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Low sets without subsets of higher many-one degree

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Given a reducibility $\leq_r$, we say that an infinite set A is r-introimmune if A is not r-reducible to any of its subsets B with $|A\backslash B| = \infty$. We consider the many-one reducibility $\leq_{\mathrm{m}}$ and we prove the existence of a low_1 m-introimmune set in Π^0_1 and the existence of a low_1 bi-m-introimmune set.

1 Introduction

The study of sets without subsets of higher Turing degree was initiated by Miller in his Master's thesis [10]. In his work, Miller raised the question of the existence of a set of natural numbers which contains no subsets of higher Turing degree. For brevity and following [4], given a reducibility $\leq_r$ we call an infinite set r-*introimmune* if it does not contain subsets of higher r-degree. Formally, given a reducibility $\leq_r$, we say that an infinite set A is r-introimmune if and only if for every $B \subseteq A$ with $|A\backslash B| = \infty$, it holds that $A \not\leq_r B$. The term "r-introimmune" was introduced in [4] to denote those sets that fail to be r-introreducible [5] in a strong way, that is, those sets that are not r-reducible to any of their co-infinite subsets.

Coming back to Miller's question, Soare [14] and Cohen (unpublished) solved it by proving independently that T-introimmune sets exist. However, such sets have a very high degree of unsolvability. In fact, Jockusch [6] showed that T-introimmune sets are Turing hard for the class of arithmetical sets. Later, Simpson [13] improved the result of Jockusch by proving that they are Turing hard for the class of hyperarithmetical sets. So, the results of Jockusch and Simpson imply that T-introimmune sets are not definable in the first order arithmetic. A natural continuation is to consider other reducibilities $\leq_r$ and to see if r-introimmune sets are arithmetical, that is definable in the first order arithmetic. For instance, [4] considered the conjunctive reducibility $\leq_{\mathrm{c}}$ and proved the existence of a c-introimmune set in Δ^0_4, the fourth Δ-level of the arithmetical hierarchy. Then, Ambos-Spies [1] extended and improved this result by proving that there are tt-introimmune sets in Δ^0_2, where tt stands for the truth table reducibility $\leq_{\mathrm{tt}}$. Therefore, [1] proves that for a reducibility $\leq_r$ stronger than $\leq_{\mathrm{tt}}$ there are arithmetical r-introimmune sets, indeed recursively approximable r-intrommune sets. On the other hand it is known that tt-introimmune sets cannot be recursively enumerable [1] (cf. also [4, Theorem 8.c]).

We conclude this brief history on introimmunity with an elegant result of Soare. Soare [15] proved an unexpected result on the topic of T-introimmunity. He observed that by the existence of non recursive sets of minimal Turing degree we directly get a set without non recursive subsets of lower Turing degree. Soare argued that the existence of such minimal Turing degrees could be combined with the existence of T-introimmune sets, in such a way to obtain a set without neither subsets of higher Turing degree nor non recursive subsets of lower Turing degree. But he proved that this is false. Let us formulate Soare's result.

Theorem 1.1 [15] *Let A be an infinite set of natural number. Then A contains a non recursive subset S such that $S <_{\mathrm{T}} A$, or $A <_{\mathrm{T}} S$.*

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Soare concluded observing that the argument used to prove the theorem "... also yields a negative answer to Sacks's question of whether there exists an infinite set all of whose subsets are either *hyperarithmetic* or of minimal *hyperdegree*" [15]. We observe that for the many-one reducibility Soare's claim is true. And even, the statement includes also the infinite recursive subsets. Precisely: *There exists an infinite set A such that for every infinite subset B of A it holds that $A \not\leq_{\mathrm{m}} B$ and $B \not\leq_{\mathrm{m}} A$.* In fact, take a cohesive co-maximal set A. Then A is m-introimmune [3]. Let B be an infinite subset of A, and let us suppose that $B \leq_{\mathrm{m}} A$. First of all, B cannot be recursive, because A is immune. Then, as $B \leq_{\mathrm{m}} A$, it follows that $\overline{B} \leq_{\mathrm{m}} \overline{A}$. But $\overline{A}$ is maximal, and every maximal set has minimal m-degree [8]. Thus $\overline{B} \equiv_{\mathrm{m}} \overline{A}$, that is $B \equiv_{\mathrm{m}} A$, and then $B \not\leq_{\mathrm{m}} A$.

