loop can also be invariant under a symmetry when the variation of the connection is an appropriate covariant derivative [16], but as there are no derivatives on the right-hand-side, this is not the case here. Multiplying from the left the second term in (2.2) by M and the fourth by $\widetilde{M}$, and both by σ_3 from the right, we find

$$\xi_{a\dot{a}}^\beta (\sigma_3)_\beta{}^\alpha + M_a{}^b \xi_{b\dot{a}}^\alpha = 0, \qquad\qquad \xi_{a\dot{a}}^\beta (\sigma_3)_\beta{}^\alpha + \widetilde{M}_{\dot{a}}{}^{\dot{b}} \xi_{a\dot{b}}^\alpha = 0,$$
$$M_a{}^b \xi_{b\dot{a}}^\alpha - \det(M) \xi_{a\dot{a}}^\beta (\sigma_3)_\beta{}^\alpha = 0, \qquad \widetilde{M}_{\dot{a}}{}^{\dot{b}} \xi_{a\dot{b}}^\alpha - \det(\widetilde{M}) \xi_{a\dot{a}}^\beta (\sigma_3)_\beta{}^\alpha = 0. \qquad (2.9)$$

Comparing the two lines, this can only be solved by all $\xi_{a\dot{a}}^\alpha = 0$ (which means no supersymmetry) or by

$$\det M = \det \widetilde{M} = -1. \qquad (2.10)$$

The supersymmetry parameters then have to satisfy the eigenvector equations

$$M_a{}^b \xi_{b\dot{a}}^\pm = \mp \xi_{a\dot{a}}^\pm, \qquad \widetilde{M}_{\dot{a}}{}^{\dot{b}} \xi_{a\dot{b}}^\pm = \mp \xi_{a\dot{a}}^\pm. \qquad (2.11)$$

If this is solved by any nonzero $\xi_{a\dot{a}}^+$, both M and $\widetilde{M}$ must have an eigenvalue -1, and if it is solved by $\xi_{a\dot{a}}^-$, one eigenvalue of both M, $\widetilde{M}$ must be 1. From (2.10) we see that regardless, M and $\widetilde{M}$ have both the eigenvalues 1 and -1.

The simplest possibility is of course when $M = \widetilde{M} = \mathrm{diag}(1, -1)$. Plugging them back into (1.1), we find the Gaiotto-Yin loop [48] with four preserved supercharges $Q_{l,r}^{\dot{1}1}, Q_{\bar{l},\bar{r}}^{\dot{2}2}$

$$\mathcal{L} = A_\varphi + \frac{i}{k}(q^1 \bar{q}_1 - q^2 \bar{q}_2 + \bar{\tilde{q}}^{\dot{1}} \tilde{q}_{\dot{1}} - \bar{\tilde{q}}^{\dot{2}} \tilde{q}_{\dot{2}}). \qquad (2.12)$$

This is indeed the most symmetric and supersymmetric bosonic loop, discussed in more details in section 3.1. Up to rotations to other matrices with eigenvalues 1 and -1, this is the only possibility with constant matrices. So in fact the main focus of this work are the cases when M or $\widetilde{M}$ have φ dependence. For example,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}, \qquad (2.13)$$

also have eigenvalues 1 and -1, but are not constant. Those are "latitude" bosonic loops [14, 30, 38], preserving two supercharges and are discussed in detail in section 3.3.1. Loops with different values of θ are not related by symmetry and are truly different (and their

expectation values are also different [49]). We find below one further inequivalent example in the same class as well as a few new classes of Wilson loops preserving two supercharges.

To study all the possible supersymmetric bosonic loops systematically, we now proceed to look for the most general configurations of M and $\widetilde{M}$ which allow for nonzero solutions to (2.11). Since any 2×2 matrix with two distinct eigenvalues is uniquely determined by its eigenvectors, it is sufficient to determine those.

