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On $\mathbb{Z}_2$-graded identities of the generalized Grassmann envelope of the upper triangular matrices $UT_{k,l}(F)$

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ABSTRACT

Let F be a field of characteristic zero and let E be the unitary Grassmann algebra generated by an infinite-dimensional F-vector space L. Denote by $\mathcal{E} = \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}$ an arbitrary $\mathbb{Z}_2$-grading on E such that the subspace L is homogeneous. Given a superalgebra $A = A^{(0)} \oplus A^{(1)}$, define its generalized Grassmann envelope $A \widehat{\otimes} \mathcal{E}$ as the superalgebra $A \widehat{\otimes} \mathcal{E} = (A^{(0)} \otimes \mathcal{E}^{(0)}) \oplus (A^{(1)} \otimes \mathcal{E}^{(1)})$. Note that when $\mathcal{E}$ is the canonical grading of E then $A \widehat{\otimes} \mathcal{E}$ is the Grassmann envelope of A. In this work we describe the generators of the T_2-ideal, $Id^{gr}(UT_{k,l}(F) \widehat{\otimes} \mathcal{E})$, of the $\mathbb{Z}_2$-graded polynomial identities of the superalgebras $UT_{k,l}(F) \widehat{\otimes} \mathcal{E}$, as well as linear bases of the corresponding relatively free graded algebras. Here, given $k \geqslant 1, l \geqslant 0$, $UT_{k,l}(F)$ is the algebra of $(k+l) \times (k+l)$ upper triangular matrices over F with the $\mathbb{Z}_2$-grading $UT_{k+l}(F) = \begin{pmatrix} UT_k(F) & 0 \\ 0 & UT_l(F) \end{pmatrix} \oplus \begin{pmatrix} 0 & M_{k\times l}(F) \\ 0 & 0 \end{pmatrix}$. In order to prove our result we obtain a similar description corresponding to the T-ideals $Id(UT_n(E))$ and $Id(UT_n(G_r))$ of ordinary polynomial identities, where G_r is the Grassmann algebra generated by an r-dimensional vector space.

1. Introduction

Superalgebras and their graded polynomial identities play a prominent role in the description of the structure of varieties of associative PI-algebras, over a field of characteristic zero, as shown in the papers by Kemer [16,17], Giambruno and Zaicev [15], Berele and Regev [2] and many other authors. A first important example of superalgebras is given by the Grassmann algebra E generated by an infinite-dimensional vector space L. A classical theorem of Krakowski and Regev determined the codimension sequence and a basis of the T-ideal $Id(E)$ of the ordinary polynomial identities of this algebra (see [18]). Later, its cocharacter sequence was obtained in [19]. Also the structure of the T_2-ideal $Id^{gr}(E)$ of all $\mathbb{Z}_2$-graded polynomial identities of E with respect to its natural $\mathbb{Z}_2$-grading is well known; see for instance [14]. Recently, we have described [9,23] the $\mathbb{Z}_2$-graded polynomial identities satisfied by E with respect to any $\mathbb{Z}_2$-grading such that L is a homogeneous subspace. On the other hand, in his celebrated results on the structure of T-ideals of the free associative algebra [16], Kemer succeeded in classifying the T-prime T-ideals over a field of characteristic zero. More precisely Kemer showed that the only non trivial T-prime ideals, in zero characteristic, are the T-ideals of the polynomial identities of the following matrix algebras: $M_m(F)$, $M_n(E)$, $M_{p,q}(E)$. Here F is the ground field and $M_{p,q}(E)$ is a certain subalgebra of $M_{p+q}(E)$. We remark that all of them have a natural superalgebra structure. Moreover, such an algebra is either simple or is the *Grassmann envelope* of a finite-dimensional $\mathbb{Z}_2$-simple superalgebra. We remark that a classification of such finite-dimensional superalgebras is given by Wall in [25]. Recall

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that the *Grassmann envelope* of the superalgebra $A = A^{(0)} \oplus A^{(1)}$ is given by

$$G(A) := (A^{(0)} \otimes \mathbf{E}^{(0)}) \oplus (A^{(1)} \otimes \mathbf{E}^{(1)})$$

where $\mathbf{E} = \mathbf{E}^{(\mathbf{0})} \oplus \mathbf{E}^{(\mathbf{1})}$ is the decomposition into the homogeneous components of the Grassmann algebra with respect to its canonical $\mathbb{Z}_2$-grading, that is, $\mathbf{E}^{(\mathbf{0})}$ and $\mathbf{E}^{(\mathbf{1})}$ are respectively the center and anticommutative part of E. As proved in [16,17], any proper T-ideal of the free algebra coincides with the ideal of the polynomial identities satisfied by the Grassmann envelope of a suitable finite-dimensional associative superalgebra. The relationship between the structure of the graded polynomial identities of a superalgebra A and those of its Grassmann envelope $G(A)$ is well understood, [16,1,3]. In particular, if $I = Id^{gr}(A)$ is the T_2-ideal of a superalgebra A, and we denote by I^* the T_2-ideal of $G(A)$, then it is proved in [16] that $I^{**} = I$.

The situation is more complicated if one considers the graded tensor product of superalgebras. Some results concerning the graded tensor product of the superalgebras listed above are contained in [21,13,8]. Besides these developments, it is natural to consider the Grassmann algebra with respect to any $\mathbb{Z}_2$-grading such that L is a homogeneous subspace and let us denote by $\mathcal{E} = \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}$ the corresponding decomposition. In this case, if $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra, we can build the superalgebra

$$A \mathbin{\widehat{\otimes}} \mathcal{E} := (A^{(0)} \otimes \mathcal{E}^{(0)}) \oplus (A^{(1)} \otimes \mathcal{E}^{(1)}),$$

which is a relevant generalization of $G(A)$. Actually, for the canonical grading of E we have $A \mathbin{\widehat{\otimes}} \mathcal{E} = G(A)$ and still all the elements of the subspace L are homogeneous of odd degree.

Besides, in the opposite situation, if the subspace L is homogeneous of even degree, then the corresponding grading of the Grassmann algebra is the trivial one, that is, $\mathcal{E}^{(0)} = E$ and $\mathcal{E}^{(1)} = 0$. In particular if, in addition, we take $A = F$ endowed with the trivial $\mathbb{Z}_2$-grading, then $A \mathbin{\widehat{\otimes}} \mathcal{E}$ and $(A \mathbin{\widehat{\otimes}} \mathcal{E}) \mathbin{\widehat{\otimes}} \mathcal{E}$ correspond to the algebras E and $E \otimes E$ endowed with the trivial $\mathbb{Z}_2$-gradings, respectively. Hence, in this case, the description of their $\mathbb{Z}_2$-graded polynomial identities can be obtained directly from the well known characterization of the T-ideals of ordinary identities $Id(E)$ and $Id(E \otimes E)$ (see [18,20]). Since $Id(E \otimes E) \neq Id(F)$, one concludes that the involutive property relating the T_2-ideal of a superalgebra A and its generalized Grassmann envelope $A \mathbin{\widehat{\otimes}} \mathcal{E}$ cannot hold in the general case.

On the other hand, it is easy to see that if a superalgebra A satisfies some monomial multilinear identity $w(y_1, \ldots, y_l, z_1, \ldots, z_m)$ then the monomial w is a graded identity also for $A \mathbin{\widehat{\otimes}} \mathcal{E}$. Hence, in order to obtain some insight concerning the relationship between the graded polynomial identities of A and those of its generalized Grassmann envelope $A \mathbin{\widehat{\otimes}} \mathcal{E}$, it is convenient to start with the case when the superalgebra A satisfies a monomial multilinear identity. Since we consider unitary superalgebras, we assume that A satisfies the monomial identity $z_1 z_2 \cdots z_m$ for some $m \geqslant 1$. In this paper we consider the generalized Grassmann envelope of a remarkable class of finite-dimensional superalgebras satisfying the monomial identity $z_1 z_2$. In this case, we notice that the superalgebras A and $G(A)$ satisfy the same $\mathbb{Z}_2$-graded polynomial identities but the situation is very different if we study the generalized Grassmann envelope $A \mathbin{\widehat{\otimes}} \mathcal{E}$. Let us consider $n = k + l$ for some integers $k \geqslant 1$, $l \geqslant 0$, and let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices with the elementary $\mathbb{Z}_2$-grading induced by the n-tuple of elements of $\mathbb{Z}_2$ $(\underbrace{0, \ldots, 0}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{l \text{ times}})$, that is:

$$UT_n(F) = \begin{pmatrix} UT_k(F) & 0 \\ 0 & UT_l(F) \end{pmatrix} \oplus \begin{pmatrix} 0 & M_{k\times l}(F) \\ 0 & 0 \end{pmatrix}.$$

We will denote by $UT_{k,l}(F)$ this superalgebra. We emphasize that $UT_{k,l}(F)$ is a minimal superalgebra according to the definition given in [15] and so it plays a fundamental role in the theory of P.I. algebras. The same result holds for any $\mathbb{Z}_2$-grading defined on $UT_n(F)$. In [4] Di Vincenzo and Drensky, as a corollary of a more general result, found a basis of the $\mathbb{Z}_2$-graded identities for $UT_{k,l}(F)$. More generally, the description of $\mathbb{Z}_2$-graded polynomial identities of $UT_n(F)$ can be found in [22,6]. We remark that $UT_{k,l}(F) \mathbin{\widehat{\otimes}} \mathcal{E}$ is isomorphic to the superalgebra

$$UT_{k,l}(\mathcal{E}) := \begin{pmatrix} UT_k(\mathcal{E}^{(0)}) & 0 \\ 0 & UT_l(\mathcal{E}^{(0)}) \end{pmatrix} \oplus \begin{pmatrix} 0 & M_{k\times l}(\mathcal{E}^{(1)}) \\ 0 & 0 \end{pmatrix}.$$

Hence these superalgebras satisfy the same $\mathbb{Z}_2$-graded identities. A first step in the study of their $\mathbb{Z}_2$-graded polynomial identities was concluded by da Silva (see [24]) with the description of the T_2-ideal $Id^{gr}(UT_{1,1}(\mathcal{E}))$. Although the description of the graded identities of $UT_{k,l}(\mathcal{E})$ for some particular cases is a consequence of [4,2], giving a description of $Id^{gr}(UT_{k,l}(\mathcal{E}))$ in the general case remains an interesting and relevant problem in PI-theory. Our main goal in this paper is to solve this problem. The key piece in our proof is the description of $Id(UT_n(G_d))$ as well as the generators of the so called "proper multilinear spaces" modulo the ideal of identities of $Id(UT_n(G_d))$. Here d denotes either a non negative integer or the symbol ∞ and G_0, G_r and G_∞ are respectively the ground field, the Grassmann algebra generated by an r-dimensional vector space and the infinite-dimensional Grassmann algebra.