A further challenge on the topic of introimmunity is the following: given a reducibility $\leq_r$ and a hierarchy of sets, to discover which is the smallest class of the hierarchy containing r-introimmune sets. In this perspective, [3] considered the many-one reducibility $\leq_{\mathrm{m}}$ and the arithmetical hierarchy, and proved the existence of a m-introimmune set in Π^0_1. This has been obtained simply by taking a cohesive co-maximal set: since every cohesive set is m-introimmune [3], the result directly follows. As Σ^0_1 cannot contain m-introimmune sets because every m-introimmune set is immune [4, Theorem 8.c], it follows that Π^0_1 is the smallest class of the arithmetical hierarchy containing such sets. Moreover, [3] proved the existence of a bi-m-introimmune set in Δ^0_2. In this case, the result cannot be proved by looking at the cohesive sets, because bi-cohesive sets do not exist. Thus, the bi-m-introimmune set of [3] has been obtained by a direct construction, using the finite-extension method. Note that $\Sigma^0_1 \cup \Pi^0_1$ cannot contain bi-m-introimmune sets, so Δ^0_2 is the smallest class of the arithmetical hierarchy containing such sets.

In this paper we continue along this line of research, and we take under consideration the many-one reducibility and the low hierarchy of sets. We observe first that from the existence of a low_2 cohesive set [7] we immediately derive the existence of a low_2 m-introimmune set. We improve this result by proving in Section 2 the existence of a low_1 m-introimmune set. Since there cannot be m-introimmune low_0 sets, the class of low_1 sets is the smallest class of the low hierarchy containing m-introimmune sets. This result refines that one obtained in [3], in the sense that our set is in the class Π^0_1. Here the tecnique is very different from that one used in [3]. In fact, we cannot take a cohesive co-maximal set, because a maximal (and then a co-maximal) set cannot be low_n for every $n \geq 0$ [9]. Thus, we get our set by a direct contruction. Namely, we construct by the finite-injury priority method a *simple* low_1 set in a way similar to that one in [16, Chapter VII.3, Theorem 1.1], with further introimmunity requirements. Finally, in Section 3 we prove our second main result by constructing, using the finite-extension method, a low_1 bi-m-introimmune set. Observe that not all low_1 sets are m-introimmune; for example, if X is any low_1 m-introimmune set, then $X \oplus X := \{2x : x \in X\} \cup \{2x+1 : x \in X\}$ is a low_1 not m-introimmune set.

Our terminology and notations are standard, so we refer to any monograph on Computability Theory like, e.g., [11, 12, 16]. For every set A and for every natural number n, we denote with $A|n$ the set $A \cap \{0, 1, \ldots, n\}$. From now on we fix an acceptable numbering $\varphi_0, \varphi_1, \ldots$ of all the Turing computable unary functions. $W_0, W_1, \ldots$ is the corresponding enumeration of all the recursively enumerable (r.e.) sets. With K we denote the halting set $\{n \in \mathbb{N} : \varphi_n(n)$ is defined$\}$. For every $e \in \mathbb{N}$ and every set $X \subseteq \mathbb{N}$, let φ^X_e be the unary function computable by the e-th oracle Turing machine with the aid of the oracle X. For every numbers e, s, x and for every oracle X we define $\varphi^X_{e,s}(x) := \varphi^X_e(x)$ if there exists $t \leq s$ such that the e-th oracle Turing machine on input x with oracle X halts in exactly t steps; in this case we say that $\varphi^X_{e,s}(x)$ is defined; $\varphi^X_{e,s}(x)$ is undefined otherwise. Whenever we write formulas like $\varphi_e(x) = \varphi_e(y)$, $\varphi_e(x) \neq x$, etc. we are assuming that $\varphi_e(x)$ is defined and $\varphi_e(y)$ is defined. Given two sets $A, B \subseteq \mathbb{N}$, A is many-one reducible to B, in short $A \leq_{\mathrm{m}} B$, if there exists a recursive function $f\colon \mathbb{N} \longmapsto \mathbb{N}$ such that for every $x \in \mathbb{N}$, $x \in A \leftrightarrow f(x) \in B$. The many-one degree of a set A is the class of sets $\{B : A \leq_{\mathrm{m}} B \wedge B \leq_{\mathrm{m}} A\} = \{B : A \equiv_{\mathrm{m}} B\}$. For the concept of a low set, we refer to the monographs [11, 16].