Note that in the two equations in (2.11) there is a free parameter. For example

$$\xi_{a\dot{a}}^- = M_a{}^b \xi_{b\dot{a}}^-, \qquad \dot{a} = \dot{1}, \dot{2}. \tag{2.14}$$

In other words, both $\xi_{a\dot{1}}^-$ and $\xi_{a\dot{2}}^-$ are eigenvectors with the eigenvalue 1, so are linearly dependent. In particular, if we view $\xi_{a\dot{a}}^-$ as a 2×2 matrix, its determinant must vanish. This already incorporates the second equation in (2.11) and likewise for $\xi_{a\dot{a}}^+$, giving

$$\det_{a\dot{a}}(\xi_{a\dot{a}}^-) = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\xi_{a\dot{a}}^-\xi_{b\dot{b}}^- = 0, \qquad \det_{a\dot{a}}(\xi_{a\dot{a}}^+) = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\xi_{a\dot{a}}^+\xi_{b\dot{b}}^+ = 0. \tag{2.15}$$

Being of rank ≤ 1, we can clearly write $\xi_{a\dot{a}}^\pm$ as the outer product of two vectors. But recall that these are linear combinations of two Killing spinors

$$\xi_{a\dot{a}}^- = \eta_{a\dot{a}}^i \xi_i^- . \tag{2.16}$$

The expression for the determinant translates to

$$2\det_{a\dot{a}}(\xi_{a\dot{a}}^-) = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\left((\xi_l^-)^2\eta_{a\dot{a}}^l\eta_{b\dot{b}}^l + 2\xi_l^-\xi_r^-\eta_{a\dot{a}}^l\eta_{b\dot{b}}^r + (\xi_r^-)^2\eta_{a\dot{a}}^r\eta_{b\dot{b}}^r\right). \tag{2.17}$$

Since ξ_i^- are different functions (as are their squares), (2.7), this vanishes only if the three terms vanish separately

$$\det_{a\dot{a}}(\eta_{a\dot{a}}^l) = \det_{a\dot{a}}(\eta_{a\dot{a}}^r) = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\eta_{a\dot{a}}^l\eta_{b\dot{b}}^r = 0. \tag{2.18}$$

The first two equations allow us to represent η as a the product of two vectors (no sum implied on the right-hand-side)

$$\eta_{a\dot{a}}^i = w_a^i z_{\dot{a}}^i, \tag{2.19}$$

where w_a^i and $z_{\dot{a}}^i$ are constants and the remaining condition is

$$(\epsilon^{ab}w_a^l w_b^r)(\epsilon^{\dot{a}\dot{b}}z_{\dot{a}}^l z_{\dot{b}}^r) = \det_{ai}(w_a^i)\det_{\dot{a}j}(z_{\dot{a}}^j) = \det_{a\dot{a}}\left(\sum_i w_a^i z_{\dot{a}}^i\right) = 0. \tag{2.20}$$

All the same arguments carry over to $\xi_{a\dot{a}}^+$, expressing

$$\bar{\eta}_{a\dot{a}}^i = \bar{w}_a^i \bar{z}_{\dot{a}}^i, \tag{2.21}$$

subject to the constraint

$$(\epsilon^{ab}\bar{w}_a^l \bar{w}_b^r)(\epsilon^{\dot{a}\dot{b}}\bar{z}_{\dot{a}}^l \bar{z}_{\dot{b}}^r) = \det_{a\dot{a}}\left(\sum_i \bar{w}_a^i \bar{z}_{\dot{a}}^i\right) = 0. \tag{2.22}$$

We may define two matrices of Killing spinors $\Xi^- = \mathrm{diag}(\xi_{\bar{l}}^-, \xi_{\bar{r}}^-)$ and $\Xi^+ = \mathrm{diag}(\xi_{\bar{r}}^+, \xi_{\bar{l}}^+)$, where the indices are properly raised and lowered with respect to (2.7). Also, it is natural to incorporate the action of σ_1, as in (2.4), such that they both have two unbarred $\imath$ indices and we can now combine (2.19), (2.21) to write $\xi_{a\dot{a}}^\pm$ as

$$\xi_{a\dot{a}}^- = w_a^\imath \Xi_{\imath\jmath}^- z_{\dot{a}}^\jmath \,, \qquad \xi_{a\dot{a}}^+ = \bar{w}_a^\imath \Xi_{\imath\jmath}^+ \bar{z}_{\dot{a}}^\jmath \,. \tag{2.23}$$

Going back to (2.20), there is a nonzero $\xi_{a\dot{a}}^-$ giving a preserved supersymmetry if either $\det_{a\imath}(w_a^\imath)$ or $\det_{\dot{a}\jmath}(z_{\dot{a}}^\jmath)$ vanish. To enumerate the different possibilities:

1. $w_a^\imath = w_a y^\imath$, but can immediately absorb $y^\imath$ in $z_{\dot{a}}^\imath$, such that $\eta_{a\dot{a}}^\imath = w_a z_{\dot{a}}^\imath$.

2. $z_{\dot{a}}^\imath$ factors, so $\eta_{a\dot{a}}^\imath = w_a^\imath z_{\dot{a}}$.