We remark that, although the description of $Id(UT_n(G_\infty))$ was obtained by Berele and Regev, our proof besides describing the generators of the proper multilinear spaces modulo $Id(UT_n(G_\infty))$, also leads to the description of $Id(UT_n(G_r))$. Our proof has the advantage of allowing us to deal with the general case. More precisely, we will describe $Id^{gr}(UT_{k,l}(\mathcal{E}))$ and the generators of the so-called "Y-proper multilinear spaces" modulo $Id^{gr}(UT_{k,l}(\mathcal{E}))$ for all $k \geqslant 1$, $l \geqslant 0$ and any $\mathbb{Z}_2$-grading $\mathcal{E}$.

2. Preliminaries

In this section, we recall some definitions and we fix some notations which will be used in this text. Let F be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra generated by a countable set X over F. We say that a polynomial $f(x_1,\dots,x_t)$ in the free associative algebra $F\langle X\rangle$ is an *(ordinary) identity* of an associative algebra A if $f(a_1,\dots,a_t)=0$ for all $a_1,\dots,a_t\in A$. The set $Id(A)$ is a T-ideal of $F\langle X\rangle$, that is, it is an ideal invariant under all endomorphisms of $F\langle X\rangle$.

Moreover, it is well known that, if the characteristic of the ground field is zero, then $Id(A)$ is determined by its multilinear polynomials. If $a,b\in A$ then the Lie commutator of length 2 is defined by $[a,b]:=ab-ba$ and the Lie commutator of length n is inductively defined by $[a_1,\dots,a_n]:=[[a_1,\dots,a_{n-1}],a_n]$ for all $n\geqslant 3$ and all $a_1,\dots,a_n\in A$. Let $\mathcal{B}$ be the unitary F-subalgebra of $F\langle X\rangle$ generated by all Lie commutators in the indeterminates of X. Let P_t be the space of the multilinear polynomials in the indeterminates $x_1,\dots,x_t$. One calls the elements of $\mathcal{B}$ *proper polynomials* and the space $P_t\cap\mathcal{B}$ is denoted by Γ_t. These spaces are very important in PI-theory. In fact, if A is also a unitary algebra, then the study of $Id(A)$ is equivalent to the study of the spaces $\Gamma_t\cap Id(A)$ (see [10]).

We say that an associative algebra A over a field F is a $\mathbb{Z}_2$-*graded algebra* (or a *superalgebra*) if there exist two subspaces $A^{(0)},A^{(1)}$ such that $A=A^{(0)}\oplus A^{(1)}$ and the following relations are satisfied:

$$A^{(0)}A^{(0)}+A^{(1)}A^{(1)}\subseteq A^{(0)}\quad\text{and}\quad A^{(0)}A^{(1)}+A^{(1)}A^{(0)}\subseteq A^{(1)}.$$

We call $A^{(i)}$ the *i-homogeneous component of* A and we say that an homogeneous element $a\in A^{(i)}$ has homogeneous degree i. A subspace $W\subseteq A$ is called *homogeneous* if and only if $W=(W\cap A^{(0)})\oplus(W\cap A^{(1)})$.

One defines a free object in the class of superalgebras by considering the free F-algebra over the disjoint union of two countable sets of variables, Y and Z, whose elements are regarded as even and odd respectively. We shall denote this free superalgebra by $F\langle Y,Z\rangle$. Its even part is the space spanned by those monomials which contain an even number of elements from Z. The remaining monomials span the odd component of $F\langle Y,Z\rangle$. A polynomial $f(y_1,\dots,y_t,z_1,\dots,z_m)\in F\langle Y,Z\rangle$ is a $\mathbb{Z}_2$-*graded identity* of the superalgebra A if $f(a_1,\dots,a_t,b_1,\dots,b_m)=0$ for all $a_1,\dots,a_t\in A^{(0)}$ and $b_1,\dots,b_m\in A^{(1)}$. The set $Id^{gr}(A)$ of all $\mathbb{Z}_2$-graded identities of A is a T_2-ideal of $F\langle Y,Z\rangle$. Furthermore, similarly to the ordinary case, we denote by $P_{t,m}$ the space of the multilinear polynomials in the indeterminates $y_1,\dots,y_t,z_1,\dots,z_m$ of the free algebra $F\langle Y,Z\rangle$. Let $\mathcal{B}(Y)$ be the unitary F-subalgebra of $F\langle Y,Z\rangle$ generated by the elements of Z and by all non trivial Lie commutators in the indeterminates of $Y\cup Z$. An element of $\mathcal{B}(Y)$ is called a *Y-proper polynomial* and the space $P_{t,m}\cap\mathcal{B}(Y)$ is denoted by $\Gamma_{t,m}$. It is well known (see [5,12]) that if A is a unitary superalgebra over a field F of characteristic zero, then $Id^{gr}(A)$ is determined by its Y-proper multilinear polynomials.

Let E be the unitary Grassmann algebra generated by an infinite-dimensional F-vector space L. Denote by $\mathcal{E}=\mathcal{E}^{(0)}\oplus\mathcal{E}^{(1)}$ an arbitrary $\mathbb{Z}_2$-grading of E such that the subspace L is homogeneous and let $\mathcal{L}$ be a linear basis of L.

Given $k\geqslant 1$, $l\geqslant 0$, let $UT_{k,l}(F)$ denote the algebra

$$UT_{k+l}(F)=\left\{\begin{pmatrix}A & B\\ 0 & C\end{pmatrix};A\in UT_k(F),B\in M_{k\times l}(F),C\in UT_l(F)\right\}$$

with the $\mathbb{Z}_2$-grading

$$UT_{k+l}(F)=\left\{\begin{pmatrix}A & 0\\ 0 & C\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\right\}.$$

Since our main goal is to characterize the $\mathbb{Z}_2$-graded polynomial identities of $UT_{k,l}(F)\widehat{\otimes}\mathcal{E}$, clearly we may assume that $\mathcal{E}$ is induced by a fixed map $\|\cdot\|:\mathcal{L}\to\mathbb{Z}_2$ which associates each basis element $e_i\in\mathcal{L}$ with its $\mathbb{Z}_2$-degree in $\mathcal{E}$. Given an integer $\epsilon\geqslant 0$, some important examples of these maps are $\|\cdot\|_\epsilon$, $\|\cdot\|_{\epsilon^*}$ and $\|\cdot\|_\infty$ defined respectively by

$$\|e_i\|_\epsilon=\begin{cases}0, & i=1,\dots,\epsilon\\ 1, & \text{otherwise}\end{cases}\qquad \|e_i\|_{\epsilon^*}=\begin{cases}1, & i=1,\dots,\epsilon\\ 0, & \text{otherwise}\end{cases}\qquad \|e_i\|_\infty=\begin{cases}0, & i\text{ even}\\ 1, & i\text{ odd.}\end{cases}$$

Note that the canonical grading is induced by $\|\cdot\|_0$, while $\|\cdot\|_{0^*}$ induces the trivial one.

Denote by E_ϵ, E_{ϵ^*} and E_∞ the Grassmann algebra endowed with the $\mathbb{Z}_2$-grading induced by the maps $\|\cdot\|_\epsilon$, $\|\cdot\|_{\epsilon^*}$ and $\|\cdot\|_\infty$, respectively. It is clear that in order to characterize the graded polynomial identities of $UT_{k,l}(F)\widehat{\otimes}\mathcal{E}$ with respect to any $\mathbb{Z}_2$-grading $\mathcal{E}$ it is enough to study them for the superalgebras $UT_{k,l}(F)\widehat{\otimes}E_\xi$, where $\xi=\epsilon,\epsilon^*,\infty$.

Moreover, given $\xi\in\{\epsilon,\epsilon^*,\infty\}$, if we denote by $UT_{k,l}(E_\xi)$ the subalgebra of $UT_{k+l}(E_\xi)$ defined by

$$UT_{k,l}(E_\xi):=\left\{\begin{pmatrix}A & B\\ 0 & C\end{pmatrix};A\in UT_k(E_\xi^{(0)}),B\in M_{k\times l}(E_\xi^{(1)}),C\in UT_l(E_\xi^{(0)})\right\}$$

and we consider the natural $\mathbb{Z}_2$-grading of $UT_{k,l}(E_\xi)$ given by:

$$UT_{k,l}(E_\xi)=\left\{\begin{pmatrix}A & 0\\ 0 & C\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\right\},$$

then clearly its graded identities coincides with the graded identities of $UT_{k,l}(F)\widehat{\otimes}E_\xi$.

Therefore we wish to describe the $\mathbb{Z}_2$-graded identities of $UT_{k,l}(E_\xi)$ for any $\xi \in \{\epsilon, \epsilon^*, \infty\}$, $k \geqslant 1$ and $l \geqslant 0$. In order to do this, for $r \geqslant 1$, let G_r denote the Grassmann algebra generated by an r-dimensional vector space and let G_∞ denote the infinite F-dimensional Grassmann algebra; that is

$$G_r = \langle 1, v_1, v_2, \ldots, v_r \,;\; v_iv_j = -v_jv_i\rangle_F$$
$$G_\infty = \langle 1, v_1, v_2, \ldots \,;\; v_iv_j = -v_jv_i\rangle_F.$$

Still, when convenient, we identify G_0 with the ground field F. Now we will describe the ordinary identities of $UT_n(G_d)$, $d = r, \infty$ with $r \geqslant 0$. This characterization will be used in the description of $UT_{k,l}(E_\xi)$.

3. The ordinary identities of $UT_n(G_d)$

First of all we note that, for any polynomial $f(x_1, \ldots, x_m) \in F\langle X\rangle$ and any matrices $A_1 = (a_{ij}^{(1)}), \ldots, A_m = (a_{ij}^{(m)}) \in UT_n(G_d)$, it is easy to check that $f(A_1, \ldots, A_m) = (b_{ij}) \in UT_n(G_d)$, where $b_{ii} = f(a_{ii}^{(1)}, \ldots, a_{ii}^{(m)})$ for all $i = 1, \ldots, n$. Furthermore, we remember that $[x_1, x_2, x_3]$ is an ordinary identity of G_d, for any $d = r, \infty$, and the Jacobson radical of $UT_n(F)$ is given by the vector space $J(UT_n(F)) = \mathrm{span}_F\{e_{ij}\,;\; 1 \leqslant i < j \leqslant n\}$ and it is a nilpotent ideal of index n. Denote by J the vector space

$$J := \mathrm{span}_F\{a_{ij}e_{ij}\,;\; 1 \leqslant i < j \leqslant n \text{ and } a_{ij} \in G_d\},$$

then it is clear that $J^n = 0$.