2 A low m-introimmune set

In this section we prove our first main result.

Theorem 2.1 *There exists a low_1 m-introimmune set in Π^0_1.*

We prove first a technical lemma which we will use in the proof of Theorem 2.1.

Lemma 2.2 *Let A and B be two sets such that $\overline{A}$ is immune, $B \subseteq \overline{A}$, $\overline{A} \backslash B$ is infinite and $\overline{A} \leq_{\mathrm{m}} B$ via f. Then, for every number z, there is a number $x \in \overline{A}$ such that $f(x) \geq z$ and $f(x) \neq x$.*

Proof. Since, by $\overline{A} \leq_{\mathrm{m}} B$ via f, $f(\overline{A}) \subseteq B$, it follows that $(\overline{A} \backslash B) \cap f(\overline{A} \backslash B) = \emptyset$, that is, for every $x \in \overline{A} \backslash B$, $f(x) \neq x$. So it suffices to show that $f(\overline{A} \backslash B)$ is infinite. For the sake of contradiction, assume that $f(\overline{A} \backslash B)$ is finite. Then $f^{-1}\big(f(\overline{A} \backslash B)\big)$ is recursively enumerable. Moreover, by $\overline{A} \leq_{\mathrm{m}} B$ via f it holds that

$$\overline{A} \backslash B \subseteq f^{-1}\big(f(\overline{A} \backslash B)\big) \subseteq \overline{A}.$$

Therefore, by infinity of $\overline{A} \backslash B$, $f^{-1}\big(f(\overline{A} \backslash B)\big)$ is an infinite r.e. subset of $\overline{A}$. But this contradicts the assumption that $\overline{A}$ is immune. □

In the proof of Theorem 2.1 we will also use the following known result on low_1 sets.

Proposition 2.3 *If $X = \bigcup_{t \geq 0} X_t$ is a recursively enumerable set and for every n,*

$$\Big[(\exists^\infty t)\varphi_{n,t}^{X_t}(n) \textit{ is defined}\Big] \rightarrow \varphi_n^X(n) \textit{ is defined},$$

then X is low_1.

We now prove Theorem 2.1.

Proof. By a finite-injury priority argument we construct an r.e. set A such that the set $\overline{A}$ will be low_1 and m-introimmune. We set $A_0 := \emptyset$ and for every $s \geq 0$ we let A_s denote the finite part of A enumerated by the end of stage s. Our final set will be $\bigcup_{s \geq 0} A_s$. The proof will be based on the construction of a low_1 simple set as it is described for instance in Soare's book [16] and we also adopt some notation introduced there. Before starting with the construction of the set A, we describe our strategy.

Strategy

It is enough to meet the following requirements, for every $e \geq 0$.

- R_{4e}: $\big[(\exists^\infty s)\big(\varphi_{e,s}^{A_s}(e) \text{ is defined}\big)\big] \rightarrow \varphi_e^A(e)$ is defined (lowness),
- R_{4e+1}: $(\exists x \geq e)(x \notin A)$ (co-infinity),
- R_{4e+2}: W_e infinite $\rightarrow W_e \cap A \neq \emptyset$ (simplicity),
- R_{4e+3}: If for any number z there is a number $x \in \overline{A}$ such that $\varphi_e(x) \geq z$ and $\varphi_e(x) \neq x$, then there is a number $u \in \overline{A}$ such that $\varphi_e(u) \in A$ (introimmunity).