3. If both determinants vanish independently, we have $\eta_{a\dot{a}}^\imath = y^\imath w_a z_{\dot{a}}$.

In the first two cases all the components of $\eta_{a\dot{a}}^\imath$ are related, representing a single preserved supercharge. In the last case the factorization to $y^\imath$ means that we have two independent solutions with $\eta_{a\dot{a}}^l = w_a z_{\dot{a}}$ and another one with $\eta_{a\dot{a}}^r = w_a z_{\dot{a}}$.

All the same analysis carries over to $\xi_{a\dot{a}}^+$ allowing to find zero, one or two independent solutions.

Lastly, because $\xi_{a\dot{a}}^-$ and $\xi_{a\dot{a}}^+$ have different eigenvalues, they cannot be proportional to each-other, so for any pair of nonzero eigenvectors

$$\epsilon^{ab} \xi_{a\dot{a}}^- \xi_{b\dot{b}}^+ \neq 0 \,, \qquad \epsilon^{\dot{a}\dot{b}} \xi_{a\dot{a}}^- \xi_{b\dot{b}}^+ \neq 0 \,. \tag{2.24}$$

When both $\xi_{a\dot{a}}^-$ and $\xi_{a\dot{a}}^+$ factorise, this translates into conditions such as $\epsilon^{ab} w_a \bar{w}_b \neq 0$.

3 Representative examples

We are now ready to write down examples of Wilson loops preserving varying number of supercharges. In this section we list the cases up to actions of symmetries explained in appendix A. We also outline the proof that the examples we present are indeed representatives of every orbit and demonstrate some details of the other elements in the orbit.

The relevant symmetries are the R-symmetry group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ (and its complexification) acting on the indices a and $\dot{a}$ respectively and the conformal group $\mathrm{SL}(2, \mathbb{R})$ (and its complexification), acting on functions of φ (and on the indices l, r, $\bar{l}$, $\bar{r}$). In addition the supersymmetry equations (2.2) have two discrete transformations relating different solutions:

- The exchange $M \leftrightarrow \widetilde{M}$ or equivalently $\mathrm{SU}(2)_L \leftrightarrow \mathrm{SU}(2)_R$ relates cases with factorised w and/or $\bar{w}$ to cases with factorised z and/or $\bar{z}$.

- The simultaneous change of sign of M and $\widetilde{M}$ exchanges the 1 and -1 eigenvalues in (2.11). Since those are matched to the spinor index and then via (2.7) to the exchange $l, r \leftrightarrow \bar{l}, \bar{r}$. This has the effect of relating cases with factorised w and/or z to cases with factorised $\bar{w}$ and/or $\bar{z}$.

The symmetry actions are explained in more detail in appendix A.

If we know two eigenvectors of M with the different eigenvalues, say $\xi^-_{a\dot{1}}$ and $\xi^+_{a\dot{2}}$, we can easily reconstruct M as

$$M_a{}^b = \left(\xi^-_{a\dot{1}}\ \xi^+_{a\dot{2}}\right)\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\left(\xi^-_{b\dot{1}}\ \xi^+_{b\dot{2}}\right)^{-1} = \frac{1}{\epsilon^{ab}\xi^-_{a\dot{1}}\xi^+_{b\dot{2}}}\begin{pmatrix}\xi^-_{1\dot{1}}\xi^+_{2\dot{2}} + \xi^-_{2\dot{1}}\xi^+_{1\dot{2}} & -2\xi^-_{1\dot{1}}\xi^+_{1\dot{2}}\\2\xi^-_{2\dot{1}}\xi^+_{2\dot{2}} & -\xi^-_{1\dot{1}}\xi^+_{2\dot{2}} - \xi^-_{2\dot{1}}\xi^+_{1\dot{2}}\end{pmatrix},$$
$$(3.1)$$

and likewise $\widetilde{M}$. In this way we can find all BPS Wilson loops by going over all possible preserved supercharges. If the loop preserves both $\xi^-_{a\dot{1}}$ and $\xi^-_{a\dot{2}}$ (and say one $\xi^+_{a\dot{1}}$), we would get the same expression from any linear combination of the two ξ^-. The cases with single ξ^- or ξ^+ arise from not fully factorised $\eta^\pm$, which is different from a linear combination of factorised ones.

In the case when the loop preserves only one or two ξ^- and no ξ^+, we can plug into the formula above any vector ξ^+ which is not proportional to the eigenvector ξ^- to get an appropriate M. As ξ^+ is any vector not necessarily made of the Killing spinors, it can have arbitrary dependence on φ, and in turn so does M. See section 3.3.3 and section 3.4 below.