We have the following lemma.

Lemma 1. *The polynomial $[x_1, x_2, x_3]\cdots[x_{3n-2}, x_{3n-1}, x_{3n}]$ is an ordinary identity of $UT_n(G_d)$, for $d = r, \infty$ with $r \geqslant 0$. Furthermore, given $r' \geqslant 0$, we have that $[x_1, x_2]\cdots[x_{2(n+r')-1}, x_{2(n+r')}]$ is an ordinary identity of $UT_n(G_{2r'})$ and $UT_n(G_{2r'+1})$.*

Proof. From the remarks above, we clearly have $[A_1, A_2, A_3] \in J$ for any matrices $A_1, A_2, A_3 \in UT_n(G_d)$ and therefore $[x_1, x_2, x_3]\cdots[x_{3n-2}, x_{3n-1}, x_{3n}]$ is an ordinary identity of $UT_n(G_d)$. In order to prove the second claim, suppose that there exists $r' \geqslant 0$ such that $[x_1, x_2]\cdots[x_{2(n+r')-1}, x_{2(n+r')}]$ is not an ordinary identity of $UT_n(G_r)$, for some $r = 2r'$ or $r = 2r' + 1$. Then there exist pairs of indices (i_u, j_u) with $1 \leqslant i_u \leqslant j_u \leqslant n$ and elements $a_{i_uj_u} \in G_r$ such that

$$[a_{i_1j_1}e_{i_1j_1}, a_{i_2j_2}e_{i_2j_2}]\cdots[a_{i_{2\tilde{r}-1}j_{2\tilde{r}-1}}e_{i_{2\tilde{r}-1}j_{2\tilde{r}-1}}, a_{i_{2\tilde{r}}j_{2\tilde{r}}}e_{i_{2\tilde{r}}j_{2\tilde{r}}}] \neq 0,$$

where $\tilde{r} = n + r'$. Now, since $J^n = 0$, at most $n - 1$ of the above commutators belong to J. In other words, at least $r' + 1$ commutators does not belong to J and therefore they are of the form $[be_{tt}, \tilde{b}e_{tt}] \neq 0$, for some $1 \leqslant t \leqslant n$, $b, \tilde{b} \in G_r$. But this is impossible since $r < 2(r' + 1)$ and we have only r distinct generators $v_i \in G_r$. □

Definition 2. Given $n \geqslant 1$ and $r' \geqslant 0$, we denote by I_n the T-ideal of $F\langle X\rangle$ generated by the polynomial $[x_1, x_2, x_3]\cdots[x_{3n-2}, x_{3n-1}, x_{3n}]$, and by $I_{n,r'}$ the T-ideal generated by I_n and the polynomial $[x_1, x_2]\cdots[x_{2(n+r')-1}, x_{2(n+r')}]$. That is,

$$I_n = \langle[x_1, x_2, x_3]\cdots[x_{3n-2}, x_{3n-1}, x_{3n}]\rangle_T$$

and

$$I_{n,r'} = \langle[x_1, x_2, x_3]\cdots[x_{3n-2}, x_{3n-1}, x_{3n}],\;\; [x_1, x_2]\cdots[x_{2(n+r')-1}, x_{2(n+r')}]\rangle_T.$$

In this section we will prove that

$$Id(UT_n(G_\infty)) = I_n$$

and

$$Id(UT_n(G_{2r'})) = Id(UT_n(G_{2r'+1})) = I_{n,r'}.$$

In order to describe a linear basis of $\Gamma_t(UT_n(G_d)) := \frac{\Gamma_t}{\Gamma_t \cap Id(UT_n(G_d))}$, let us give some lemmas, definitions and remarks. In particular, the next lemma means that $I^n = I_n$, where $I = \langle[x_1, x_2, x_3]\rangle_T$.

Lemma 3. *Let $\tilde{f} = f_1f_2\cdots f_n$ be a multilinear polynomial of $F\langle X\rangle$ such that each f_i belongs to the T-ideal, I, generated by $[x_1, x_2, x_3]$. Then $\tilde{f} \in I_n$.*

Proof. It is enough to prove that I_n contains all products $\tilde{c} = w_1c_1w_2c_2w_3\cdots w_nc_nw_{n+1}$, where each c_i is a commutator of length 3 and each w_i is either equal to 1 or $w_i \in X$.

If $w_i = 1$ for all $i = 2, \ldots, n$, then it is clear that $\tilde{c} \in I_n$. Otherwise, consider $i := \min\{j\,;\; 2 \leqslant j \leqslant n \text{ and } w_j \neq 1\}$. Then $\tilde{c} = w_1c_1\cdots c_{i-1}\underbrace{w_ic_i}w_{i+1}\cdots w_nc_nw_{n+1}$ and we can write

$$\tilde{c} = w_1c_1\cdots c_{i-1}\underbrace{c_iw_i}w_{i+1}\cdots w_nc_nw_{n+1} - w_1c_1\cdots c_{i-1}\underbrace{[c_i, w_i]}w_{i+1}\cdots w_nc_nw_{n+1}.$$

By proceeding inductively we obtain that $\tilde{c}$ is a linear combination of products of the form $w_1\bar{c}_1\bar{c}_2\cdots\bar{c}_n\bar{w}_{n+1}$, where either $\bar{c}_i = c_i$ or $\bar{c}_i = [c_i, \bar{w}_i]$, for a certain $\bar{w}_i \in F\langle X\rangle$. In any case, it is clear that $w_1\bar{c}_1\bar{c}_2\cdots\bar{c}_n\bar{w}_{n+1} \in I_n$ and the same holds for $\tilde{c}$. □

Definition 4. (1) We say that a commutator $c = [x_{i_1}, x_{i_2}, \ldots, x_{i_u}]$ is *semistandard* if the indices $i_1, i_2, \ldots, i_u$ satisfy the inequalities $i_1 > i_2$ and $i_2 < \cdots < i_u$ if $u > 2$, and $i_1 < i_2$ if $u = 2$. If c is a semistandard commutator, we take $\mathcal{S} = \{i_2, \ldots, i_u\}$ and we write $c = c_{i_1,\mathcal{S}}$.

(2) We say that a product $\tilde{c} = c_1\cdots c_m$ of commutators is *m-semistandard* if it is a product of m semistandard commutators, that is, if each $c_i = c_{k_i,\mathcal{S}_i}$, for certain positive integer k_i and ordered set $\mathcal{S}_i = \{s_{1,i}, \ldots, s_{S_i,i}\}$. In this case we put $\tilde{k} = (k_1, \ldots, k_m)$, $\tilde{\mathcal{S}} = (\mathcal{S}_1, \ldots, \mathcal{S}_m)$ and we write $\tilde{c} = c_{\tilde{k},\tilde{\mathcal{S}}}$.

(3) Given an m-semistandard product $\tilde{c} = c_1\cdots c_m = c_{\tilde{k},\tilde{\mathcal{S}}}$ we say that c_i is a *mark commutator* if it satisfies one of the conditions:
- its length is at least 3
- $i > 1$, both c_i and c_{i-1} have length 2, c_{i-1} is not a mark commutator and $k_i < s_{1,i-1}$.

(4) Given an m-semistandard product $\tilde{c} = c_{\tilde{k},\tilde{\mathcal{S}}}$, we say that $\tilde{c}$ is *m'-marked* if it has exactly m' mark commutators.

The reader can find some examples of m-semistandard m'-marked commutators in Remark 6.

Remark 5. It is important to note that the definition of "mark commutator" (and therefore the definition of "m'-marked") is given (for commutators with length 2) by looking at the product of commutators from left to right.

Remark 6. Given an m-semistandard product $\tilde{c} \in \Gamma_t$, it is easy to verify that there exists $m' \geqslant 0$ such that $\tilde{c}$ is an m'-marked product. If $t \geqslant 2$ and $m' \geqslant 1$, it is clear that there exists positive integers $h_1, \ldots, h_{m'}$ and ordered sets $\mathcal{A}_1, \ldots, \mathcal{A}_{m'+1}$, $\mathcal{B}_1, \ldots, \mathcal{B}_{m'}$ such that

$$\begin{aligned}\tilde{c} = {} & \underbrace{[x_{a_{1,1}}, x_{a_{2,1}}]\cdots[x_{a_{A_1-1,1}}, x_{a_{A_1,1}}]}_{\text{elements of } \mathcal{A}_1}[x_{h_1}, \underbrace{x_{b_{1,1}}, \ldots, x_{b_{B_1,1}}}_{\text{elements of } \mathcal{B}_1}] \\ & \cdots\underbrace{[x_{a_{1,m'}}, x_{a_{2,m'}}]\cdots[x_{a_{A_{m'}-1,m'}}, x_{a_{A_{m'},m'}}]}_{\text{elements of } \mathcal{A}_{m'}}[x_{h_{m'}}, \underbrace{x_{b_{1,m'}}, \ldots, x_{b_{B_{m'},m'}}}_{\text{elements of } \mathcal{B}_{m'}}] \\ & \cdot\underbrace{[x_{a_{1,m'+1}}, x_{a_{2,m'+1}}]\cdots[x_{a_{A_{m'+1}-1,m'+1}}, x_{a_{A_{m'+1},m'+1}}]}_{\text{elements of } \mathcal{A}_{m'+1}}\end{aligned}$$

where $\mathcal{A}_i = \{a_{1,i}, \ldots, a_{A_i,i}\}$ with $A_i \geqslant 0$ an even number, $\mathcal{B}_j = \{b_{1,j}, \ldots, b_{B_j,j}\}$ with $B_j \geqslant 1$, for $i \in \{1, \ldots, m'+1\}$, $j \in \{1, \ldots, m'\}$. Still, for $j \in \{1, \ldots, m'\}$, we have $A_j \geqslant 2$ and $h_j < b_{1,j},\ a_{A_j,j}$ if $B_j = 1$; and $h_j > b_{1,j}$ if $B_j > 1$.

Note that the commutators $[x_{h_j}, x_{b_{1,j}}, \ldots, x_{b_{B_j,j}}]$, with $j = 1, \ldots, m'$, are exactly the m' mark commutators in the product $\tilde{c}$.

Notation: $\tilde{c} = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\mathcal{A}_2(h_2\mathcal{B}_2)\cdots\mathcal{A}_{m'}(h_{m'}\mathcal{B}_{m'})\mathcal{A}_{m'+1}}$. When convenient, we will use the notation $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}$, where $\tilde{\mathcal{A}} = (\mathcal{A}_1, \ldots, \mathcal{A}_{m'+1})$, $\tilde{h} = (h_1, \ldots, h_{m'})$ and $\tilde{\mathcal{B}} = (\mathcal{B}_1, \ldots, \mathcal{B}_{m'})$.

