A class of groups with inert subgroups *

to the memory of Jim Wiegold

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Abstract. Two subgroups H and K of a group are commensurable iff $H \cap K$ has finite index in both H and K. We prove that hyper-(abelian or finite) groups with finite abelian total rank in which every subgroup is commensurable to a normal one are finite-by-abelian-by-finite.
Keywords: *normal, commensurable, inert, finite index, subgroup.*

Introduction

In [2] authors study CF-groups (core finite), i.e. groups G in which $|H : H_G|$ is finite for each subgroup H. In other words, each H contains a normal subgroup of G with finite index in H. This class arises in a natural way as the dual of the class of groups G with $|H^G : H|$ finite for each $H \leq G$. The latter class was earlier considered in a very celebrated paper of B.H.Neumann [8] and revealed to be the class of finite-by-abelian groups, i.e. groups with finite derived group. In fact in [2] it is proved that a locally finite CF-groups are abelian-by-finite (i.e. they have an abelian subgroup with finite index) and are BCF too. This means that they are CF in a bounded way, i.e. the above index is bounded independently of H. As Tarski groups are CF, a complete classification of CF-groups seems to be much difficult. Anyway, in [12] it is proved that a locally radical CF-group is abelian-by-finite indeed, while an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given.

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To consider the above two classes in the same framework, we consider *CN-groups*, that is groups in which each subgroup H is cn (commensurable to a normal one). Recall that two subgroups H and K of a group G are told *commensurable* iff $H \cap K$ has finite index in both H and K. This is an equivalence relation and will be denoted by $\sim$. Clearly, if $H \,\mathrm{cn}\, \mathrm{G}$, then H is *inert* in G, that is commensurable with all conjugates of its. Groups whose all subgroups are inert are called *inertial (or also totally inertial) groups* and have received much attention (see [1] and [11]).

From above quoted results following questions arise.

Question 1: Given a group theoretical class $\mathbf{X}$, is a CN-group in $\mathbf{X}$ finite-by-abelian-by-finite?

We show that *Question 1 has a positive answer for the class of hyper-(abelian or finite) groups with finite abelian total rank.* As costumary we denote this class by $\mathfrak{S}_1\mathfrak{F}$.

Theorem 1 *A CN-group G in the class $\mathfrak{S}_1\mathfrak{F}$ is finite-by-abelian-by-finite.*

Question 2: When is a finite-by-abelian-by-finite CN-group BCN?

Recall that B*CN means CN in a bounded way*, that is
$\exists n \in \mathbb{N}\ \forall H \leq G\ \exists N \lhd G$ such that $|H/(H \cap N)| \cdot |N/(H \cap N)| \leq n$
(and n is independent of H). This is true for CN-groups G that following [11] we call of *elementary type* that is with normal subgroups $G_1 \leq G_0$ of G, finite and of finite index in G resp., such that elements of G act as power automorphisms on the abelian factor G_0/G_1. Note that such a group is clearly BCN, as each subgroup H of G is commensurable to $N := (H \cap G_0)G_1$ and $|H/(H \cap N)| \cdot |N/(H \cap N)| \leq |G/G_0| \cdot |G_1|$. Recall that an automorphism is said *power* iff it fixes setwise each subgroup.
In Theorem 2 we see that *for abelian-by-finite groups the two classes CN and CF do coincide*, and their structure follows from results in [6]. Theorem 2 also gives complete description of abelian-by-finite BCF-groups, which are precisely abelian-by-finite BCN-groups.

For terminology, notation and basic facts we refer to [10] and [11]

Recall that soluble-by-finite groups with finite abelian total rank are precisely *groups G with a series whose factors are either subgroups of direct* *products of finitely many either Prüfer groups or copies of $\mathbb{Q}$, the last factor being possibly a finite non-solvable group* (see [7]). Notice that abelian p-sections of such a group are Chernikov. Also if $D = Div(G)$ is the biggest normal abelian divisible periodic subgroup of G, then G/D has a finite series whose factors are finite or copies of $\mathbb{Q}$. If G is CN, and hence totally inertial, then its elements act on D as power automorphisms (see [11]).

Proof of Theorem 1 Since the set $D = Div(G)$ is a Chernikov group, we may factor out by $D_{p'}$ ($p \in \pi(D)$) and assume that D is a p-group of finite rank. In fact suppose that for each prime p in the finite $\pi := \pi(D)$, $G/D_{p'}$ is finite-by-abelian-by-finite. Then G has a subgroup of finite index $G_p \geq D_{p'}$ such that $G'_p D_{p'}/D_{p'}$ is finite. So $H := \bigcap_{p\in\pi} G_p$ has finite index in G and $H'/H' \cap D_{p'}$ is finite for any $p \in \pi$. Thus H' is finite and G itself is finite-by-abelian-by-finite.