3.1   1/4 BPS loops

The most supersymmetric bosonic loops preserve two components of $\xi^-_{a\dot{a}}$ and two of $\xi^+_{a\dot{a}}$. They are given by $\eta^\imath_{a\dot{a}} = y^\imath w_a z_{\dot{a}}$ and $\bar{\eta}^\imath_{a\dot{a}} = \bar{y}^\imath \bar{w}_a \bar{z}_{\dot{a}}$. We can choose $w_a = \delta^1_a$, $z_{\dot{a}} = \delta^{\dot{1}}_{\dot{a}}$, $\bar{w}_a = \delta^2_a$, $\bar{z}_{\dot{a}} = \delta^{\dot{2}}_{\dot{a}}$ such that the four independent supercharges have the nonzero parameters

$$\eta^l_{1\dot{1}}, \qquad \eta^r_{1\dot{1}}, \qquad \bar{\eta}^l_{2\dot{2}}, \qquad \bar{\eta}^r_{2\dot{2}}. \tag{3.2}$$

Clearly they are eigenvector with eigenvalues ± 1 of

$$M = \widetilde{M} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{3.3}$$

which is also what one gets from (3.1), matching (2.12).

This is the $\mathcal{N} = 4$ avatar of the Wilson loops first constructed by Gaiotto and Yin [48] and then rediscovered as the 1/6 BPS loops of ABJM theory [2–4]. These M and $\widetilde{M}$ break $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ to $\mathrm{U}(1)^2$. Conjugation of M by elements of the complexified $\mathrm{SL}(2,\mathbb{C})_L$ can produce any other constant matrix with eigenvalues ± 1, and likewise for $\widetilde{M}$. So this is the orbit of the broken symmetry group.

Looking at it from the point of view of the parameters w_a, $\bar{w}_a$, $z_{\dot{a}}$, $\bar{z}_{\dot{a}}$, $y^\imath$ and $\bar{y}^\imath$, starting with any other factorised choice, we can use two $\mathrm{SL}(2,\mathbb{C})_{L,R}$ actions to set $w_a = \delta^1_a$ and $z_{\dot{a}} = \delta^{\dot{1}}_{\dot{a}}$ as in the above example. This on its own does not fix the barred parameters. But as they cannot be linearly dependent on the unbarred ones (2.24), we can use the Gram-Schmidt process to produce $\bar{w}_a \propto \delta^2_a$, $\bar{z}_{\dot{a}} \propto \delta^{\dot{2}}_{\dot{a}}$, and then can rescale them to reproduce (3.2).

The superalgebra generated by the four preserved supercharges in (3.2) includes the $\mathrm{SO}(2,1)_C$ conformal group of the circle, guaranteeing that these loops are conformal [15, 43]. In fact, of all the loops discussed in this paper, these are the only ones that are conformal and all the other ones have explicit φ dependence in M and/or $\widetilde{M}$.

3.2 3/16 BPS loops

This case arises when three of w, z, $\bar{w}$, $\bar{z}$ factorise while the remaining one doesn't. Let us focus on the case when $\bar{z}$ does not factorise. As above, we take $w_a = \delta_a^1$, $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$, $\bar{w}_a = \delta_a^2$ and $\bar{z}_{\dot{a}}^i = \delta_{\dot{a}}^i$ (with $l \simeq \dot{1}$).

The preserved supercharges are then linear combinations of

$$Q_l^{1\dot{1}}, \qquad Q_r^{1\dot{1}}, \qquad Q_{\bar{r}}^{2\dot{1}} + Q_{\bar{l}}^{2\dot{2}}, \tag{3.4}$$

and the connection is as in (1.1) with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

This is a new example of a Wilson loop with an exotic number of preserved supercharges. Unlike the 1/4 BPS loops, this is not conformally invariant, as is evident by the explicit φ dependence in $\widetilde{M}$. Acting with the broken $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SO}(2,1)_C$ (possibly complexified) gives a rich orbit of further examples. Note that the action of the complexified conformal group $\mathrm{SL}(2,\mathbb{C})$ can transform $e^{i\varphi}$ to any fractional linear function. Other examples include the replacement of $M \leftrightarrow \widetilde{M}$ as well as changing their signs. Those correspond to a choice of a different parameter among w, z, $\bar{w}$, $\bar{z}$ that doesn't factorise. In particular the example with lower triangular $\widetilde{M}$ is related by symmetry to the case where M and $\widetilde{M}$ have opposite signs.