For instance,

- $\tilde{c} = [x_1, x_4][x_2, x_5][x_3, x_6][x_7, x_8]$ is a 1-marked product ($[x_2, x_5]$ is the only mark commutator) with $\mathcal{A}_1 = \{1, 4\}$, $h_1 = 2$, $\mathcal{B}_1 = \{5\}$ and $\mathcal{A}_2 = \{3, 6, 7, 8\}$.
- $[x_1, x_3][x_2, x_4][x_5, x_6][x_7, x_8]$ is a 1-marked product ($[x_2, x_4]$ is the mark commutator) with $\mathcal{A}_1 = \{1, 3\}$, $h_1 = 2$, $\mathcal{B}_1 = \{4\}$ and $\mathcal{A}_2 = \{5, 6, 7, 8\}$.
- $[x_2, x_3][x_1, x_4][x_7, x_{10}][x_6, x_9][x_5, x_8]$ is a 2-marked product ($[x_1, x_4]$ and $[x_6, x_9]$ are the only mark commutators) with $\mathcal{A}_1 = \{2, 3\}$, $h_1 = 1$, $\mathcal{B}_1 = \{4\}$, $\mathcal{A}_2 = \{7, 10\}$, $h_2 = 6$, $\mathcal{B}_2 = \{9\}$ and $\mathcal{A}_3 = \{5, 8\}$.
- $[x_1, x_4][x_2, x_5, x_8][x_3, x_7][x_6, x_8]$ is a 2-marked product ($[x_2, x_5, x_8]$ and $[x_6, x_8]$ are the mark commutators) with $\mathcal{A}_1 = \{1, 4\}$, $h_1 = 2$, $\mathcal{B}_1 = \{5, 8\}$, $\mathcal{A}_2 = \{3, 7\}$, $h_2 = 6$, $\mathcal{B}_2 = \{8\}$ and $\mathcal{A}_3 = \emptyset$.

If $m' = 0$ then clearly t is an even number and $\tilde{c} = [x_1, x_2]\cdots[x_{t-1}, x_t]$. In this case we also denote $\tilde{c} = c_{\mathcal{A}_1} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}$, where $\tilde{\mathcal{A}} = \mathcal{A}_1 = \{1, \ldots, t\}$, $\tilde{h} = \emptyset$ and $\tilde{\mathcal{B}} = \emptyset$.

We will prove that, given $t \geqslant 2$, the space Γ_t is spanned, modulo I_n, by products $\tilde{c} = c_1\cdots c_m \in \Gamma_t$ which are m-semistandard m'-marked products of commutators such that $m \geqslant 1$ and $0 \leqslant m' \leqslant n-1$. On the other hand, the space Γ_t is spanned, modulo $I_{n,r'}$, by products $\tilde{c} = c_1\cdots c_m \in \Gamma_t$ which are m-semistandard m'-marked products of commutators such that $1 \leqslant m \leqslant n + r' - 1$ and $0 \leqslant m' \leqslant n-1$.

For our purposes it is extremely important to understand what happens when we consider in Γ_t an m-semistandard m'-marked product of commutators in case all of the commutators are of length 2 and $m' \geqslant n$. If we try to write it as a linear combination, modulo I_n, of m-semistandard $\overline{m'}$-marked products of commutators with $\overline{m'} < n$, then the well known fact (see Lemma 1.4.2 of [11]) that

$$[x_{\sigma(l_1)}, x_{\sigma(l_2)}]\cdots[x_{\sigma(l_{u-1})}, x_{\sigma(l_u)}] - (-1)^{\sigma}[x_{l_1}, x_{l_2}]\cdots[x_{l_{u-1}}, x_{l_u}] \in I \quad (1)$$

plays an important role (here $(-1)^{\sigma}$ is the sign of the permutation $\sigma \in S_u$).

Let us see an example.

Example 7. Consider the 5-semistandard 2-marked product

$$\tilde{c} = [x_2, x_3][x_1, x_4][x_7, x_{10}][x_6, x_9][x_5, x_8].$$

Then $\tilde{c}$ can be written, modulo $I_2 = \langle [x_1, x_2, x_3][x_4, x_5, x_6]\rangle_T$, as a linear combination of the 5-semistandard 1-marked products

$$[x_1, x_2][x_3, x_4][x_7, x_{10}]\underline{[x_6, x_9]}[x_5, x_8], \qquad [x_1, x_2][x_3, x_4][x_6, x_7][x_9, x_{10}]\underline{[x_5, x_8]}$$
$$\text{and} \quad [x_2, x_3]\underline{[x_1, x_4]}[x_5, x_6][x_7, x_8][x_9, x_{10}],$$

and of the 5-semistandard 0-marked product

$$[x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8][x_9, x_{10}],$$

where the mark commutators are underlined.

In fact, since $[x_2, x_3][x_1, x_4] - [x_1, x_2][x_3, x_4] = [x_2, x_3][x_1, x_4] - (-1)^{(1\,2\,3)}[x_1, x_2][x_3, x_4]$ and $[x_7, x_{10}][x_6, x_9] + [x_6, x_7][x_9, x_{10}] = [x_7, x_{10}][x_6, x_9] - (-1)^{(6\,7\,10\,9)}[x_6, x_7][x_9, x_{10}]$ belong to $I = \langle [x_1, x_2, x_3]\rangle_T$, then by Lemma 3 we get

$$([x_2, x_3][x_1, x_4] - [x_1, x_2][x_3, x_4])\,([x_7, x_{10}][x_6, x_9] + [x_6, x_7][x_9, x_{10}])\,[x_5, x_8] \in I_2 \tag{2}$$

and then modulo I_2 we have

$$\tilde{c} \equiv -[x_2, x_3]\underline{[x_1, x_4]}[x_6, x_7][x_9, x_{10}]\underline{[x_5, x_8]} + [x_1, x_2][x_3, x_4][x_7, x_{10}]\underline{[x_6, x_9]}[x_5, x_8]$$
$$+ [x_1, x_2][x_3, x_4][x_6, x_7][x_9, x_{10}]\underline{[x_5, x_8]}.$$

Since the polynomial $\bar{c} = [x_2, x_3][x_1, x_4][x_6, x_7][x_9, x_{10}][x_5, x_8]$ is a 5-semistandard 2-marked product, then by using that

$$([x_2, x_3][x_1, x_4] - [x_1, x_2][x_3, x_4])\,([x_6, x_7][x_9, x_{10}][x_5, x_8] - [x_5, x_6][x_7, x_8][x_9, x_{10}]) \in I_2,$$

we obtain

$$\bar{c} \equiv [x_2, x_3]\underline{[x_1, x_4]}[x_5, x_6][x_7, x_8][x_9, x_{10}] + [x_1, x_2][x_3, x_4][x_6, x_7][x_9, x_{10}]\underline{[x_5, x_8]}$$
$$- [x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8][x_9, x_{10}].$$

Note that by using Eq. (2) we could write the 5-semistandard 2-marked product $\tilde{c} = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\mathcal{A}_2(h_2\mathcal{B}_2)\mathcal{A}_3}$ (where $\mathcal{A}_1 = \{2, 3\}$, $h_1 = 1$, $\mathcal{B}_1 = \{4\}$, $\mathcal{A}_2 = \{7, 10\}$, $h_2 = 6$, $\mathcal{B}_2 = \{9\}$ and $\mathcal{A}_3 = \{5, 8\}$) as a linear combination of the 5-semistandard 2-marked product $\bar{c} = c_{\bar{\mathcal{A}}_1(\bar{h}_1\bar{\mathcal{B}}_1)\bar{\mathcal{A}}_2(\bar{h}_2\bar{\mathcal{B}}_2)\bar{\mathcal{A}}_3}$ (where $\bar{\mathcal{A}}_1 = \{2, 3\}$, $\bar{h}_1 = 1$, $\bar{\mathcal{B}}_1 = \{4\}$, $\bar{\mathcal{A}}_2 = \{6, 7, 9, 10\}$, $\bar{h}_2 = 5$, $\bar{\mathcal{B}}_2 = \{8\}$ and $\bar{\mathcal{A}}_3 = \emptyset$) and two 5-semistandard 1-marked products. Nevertheless if we compare the two 5-semistandard 2-marked products $\tilde{c}$ and $\bar{c}$ then we have $\mathcal{A}_1 = \bar{\mathcal{A}}_1$, $h_1 = \bar{h}_1$, $\mathcal{B}_1 = \bar{\mathcal{B}}_1$ and $\mathcal{A}_2 \cup \{h_2\} \cup \mathcal{B}_2 = \bar{\mathcal{A}}_2$ (that is, if we look at the product of commutators from left to right then the second mark commutator of $\bar{c}$ is in a later position than the second mark commutator of $\tilde{c}$). On the other hand, by using a similar idea, we could write $\bar{c}$ as a linear combination of two 5-semistandard 1-marked products and one 5-semistandard 0-marked product.

In general, given an m-semistandard m'-marked product $\tilde{c}$ of commutators with all of them of length 2 and $m' \geqslant n$, by proceeding as above we can move, "in some sense", at least one mark commutator of each product to the right. If we go on with this procedure, we can write $\tilde{c}$ as a linear combination, modulo I_n, of m-semistandard $\overline{m'}$-marked products of commutators with $\overline{m'} < n$. We will formulate this result in next lemma.

