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NUMERICAL TREATMENT OF A CLASS OF SYSTEMS OF FREDHOLM INTEGRAL EQUATIONS ON THE REAL LINE

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ABSTRACT. In this paper the authors propose a Nyström method based on a "truncated" Gaussian rule to solve systems of Fredholm integral equations on the real line. They prove that it is stable and convergent and that the matrices of the solved linear systems are well conditioned. Moreover, they give error estimates in weighted uniform norm and show some numerical tests.

1. INTRODUCTION

We consider systems of Fredholm integral equations of the following type:

$$f_r(x) - \tau \sum_{s=1}^{n} \int_{-\infty}^{+\infty} h^{r,s}(x,y) f_s(y) w^{\alpha_s,\beta_s}(y)dy = g_r(x), \quad r = 1,\dots,n, \tag{1}$$

where $x \in \mathbb{R}, w^{\alpha_s,\beta_s}(y) = |y|^{\alpha_s} e^{-|y|^{\beta_s}}, \alpha_s > -1, \beta_s > 1$, are generalized Freud weights, $\tau \in \mathbb{R}$, $g_r, h^{r,s}, r,s = 1,\dots,n$, are given functions and $f_r, r = 1,\dots,n$, are the unknowns.

With the notations,

$$\mathbf{f}(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad \mathbf{g}(x) := \begin{pmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

$$\mathbf{K} := \begin{pmatrix} K^{1,1} & K^{1,2} & \dots & K^{1,n} \\ K^{2,1} & K^{2,2} & \dots & K^{2,n} \\ \vdots & & \ddots & \vdots \\ K^{n,1} & K^{n,2} & \dots & K^{n,n} \end{pmatrix}, \quad \mathbf{I} := \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I \end{pmatrix}, \tag{2}$$

where

$$K^{r,s} f(x) = \tau \int_{-\infty}^{+\infty} h^{r,s}(x,y) f(y) w^{\alpha_s,\beta_s}(y)dy, \quad r,s = 1,\dots,n, \tag{3}$$

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and I denotes the identity operator, the system (1) can be rewritten as follows:

$$(\mathbf{I}-\mathbf{K})\mathbf{f}=\mathbf{g}. \tag{4}$$

The first difficulty in this kind of problem is the choice of the space of functions where the possible solutions live and/or the operator $\mathbf{K}$ is completely continuous. Moreover, the integrals on the real line involving non-standard weights do not help to find efficient numerical procedures. In fact, both the Gaussian rules and the product rules have a "poor" behaviour in the approximation of the operator $\mathbf{K}$ (see, for example, [3]). On the other hand, if we want to construct a global approximation of the possible solution of (4), the study of the stability and the convergence requires the knowledge of results on the polynomial approximation with generalized Freud weights. Such results only recently appeared in [9, 10]. For these reasons the numerical treatment of (4) has received little attention until now. To our knowledge there is only [8].

In this paper we will study (4) in a product space of continuous functions with weighted uniform metric and we will assume that the kernels and the right-hand sides of (1) can have singularities of algebraic-type at the origin and of exponential-type in $\pm\infty$.

Concerning the approximation of the solutions of (4) (when they exist), we propose a Nyström method based on a simple "truncated" Gaussian quadrature rule. The procedure can be easily implemented, and we will prove that it is stable and convergent. Moreover, the linear systems related to the Nyström interpolation are well conditioned.

The paper is organized as follows. In Section 2 we introduce the function spaces in which the systems are studied and some preliminary notation and results. Section 3 is dedicated to the description of the numerical method. The stability and convergence is proved in Section 4. In Section 5 we show some numerical tests.

2. Preliminaries

In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to indicate that $\mathcal{C}$ is independent of the parameters $a, b, \dots$. If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$ if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

In order to introduce the function spaces where we are going to study (4), we give some preliminary definitions.

First of all we denote by $\mathbb{P}_m$ the set of all polynomials of degree at most m and by $C^0(A)$ the collection of the continuous functions on $A \subset \mathbb{R}$. We recall that to every exponential weight $\sigma(x) = e^{-Q(x)}$, where Q is even and satisfies certain conditions (see [4, p. 7]), can be associated the so-called Maskar-Rakmanov-Saff number (M-R-S number) $B_t = B_t(\sigma)$ that is defined as the positive solution of the equation

$$t = \frac{1}{\pi}\int_0^1 \frac{xB_tQ'(xB_t)}{\sqrt{1-x^2}}dx, \quad t > 0;$$

B_t is an increasing function of t and, for every polynomial $P_m \in \mathbb{P}_m$, one has

$$\max_{x\in\mathbb{R}} |P_m(x)e^{-Q(x)}| = \max_{x\in[-B_m, B_m]} |P_m(x)e^{-Q(x)}|.$$

In this paper we will consider weight functions of the type $u(x) = (1+|x|)^\lambda |x|^\gamma e^{-\frac{|x|^\beta}{2}}$, $x \in \mathbb{R}, \lambda, \gamma \geq 0, \beta > 1$, and, since to our aims we do not need the exact value of B_t, we will use the equivalence $B_t(u) \sim t^{1/\beta}$.

Now, we define the space

$$C_u = \begin{cases} \left\{ f \in C^0(\mathbb{R} \setminus \{0\}) : \lim\limits_{\substack{x \to \pm\infty \\ x \to 0}} (fu)(x) = 0 \right\}, & \gamma > 0, \\ \\ \left\{ f \in C^0(\mathbb{R}) : \lim\limits_{x \to \pm\infty} (fu)(x) = 0 \right\}, & \gamma = 0. \end{cases}$$

The space C_u equipped with the weighted norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{x \in \mathbb{R}} |(fu)(x)|$$

is complete. For the sake of brevity, we will write $\|f\|_A = \max_{x \in A} |f(x)|$.