To prove that there is a single orbit of the group and (3.5) is indeed a representative, note that any unfactorised $\bar{z}$ is a rank 2 matrix which can be brought into the form above with the action of either $\mathrm{SL}(2,\mathbb{C})_R$ or $\mathrm{SL}(2,\mathbb{C})_C$. It is still invariant under conjugation by any identical element of the two groups, which means we have $\mathrm{SL}(2,\mathbb{C})_R$ freedom to set $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$. Then the $\mathrm{SL}(2,\mathbb{C})_L$ action can bring $\bar{w}$ into the desired form. This procedure leaves w as a vector not parallel to $\bar{w}$, so we can choose a linear combination producing $w_a = \delta_a^1$.

Supersymmetry enhancement can be easily seen by exploiting the action of $\mathrm{SL}(2,\mathbb{C})_C$, which allows us to apply arbitrary constant rescalings to $e^{i\varphi}$. In the limit where the phase in (3.5) vanishes, supersymmetry is enhanced to 1/4 BPS.

3.3 1/8 BPS loops

There are of course $\binom{4}{2} = 6$ different pairs out of w, z, $\bar{w}$, $\bar{z}$ to factorise, but they are pairwise related by extra symmetries. We discuss the three inequivalent classes below.

3.3.1 Factorised w and $\bar{w}$

In this case we can take representatives with $w_a = \delta_a^1$, $\bar{w}_a = \delta_a^2$ and $z_{\dot{a}}^i = \delta_{\dot{a}}^i$. To get to this form from an arbitrary unfactorised $z_{\dot{a}}^i$, we can either act from the left with $\mathrm{SL}(2,\mathbb{C})_R$ or from the right with $\mathrm{SL}(2,\mathbb{C})_C$. This form is still invariant under conjugation by the same elements of the two groups, so that is the remaining freedom we have to act on $\bar{z}$, as well as overall rescaling, which is immaterial. By rescaling we can make $\det_{\dot{a}i} \bar{z} = 1$ (since it has rank 2) and then by conjugation bring it to Jordan normal form

$$\bar{z}_\lambda = \begin{pmatrix} \bar{z}_{\dot{1}}^l & \bar{z}_{\dot{1}}^r \\ \bar{z}_{\dot{2}}^l & \bar{z}_{\dot{2}}^r \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \bar{z}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

Plugging the diagonal case into (3.1) gives

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \frac{1}{\lambda - \lambda^{-1}} \begin{pmatrix} -\lambda - \lambda^{-1} & -2\lambda e^{-i\varphi} \\ 2\lambda^{-1} e^{i\varphi} & \lambda + \lambda^{-1} \end{pmatrix}. \tag{3.7}$$

These loops are in fact related to the "bosonic latitude loops" (2.13) of [14, 30, 38]. To see that, we take $\cos\theta = -(\lambda + \lambda^{-1})/(\lambda - \lambda^{-1})$, such that

$$\widetilde{M}_\theta = \begin{pmatrix} \cos\theta & i\lambda e^{-i\varphi}\sin\theta \\ -i\lambda^{-1} e^{i\varphi}\sin\theta & -\cos\theta. \end{pmatrix}. \tag{3.8}$$

Conformal symmetry acts on $e^{\pm i\varphi}$ as Möbius transformations, which in particular includes the rescaling that eliminates $i\lambda$ from the matrix above, reproducing (2.13). This can also be realised by conjugating $\widetilde{M}$ by

$$\begin{pmatrix} 1/\sqrt{i\lambda} & 0 \\ 0 & \sqrt{i\lambda} \end{pmatrix}, \tag{3.9}$$

which is an $\mathrm{SL}(2,\mathbb{C})_R$ transformation.

In terms of z and $\bar{z}$, this $\mathrm{SL}(2,\mathbb{C})_R$ acts on them from the left giving

$$z \to \begin{pmatrix} 1/\sqrt{i\lambda} & 0 \\ 0 & \sqrt{i\lambda} \end{pmatrix}, \qquad \bar{z}_\lambda \to \begin{pmatrix} -i\sqrt{i\lambda} & 0 \\ 0 & i/\sqrt{i\lambda} \end{pmatrix}. \tag{3.10}$$

Inserting the resulting $\xi_{a\dot{a}}^{\pm}$ into (3.1) and taking the above relation between λ and θ produces the same result (2.13). Since the original and new z and $\bar{z}$ are all diagonal, the exact same result can be achieved by right multiplication, which is an $\mathrm{SL}(2,\mathbb{C})_C$ transformation. The fact that we can act with either of the groups indicates that these loops are invariant under a particular combination of the two group actions, which is a known symmetry of the latitude [15, 49].