Lemma 8. *Let $\tilde{c}$ be a polynomial in Γ_t such that $\tilde{c}$ is an m-semistandard m'-marked product of commutators with length 2. Then $\tilde{c}$ can be written, modulo I_n, as a linear combination of some multilinear polynomials $\bar{c} \in \Gamma_t$, where $\bar{c}$ is an m-semistandard $\overline{m'}$-marked product of commutators with length 2 such that $0 \leqslant \overline{m'} \leqslant n - 1$.*

Proof. The result is clearly true for $0 \leqslant m' \leqslant n - 1$. Suppose that $m' \geqslant n$ and write $\tilde{c} = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\mathcal{A}_2(h_2\mathcal{B}_2)\cdots\mathcal{A}_{m'}(h_{m'}\mathcal{B}_{m'})\mathcal{A}_{m'+1}}$, with $A_j \geqslant 2$ and $h_j < b_{1,j},\ a_{A_j,j}$ for all $j \in \{1, \ldots, m'\}$ (see Remark 6). By applying induction we may assume that the result holds for all polynomials $w \in \Gamma_t$ such that $w = c_{\bar{\mathcal{A}}_1(\bar{h}_1\bar{\mathcal{B}}_1)\bar{\mathcal{A}}_2(\bar{h}_2\bar{\mathcal{B}}_2)\cdots\bar{\mathcal{A}}_s(\bar{h}_s\bar{\mathcal{B}}_s)\bar{\mathcal{A}}_{s+1}}$ is an m-semistandard s-marked product of commutators with length 2 such that either $0 \leqslant s \leqslant m' - 1$ or $s = m'$ and there exists $l \in \{1, \ldots, m'\}$ such that $\mathcal{A}_i = \bar{\mathcal{A}}_i$, $h_i = \bar{h}_i$, $\mathcal{B}_i = \bar{\mathcal{B}}_i$ for all $i \in \{1, \ldots, l-1\}$ and $\mathcal{A}_l \cup \{h_l\} \cup \mathcal{B}_l \subseteq \bar{\mathcal{A}}_l$, that is, $w = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\cdots\mathcal{A}_{l-1}(h_{l-1}\mathcal{B}_{l-1})\bar{\mathcal{A}}_l(\bar{h}_l\bar{\mathcal{B}}_l)\cdots\bar{\mathcal{A}}_{m'}(\bar{h}_{m'}\bar{\mathcal{B}}_{m'})\bar{\mathcal{A}}_{m'+1}}$ and $\mathcal{A}_l \cup \{h_l\} \cup \mathcal{B}_l \subseteq \bar{\mathcal{A}}_l$ for some $l \in \{1, \ldots, m'\}$.

Now, for any D-tuple $\mathcal{D} = (d_1, \ldots, d_D)$ such that D is even, we denote by $u_{\mathcal{D}}$ the product $[x_{d_1}, x_{d_2}] \cdots [x_{d_{D-1}}, x_{d_D}]$. Further, for each $j \in \{1, \ldots, m'\}$, denote by $\mathcal{C}_j = \{c_{1,j}, \ldots, c_{A_j+2,j}\}$ the ordered set formed by the elements of $\mathcal{A}_j \cup \{h_j\} \cup \mathcal{B}_j$, that is, $c_{1,j} < c_{2,j} < \cdots < c_{A_j+2,j}$ and $c_{i,j} \in \mathcal{A}_j \cup \{h_j\} \cup \mathcal{B}_j$ for all $i = 1, \ldots, A_j + 2$. If σ_j is the permutation of the elements of $\mathcal{C}_j$ given by

$$\sigma_j(c_{i,j}) = \begin{cases} a_{i,j} & \text{if } 1 \leqslant i \leqslant A_j \\ h_j & \text{if } i = A_j + 1 \\ b_{1,j} & \text{if } i = A_j + 2, \end{cases}$$

then $\sigma_j(\mathcal{C}_j) := \{\sigma_j(c_{1,j}), \ldots, \sigma_j(c_{A_j+2,j})\} = \mathcal{A}_j \cup \{h_j\} \cup \mathcal{B}_j$ and

$$\tilde{c} = u_{\sigma_1(\mathcal{C}_1)} \cdots u_{\sigma_{m'}(\mathcal{C}_{m'})} u_{\mathcal{A}_{m'+1}}.$$

Since $m' \geqslant n$, by Eq. (1) and Lemma 3, we get

$$(u_{\sigma_1(\mathcal{C}_1)} - (-1)^{\sigma_1} u_{\mathcal{C}_1}) \cdots (u_{\sigma_{m'}(\mathcal{C}_{m'})} - (-1)^{\sigma_{m'}} u_{\mathcal{C}_{m'}})\, u_{\mathcal{A}_{m'+1}} \in I_n$$

and thus $\tilde{c}$ is, modulo I_n, a linear combination of polynomials $\varpi \in \Gamma_t$ such that $\varpi = u_{\mathcal{D}_1} \cdots u_{\mathcal{D}_{m'}} u_{\mathcal{A}_{m'+1}}$, where each $\mathcal{D}_i$ is either equal to $\mathcal{C}_i$ or $\sigma_i(\mathcal{C}_i)$, and there exists at least one index l such that $\mathcal{D}_l = \mathcal{C}_l$. Now, let θ denote the quantity of mark commutators of ϖ and write $\varpi = c_{\mathcal{A}_1^{\varpi}(h_1^{\varpi}\mathcal{B}_1^{\varpi})\mathcal{A}_2^{\varpi}(h_2^{\varpi}\mathcal{B}_2^{\varpi})\cdots\mathcal{A}_\theta^{\varpi}(h_\theta^{\varpi}\mathcal{B}_\theta^{\varpi})\mathcal{A}_{\theta+1}^{\varpi}}$. It is clear that $0 \leqslant \theta \leqslant m'$. If $0 \leqslant \theta \leqslant m'-1$ then the result follows directly from the induction hypothesis. Then assume that $\theta = m'$ and let ℓ be the smallest index l such that $\mathcal{D}_l = \mathcal{C}_l$. Then $\mathcal{D}_i = \sigma_i(\mathcal{C}_i)$ for all $i = 1, \ldots, \ell-1$, and $\mathcal{D}_\ell = \mathcal{C}_\ell$, which implies $\mathcal{A}_i^{\varpi} = \mathcal{A}_i$, $h_i^{\varpi} = h_i$ and $\mathcal{B}_i^{\varpi} = \mathcal{B}_i$ for all $i \in \{1, \ldots, \ell-1\}$ and $\mathcal{A}_\ell \cup \{h_\ell\} \cup \mathcal{B}_\ell \subseteq \mathcal{A}_\ell^{\varpi}$, that is, $\varpi = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\cdots\mathcal{A}_{\ell-1}(h_{\ell-1}\mathcal{B}_{\ell-1})\mathcal{A}_\ell^{\varpi}(h_\ell^{\varpi}\mathcal{B}_\ell^{\varpi})\cdots\mathcal{A}_{m'}^{\varpi}(h_{m'}^{\varpi}\mathcal{B}_{m'}^{\varpi})\mathcal{A}_{m'+1}^{\varpi}}$ and $\mathcal{A}_\ell \cup \{h_\ell\} \cup \mathcal{B}_\ell \subseteq \mathcal{A}_\ell^{\varpi}$. Therefore also in this case the result follows by the induction hypothesis. □

Remark 9. By proceeding similarly as in Lemma 8, we can prove that if $\tilde{c} = w_1 v w_2$ is a polynomial in Γ_t such that $v \in I_\alpha$ and, for $i = 1, 2$, w_i is an m_i-semistandard m_i'-marked product of commutators with length 2; then $\tilde{c}$ can be written, modulo I_n, as a linear combination of some multilinear polynomials $\bar{w}_1 v \bar{w}_2 \in \Gamma_t$, where, for $i = 1, 2$, $\bar{w}_i$ is an m_i-semistandard $\overline{m_i'}$-marked product of commutators with length 2 such that $0 \leqslant \overline{m_1'} + \overline{m_2'} \leqslant n - \alpha - 1$.

Note that the crucial point here is that, by using notation similar to the above lemma, if we write, for $i = 1, 2$,

$$w_i = c_{\mathcal{A}_1^{(i)}(h_1^{(i)}\mathcal{B}_1^{(i)})\cdots\mathcal{A}_{m_i'}^{(i)}(h_{m_i'}^{(i)}\mathcal{B}_{m_i'}^{(i)})\mathcal{A}_{m_i'+1}^{(i)}} = u_{\sigma_1^{(i)}(\mathcal{C}_1^{(i)})} \cdots u_{\sigma_{m_i'}^{(i)}(\mathcal{C}_{m_i'}^{(i)})} u_{\mathcal{A}_{m_i'+1}^{(i)}}$$

and if $m_1' + m_2' \geqslant n - \alpha$, then we need to use that

$$f_{w_1} v f_{w_2} \in I_n,$$

where

$$f_{w_i} = (u_{\sigma_1^{(i)}(\mathcal{C}_1^{(i)})} - (-1)^{\sigma_1^{(i)}} u_{\mathcal{C}_1^{(i)}}) \cdots (u_{\sigma_{m_i'}^{(i)}(\mathcal{C}_{m_i'}^{(i)})} - (-1)^{\sigma_{m_i'}^{(i)}} u_{\mathcal{C}_{m_i'}^{(i)}})\, u_{\mathcal{A}_{m_i'+1}^{(i)}}, \quad i = 1, 2.$$

Proposition 10. *Given $t \geqslant 2$, the space Γ_t is spanned, modulo I_n, by products $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ which are m-semistandard m'-marked products of commutators such that $m \geqslant 1$ and $0 \leqslant m' \leqslant n-1$.*

Proof. Let $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ be a product of commutators. Then it is well known (see [7]) that we can assume that $\tilde{c}$ is an m-semistandard product. Denote by $p_{\tilde{c}}$ the number of commutators c_i in $\tilde{c}$ with length at least 3. By Lemma 3 we may assume $p_{\tilde{c}} \leqslant n-1$. Thus we can write

$$\tilde{c} = w_1 c_1' w_2 c_2' \cdots w_{p_{\tilde{c}}} c_{p_{\tilde{c}}}' w_{p_{\tilde{c}}+1},$$

where each c_i' is a semistandard commutator with length at least 3 and each w_i is either equal to 1 or is a product of m_i-semistandard commutators with length 2. It follows from Remark 6 that for each $i \in \{1, \ldots, p_{\tilde{c}}+1\}$ there exists m_i' such that w_i is an m_i'-marked product. Thus $m' = p_{\tilde{c}} + m_1' + \cdots + m_{p_{\tilde{c}}+1}'$ and we need to prove that we may assume $m_1' + \cdots + m_{p_{\tilde{c}}+1}' \leqslant n - 1 - p_{\tilde{c}}$.

If $w_i = 1$ for all $i = 1, \ldots, p_{\tilde{c}}+1$, then w_i is a 0-marked product for all $i = 1, \ldots, p_{\tilde{c}}+1$, which implies that $m_1' + \cdots + m_{p_{\tilde{c}}+1}' = 0 \leqslant n - 1 - p_{\tilde{c}}$.

If there exists j such that $w_j \neq 1$ and $w_i = 1$ for all $i \neq j$, then by applying Lemma 8 we obtain that w_j can be written, modulo $I_{n-p_{\tilde{c}}}$, as a linear combination of some multilinear polynomials $\bar{w}_j$, where $\bar{w}_j$ is an m_j-semistandard $\overline{m_j'}$-marked product of commutators with length 2 such that $0 \leqslant \overline{m_j'} \leqslant n - p_{\tilde{c}} - 1$. Since $I_n = I^n$, and each $c_u' \in I$ for $u = 1, \ldots, p_{\tilde{c}}$, we concluded that $\tilde{c} = c_1' \cdots c_{j-1}' w_j c_j' \cdots c_{p_{\tilde{c}}}'$ can be written, modulo I_n, as a linear combination of $c_1' \cdots c_{j-1}' \bar{w}_j c_j' \cdots c_{p_{\tilde{c}}}'$, where $\bar{w}_j$ is an m_j-semistandard $\overline{m_j'}$-marked product of commutators with length 2 such that $0 \leqslant \overline{m_j'} \leqslant n - 1 - p_{\tilde{c}}$.