Moreover, we define the Sobolev-type spaces

$$W_\mu(u) = \left\{ f \in C_u : f^{(\mu-1)} \in C^0(\mathbb{R} \setminus \{0\}) \text{ and } \|f^{(\mu)} u\|_\infty < +\infty \right\}, \quad \mu \geq 1,$$

and we equip them with the norm

$$\|f\|_{W_\mu(u)} := \|fu\|_\infty + \|f^{(\mu)} u\|_\infty.$$

With

$$E_m(f)_u = \inf_{P \in \mathbb{P}_m} \|(f - P)u\|_\infty$$

the error of best approximation in C_u, in [9, 10] the authors proved that, for $1 \leq k < m$,

$$E_m(f)_u \leq \mathcal{C} \omega^k \left(f, \frac{B_m}{m} \right)_u^*, \quad \mathcal{C} \neq \mathcal{C}(m, f), \tag{5}$$

where

$$\omega^k(f, t)_u^* = \Omega^k(f, t)_u^* + \sum_{i=1}^{3} \inf_{P \in \mathbb{P}_{k-1}} \|(f - P)u\|_{I_i},$$

with

$$\Omega^k(f, t)_u^* = \sup_{0 < h \leq t} \|u \Delta_h^k f\|_{I_{kh}},$$

$t < t_0$ (t_0 sufficiently small), $I_1 =]-\infty, -Akh^{-1/(\beta-1)}[$, $I_2 =]-4kh, 4kh[$, $I_3 =]Akh^{-1/(\beta-1)}, +\infty[$, $I_{kh} = \left[-Akh^{-1/(\beta-1)}, -4kh\right] \bigcup \left[4kh, Akh^{-1/(\beta-1)}\right]$, $A > 0$, a positive fixed constant,

$$\Delta_h f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right), \quad \Delta_h^k = \Delta_h(\Delta_h^{k-1})$$

and $B_m = B_m(u) \sim m^{\frac{1}{\beta}}$.

Moreover, for all $f \in W_\mu(u)$, with u having the parameter $\gamma \geq 0$ not an integer, we have [9, 10]

$$E_m(f)_u \leq \mathcal{C} \left(\frac{B_m}{m} \right)^\mu \|f\|_{W_\mu(u)}, \quad m > \mu, \tag{6}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Now we introduce the product space $\mathbf{C_u}$ where we will consider (4). Letting $\mathbf{u} = (u_1, \dots, u_n)$ with $u_s(x) = (1+|x|)^{\lambda_s}|x|^{\gamma_s}e^{-\frac{|x|^{\beta_s}}{2}}$, we set

$$\mathbf{C_u} = \{(f_1, \dots, f_n) \ : \ f_s \in C_{u_s}, \ s = 1, \dots, n\}$$

and we equip it with the norm

$$\|\mathbf{f}\|_{\mathbf{C_u}} = \max_{s=1,\dots,n}\{\|f_s\|_{C_{u_s}}\}.$$

We will also use the Sobolev space

$$\mathbf{W}_\mu(\mathbf{u}) = \{(f_1, \dots, f_n) \ : \ f_s \in W_\mu(u_s), \ s = 1, \dots, n\}$$

with the norm

$$\|\mathbf{f}\|_{\mathbf{W}_\mu(\mathbf{u})} = \max_{s=1,\dots,n}\{\|f_s\|_{W_\mu(u_s)}\}.$$

Recalling the definition (2), we have

$$\|\mathbf{Kf}\|_{\mathbf{C_u}} = \max_{r=1,\dots,n}\max_{x\in\mathbb{R}}\left|\sum_{s=1}^{n} u_r(x)(K^{r,s}f_s)(x)\right| \le n \max_{r,s=1,\dots,n}\|u_r K^{r,s}f_s\|_\infty. \tag{7}$$

Since

$$\begin{aligned}
|u_r(x)(K^{r,s}f_s)(x)| &\le |\tau|\|f_s u_s\|_\infty \int_{-\infty}^{+\infty} u_r(x)|h^{r,s}(x,y)|u_s(y)\frac{w^{\alpha_s,\beta_s}(y)}{u_s^2(y)}\,dy \\
&= |\tau|\|f_s\|_{C_{u_s}} \int_{-\infty}^{+\infty} u_r(x)|h^{r,s}(x,y)|u_s(y)\frac{|y|^{\alpha_s-2\gamma_s}}{(1+|y|)^{2\lambda_s}}dy,
\end{aligned}$$

setting

$$\overline{M} := \max_{r,s=1,\dots,n}\max_{x,y\in\mathbb{R}} u_r(x)|h^{r,s}(x,y)|u_s(y)$$

and

$$\rho := \max_{s=1,\dots,n}\int_{-\infty}^{+\infty}\frac{|y|^{\alpha_s-2\gamma_s}}{(1+|y|)^{2\lambda_s}}dy,$$

we deduce

$$\|\mathbf{Kf}\|_{\mathbf{C_u}} \le |\tau|(n\rho\overline{M})\|\mathbf{f}\|_{\mathbf{C_u}}, \tag{8}$$

i.e., $\mathbf{K}$ is a bounded map from $\mathbf{C_u}$ into itself if the quantities $\overline{M}$ and ρ are bounded.

In order to solve the systems (4) by means of a Nyström method we need a quadrature rule. Before introducing it we give some notation.