Turning to the non-diagonal case in (3.6). This is a new example, which to our knowledge has not been previously described. We find the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} -1 + 2e^{-i\varphi} & -2e^{-i\varphi} + 2e^{-2i\varphi} \\ -2 & 1 - 2e^{-i\varphi} \end{pmatrix}. \tag{3.11}$$

Another representative of this orbit is given by

$$z = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \bar{z} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \tag{3.12}$$

which leads to

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 - 2e^{2i\varphi} & 2e^{i\varphi}\left(1 + e^{i\varphi}\right) \\ 2e^{i\varphi}\left(1 - e^{i\varphi}\right) & -1 + 2e^{2i\varphi} \end{pmatrix}. \tag{3.13}$$

Instead of constructing the detailed map between the two cases, as we did for the latitudes, note that z and $\bar{z}$ offer a simple way to identify the orbit. Since $z, \bar{z}$ both transform in

the fundamental of $\mathrm{SL}(2,\mathbb{C})_R$ and the antifundamental of $\mathrm{SL}(2,\mathbb{C})_C$. The matrix $z^{-1}\bar{z}$ transforms in the adjoint of $\mathrm{SL}(2,\mathbb{C})_C$ and is invariant under the other group. Conversely $\bar{z}z^{-1}$ is in the adjoint of $\mathrm{SL}(2,\mathbb{C})_R$ and invariant under the other group. The eigenvalues of these matrices are then invariant under both groups, but these two matrices have the same eigenvalues. Since any nonzero rescaling is immaterial, we can always set them to λ and $1/\lambda$ and compare with (3.6).

Clearly for the latitudes in (3.10), we reproduce the same eigenvalues as $\bar{z}_\lambda$ in (3.6). Likewise, the Jordan form of those for (3.12) is the same as z'.

The case with factorized z and $\bar{z}$ gives similar loops under the exchange $M \leftrightarrow \widetilde{M}$.

To see how to get supersymmetry enhancement to the previous examples, the case of enhancement to 4 supercharges in section 3.1 is obvious, by taking $\theta \to 0$ in (2.13).

To get to the 3/16 BPS case in section 3.2, we need to take $\lambda \to \infty$ in (3.7). In terms of θ, this is a double scaling limit by first using a complexified conformal transformation that scales $e^{i\varphi} \to 2e^{i\varphi}/\theta$ and $e^{-i\varphi} \to \theta\,e^{-i\varphi}/2$ and then take $\theta \to 0$. The expression in (2.13) then clearly becomes the transpose of (3.5), which is another example of a 3/16 BPS loop.

The nondiagonalisable case admits similar limits. Rescaling $e^{-i\varphi}$ in (3.11) allows us to tune out the φ dependent terms entirely, which brings us back to the 1/4 BPS case. To hit a 3/16 BPS orbit instead, we first rescale the phases by x^2, and then conjugate $\widetilde{M}$ with $\mathrm{diag}(x,x^{-1})$, which shifts one factor of x^2 from the bottom left to the top right. The limit $x \to 0$ then removes all but one phase, and we recover the 3/16 BPS loop (3.5), as before.

3.3.2 Factorised w and $\bar{z}$

As an illustration of this case, we can take $w_a = \delta_a^1$, $\bar{z}_{\dot{a}} = \delta_{\dot{a}}^{\dot{2}}$, $\bar{w}_a^i = \delta_a^1 \delta_l^i - \delta_a^2 \delta_r^i$ and $z_{\dot{a}}^i = \delta_{\dot{a}}^i$ to get

$$M = \begin{pmatrix} 1 & -2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 0 \\ -2e^{i\varphi} & -1 \end{pmatrix}. \tag{3.14}$$

Unlike the previous example, here we could use the symmetry to choose a unique representative, so there is only one conjugacy class (the argument follows the same logic as in the previous examples). To our knowledge, such loops have not been previously described. The action of the conformal group on these loops produces more loops with fractional linear functions in both M and $\widetilde{M}$.

The case when $\bar{w}$ and z are instead factorised is related again by $M \leftrightarrow \widetilde{M}$.