If there exist j_1, j_2 such that $w_{j_1}, \neq 1$, $w_{j_2} \neq 1$ and $w_i = 1$ for all $i \neq j_1, j_2$, then the result follows by proceeding similarly as above and using Remark 9.

The other cases are similar. □

Proposition 11. *Given $t \geqslant 2$, the space Γ_t is spanned, modulo $I_{n,r'}$, by products $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ which are m-semistandard m'-marked products of commutators such that $1 \leqslant m \leqslant n + r' - 1$ and $0 \leqslant m' \leqslant n-1$.*

Proof. Let $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ be a product of commutators. Since $I_n \subset I_{n,r'}$, by using Proposition 10 we can assume that $\tilde{c}$ is an m-semistandard m'-marked product of commutators such that $m \geqslant 1$ and $0 \leqslant m' \leqslant n-1$. Since $[x_1, x_2] \cdots [x_{2m-1}, x_{2m}]$ lies in $I_{n,r'}$ when $m \geqslant n + r'$, the same conclusion holds for $\tilde{c}$. Therefore we must have $m \leqslant n + r' - 1$ and the proof is complete. □

The following theorem is the main result of this section.

Theorem 12. *Given $n \geqslant 2$ and $r' \geqslant 0$ let I_n and $I_{n,r'}$ be the T-ideals of $F\langle X\rangle$ introduced in Definition 2 and let $t \geqslant 2$. Then the following hold:*

(a) $Id(UT_n(G_\infty)) = I_n$.
Moreover, a linear basis of $\Gamma_t(UT_n(G_\infty))$ consists of the products $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ of commutators such that each $\tilde{c}$ is an m-semistandard m'-marked product with $m \geqslant 1$ and $0 \leqslant m' \leqslant n-1$.

(b) $Id(UT_n(G_{2r'})) = Id(UT_n(G_{2r'+1})) = I_{n,r'}$.
Furthermore, for $r = 2r'$ or $r = 2r'+1$ a linear basis of $\Gamma_t(UT_n(G_r))$ consists of the products $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ of commutators such that each $\tilde{c}$ is an m-semistandard m'-marked product with $1 \leqslant m \leqslant n+r'-1$ and $0 \leqslant m' \leqslant n-1$.

Proof. Consider the T-ideal I_n given in Definition 2. It follows from Lemma 1 that $I_n \subseteq Id(UT_n(G_\infty))$ and, by Proposition 10, the space Γ_t is spanned, modulo I_n, by products $\tilde{c} = c_1 \cdots c_m \in \Gamma_t$ of commutators such that each $\tilde{c}$ is an m-semistandard m'-marked product with $m \geqslant 1$ and $0 \leqslant m' \leqslant n-1$. Therefore, it is sufficient to show that these polynomials are linearly independent modulo $Id(UT_n(G_\infty))$.

By using the notation $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}$, given in Remark 6 (from now on in this proof, it is extremely important to have in mind the decomposition given in that remark), let

$$f = \sum_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}} \alpha_{\tilde{c}}\, c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}$$

be a linear combination of these polynomials and assume that $f \in Id(UT_n(G_\infty))$. We must prove that every coefficient is zero.

If $t = 2$ then $[x_1, x_2] = c_{\{1,2\},\emptyset,\emptyset}$ is the only polynomial $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}} \in \Gamma_t$. Then $f(x_1, x_2) = \alpha_{\tilde{c}} c_{\{1,2\},\emptyset,\emptyset} = \alpha_{\tilde{c}}[x_1, x_2]$ and thus $0 = f(e_{11}, e_{12}) = \alpha_{\tilde{c}} e_{12}$ which implies $\alpha_{\tilde{c}} = 0$.

Assume then that $t \geqslant 3$. Given a product $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}} = c_{\mathcal{A}_1(h_1\mathcal{B}_1)\cdots\mathcal{A}_{m'}(h_{m'}\mathcal{B}_{m'})\mathcal{A}_{m'+1}}$, we remember that $A_i = |\mathcal{A}_i|$ and $B_j = |\mathcal{B}_j|$, for $i = 1, \ldots, m'+1$ and $j = 1, \ldots, m'$. For convenience, if $m' < n-1$, we will set $A_i = 0$ and $B_j = 0$ for $i = m'+2, \ldots, n$ and $j = m'+1, \ldots, n-1$. In this notation, we can associate with $\tilde{c}$ the $(2n-1)$-tuple

$$\mathbf{O}_{\tilde{c}} := (A_1, B_1, \ldots, A_{n-1}, B_{n-1}, A_n).$$

Given two products $w = c_{\tilde{\mathcal{A}}^w,\tilde{h}^w,\tilde{\mathcal{B}}^w}$, $\varpi = c_{\tilde{\mathcal{A}}^\varpi,\tilde{h}^\varpi,\tilde{\mathcal{B}}^\varpi} \in \Gamma_t$, we write $w =_{\mathbf{o}} \varpi$ if $\mathbf{O}_w = \mathbf{O}_\varpi$. Moreover, we write $w <_{\mathbf{o}} \varpi$ if there exist $\ell \geqslant 1$ such that either

- the $(2\ell-2)$-tuples $(A_1^w, B_1^w, \ldots, A_{\ell-1}^w, B_{\ell-1}^w)$ and $(A_1^\varpi, B_1^\varpi, \ldots, A_{\ell-1}^\varpi, B_{\ell-1}^\varpi)$ coincide and $A_\ell^w < A_\ell^\varpi$
- or the $(2\ell-1)$-tuples $(A_1^w, B_1^w, \ldots, A_\ell^w)$ and $(A_1^\varpi, B_1^\varpi, \ldots, A_\ell^\varpi)$ coincide and $B_\ell^w > B_\ell^\varpi$.

Obviously the notation $w \leqslant_{\mathbf{o}} \varpi$ means $w <_{\mathbf{o}} \varpi$ or $w =_{\mathbf{o}} \varpi$. Note that $w \leqslant_{\mathbf{o}} \varpi$ for all products ϖ if and only if $\mathbf{O}_w = (0, t-1, 0, \ldots, 0)$.

Consider a product $w = c_{\tilde{\mathcal{A}}^w,\tilde{h}^w,\tilde{\mathcal{B}}^w} = c_{(h_1^w\mathcal{B}_1^w)}$ (which there exists because $t \geqslant 3$). We evaluate this product by substituting

$$\bar{x}_{h_1^w} = e_{12} \quad \text{and} \quad \bar{x}_{b_{q,1}^w} = e_{11}$$

for all $1 \leqslant q \leqslant B_1^w$, and obtain that $c_{\tilde{\mathcal{A}}^\varpi,\tilde{h}^\varpi,\tilde{\mathcal{B}}^\varpi}(\bar{x}_1, \ldots, \bar{x}_t) \neq 0$ if and only if $\tilde{\mathcal{A}^\varpi} = \emptyset$, $\tilde{h}^\varpi = h_1^w$ and $\tilde{\mathcal{B}}^\varpi = \mathcal{B}_1^w$. Thus $0 = f(\bar{x}_1, \ldots, \bar{x}_t) = \alpha_w c_{\tilde{\mathcal{A}}^w,\tilde{h}^w,\tilde{\mathcal{B}}^w}(\bar{x}_1, \ldots, \bar{x}_t) = (-1)^{t-1}\alpha_w e_{12}$ which implies $\alpha_w = 0$. Thus $\alpha_{\tilde{c}} = 0$ for all products $\tilde{c} = c_{\emptyset,h_1,\mathcal{B}_1}$.

Consider now an m-semistandard m'-marked product $\varpi = c_{\tilde{\mathcal{A}}^\varpi,\tilde{h}^\varpi,\tilde{\mathcal{B}}^\varpi} \in \Gamma_t$ such that $\mathbf{O}_\varpi \neq (0, t-1, 0, \ldots, 0)$. By applying induction we may assume that $\alpha_{\tilde{c}} = 0$ for all products $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}} \in \Gamma_t$ such that $\tilde{c} <_{\mathbf{o}} \varpi$.

Consider the evaluation

$$\bar{x}_{a_{i,j}^\varpi} = v_{A'_1+\cdots+A'_{j-1}+i}\, e_{jj}, \qquad \bar{x}_{h_l^\varpi} = e_{l,l+1} \quad \text{and} \quad \bar{x}_{b_{q,l}^\varpi} = e_{l,l} \tag{3}$$

for all $1 \leqslant i \leqslant A_j^\varpi$, $1 \leqslant j \leqslant m'+1$, $1 \leqslant q \leqslant B_l^\varpi$ and $1 \leqslant l \leqslant m'$. We will show that $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) = 0$ for all products $\tilde{c} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}} \in \Gamma_t$ such that either $\varpi <_{\mathbf{o}} \tilde{c}$ or $\tilde{c} =_{\mathbf{o}} \varpi$ and $\tilde{c} \neq \varpi$.

Consider first the case $\tilde{c} =_{\mathbf{o}} \varpi$ and suppose that $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) \neq 0$. We will prove that $\tilde{c} = \varpi$. Note that (3) together with $\mathbf{O}_{\tilde{c}} = \mathbf{O}_\varpi$ and $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) \neq 0$ imply $\mathcal{A}_1 \cup \{h_1\} \cup \mathcal{B}_1 = \mathcal{A}_1^\varpi \cup \{h_1^\varpi\} \cup \mathcal{B}_1^\varpi$ (because all the $A_1^\varpi + B_1^\varpi + 1$ variables of the form $v_u e_{11}$, e_{12} or e_{11} must appear in the first $A_1 + B_1 + 1$ positions of $\tilde{c}$) and thus $\mathcal{A}_2 \cup \{h_2\} \cup \mathcal{B}_2 = \mathcal{A}_2^\varpi \cup \{h_2^\varpi\} \cup \mathcal{B}_2^\varpi$ and so forth. Hence $\mathcal{A}_i \cup \{h_i\} \cup \mathcal{B}_i = \mathcal{A}_i^\varpi \cup \{h_i^\varpi\} \cup \mathcal{B}_i^\varpi$ for all $i = 1, \ldots, m'$, and $\mathcal{A}_{m'+1} = \mathcal{A}_{m'+1}^\varpi$. Moreover, it is clear that the B_1^ϖ elements e_{11} must appear in the same commutator in which e_{12} appears, and this commutator must be the last one of the first $\frac{A_1}{2}+1$ commutators of $\tilde{c}$. Thus $\{h_1\} \cup \mathcal{B}_1 = \{h_1^\varpi\} \cup \mathcal{B}_1^\varpi$ and $\mathcal{A}_1 = \mathcal{A}_1^\varpi$, still, since all the commutators of $\tilde{c}$ and ϖ are semistandard, we get that $h_1 = h_1^\varpi$ and $\mathcal{B}_1 = \mathcal{B}_1^\varpi$. By proceeding inductively we conclude that $\mathcal{A}_i = \mathcal{A}_i^\varpi$, $h_i = h_i^\varpi$ and $\mathcal{B}_i = \mathcal{B}_i^\varpi$ for all $i = 1, \ldots, m'$ and therefore $\tilde{c} = \varpi$.