Note that, by the effectivity of the construction and Proposition 2.3, the lowness requirements imply that A is low_1. The co-infinity and simplicity requirements imply that A is simple, hence $\overline{A}$ is immune. So, by Lemma 2.2 the introimmunity requirements guarantee that $\overline{A}$ is m-introimmune.

As usual we call a requirement R_n *positive* if the strategy for meeting R_n will enumerate numbers into A and *negative* if the strategy will keep numbers out of A. In order to model the restraints imposed by the negative requirements we define the *restraint* function $r\colon \mathbb{N} \times \mathbb{N} \longmapsto \mathbb{N}$, where $r(n, s-1)$ is a strict upper bound on the numbers which requirement R_n wants to keep out of A at stage $s > 0$. We say that requirement R_n is *injured* at stage $s > 0$ if a number $x < r(n, s-1)$ is enumerated into A at stage s, i.e., if $A_s | r(n, s-1) \neq A_{s-1} | r(n, s-1)$. We will ensure that any requirement R_n will be injured only finitely often by allowing the positive action for a requirement $R_{n'}$ to injure R_n only if $R_{n'}$ has higher priority than R_n, i.e. $n' < n$, and by guaranteeing that each requirement acts at most finitely often. Since at a given stage there might be more than one requirement which wants to act, we will specify when a requirement requires attention (i.e. wants to act); then, we let act the highest priority requirement which requires attention.

We next describe the strategies for meeting the different types of requirements. The lowness and co-infinity requirements are purely negative, the simplicity requirements are purely positive, and the introimmunity requirements are negative and positive. So, the lowness and co-infinite requirements will never require attention, while for the simplicity requirements R_{4e+2} we can set $r(4e+2, s) := 0$ for all $s \geq 0$, whence these requirements will never be injured.

In order to meet a negative lowness requirement R_{4e} it is enough to define $r(4e, s)$ as the *use function* of $\varphi_{e,s}^{A_s}(e)$, i.e., $r(4e, s) :=$ the least strict upper bound >0 of the set of oracles queries made by the e-th oracle Turing machine on input e with oracle A_s, if such Turing machine halts in $t \leq s$ steps, $r(4e, s) := 0$ otherwise.

Thus, by restraining A on $r(4e, s)$ if $\varphi_{e,s}^{A_s}(e)$ is defined, the relative computation will be preserved, guaranteeing that $\varphi_e^A(e)$ will be defined, unless R_{4e} will be injured later.

In order to meet a negative co-infinity requirement R_{4e+1} it is sufficient to define $r(4e+1, s) := \min x[x \geq e \wedge x \notin A_s] + 1$.

Then, assuming that requirement R_{4e+1} is not injured after stage s_0, the least number $x \geq e$ which has not yet entered A_{s_0} will be kept out of A for ever.

In order to meet a positive simplicity requirement R_{4e+2} it suffices to wait for a number x to show up in W_e which is not restrained by any higher priority requirement and then put x into A. If W_e is infinite, this will eventually happen since the restraint by the higher priority requirements will be finite. In what follows, $W_{e,s}$ denotes the finite approximation of W_e obtained by performing s steps in the enumeration of W_e. Formally, we say that:

- requirement R_{4e+2} is satisfied at stage s, if there is a number $x \leq s$ such that $x \in W_{e,s} \cap A_s$,
- requirement R_{4e+2} requires attention at stage s, if $s > 4e+2$, R_{4e+2} is not satisfied at stage $s-1$, and there is a number $x \leq s$ such that

(1) $$x \in W_{e,s} \wedge x \geq \max_{n<4e+2} r(n, s-1).$$

Finally, the strategy for meeting a positive and negative introimmunity requirement R_{4e+3} is as follows. The goal is to find a number x such that:

- x has not yet been put into A, and
- $\varphi_e(x)$ is defined and $\varphi_e(x) \neq x$.

Then, put $\varphi_e(x)$ into A and at the time restrain x from A.