Let $\{p_m^{\alpha,\beta}\}_m$ be the sequence of the orthonormal polynomials w.r.t. the weight $w^{\alpha,\beta}(x) = |x|^\alpha e^{-|x|^\beta}$, i.e.

$$p_m^{\alpha,\beta}(x) = \gamma_m x^m + \cdots, \quad \gamma_m > 0,$$

and

$$\int_{-\infty}^{+\infty} p_n^{\alpha,\beta}(x)p_m^{\alpha,\beta}(x)|x|^\alpha e^{-|x|^\beta}dx = \delta_{n,m}.$$

We denote by $x_k^{\alpha,\beta} := x_{m,k}(w^{\alpha,\beta})$, $k = 1, 2, \dots, \left\lfloor\frac{m}{2}\right\rfloor$, the positive zeros of $p_m^{\alpha,\beta}$ and by $x_{-k}^{\alpha,\beta} := x_{m,-k}(w^{\alpha,\beta})$, $k = 1, 2, \dots, \left\lfloor\frac{m}{2}\right\rfloor$, the corresponding negative ones ($\lfloor a\rfloor$ stands for the largest integer smaller than or equal to $a \in \mathbb{R}^+$). If m is odd, we set $x_0 = 0$. All the zeros belong to the interval $(-B_m(\sqrt{w^{\alpha,\beta}}),\ B_m(\sqrt{w^{\alpha,\beta}}))$.

Then, also following an idea in [5, 6], the following "truncated" Gaussian rule has been introduced in [7]:

$$\text{(9)} \qquad \int_{-\infty}^{+\infty} F(x) w^{\alpha,\beta}(x)\, dx = \sum_{|k|\le j} \lambda_k^{\alpha,\beta} F(x_k^{\alpha,\beta}) + e_m^*(F), \quad F \in C^0(\mathbb{R}),$$

where $\lambda_k^{\alpha,\beta} := \lambda_{m,k}^{\alpha,\beta}$ are the Christoffel numbers related to the weight $w^{\alpha,\beta}$,

$$\text{(10)} \qquad j := j(m) = \min_{k=1,\dots,\lfloor \frac{m}{2} \rfloor} \left\{ k \ : \ x_k \ge \theta\, B_m(\sqrt{w^{\alpha,\beta}}) \right\}, \quad 0 < \theta < 1,$$

and $e_m^*(F)$ is the remainder term. Notice that $e_m^*(1) \neq 0$, but the following proposition holds true [7].

Proposition 1. *Let m be even and consider the weight function $\sigma(x) = (1 + |x|)^\lambda |x|^\gamma e^{-a|x|^\beta}$, with $x \in \mathbb{R}, \lambda \ge 0$, $\gamma \ge 0$ not an integer, $\beta > 1$ and $0 < a \le 1$. If*

$$\int_{-\infty}^{+\infty} w^{\alpha,\beta}(x) \sigma^{-1}(x)\, dx < +\infty,$$

then, for every $F \in C_\sigma$, we have

$$|e_m^*(F)| \le \mathcal{C}[E_M(F)_\sigma + e^{-Am} \|F\|_{C_\sigma}],$$

where $M = \left\lfloor \left(\frac{\theta}{1+\theta}\right)^\beta \frac{m}{2} \right\rfloor$, with $\theta \in (0,1)$ fixed, and the constants $\mathcal{C}$ and A are independent of m and F.

We point out that the assumption that m is even is due to the assumptions that $F \in C_\sigma$ and that σ can have a zero at the origin.

As one can see, the quadrature sum in (9) uses a finite section of F. Therefore, possible overflows when F increases in an exponential way are avoided and the dimensions of the linear systems connected with the Nyström method are drastically reduced.

For the sake of completeness, we also note that (see [3])

$$\begin{aligned} |e_m^*(F)| \quad &\le \quad \mathcal{C} \left[\frac{B_m(\sqrt{w^{\alpha,\beta}})}{m} \int_{-\infty}^{+\infty} |F'(x)| w^{\alpha,\beta}(x) dx \right. \\ &+ \quad \left. e^{-Am} \int_{-\infty}^{+\infty} |F(x)| w^{\alpha,\beta}(x) dx \right], \end{aligned}$$

where the constants $\mathcal{C}$ and A are independent of m and F, and the same estimate is not true for the error of the ordinary Gaussian rule.

3. Numerical method

The Nyström method we propose is the following. We introduce the sequence of operators $\{\mathbf{K}_m\}_m$ defined by

$$\mathbf{K}_m := \begin{pmatrix} K_m^{1,1} & K_m^{1,2} & \dots & K_m^{1,n} \\ K_m^{2,1} & K_m^{2,2} & \dots & K_m^{2,n} \\ \vdots & & \ddots & \vdots \\ K_m^{n,1} & K_m^{n,2} & \dots & K_m^{n,n} \end{pmatrix}, \quad m \in \mathbb{N},$$

where

$$(11)\quad (K_m^{r,s}f)(x)=\tau\sum_{\substack{|k|\le j\\k\ne 0}}h^{r,s}(x,x_k^{\alpha_s,\beta_s})f(x_k^{\alpha_s,\beta_s})\lambda_k^{\alpha_s,\beta_s},\quad r,s=1,\dots,n,\quad m\in\mathbb{N},$$

are the approximations, by the quadrature rules (9), of $K^{r,s}f$, $r,s=1,\dots,n$, defined in (3). Then, we proceed to solve the approximating systems

$$(\mathbf{I}-\mathbf{K}_m)\mathbf{f}_m=\mathbf{g},\quad m\in\mathbb{N}, \tag{12}$$

in the unknowns

$$\mathbf{f}_m=\begin{pmatrix} f_{m,1}\\ f_{m,2}\\ \vdots\\ f_{m,n}\end{pmatrix},$$

in place of the equation $(\mathbf{I}-\mathbf{K})\mathbf{f}=\mathbf{g}$.