The loop (3.14) admits 3/16 BPS limits. Conjugation with $\mathrm{diag}(x,x^{-1}) \in \mathrm{SL}(2,\mathbb{C})_L$ and taking $x \to \infty$ allows us to tune out the phase in M and the same can be done with $\widetilde{M}$.

3.3.3 Factorised w and z

When w and z both factorise, we have two preserved ξ^- supercharges and no ξ^+ ones. We choose representative supercharges with $w_a = \delta_a^1$ and $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$. This does not completely fix M and $\widetilde{M}$, as (3.1) requires also to specify ξ^+. From the above information alone, we find

$$M_1{}^1 = \widetilde{M}_1{}^1 = 1, \qquad M_2{}^1 = \widetilde{M}_2{}^1 = 0. \tag{3.15}$$

Then, since $\det M = \det \widetilde{M} = -1$, we get $M_2{}^2 = \widetilde{M}_2{}^2 = -1$, which leaves $M_1{}^2$ and $\widetilde{M}_1{}^2$ as completely arbitrary periodic functions of φ. We denote them $2\mu(\varphi)$ and $2\tilde{\mu}(\varphi)$ respectively

to get

$$M = \begin{pmatrix} 1 & 2\mu(\varphi) \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 2\tilde{\mu}(\varphi) \\ 0 & -1 \end{pmatrix}. \qquad (3.16)$$

Alternatively, we can arrive at the same result by choosing the second eigenvectors for M and $\widetilde{M}$ as

$$\begin{pmatrix} \mu(\varphi) \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\mu}(\varphi) \\ -1 \end{pmatrix}. \qquad (3.17)$$

Of course, if $\mu(\varphi)$ or $\tilde{\mu}(\varphi)$ are constants or $e^{\pm i\varphi}$, these eigenvectors are in fact Killing spinors and the loop will have enhanced supersymmetry to 3/16 or 1/4 and will match the forms in section 3.2 or section 3.1 up to symmetry action.

To understand the reason for this freedom of arbitrary functions in M and $\widetilde{M}$, it is instructive to reconstruct the general supercharge preserved by these loops, that is (2.3)

$$\mathcal{Q} = \eta_{1\dot{1}}^{\imath} Q_{\imath}^{1\dot{1}}. \qquad (3.18)$$

Examining the supersymmetry variations (2.1), we clearly see that $\mathcal{Q}(q^1 \bar{q}_2) = \mathcal{Q}(\bar{\tilde{q}}^{\dot{1}} \tilde{q}_{\dot{2}}) = 0$. So this Wilson loop is simply the 1/4 BPS Wilson loop of section 3.1 with arbitrary insertions of these bilinears that are chiral under this pair of supercharges.

3.4 1/16 BPS loops

Among the four possible cases, we choose the one with factorised w, so in particular no ξ^+ supercharges. If we choose $w_a = \delta_a^1$ and $z_{\dot{a}}^{\imath} = \delta_{\dot{a}}^{\imath}$, then for M things are similar to the last case, where we get $M_1{}^1 = 1$, $M_2{}^1 = 0$, and by $\det M = 1$ we have the other two entries $M_1{}^2 = 2\mu(\varphi)$, $M_2{}^2 = -1$. The second matrix, $\widetilde{M}$, is different. Its entries satisfy the two equations

$$\widetilde{M}_{\dot{1}}{}^{\dot{1}} e^{-i\varphi} - \widetilde{M}_{\dot{1}}{}^{\dot{2}} = e^{-i\varphi}, \qquad \widetilde{M}_{\dot{2}}{}^{\dot{1}} e^{-i\varphi} - \widetilde{M}_{\dot{2}}{}^{\dot{2}} = -1, \qquad (3.19)$$

and there is the extra condition $\det \widetilde{M} = -1$, $\operatorname{tr} \widetilde{M} = 0$. It therefore still has one completely free parameter and we take $\widetilde{M}_{\dot{1}}{}^{\dot{1}} = \tilde{\mu}(\varphi)$. Then

$$M = \begin{pmatrix} 1 & 2\mu(\varphi) \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \tilde{\mu}(\varphi) & e^{-i\varphi}(\tilde{\mu}(\varphi) - 1) \\ -e^{i\varphi}(\tilde{\mu}(\varphi) + 1) & -\tilde{\mu}(\varphi) \end{pmatrix}. \qquad (3.20)$$

4 Theories without twisted hypers

In the discussion so far, we assumed that the theory contains both hypermultiplets and twisted hypermultiplets, but this is not necessary for $\mathcal{N} = 4$ supersymmetry. The discussion follows through if the theory has only hypermultiplets. The supersymmetry variations are as in (2.1) except that we should remove all the twisted hypermultiplet fields.