In order to be compatible with the other strategies, $\varphi_e(x)$ must not be restrained by some higher priority requirement. But, if the hypothesis of the requirement R_{4e+3} is satisfied, this will not be a problem since the set of restrained numbers will be finite.

Formally, we say that:

- requirement R_{4e+3} is satisfied at stage s, if there is a number $x \leq s$ such that
$$x \notin A_s \wedge x < r(4e+3, s) \wedge \varphi_{e,s}(x) \text{ is defined } \wedge \varphi_{e,s}(x) \in A_s;$$
- requirement R_{4e+3} requires attention at stage s, if $s \geq 4e+3$, R_{4e+3} is not satisfied at stage $s-1$ and there is a number $x \leq s$ such that

(2) $$x \notin A_{s-1} \wedge \varphi_{e,s}(x) \neq x \wedge \varphi_{e,s}(x) \geq \max_{n<4e+3} r(n, s-1).$$

For requirements R_{4e+3} we initially set $r(4e+3, 0) := 0$ for every $e \geq 0$. At every stage $s > 0$ we set

$$r(4e+3, s) := \begin{cases} 0 & \text{if } \quad R_{4e+3} \text{ is injured at stage } s, \\ x+1 & \text{if } \quad R_{4e+3} \text{ becames active at stage } s, \text{ where } x \text{ is minimal such that (2) holds,} \\ r(4e+3, s-1) & \text{otherwise.} \end{cases}$$

Now, using the above introduced notations, the construction of the set A is as follows.

Construction of the set A.

Stage $s > 0$. Let A_{s-1} be the set constructed up to the end of stage $s-1$.

If there is no requirement which requires attention, then set $A_s := A_{s-1}$ and go to the next stage $s+1$. If there is a requirement R_n which requires attention, then let n_0 be the least such n. Call R_{n_0} *active* and distinguish the following two cases on n_0:

- $n_0 = 4e+2$. Pick x minimal such that (1) holds and set $A_s := A_{s-1} \cup \{x\}$. Go to the next stage $s+1$.
- $n_0 = 4e+3$. Pick x minimal such that (2) holds, set $A_s := A_{s-1} \cup \{\varphi_{e,s}(x)\}$. Go to the next stage $s+1$.

End construction of the set A. □

Note that the construction is effective. So, A is r.e. and, in order to show that A has the required properties, it is sufficient to show that all the requirements are met. We establish this by proving the following two claims.

Claim 2.4 *For any* $n \geq 0$, $\lim_{s\to\infty} r(n,s) < \infty$ *exists and* R_n *requires attention at most finitely often.*

Proof. The proof is by induction. Fix n and, by inductive hypothesis, choose a stage s_n such that, for $n' < n$, it holds that $r(n', s) = r(n', s_n)$ for all $s \geq s_n$ and $R_{n'}$ does not require attention at stage $s \geq s_n$. This means that R_n will not be injured at any stage $s \geq s_n$ and R_n will become active at any such stage at which it requires attention.

Now, if $n = 4e$ or $n = 4e + 1$ then it suffices to show that $\lim_{s\to\infty} r(n,s) < \infty$ exists since such a requirement R_n will never require attention. But if $n = 4e$, then $\lim_{s\to\infty} r(n,s) = r(n,\tilde{s})$ for the least number $\tilde{s} \geq s_n$ such that $r(n,\tilde{s}) > 0$ (and, of course, $\lim_{s\to\infty} r(n,s) = 0$ if there is no such number). If $n = 4e + 1$, then $\lim_{s\to\infty} r(n,s) = r(n,s_n)$.

If $n = 4e + 2$, then the first part of the claim is trivial since $r(n,s) = 0$ for every $s \geq 0$. Moreover, if R_{4e+2} requires attention at some stage $\hat{s} \geq s_n$, then it will become active at stage $\hat{s}$ and will be satisfied at all the later stages. Thus R_{4e+2} will require attention at most once after stage s_n.