From (11), we get

$$|u_r(x)(K_m^{r,s}f_s)(x)|\le|\tau|\|f_su_s\|_\infty\max_{r,s=1,\dots,n}\max_{x,y\in\mathbb{R}}u_r(x)|h^{r,s}(x,y)|u_s(y)$$
$$\times\max_{s=1,\dots,n}\sum_{\substack{|k|\le j\\k\ne 0}}\frac{\lambda_k^{\alpha_s,\beta_s}}{u_s^2(x_k^{\alpha_s,\beta_s})}.$$

But, recalling that (see, for instance, [7]) $\lambda_k^{\alpha_s,\beta_s}\sim\Delta x_k^{\alpha_s,\beta_s}w^{\alpha_s,\beta_s}(x_k^{\alpha_s,\beta_s})$, where $\Delta x_k^{\alpha_s,\beta_s}=x_{k+1}^{\alpha_s,\beta_s}-x_k^{\alpha_s,\beta_s}$ and the constants in "$\sim$" are independent of m and k, the sum on the right-hand side is dominated by

$$\sum_{\substack{|k|\le j\\k\ne 0}}\Delta x_k^{\alpha_s,\beta_s}\frac{|x_k^{\alpha_s,\beta_s}|^{\alpha_s-2\gamma_s}}{(1+|x_k^{\alpha_s,\beta_s}|)^{2\lambda_s}}\le\mathcal{C}\int_{-\infty}^{+\infty}\frac{|y|^{\alpha_s-2\gamma_s}}{(1+|y|)^{2\lambda_s}}dy,\quad\mathcal{C}\ne\mathcal{C}(m,s), \tag{13}$$

since it is a Riemann sum of a piecewise monotonic function. Then, with the same notation used in (8), recalling (7), we deduce

$$\|\mathbf{K}_m\mathbf{f}\|_{\mathbf{C_u}}\le\mathcal{C}(n\rho\overline{M})\|\mathbf{f}\|_{\mathbf{C_u}},\quad\mathcal{C}\ne\mathcal{C}(m,\mathbf{f},r,s). \tag{14}$$

Now, for any fixed m, (12) can be written as follows:

$$f_{m,r}(x)-\tau\sum_{s=1}^n\sum_{\substack{|k|\le j\\k\ne 0}}h^{r,s}(x,x_k^{\alpha_s,\beta_s})f_{m,s}(x_k^{\alpha_s,\beta_s})\lambda_k^{\alpha_s,\beta_s}=g_r(x),\quad r=1,\dots,n,$$

and the unknowns are $f_{m,s}(x_k^{\alpha_s,\beta_s})$. Since we will proceed to compare the possible solution $\mathbf{f}_m$ with the exact solution in the metric of $\mathbf{C_u}$, we need to compute the quantities $c_{s,k}=f_{m,s}(x_k^{\alpha_s,\beta_s})u_s(x_k^{\alpha_s,\beta_s})$, $s=1,\dots,n$. To this end, we multiply the $r-$th equation, $r=1,\dots,n$, by $u_r(x)=(1+|x|)^{\lambda_r}|x|^{\gamma_r}e^{-\frac{|x|^{\beta_r}}{2}}$ to obtain

$$(f_{m,r}u_r)(x)-\tau\sum_{s=1}^n\sum_{\substack{|k|\le j\\k\ne 0}}h^{r,s}(x,x_k^{\alpha_s,\beta_s})\frac{u_r(x)}{u_s(x_k^{\alpha_s,\beta_s})}(f_{m,s}u_s)(x_k^{\alpha_s,\beta_s})\lambda_k^{\alpha_s,\beta_s}$$
$$=(g_ru_r)(x),\quad r=1,\dots,n.$$

Collocating the $r-$th equation, $r = 1, \dots, n$, at the zeros $x_i^{\alpha_r,\beta_r}$, $|i| \le j, i \neq 0$, we get the linear system

$$(15)\qquad c_{r,i} - \tau \sum_{s=1}^{n} \sum_{\substack{|k|\le j\\ k\neq 0}} h^{r,s}(x_i^{\alpha_r,\beta_r}, x_k^{\alpha_s,\beta_s}) \lambda_k^{\alpha_s,\beta_s} \frac{u_r(x_i^{\alpha_r,\beta_r})}{u_s(x_k^{\alpha_s,\beta_s})} c_{s,k} = (g_r u_r)(x_i^{\alpha_r,\beta_r}),$$

$$r = 1, \dots, n, \ \ |i| \le j, i \neq 0,$$

in the unknowns $c_{s,k}$.

If the system (15) admits the unique solution $[c^*_{1,-j}, \dots, c^*_{1,-1}, c^*_{1,1}, \dots, c^*_{1,j}, \ \dots, c^*_{n,-j}, \dots, c^*_{n,-1}, c^*_{n,1}, \dots, c^*_{n,j}]^T$, then we construct the Nyström interpolant

$$\mathbf{f}^*_m = \begin{pmatrix} f^*_{m,1} \\ f^*_{m,2} \\ \vdots \\ f^*_{m,n} \end{pmatrix},$$

where

$$f^*_{m,r}(x) = \tau \sum_{s=1}^{n} \sum_{\substack{|k|\le j\\ k\neq 0}} h^{r,s}(x, x_k^{\alpha_s,\beta_s}) \frac{\lambda_k^{\alpha_s,\beta_s}}{u_s(x_k^{\alpha_s,\beta_s})} \, c_{s,k} + g_r(x), \qquad r = 1, \dots, n,$$

and we will estimate the error $\|\mathbf{f}^* - \mathbf{f}^*_m\|_{\mathbf{C_u}}$.