So assume that D *is a p-group of finite rank.* By results of Robinson [11], if G is not elementary, there are a finite normal subgroup F of G and a normal subgroup K of finite index of G such that K/DF is finite-by-(torsion-free abelian) and either K/F splits on $Div(K/F)$ (type I in [11]), or $Div(K/F) \leq Z(K/F)$(type II in [11]).

We may assume $F = 1$ and $K = G$, and so G/D is finite-by-(torsion-free abelian) and either G splits on D or $D \leq Z(G)$.

In the former case $G = D \rtimes Q$, and there is a finite normal subgroup L of Q such that Q/L is torsion-free abelian. Let $g \in Q$, $H := \langle g \rangle$ and let $N \lhd G$ commensurable to H. Then $[H \cap N, D] \leq N$. If $[H \cap N, D] \neq 1$, then $[H \cap N, D] = D$ (recall that elements of G act as power automorphisms on D) and hence $D \leq N$, contradicting the fact that $H \sim N$. Hence $[H \cap N, D] = 1$, and so there is a subgroup of finite index of H centralizing A. Hence $Q/C_Q(D)$ is finite, being a periodic group of automorphisms of the Chernikov group D. We may assume now that $Q = C_Q(D)$ and so $G = D \times Q$ is finite-by-abelian, as wished.

In the latter case (D central in G), we claim that G' *is finite.* We will show that $|H^G : H|$ is finite for any $H \leq G$, so by the above quoted result of B.H.Neumann (see [8]), G' is finite. We may assume $D \not\leq H$, and $D \cap H = 1$, as it is normal in G and we can factor out by it. So H' is finite. Take $N \lhd G$ commensurable to H and let $H_1 := H \cap N$. As $|H : H_1|$ is finite, $H^n \leq H_1 H'$ for a suitable $n \in \mathbb{N}$. As G is nilpotent of class 2, $[H_1 H', G] = [H_1, G] \leq N \cap D$, which is finite (as $H \cap D = 1$ and $H \sim N$). Moreover $[H, G]^n = [H^n, G] \leq [H_1 H', G]$ is finite, too. As $[H, G] \leq D$ has finite rank, we have that $[H, G]$ is finite and so $|H^G : H|$ is finite, as wished. $\square$

Recall that an automorphism γ of a group G setwise mapping each subgroup to a commensurable one is told *inertial*. Moreover γ called a *boundedly inertial*, or simply *bin* iff $\exists n \in \mathbb{N} \quad |H/(H \cap H^\gamma)| \cdot |H^\gamma/(H \cap H^\gamma)| \leq n$ for all subgroup H of G (and n is independent of H). Such automorphisms are studied in [5] in the case G is abelian. Thus a CN-group (BCN, resp.) G with an abelian normal subgroup A of finite index acts on A as a finite group of inertial automorphisms (bin-automorphisms, resp.).

Theorem 2 *Let G be an abelian-by-finite group. Then G is CN (resp. BCN) iff it is CF (resp. BCF). Moreover:*
- If G is periodic, then G is a BCF-group iff G is a CF-group .
- If G is non-periodic, then G is a BCF-group iff there is a a normal series

$$1 \leq V \leq K \leq A \leq G$$

where:
i) A is abelian with finite index,
ii) G/K has finite exponent,
iii) each element of G acts on K as the identity or the inversion map,
iv) V is free abelian ,
v) G acts the periodic group A/V by means of almost power automorphisms,
vi) either $K = A$ (elementary case) or V has finite rank.

Proof. Let A be an abelian normal subgroup of G. The group G acts as a finite group of automorphisms of A and hence the properties CN and CF (BCN and BCF, resp.) are obviously equivalent, and they are equivalent to the fact that every element g of G acts on A as an inertial automorphism.

Let now G be BFC. If G is periodic, by results in [5] and [6] it follows that there is n such that $|H/H_G| \leq n$ for each $H \leq G$.

If G is a non-periodic CF-group, then G is BCF iff every element g of the finite group $\bar{G} = G/C_G(A)$ acts as a bin-automorphism on A. By Th. 3 and Cor.1 of [5], this is equivalent to saying that for each g the subgroup $E_g = A^{g-\epsilon}$ has finite exponent (where $g = \epsilon = \pm 1$ on A/T, where $T \neq A$ is the torsion subgroup of A). Take $E := \langle E_g \,|\, g \in \bar{G} \rangle$ and, for any $g \in \bar{G}$, let K_g be the kernel of the endomorphism $g - \epsilon$ of A and put $K := \bigcap_{g \in \bar{G}} K_g$. As E_g is the image of $g - \epsilon$, it is clear that E has finite exponent iff A/K has. Moreover K is G-hamiltonian. The statement follows from [5], Th. 3 and [6], as $V \leq K$. $\square$

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