Finally, assume that $n = 4e + 3$. Since $r(4e + 3, s) > r(4e + 3, s - 1)$ only if R_{4e+3} is active at stage s, it is sufficient to show that R_{4e+3} will require attention at most once after stage s_n. Assume that $s \geq s_n$ is the least stage at which R_{4e+3} requires attention. Then R_{4e+3} becomes active at stages s. Hence there is a number $x \notin A_s$ such that $\varphi_{e,s}(x) \in A_s$ and $x < r(4e + 3, s)$. Since R_{4e+3} will not be injured after stage s_n it follows by induction that, for $t \geq s$, $r(4e + 3, t) = r(4e + 3, s)$ and R_{4e+3} is satisfied at stage t. Thus R_{4e+3} will not require attention after stage s. □

Claim 2.5 *For any* $n \geq 0$, *requirement* R_n *is met.*

Proof. By Claim 2.4 choose a stage s_0 such that, for all $n' < n$, $R_{n'}$ does not require attention after stage s_0 and $r(n', s) = r(n', s_0)$ for every $s \geq s_0$, and let $z = \max_{n' \leq n} r(n', s_0)$. We analyse the four cases for requirement R_n.

(1) If R_n is a lowness requirement R_{4e}, then w.l.o.g. we may assume that $\varphi_{e,s}^{A_s}(e)$ is defined for infinitely many s. So, we may fix $s \geq s_0$ such that $\varphi_{e,s}^{A_s}(e)$ is defined. By definition, $r(4e, s)$ is the use function of $\varphi_{e,s}^{A_s}(e)$. Moreover, by choice of s_0, for any $t \geq s$, $r(4e, t) = r(4e, s)$ and R_{4e} will not be injured at stage t. This implies that $A|r(4e, s) = A_s|r(4e, s)$, therefore $\varphi_e^A(e) = \varphi_{e,s}^{A_s}(e)$, and this proves that R_{4e} is met.

(2) If R_n is a co-infinity requirement R_{4e+1}, then, for every $s \geq s_0$,

$$r(4e + 1, s) = r(4e + 1, s_0) = \min x[x \geq e \wedge x \notin A_{s_0}] + 1.$$

Since R_{4e+1} will not be injured at any stage $\geq s_0$, it follows that the least number $x \geq e$ in $\overline{A}_{s_0}$ will never enter A. So R_{4e+1} will be met.

(3) If R_n is a simplicity requirement R_{4e+2}, then for the sake of contradiction assume that R_n is not met. Then W_e is infinite and $W_e \cap A = \emptyset$, whence R_{4e+2} is never satisfied. By infinity of W_e we may fix a number $x \in W_e$ such that $x > z$ and a stage $s \geq \max\{s_0, 4e + 2\}$ such that $x \in W_{e,s}$. Then R_{4e+2} requires attention at stage s, contrary to the choice of s_0.

(4) If R_n is an introimmunity requirement R_{4e+3}, then for the sake of contradiction assume that R_{4e+3} is not met. By the latter we may fix a number $x \notin A$ such that $\varphi_e(x) \neq x$, $\varphi_e(x) > z$ and $\varphi_e(x) \notin A$. Moreover, since an introimmunity requirement which is satisfied at some stage s and not injured at any stage $t \geq s$ is met, R_{4e+3} is not met at any stage $s \geq s_0$. So, at the least stage $s \geq \max\{s_0, x\}$ such that $\varphi_{e,s}(x)$ is defined, R_{4e+3} will require attention. But this contradicts the choice of s_0.

This completes the proof of Claim 2.5 and of the theorem. □

3 A low bi-m-introimmune set

In this section we prove our second main result by constructing a low_1 bi-m-introimmune set. We use a sufficient condition for the bi-m-introimmunity based on *strongly bi-*m*-immune* sets. This condition is formalized in Lemma 3.2 below. First, we state the concept of strong bi-m-immunity, whose polynomial-time version was introduced in [2].

Definition 3.1 A set $X \subseteq \mathbb{N}$ is strongly bi-m-immune if and only if every m-reduction of X to any set Y is one-one almost everywhere.