Note that, for the sake of simplicity of notation, we set $\mathbf{f}^*_m := \mathbf{f}^*_{2j}$ and $f^*_{m,r} := f^*_{2j,r}, r = 1, \dots, n$, being $j = j(m)$ defined in (10). Moreover, the matrix of coefficients A_{n2j} of the linear system (15) is of order $n(2j)$ instead of nm if we used a rule that is not truncated.

The proposed numerical method is easy to implement. In fact, the construction of the linear systems (15) only requires the computation of the zeros $x_i^{\alpha_s,\beta_s}$ and of the weights $\lambda_i^{\alpha_s,\beta_s}$. We performed it by using the software package `OrthogonalPolynomials` (see [2]).

In the next section we will prove the stability and the convergence of the proposed procedure.

4. Stability and convergence analysis

We first make some assumptions on the kernels and the weights appearing in equation (1) and in the definition of the space $\mathbf{C_u}$. Concerning the weights $u_s(x) = (1 + |x|)^{\lambda_s} |x|^{\gamma_s} e^{-\frac{|x|^{\beta_s}}{2}}$ and $w^{\alpha_s,\beta_s} = |x|^{\alpha_s} e^{-|x|^{\beta_s}}$, $s = 1, \dots, n$, we will assume that

$$(16)\qquad \lambda_s > \frac{1}{2}, \quad 0 \le \gamma_s < \frac{\alpha_s + 1}{2}, \quad \gamma_s \notin \mathbb{N}, \quad \beta_s > 1.$$

Moreover, if we set $H_y^{r,s}(x) = u_s(y) h^{r,s}(x, y)$ and $H_x^{r,s}(y) = u_r(x) h^{r,s}(x, y)$, $r, s = 1, \dots, n$, we will suppose that

$$(17)\qquad \begin{cases} \lim\limits_{\substack{x\to\pm\infty\\ x\to 0}} \sup\limits_{y\in\mathbb{R}} H_y^{r,s}(x) u_r(x) = 0, \\ \lim\limits_{h\to 0} \sup\limits_{y\in\mathbb{R}} |H_y^{r,s}(x+h) - H_y^{r,s}(x)| = 0, \quad x \in [-b, -a] \cup [a, b], \end{cases}$$

$$\left\{ \begin{array}{l} \lim\limits_{\substack{y\to\pm\infty \\ y\to 0}} \sup\limits_{x\in\mathbb{R}} H_x^{r,s}(y) u_s(y) = 0, \\ \lim\limits_{h\to 0} \sup\limits_{x\in\mathbb{R}} |H_x^{r,s}(y+h) - H_x^{r,s}(y)| = 0, \quad y \in [-b,-a]\cup[a,b], \end{array} \right. \tag{18}$$

for $r,s = 1,\dots,n$ and for arbitrary $0 < a < b < +\infty$.

In other words, we will assume that $H_y^{r,s} \in C_{u_r}$ and $H_x^{r,s} \in C_{u_s}$ uniformly with respect to y and x, respectively. Thus, the kernels $h^{r,s}(x,y)$ can have singularities of algebraic-type at the origin and of exponential-type in $\pm\infty$.

Notice also that, recalling (8) and (14), the hypotheses (16) and (17) imply

$$\|\mathbf{K}\|_{\mathbf{C_u}\to\mathbf{C_u}} < +\infty$$

and

$$\sup_m \|\mathbf{K}_m\|_{\mathbf{C_u}\to\mathbf{C_u}} < +\infty. \tag{19}$$

Moreover, we have the following lemma.

Lemma 1. *Under the assumptions* $(16)-(18)$*, the sequence* $\{\mathbf{K}_m\}_m$ *strongly converges to* $\mathbf{K}$ *and is collectively compact.*

As a consequence of the previous lemma we deduce the following theorem.

Theorem 1. *Assume that* $Ker(\mathbf{I}-\mathbf{K}) = \{0\}$ *in* $\mathbf{C_u}$*. Under the assumptions* $(16)-(18)$ *and*

$$\mathbf{g} \in \mathbf{C_u},$$

for a sufficiently large m *(say* $m \geq m_0$*), the systems* (15) *are unisolvent. Moreover, the condition numbers in uniform norm of their matrices* $A_{n2j}, j = j(m)$*, are independent of the dimension* $n2j$ *and the Nyström interpolants* $\mathbf{f}_m^*$ *converge to the exact solution* $\mathbf{f}^*$ *of* (4)*, i.e.,*

$$\lim_m \|\mathbf{f}^* - \mathbf{f}_m^*\|_{\mathbf{C_u}} = 0.$$

In particular, if the kernels and the right-hand sides are μ *times differentiable with respect to both the variables and, for all* $r,s = 1,\dots,n$,

$$\sup_{y\in\mathbb{R}} \|H_y^{r,s}\|_{W_\mu(u_r)} < +\infty,$$

$$\sup_{x\in\mathbb{R}} \|H_x^{r,s}\|_{W_\mu(u_s)} < +\infty, \tag{20}$$

and

$$\mathbf{g} \in \mathbf{W}_\mu(\mathbf{u}), \tag{21}$$

then the estimate

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{\mathbf{C_u}} \leq \mathcal{C} \left(\frac{\bar{B}_m}{m}\right)^\mu \|\mathbf{f}^*\|_{\mathbf{W}_\mu(\mathbf{u})} \tag{22}$$

holds true, where $\mathcal{C} \neq \mathcal{C}(m,\mathbf{f})$ *and* $\bar{B}_m = \max\limits_{s=1,\dots,n} B_m(u_s) \sim \max\limits_{s=1,\dots,n} m^{\frac{1}{\beta_s}}$.