Consider now the case $\varpi <_{\mathbf{o}} \tilde{c}$. Then there exists $\ell \geqslant 1$ such that one of the following holds:

- the $(2\ell-2)$-tuples $(A_1^{\varpi}, B_1^{\varpi}, \ldots, A_{\ell-1}^{\varpi}, B_{\ell-1}^{\varpi})$ and $(A_1, B_1, \ldots, A_{\ell-1}, B_{\ell-1})$ coincide and $A_\ell^{\varpi} < A_\ell$
- or the $(2\ell-1)$-tuples $(A_1^{\varpi}, B_1^{\varpi}, \ldots, A_\ell^{\varpi})$ and $(A_1, B_1, \ldots, A_\ell)$ coincide and $B_\ell^{\varpi} > B_\ell$.

In the first case, following the ideas discussed above we get $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) = 0$ if we do not have equality between the $(2\ell-1)$-tuples of sets $(\mathcal{A}_1^{\varpi}, \mathcal{B}_1^{\varpi}, \ldots, \mathcal{A}_{\ell-1}^{\varpi}, \mathcal{B}_{\ell-1}^{\varpi}, \mathcal{A}_\ell^{\varpi})$ and $(\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{A}_{\ell-1}, \mathcal{B}_{\ell-1}, \{a_{1,\ell}, \ldots, a_{A_\ell^{\varpi},\ell}\})$. Thus we can assume

$$\mathcal{A}_i = \mathcal{A}_i^{\varpi}, \qquad h_i = h_i^{\varpi}, \qquad \mathcal{B}_i = \mathcal{B}_i^{\varpi} \quad \text{for all } i = 1, \ldots, \ell-1, \quad \text{and} \quad \{a_{1,\ell}, \ldots, a_{A_\ell^{\varpi},\ell}\} = \mathcal{A}_\ell^{\varpi}.$$

If $B_\ell^{\varpi} > 1$ then we have more than one element $e_{\ell\ell}$ while the $(\frac{A_1+\cdots+A_{\ell-1}+A_\ell^{\varpi}}{2} + \ell)$-th commutator of $\tilde{c}$ has only length 2, thus $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) = 0$. If $B_\ell^{\varpi} = 1$ then it follows from the definition of mark commutators that we cannot have $\{h_\ell^{\varpi}\} \cup \mathcal{B}_\ell^{\varpi} = \{a_{A_\ell^{\varpi}+1,\ell}, a_{A_\ell^{\varpi}+2,\ell}\}$ and thus we also have $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) = 0$.

By proceeding similarly in the second case we also conclude that $\tilde{c}(\bar{x}_1, \ldots, \bar{x}_t) = 0$. Thus $0 = f(\bar{x}_1, \ldots, \bar{x}_t) = \alpha_{\varpi} c_{\tilde{\mathcal{A}}^{\varpi}, \tilde{h}^{\varpi}, \tilde{\mathcal{B}}^{\varpi}}(\bar{x}_1, \ldots, \bar{x}_t)$ and therefore $\alpha_{\varpi} = 0$.

The proof of part (b) is similar to part (a). Clearly, in this case we will use Proposition 11 instead of Proposition 10. However, it is important to note that, given an m-semistandard m'-marked product $\varpi = c_{\tilde{\mathcal{A}}^{\varpi}, \tilde{h}^{\varpi}, \tilde{\mathcal{B}}^{\varpi}} \in \Gamma_t$, in order to prove that $\alpha_{\varpi} = 0$, we can use the substitution (3) only when $m - m' \leqslant r'$. In other cases we properly evaluate each one of the remaining $m - m' - r'$ commutators $[\bar{x}_{a_{i,j}^{\varpi}}, \bar{x}_{a_{i+1,j}^{\varpi}}]$ with substitutions of the form $\bar{x}_{a_{i,j}^{\varpi}} = e_{\theta,\theta+1}$ and $\bar{x}_{a_{i,j+1}^{\varpi}} = e_{\theta,\theta}$ for some appropriate $\theta \in \{1, \ldots, n-1\}$. $\square$

It is worth noting that when $d = 0$ we work with the algebra $UT_n(F)$ and therefore Theorem 12 is a generalization of Theorem 5.2.1 of [10] for fields F with characteristic zero. On the other hand, if $d = \infty$ then we work with $UT_n(E)$, and thus we have proved, in a different way from [2], that

$$Id(UT_n(E)) = \langle [x_1, x_2, x_3] \cdots [x_{3n-2}, x_{3n-1}, x_{3n}] \rangle_T.$$

If we identify G_d with $UT_1(G_d)$ then it is clear that we have the following.

Remark 13. Given $t \geqslant 2$, $d = r, \infty$ where $r = 2r'$ or $r = 2r' + 1$ with $r' \geqslant 0$, then:

(a) $Id(UT_1(G_\infty)) = I_1 = \langle [x_1, x_2, x_3] \rangle_T$.
Moreover, $\Gamma_t(UT_1(G_\infty)) = 0$ if t is odd, and a linear basis of $\Gamma_t(UT_1(G_\infty))$ is given by the product $[x_1, x_2] \cdots [x_{t-1}, x_t]$ if t is even.

(b) $Id(UT_1(G_r)) = I_{1,r'} = \langle [x_1, x_2, x_3], [x_1, x_2] \cdots [x_{2r'+1}, x_{2(r'+1)}] \rangle_T$.
Furthermore, if t is even and $2 \leqslant t \leqslant 2r'$ then a linear basis of $\Gamma_t(UT_1(G_r))$ is given by the product $[x_1, x_2] \cdots [x_{t-1}, x_t]$. Otherwise $\Gamma_t(UT_1(G_r)) = 0$.

4. The $\mathbb{Z}_2$-graded identities of $UT_{k,l}(E_\xi)$

In this section we characterize the $\mathbb{Z}_2$-graded identities of $UT_{k,l}(E_\xi)$ for any $k \geqslant 1$, $l \geqslant 0$ and $\xi = \epsilon, \epsilon^*, \infty$. Note that the case $l = 0$ is an easy corollary of Theorem 12 and Remark 13.

Corollary 14. *Given $k \geqslant 1$ and $\xi \in \{\epsilon, \epsilon^*, \infty\}$, where $\epsilon = 2\epsilon'$ or $\epsilon = 2\epsilon' + 1$ with $\epsilon' \geqslant 0$, let $Id^{gr}(UT_{k,0}(E_\xi))$ be the T_2-ideal of graded polynomial identities for the superalgebra $UT_{k,0}(E_\xi)$. Then*

(a) *$Id^{gr}(UT_{k,0}(E_\infty))$ and $Id^{gr}(UT_{k,0}(E_{\epsilon^*}))$ are both generated by the following polynomials:*
- z_1
- $[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$.

(b) *$Id^{gr}(UT_{k,0}(E_\epsilon))$ is generated by the following polynomials*
- z_1
- $[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$
- $[y_1, y_2] \cdots [y_{2(k+\epsilon')-1}, y_{2(k+\epsilon')}]$.

Proof. It is enough to note that $Id^{gr}(UT_{k,0}(E_\xi))$ coincides with the set

$$\langle z_1, f(y_1, \ldots, y_u); f(x_1, \ldots, x_u) \text{ is a generator of } Id(UT_k(G_d)) \rangle_{T_2},$$

where

$$d = \begin{cases} \epsilon & \text{if } \xi = \epsilon \\ \infty & \text{otherwise.} \end{cases} \qquad (4)$$

$\square$

Note that z_1 is also a graded identity for the superalgebra $UT_{k,l}(E_{0^*})$ for any $k, l \geqslant 1$, since E_{0^*} has the trivial grading. Moreover by taking $n = \max\{k, l\}$ it is clear that

$$Id^{gr}(UT_{k,l}(E_{0^*})) \cap \Gamma_{t,0} = \{f(y_1, \ldots, y_t) \in \Gamma_{t,0}; f(x_1, \ldots, x_t) \in Id(UT_n(G_\infty)) \cap \Gamma_t\}.$$

Therefore we have the following result.

Proposition 15. *Given $k, l \geqslant 1$, set $n = \max\{k, l\}$. Then*

$$Id^{gr}(UT_{k,l}(E_{0^*})) = \langle z_1, [y_1, y_2, y_3] \cdots [y_{3n-2}, y_{3n-1}, y_{3n}]\rangle_{T_2}.$$

Thus we need to study the cases $k, l \geqslant 1$ and $\xi \neq 0^*$. If $A, A' \in UT_k(E_\xi^{(0)})$, $B, B' \in M_{k\times l}(E_\xi^{(1)})$ and $C, C' \in UT_l(E_\xi^{(0)})$, then

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BC' \\ 0 & CC' \end{pmatrix}.$$

Thus it is clear that

$$z_1z_2 \in Id^{gr}(UT_{k,l}(E_\xi)). \tag{5}$$

Moreover, by taking $n = \max\{k, l\}$ and d as in (4), we have

$$Id^{gr}(UT_{k,l}(E_\xi)) \cap \Gamma_{t,0} = \{f(y_1, \ldots, y_t) \in \Gamma_{t,0}; f(x_1, \ldots, x_t) \in Id(UT_n(G_d)) \cap \Gamma_t\}, \tag{6}$$

for $k > l$:

$$\{z_1f(y_1, \ldots, y_t) \in \Gamma_{t,1}; f(x_1, \ldots, x_t) \in Id(UT_l(G_d)) \cap \Gamma_t\} \subseteq Id^{gr}(UT_{k,l}(E_\xi)) \cap \Gamma_{t,1}, \tag{7}$$

and for $k < l$:

$$\{f(y_1, \ldots, y_t)z_1 \in \Gamma_{t,1}; f(x_1, \ldots, x_t) \in Id(UT_k(G_d)) \cap \Gamma_t\} \subseteq Id^{gr}(UT_{k,l}(E_\xi)) \cap \Gamma_{t,1}. \tag{8}$$

Furthermore, by proceeding similarly as in the proof of Lemma 1 we find:

Lemma 16. *Given $k, l \geqslant 1$ and either $\epsilon = 2\epsilon'$ or $\epsilon = 2\epsilon' + 1$, then the polynomial*

$$[y_1, y_2] \cdots [y_{2(k+u)-1}, y_{2(k+u)}]z[y_{2(k+u)+1}, y_{2(k+u+1)}] \cdots [y_{2(k+l+\epsilon'-1)-1}, y_{2(k+l+\epsilon'-1)}]$$

is a graded identity of $UT_{k,l}(E_\epsilon)$ for any $0 \leqslant u \leqslant \epsilon' - 1$.