Lemma 3.2 [3] *Let* $f\colon \mathbb{N} \longmapsto \mathbb{N}$ *be inductively defined by* $f(0) := 1$ *and* $f(n+1) := \max_{u,v \leq n+1}\{f(n), \varphi_u(v) : \varphi_u(v)$ *is defined*$\} + 1$. *Then any strongly bi*-m-*immune set* A *satisfying*

$$(\exists^\infty n)(A \cap \{x : n \leq x \leq f(n)\} = \emptyset) \wedge (\exists^\infty n)(\overline{A} \cap \{x : n \leq x \leq f(n)\} = \emptyset) \tag{3}$$

is bi-m-*introimmune*.

Theorem 3.3 *There exists a low*$_1$ *bi*-m-*introimmune set*.

Proof. By a finite-extension method we construct a strongly bi-m-immune set A which satisfies condition (3). We define A in stages, where the characteristic sequence of A of length l_s is defined at stage s of the construction, for appropriate numbers $l_0 < l_1 < l_2 \cdots$. Let f be the function defined in Lemma 3.2. For ease of notation in the following we denote the initial segment of the characteristic sequence of A of length l_s by the binary string σ_s. We denote the length of a binary string σ with $|\sigma|$. For every natural number x with $x \leq |\sigma|$, $\sigma(x)$ is the x-th symbol of σ. The construction will be recursive in the halting problem, i.e., the binary strings σ_s can be uniformly computed with the aid of the oracle K, thus $A \leq_{\mathrm{T}} K$. It is sufficient to meet the following requirements (for every $e \geq 0$):

- $R_{4e} : (\exists n \geq e)(A \cap \{x : n \leq x \leq f(n)\} = \emptyset)$,
- $R_{4e+1} : (\exists n \geq e)(\overline{A} \cap \{x : n \leq x \leq f(n)\} = \emptyset)$,
- R_{4e+2} : If the set $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x \neq y$ and $\varphi_e(x) = \varphi_e(y)\}$ is infinite, then there are numbers u and v such that $\varphi_e(u) = \varphi_e(v)$ and $u \in A \leftrightarrow v \notin A$,
- $R_{4e+3} : \varphi_e^A(e)$ is defined $\leftrightarrow \varphi_e^{\sigma_{4e+3}}(e)$ is defined.

Namely, the requirements R_{4e} and R_{4e+1} guarantee (3), while the requirements R_{4e+2} guarantee that A is strongly bi-m-immune. Therefore, by Lemma 3.2, A is bi-m-introimmune. Finally, since the construction will be recursive in K, the requirements R_{4e+3} ensure that A is low$_1$. In fact, by R_{4e+3}, $e \in A' \leftrightarrow \varphi_e^A(e)$ is defined $\leftrightarrow \varphi_e^{\sigma_{4e+3}}(e)$ is defined. But the latter can be decided with the oracle K, so $A' \leq_{\mathrm{T}} K$.

The definition of the initial segment σ_s of A, where the extension σ_s of σ_{s-1} is chosen so that requirement R_s will be satisfied, is as follows. Let σ_{-1} be the empty string. Given $s \geq 0$ and σ_{s-1}, for the definition of σ_s distinguish the following four cases:

- $s = 4e$. Then set $\sigma_s := \sigma_{s-1}0^{f(|\sigma_{s-1}|)}$.
- $s = 4e + 1$. Then set $\sigma_s := \sigma_{s-1}1^{f(|\sigma_{s-1}|)}$.
- $s = 4e + 2$. If there is a proper extension τ of σ_{s-1} such that there are numbers x and y with $x < y < |\tau|$ satisfying $\tau(x) \neq \tau(y)$ and $\varphi_e(x) = \varphi_e(y)$, then set $\sigma_s := \tau$ for the least such τ. Otherwise set $\sigma_s := \sigma_{s-1}0$.
- $s = 4e + 3$. If there is a proper extension τ of σ_{s-1} such that $\varphi_e^\tau(e)$ is defined, then set $\sigma_s := \tau$ for the least such τ. Otherwise set $\sigma_s := \sigma_{s-1}0$.