Hence, the convergence order of $\mathbf{f}_m^*$ to $\mathbf{f}^*$ depends on the smoothness of the kernels and the right-hand sides.

5. Proofs

Proof of Lemma 1. We first prove the strong convergence of $\{\mathbf{K}_m\}_m$ to $\mathbf{K}$. We have

$$\|(\mathbf{K}-\mathbf{K}_m)\mathbf{f}\|_{\mathbf{C_u}} \le n \max_{r,s=1,\dots,n} \|(K^{r,s}-K^{r,s})f_s\|_{C_{u_r}} \tag{23}$$

and

$$\begin{aligned} |u_r(x)(K^{r,s}-K_m^{r,s})f_s(x)| &= u_r(x)|\tau| \Bigg| \int_{-\infty}^{+\infty} h^{r,s}(x,y)f_s(y)w^{\alpha_s,\beta_s}(y)dy \\ &- \sum_{\substack{|k|\le j\\ k\neq 0}} h^{r,s}(x,x_k^{\alpha_s,\beta_s})f_s(x_k^{\alpha_s,\beta_s})\lambda_k^{\alpha_s,\beta_s}\Bigg|. \end{aligned}$$

The quantity in the absolute value is the error of "truncated" Gaussian rule (9) related to the weight w^{α_s,β_s} and the function $F_x^{r,s}(y) = f_s(y)u_r(x)h^{r,s}(x,y) = f_s(y)H_x^{r,s}(y)$. Now, applying Proposition 1 with $\sigma := u_s^2$ and $F(y) := F_x^{r,s}(y)$, if we prove that the integrals

$$\int_{-\infty}^{+\infty} \frac{w^{\alpha_s,\beta_s}(y)}{u_s^2(y)}\,dy, \quad s=1,\dots,n,$$

are bounded and the functions $F_x^{r,s}, r,s=1,\dots,n$, belong to $C_{u_s^2}$ uniformly with respect to x, then we get

$$|u_r(x)(K^{r,s}-K_m^{r,s})f_s(x)| \le \mathcal{C}\left[E_M(F_x^{r,s})_{u_s^2} + e^{-Am}\|F_x^{r,s}\|_{C_{u_s^2}}\right], \tag{24}$$

with $M=\lfloor cm \rfloor, c\in(0,1)$ fixed, $\mathcal{C}\neq\mathcal{C}(m,F_x^{r,s})$ and $A\neq A(m,F_x^{r,s})$. Obviously, $\lim_m e^{-Am}\sup_{x\in\mathbb{R}}\|F_x^{r,s}\|_{C_{u_s^2}}=0$ and, by the Jackson inequality (5), $\lim_m \sup_{x\in\mathbb{R}} E_M(F_x^{r,s})_{u_s^2} = 0$.

Since, by (16),

$$\int_{-\infty}^{+\infty} \frac{w^{\alpha_s,\beta_s}(y)}{u_s^2(y)}\,dy = \int_{-\infty}^{+\infty} \frac{|y|^{\alpha_s-2\gamma_s}}{(1+|y|)^{2\lambda_s}}dy < +\infty,$$

it remains to prove that $F_x^{r,s}, r,s=1,\dots,n$, belong to $C_{u_s^2}$ uniformly with respect to x, i.e., recalling the definition of $C_{u_s^2}$,

$$\begin{cases} \lim\limits_{\substack{y\to\pm\infty\\ y\to 0}} \sup\limits_{x\in\mathbb{R}} F_x^{r,s}(y)u_s^2(y) = 0, \\ \lim\limits_{h\to 0}\sup\limits_{x\in\mathbb{R}} |F_x^{r,s}(y+h)-F_x^{r,s}(y)| = 0, \quad y\in[-b,-a]\cup[a,b], \end{cases} \tag{25}$$

for arbitrary $0<a<b<+\infty$.

We have

$$|F_x^{r,s}(y)u_s^2(y)| = |f_s(y)u_s(y)H_x^{r,s}(y)u_s(y)| \le \|f_s\|_{C_{u_s}} \sup_{x\in\mathbb{R}} |H_x^{r,s}(y)|u_s(y)$$

and, under assumption (18) and for all $f_s\in C_{u_s}$, we get the first limit condition in (25). Moreover, if $y\in[a,b]$ with $0<a'<a<b<b'<+\infty$, we can choose h such that $y+h\in[a',b']$. Then we obtain

$$\begin{aligned} |F_x^{r,s}(y+h)-F_x^{r,s}(y)| &\le |f_s(y+h)||H_x^{r,s}(y+h)-H_x^{r,s}(y)| \\ &+ |H_x^{r,s}(y)||f_s(y+h)-f_s(y)| \end{aligned}$$

and

$$\begin{aligned}\sup_{x\in\mathbb{R}}|F_x^{r,s}(y+h)-F_x^{r,s}(y)| &\leq \|f_s\|_{[a',b']}\sup_{x\in\mathbb{R}}|H_x^{r,s}(y+h)-H_x^{r,s}(y)| \\ &+ \sup_{x\in\mathbb{R}}\|H_x^{r,s}\|_{[a,b]}|f_s(y+h)-f_s(y)|.\end{aligned}$$

Consequently, again using assumption (18), for all $f_s \in C_{u_s}$, we deduce the second limit condition in (25) for $y \in [a,b]$. Analogously if $y \in [-b,-a]$.