The ansatz for the Wilson loops is as in (1.1), but without the term involving $\widetilde{M}$. The theory has the same 16 supercharges and Killing spinors, but the supersymmetry conditions impose far fewer constraints. Specifically, only the left equation in (2.11) remains, meaning that there is no constraint on the dotted indices of the supercharges.

In terms of the conditions on η and $\bar{\eta}$, the z and $\bar{z}$ parameters are always factorised, so equation (2.19) becomes

$$\eta^i_{a\dot{a}} = w^i_a z_{\dot{a}}\,, \tag{4.1}$$

and the requirement for supersymmetry is that $\det_{a\imath} w^i_a = 0$ or $\det_{a\imath} \bar{w}^i_a = 0$, meaning one of them also factorises.

The completely factorised case is as in section 3.1, with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

Now this loop preserves 8 supercharges, so is 1/2 BPS.

When only w factorises but not $\bar{w}$ we have loops similar to the construction in section 3.2

$$M = \begin{pmatrix} 1 & 2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}, \tag{4.3}$$

which are now 1/4 BPS. Likewise, when only $\bar{w}$ factorises we can get a similar M with the phase in the bottom left.

In a linear quiver theory with two nodes, bifundamental hypermultiplets and no twisted hypermultiplets, there are then two bosonic 1/2 BPS loops, one at each of the nodes as well as fermionic 1/2 BPS loops. The bosonic loops break $\mathrm{SU}(2)_L$ and preserve $\mathrm{SU}(2)_R$. The 1/2 BPS fermionic loops break $\mathrm{SU}(2)_R$ and preserve $\mathrm{SU}(2)_L$.

Acknowledgments

A   Symmetries

The theories studied here have $\mathrm{SO}(4) \cong (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathbb{Z}_2$ R-symmetry and $\mathrm{SO}(4,1)$ conformal symmetry, which are packaged together into an $\mathrm{OSp}(4|4)$ supergroup. The geometry of the circle breaks the conformal group to $\mathrm{SO}(2,1)_C \times \mathrm{SO}(2)$, and the particular choice of line operator can break the symmetry further. However, the bosonic loops all are invariant under the transverse $\mathrm{SO}(2)$, so what we focus on is the action of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SO}(2,1)_C$ on the loops constructed in the body of the paper.

To be precise, we are not imposing any reality or hermiticity conditions, so we allow for the action of the complexified group $\mathrm{SL}(2,\mathbb{C})_L \times \mathrm{SL}(2,\mathbb{C})_R \times \mathrm{SL}(2,\mathbb{C})_C$. The scalars q_a

are in the fundamental of $\mathrm{SU}(2)_L$ and $\bar{q}^a$ in the anti-fundamental. The scalars from the twisted hyper are charged under $\mathrm{SU}(2)_R$. Clearly the matrices M and $\widetilde{M}$ are in the adjoint of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively. The action on the supercharges as well as on $\eta, \bar{\eta}$ and their decomposition into the parameters w, z, y, $\bar{w}$, $\bar{z}$ and $\bar{y}$ can be read off from their index structure.

The action of the conformal group is more involved. The parameters $\eta^i_{a\dot{a}}$ and $\bar{\eta}^i_{a\dot{a}}$ are doublets, with the indices i interchanged under the action of the conformal generators $J_\pm$ satisfying the algebra

$$[J_0, J_\pm] = \pm J_\pm \,, \qquad [J_+, J_-] = 2J_0 \,. \tag{A.1}$$

Under a finite conformal transformation $U \in \mathrm{SL}(2, \mathbb{C})_C$, they transform as

$$\eta^i_{a\dot{a}} \longmapsto \eta^j_{a\dot{a}} \left(U^T\right)_j^{\ i} \,, \qquad \bar{\eta}^i_{a\dot{a}} \longmapsto \bar{\eta}^j_{a\dot{a}} \left(U^T\right)_j^{\ i} \,. \tag{A.2}$$

The action on M and $\widetilde{M}$ is set by their φ dependence. The 1/4 BPS loops in section 3.1 are all invariant. In all the other cases we know that the conformal group acts on the unit circle via Möbius transformations

$$U: \ e^{i\varphi} \ \longmapsto \ \frac{de^{i\varphi} - b}{-ce^{i\varphi} + a} \,, \qquad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1 \,. \tag{A.3}$$

This determines the transformations of all the Wilson loops discussed in section 3.

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