In order to show the correctness of the construction, first observe that σ_s is a proper extension of σ_{s-1}. Thus $A := \lim_{s\to\infty} \sigma_s$ is well defined. Moreover, since the function f is computable with oracle K and since the required properties of the string τ in case of $s = 4e+2$ or $s = 4e+3$ above are Σ_1^0-properties, the construction is recursive in K. Hence, it only remains to show that for every $e \geq 0$ the requirement R_e is met. In case of $s = 4e$ or $s = 4e + 1$ this is immediate by construction. In case of $s = 4e + 2$, w.l.o.g. assume that there are infinitely many pairs of numbers (x, y) such that $x < y$ and $\varphi_e(x) = \varphi_e(y)$. Then there is such a pair where $|\sigma_s| < y$. So, there is a proper extension τ of σ_{s-1} such that $x < y < |\tau|$ and $\tau(x) \neq \tau(y)$. It follows by construction that $\sigma_s(x) \neq \sigma_s(y)$, hence $x \in A \leftrightarrow y \notin A$, for a pair (x, y) with $\varphi_e(x) = \varphi_e(y)$. This proves that requirement R_{4e+2} is met. Finally, if $s = 4e + 3$ then either for all τ extending σ_{s-1} it holds that $\varphi_e^\tau(e)$ is undefined or σ_s is chosen so that $\varphi_e^{\sigma_s}(e)$ is defined. Obviously, in the first case both $\varphi_e^A(e)$ and $\varphi_e^{\sigma_s}(e)$ are undefined, while in the second case both $\varphi_e^A(e)$ and $\varphi_e^{\sigma_s}(e)$ are defined . Thus requirement R_{4e+3} is met too, which completes the proof. □

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References

[1] K. Ambos-Spies, Problems which cannot be reduced to any proper subproblems, in: Mathematical Foundations of Computer Science 2003, 28th International Symposium, MFCS 2003, Bratislava, Slovakia, August 25–29, 2003, Proceedings, edited by B. Rovan, P. Vojtáš. Lecture Notes in Computer Science Vol. 2747 (Springer, Berlin, 2003), pp. 162–168.

[2] J. L. Balcázar and U. Schöning, Bi-immune sets for complexity classes, Math. Syst. Theory **18**, 1–10 (1985).

[3] P. Cintioli, Sets without subsets of higher many-one degree, Notre Dame J. Form. Log. **46**(2), 207–216 (2005).

[4] P. Cintioli and R. Silvestri, Polynomial time introreducibility, Theory Comput. Syst. **36**(1), 1–15 (2003).

[5] J. C. E. Dekker and J. Myhill, Retraceable sets, Can. J. Math. **10**, 357–373 (1958).

[6] C. G. Jockusch Jr., Upward closure and cohesive degrees, Isr. J. Math. **15**, 332–335 (1973).

[7] C. Jockusch and F. Stephan, A cohesive set which is not high, Math. Log. Q. **39**(4), 515–530 (1993); correction ibid. **43**(4), 569 (1997).

[8] A. H. Lachlan, Recursively enumerable many-one degrees, Alg. Log. **11**(362), 326–358 (1972).

[9] D. A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, Z. Math. Log. Grundlagen Math. **12**, 295–310 (1966).

[10] W. Miller, Sets of integer and degrees of unsolvability, M. A. Thesis, University of Washington (1967)

[11] P. Odifreddi, Classical Recursion Theory, in: Studies in Logic and the Foundations of Mathematics Vol. 125 (North Holland, Amsterdam, 1989).

[12] H. Rogers Jr., Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).

[13] S. G. Simpson, Sets which do not have subsets of every higher degree, J. Symb. Log. **43**(1), 135–138 (1978).

[14] R. I. Soare, Sets with no subset of higher degree, J. Symb. Log. **34**(1), 53–56 (1969).

[15] R. I. Soare, A note on degrees of subsets, J. Symb. Log. **34**(2), 256 (1969).

[16] R. I. Soare, Recursively enumerable sets and degrees, Perspectives in Mathematical Logic (Springer-Verlag, 1987).