Finally we have our main result:

Theorem 17. *Given $k, l \geqslant 1$ and $\xi \in \{\epsilon, \epsilon^*, \infty\}$, where $\epsilon = 2\epsilon'$ or $\epsilon = 2\epsilon' + 1$, then:*

(a) *The generators of the T_2-ideal $Id^{gr}(UT_{k,l}(E_\infty))$ and $Id^{gr}(UT_{k,l}(E_{\epsilon^*}))$, with $\epsilon \geqslant 1$, depend on k, l and are given by the following table:*

$k > l$	$k = l$	$k < l$
z_1z_2	z_1z_2	z_1z_2
$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$	$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$	$[y_1, y_2, y_3] \cdots [y_{3l-2}, y_{3l-1}, y_{3l}]$
$z[y_1, y_2, y_3] \cdots [y_{3l-2}, y_{3l-1}, y_{3l}]$		$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]z$

(b) *The generators of the T_2-ideal $Id^{gr}(UT_{k,l}(E_\epsilon))$ with $\epsilon \geqslant 0$ depend on k, l, ϵ' and are given by the table below together with the polynomials*

$$[y_1, y_2] \cdots [y_{2(k+u)-1}, y_{2(k+u)}]z[y_{2(k+u)+1}, y_{2(k+u+1)}] \cdots [y_{2(k+l+\epsilon'-1)-1}, y_{2(k+l+\epsilon'-1)}],$$

for $0 \leqslant u \leqslant \epsilon' - 1$.

$k > l$	$k = l$	$k < l$
z_1z_2	z_1z_2	z_1z_2
$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$	$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]$	$[y_1, y_2, y_3] \cdots [y_{3l-2}, y_{3l-1}, y_{3l}]$
$z[y_1, y_2, y_3] \cdots [y_{3l-2}, y_{3l-1}, y_{3l}]$		$[y_1, y_2, y_3] \cdots [y_{3k-2}, y_{3k-1}, y_{3k}]z$
$[y_1, y_2] \cdots [y_{2(k+\epsilon')-1}, y_{2(k+\epsilon')}]$	$[y_1, y_2] \cdots [y_{2(k+\epsilon')-1}, y_{2(k+\epsilon')}]$	$[y_1, y_2] \cdots [y_{2(l+\epsilon')-1}, y_{2(l+\epsilon')}]$
$z[y_1, y_2] \cdots [y_{2(l+\epsilon')-1}, y_{2(l+\epsilon')}]$		$[y_1, y_2] \cdots [y_{2(k+\epsilon')-1}, y_{2(k+\epsilon')}]z$

Proof. First of all, in order to distinguish between the elements of the basis $\mathcal{L} = \{e_1, e_2, \dots\}$ of L with respect their $\mathbb{Z}_2$-degree, we set $\eta_i := e_{2i}$ and $\zeta_i := e_{2i-1}$ for all $i = 1, 2, \dots$ in the superalgebra E_∞. Similarly we write $\eta_i := e_{\epsilon+i}$ for all $i = 1, 2, \dots$ and $\zeta_i := e_i$, for $i = 1, \dots, \epsilon$, in the superalgebra E_{ϵ^*}. Finally, in the superalgebra E_ϵ we put $\eta_i := e_i$, for $i = 1, \dots, \epsilon$ and $\zeta_i := e_{\epsilon+i}$ for all $i = 1, 2, \dots$.

Consider first the case $k > l$ and $\xi = \infty$. Denote by I the T_2-ideal generated by $z_1z_2, [y_1, y_2, y_3]\cdots[y_{3k-2}, y_{3k-1}, y_{3k}]$ and $z[y_1, y_2, y_3]\cdots[y_{3l-2}, y_{3l-1}, y_{3l}]$. By (5), (6) and (7) we get $I \subseteq Id^{gr}(UT_{k,l}(E_\infty))$ and, in order to characterize $Id^{gr}(UT_{k,l}(E_\infty))$, it is enough to consider the spaces $\Gamma_{t,0}$ and $\Gamma_{t,1}$ only. Moreover it follows from (6), Theorem 12 and Remark 13 that it is enough to study the space $\Gamma_{t,1}$.

By Lemma 3 of [24] and Proposition 10, the space $\Gamma_{t,1}$ is spanned, modulo I, by polynomials

$$\tilde{u} = \tilde{v}[z_1, y_{l_1}, \dots, y_{l_s}]\tilde{w} \in \Gamma_{t,1}$$

such that $\tilde{v}$ (respect. $\tilde{w}$) is an m-semistandard (respect. μ-semistandard) m'-marked (respect. μ'-marked) product of commutators in the variables of Y with

$$0 \leqslant m' \leqslant k-1 \quad \text{and} \quad 0 \leqslant \mu' \leqslant l-1;$$

and further $l_1 < \cdots < l_s$, for $s \geqslant 0$. Therefore, it is sufficient to show that these polynomials are linearly independent modulo $Id^{gr}(UT_{k,l}(E_\infty))$.

By using the notation $\tilde{v} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}$ and $\tilde{w} = c_{\tilde{\mathcal{C}},\tilde{g},\tilde{\mathcal{D}}}$, given in Remark 6, and by writing $c_{z,\tilde{l}} = [z_1, y_{l_1}, \dots, y_{l_s}]$, where $\tilde{l} = \{l_1, \dots, l_s\}$, we have $\tilde{u} = c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}\, c_{z,\tilde{l}}\, c_{\tilde{\mathcal{C}},\tilde{g},\tilde{\mathcal{D}}}$. Let

$$f = \sum_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}},\tilde{l},\tilde{\mathcal{C}},\tilde{g},\tilde{\mathcal{D}}} \alpha_{\tilde{u}}\, c_{\tilde{\mathcal{A}},\tilde{h},\tilde{\mathcal{B}}}\, c_{z,\tilde{l}}\, c_{\tilde{\mathcal{C}},\tilde{g},\tilde{\mathcal{D}}}$$

be a linear combination of these polynomials and assume that $f \in Id^{gr}(UT_{k,l}(E_\infty))$. We must prove that every coefficient is zero.

In order to do this, it is enough to proceed by induction as in Theorem 12. In this case, given $\tilde{u} = \tilde{v}[z_1, y_{l_1}, \dots, y_{l_s}]\tilde{w}$ and $\tilde{u'} = \tilde{v'}[z_1, y_{l'_1}, \dots, y_{l'_{s'}}]\tilde{w'}$ we write $\tilde{u} =_{\mathbf{o}} \tilde{u'}$ if

$$\tilde{v} =_{\mathbf{o}} \tilde{v'}, \qquad s = s' \quad \text{and} \quad \tilde{w} =_{\mathbf{o}} \tilde{w'};$$

and we write $\tilde{u} <_{\mathbf{o}} \tilde{u'}$ if $\tilde{u}$ and $\tilde{u'}$ satisfy one of the following conditions:

- $\tilde{v} <_{\mathbf{o}} \tilde{v'}$
- $\tilde{v} =_{\mathbf{o}} \tilde{v'}$ and $s > s'$
- $\tilde{v} =_{\mathbf{o}} \tilde{v'}$, $s = s'$ and $\tilde{w} <_{\mathbf{o}} \tilde{w'}$.

Moreover we work with evaluations of $\tilde{u}$ where we substitute

$$\bar{y}_{a_{i,j}} = \eta_{A_1+\cdots+A_{j-1}+i}\, e_{jj}, \qquad \bar{y}_{h_q} = e_{q,q+1} \quad \text{and} \quad \bar{x}_{b_{p,q}} = e_{qq}$$

for all $1 \leqslant i \leqslant A_j$, $1 \leqslant j \leqslant m'+1$, $1 \leqslant p \leqslant B_q$, $1 \leqslant q \leqslant m'$,

$$\bar{z}_1 = \zeta_1 e_{m'+1,k+1} \quad \text{and} \quad \bar{y}_{l_q} = e_{m'+1,m'+1}$$

for all $1 \leqslant q \leqslant s$ and, by putting $A = A_1 + \cdots + A_{m'+1}$,

$$\bar{y}_{c_{i,j}} = \eta_{A+C_1+\cdots+C_{j-1}+i}\, e_{k+j,k+j}, \qquad \bar{y}_{g_q} = e_{k+q,k+q+1} \quad \text{and} \quad \bar{y}_{d_{p,q}} = e_{k+q,k+q}$$

for all $1 \leqslant i \leqslant C_j$, $1 \leqslant j \leqslant \mu'+1$, $1 \leqslant p \leqslant D_q$, $1 \leqslant q \leqslant \mu'$.

The cases $k = l$ and $k \geqslant l$ with $\xi = \infty$ are similar.

Now, since we have an infinite number of distinct elements η_i and at least one element ζ_j in ϵ^* for $\epsilon \geqslant 1$, then it is clear that we can repeat the proof above also in the case when $\xi = \epsilon^*$.

The proof of part (b) is similar to part (a). Nevertheless, similarly to part (b) of Theorem 12, we have that some additional conditions must be satisfied by the generators $\tilde{u}$ of the quotient space $\Gamma_{t,1}(Id^{gr}(UT_{k,l}(E_\epsilon))$. Namely, m and μ must satisfy also the conditions:

$$0 \leqslant m \leqslant k+\epsilon'-1, \qquad 0 \leqslant \mu \leqslant l+\epsilon'-1 \quad \text{and} \quad 0 \leqslant m+\mu \leqslant k+l+\epsilon'-2. \quad \square$$

It is worth noting that Corollary 14, Proposition 15 and Theorem 17 together constitute an important generalization of Theorems 3.5 and 3.6 of [6]. Furthermore, Proposition 15 and Theorem 17 together generalize Theorems 8, 9 and 12 of [24].

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