Now we prove the collectively compactness of the sequence $\{\mathbf{K}_m\}_m$, i.e., the relatively compactness in $\mathbf{C_u}$ of the set

$$\{\mathbf{K}_m\mathbf{f} \in \mathbf{C_u} \ : \ m \geq 1 \text{ and } \|\mathbf{f}\|_{\mathbf{C_u}} \leq 1\}.$$

Using the Ascoli-Arzelà Theorem, it is sufficient to prove

$$\sup_m \|\mathbf{K}_m\|_{\mathbf{C_u}} \leq \mathcal{C} < +\infty$$

and

$$\lim_{h\to 0}\sup_m \sup_{\|\mathbf{f}\|_{\mathbf{C_u}}=1}\|(\mathbf{K}_m\mathbf{f})(\cdot+h)-(\mathbf{K}_m\mathbf{f})(\cdot)\|_{\mathbf{C_u}} = 0.$$

Under the assumptions of the lemma the former condition is fulfilled (see (19)). In order to prove the latter one, we recall that

$$\|(\mathbf{K}_m\mathbf{f})(\cdot+h)-(\mathbf{K}_m\mathbf{f})\|_{\mathbf{C_u}} \leq n \max_{r,s=1,\dots,n}\|(K_m^{r,s}f_s)(\cdot+h)-(K_m^{r,s}f_s)(\cdot)\|_{C_{u_r}}.$$

Using the definition (11) of $K_m^{r,s}$, $\lambda_k^{\alpha_s,\beta_s} \sim \Delta x_k^{\alpha_s,\beta_s} w^{\alpha_s,\beta_s}(x_k^{\alpha_s,\beta_s})$ and (13), from (16) we deduce

$$\begin{aligned}&u_r(x)|(K_m^{r,s}f_s)(x+h)-(K_m^{r,s}f_s)(x)| \\ &\leq \mathcal{C}\sum_{\substack{|k|\leq j\\ k\neq 0}}\Delta x_k^{\alpha_s,\beta_s}\frac{|x_k^{\alpha_s,\beta_s}|^{\alpha_s-2\gamma_s}}{(1+|x_k^{\alpha_s,\beta_s}|)^{2\lambda_s}}|(f_s u_s)(x_k^{\alpha_s,\beta_s})| \\ &\times u_r(x)u_s(x_k^{\alpha_s,\beta})|h^{r,s}(x+h,x_k^{\alpha_s,\beta_s})-h^{r,s}(x,x_k^{\alpha_s,\beta_s})| \\ &\leq \mathcal{C}\|\mathbf{f}\|_{\mathbf{C_u}}\left(\int_{-\infty}^{+\infty}\frac{|y|^{\alpha_s-2\gamma_s}}{(1+|y|)^{2\lambda_s}}dy\right)\sup_{y\in\mathbb{R}}\|H_y^{r,s}(\cdot+h)-H_y^{r,s}(\cdot)\|_{C_{u_r}} \\ &\leq \mathcal{C}\|\mathbf{f}\|_{\mathbf{C_u}}\sup_{y\in\mathbb{R}}\|H_y^{r,s}(\cdot+h)-H_y^{r,s}(\cdot)\|_{C_{u_r}},\end{aligned}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$. Then, under assumption (17), i.e., $H_y^{r,s} \in C_{u_r}$ uniformly with respect to y, the latter limit condition is also verified. The proof is complete. □

Proof of Theorem 1. From Lemma 1 and (19), we deduce that the operator $\mathbf{K}$ is compact and

$$\lim_m \|(\mathbf{K}-\mathbf{K}_m)\mathbf{K}_m\|_{\mathbf{C_u}} = 0.$$

Therefore, the Fredholm alternative holds true in $\mathbf{C_u}$ and, by virtue of the assumption $Ker(\mathbf{I}-\mathbf{K}) = \{0\}$ in $\mathbf{C_u}$, we deduce that the system (4) has a unique solution $\mathbf{f}^* \in \mathbf{C_u}$.

Moreover, applying [1, Theorem 4.1.1, p. 106] or [11, Theorem 2.1], for $m \geq m_0$, the inverse operators $(\mathbf{I}-\mathbf{K}_m)^{-1}$ exist and

$$\begin{aligned}\|(\mathbf{I}-\mathbf{K}_m)^{-1}\|_{\mathbf{C_u}\to\mathbf{C_u}} &\leq \frac{1+\|(\mathbf{I}-\mathbf{K})^{-1}\|_{\mathbf{C_u}\to\mathbf{C_u}}\|\mathbf{K}_m\|_{\mathbf{C_u}\to\mathbf{C_u}}}{1-\|(\mathbf{I}-\mathbf{K})^{-1}\|_{\mathbf{C_u}\to\mathbf{C_u}}\cdot\|(\mathbf{K}-\mathbf{K}_m)\mathbf{K}_m\|_{\mathbf{C_u}\to\mathbf{C_u}}} \\ &\leq \mathcal{C}. \qquad (26)\end{aligned}$$

Consequently, for $m \geq m_0$, the linear system (15) is unisolvent and

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{\mathbf{C_u}} \sim \|(\mathbf{K} - \mathbf{K}_m)\mathbf{f}^*\|_{\mathbf{C_u}}. \tag{27}$$

Hence, by virtue of Lemma 1, the Nyström interpolants $\mathbf{f}_m^*$ converge to the exact solution $\mathbf{f}^*$.

Now, proceeding as in [1, pp. 112-113], by (19) and (26), we deduce

$$\mathrm{cond}(A_{n2j}) \leq \mathrm{cond}(\mathbf{I} - \mathbf{K}_m) \leq \mathcal{C} < +\infty, \quad \mathcal{C} \neq \mathcal{C}(m),$$

where $\mathrm{cond}(A_{n2j}) = \|A_{n2j}\|_\infty \|A_{n2j}^{-1}\|_\infty$ and $\mathrm{cond}(\mathbf{I} - \mathbf{K}_m) = \|\mathbf{I} - \mathbf{K}_m\|_{\mathbf{C_u}\to\mathbf{C_u}} \times \|(\mathbf{I} - \mathbf{K}_m)^{-1}\|_{\mathbf{C_u}\to\mathbf{C_u}}$.

It remains to prove (22). From (27), (23) and (24), we have

$$\begin{aligned} \|\mathbf{f}^* - \mathbf{f}_m^*\|_{\mathbf{C_u}} &\leq n \max_{r,s=1,\dots,n} \max_{x\in\mathbb{R}} |u_r(x)(K^{r,s} - K_m^{r,s}) f_s^*(x)| \\ &\leq \mathcal{C} \max_{r,s=1,\dots,n} \sup_{x\in\mathbb{R}} \left[E_M(F_x^{r,s})_{u_s^2} + e^{-Am} \|F_x^{r,s}\|_{C_{u_s^2}} \right], \end{aligned} \tag{28}$$

where $F_x^{r,s} = f_s^*(y) u_r(x) h^{r,s}(x,y) = f_s^*(y) H_x^{r,s}(y)$, $M = \lfloor cm \rfloor$, with $c \in (0,1)$ fixed, $\mathcal{C} \neq \mathcal{C}(m, F_x^{r,s})$ and $A \neq A(m, F_x^{r,s})$. Concerning the second addendum, we have

$$e^{-Am} \max_{r,s=1,\dots,n} \sup_{x\in\mathbb{R}} \|F_x^{r,s}\|_{C_{u_s^2}} \leq e^{-Am} \max_{r,s=1,\dots,n} \|f_s^*\|_{\mathcal{C}_s} \|H_x^{r,s}\|_{C_{u_s}}. \tag{29}$$

Moreover, for the first addendum we deduce

$$\begin{aligned} \max_{r,s=1,\dots,n} \sup_{x\in\mathbb{R}} E_M(F_x^{r,s})_{u_s^2} &= \max_{r,s=1,\dots,n} \sup_{x\in\mathbb{R}} E_M(f_s^* H_x^{r,s})_{u_s^2} \\ &\leq \max_{r,s=1,\dots,n} \|f_s^*\|_{C_{u_s}} \sup_{x\in\mathbb{R}} E_{\lfloor M/2 \rfloor}(H_x^{r,s})_{u_s} \\ &+ 2 \max_{r,s=1,\dots,n} E_{\lfloor M/2 \rfloor}(f_s^*)_{u_s} \sup_{x\in\mathbb{R}} \|H_x^{r,s}\|_{C_{u_s}}. \end{aligned}$$

Since, by (20) and (21), g_r and $H_x^{r,s}$, for all $r,s = 1,\dots,n$, belong to the Sobolev space of index $\mu \geq 1$, with respect to y, in the same space where the solutions $f_s^*, s = 1,\dots,n$, live. Therefore, applying (6), we obtain

$$\begin{aligned} \max_{r,s=1,\dots,n} \sup_{x\in\mathbb{R}} E_M(F_x^{r,s})_{u_s^2} &\leq \frac{\mathcal{C}}{m^\mu} \max_{r,s=1,\dots,n} (B_m(u_s))^\mu \left\{ \|f_s^*\|_{C_{u_s}} \sup_{x\in\mathbb{R}} \|H_x^{r,s}\|_{W_\mu(u_s)} \right. \\ &+ \left. \|f_s^*\|_{W_\mu(u_s)} \sup_{x\in\mathbb{R}} \|H_x^{r,s}\|_{C_{u_s}} \right\}. \end{aligned} \tag{30}$$

Substituting (30) and (29) into (28), we conclude

$$\|\mathbf{f}^* - \mathbf{f}_m^*\|_{\mathbf{C_u}} \leq \frac{\mathcal{C}}{m^\mu} \max_{r,s=1,\dots,n} \left\{ (B_m(u_s))^\mu \|f_s^*\|_{W_\mu(u_s)} \sup_{x\in\mathbb{R}} \|H_x^{r,s}\|_{W_\mu(u_s)} \right\}. \qquad \square$$

6. Numerical examples

Now we apply our numerical method to some systems of integral equations.

In the examples that follow we will use the following definition of the truncation index. Since $\lambda_k^{\alpha_s,\beta_s} \sim \Delta x_k^{\alpha_s,\beta_s} w^{\alpha_s,\beta_s}(x_k^{\alpha_s,\beta_s})$, $s = 1,\dots,n$, for every fixed m even, we set

$$j_s = \min_{k=1,\dots,m/2} \{k \ : \ \lambda_{m,k}^{\alpha_s,\beta_s} < tol\} \tag{31}$$

and

$$j = \min_{s=1,\dots,n} \{j_s\},$$

tol being the precision to be achieved in the computations. The above definition is equivalent to (10) in the sense that there exists a $\theta \in (0,1)$ such that $x_{j_s-1} < \theta B_m(\sqrt{w^{\alpha_s,\beta_s}}) \le x_{j_s}$, for every $s = 1,\dots,n$, where j_s is given by (31).

In particular, we have chosen $tol = 2.22e-16$ because all the computations have been performed in double-precision arithmetic. Therefore the truncation has no effect for small values of m.

Notice that, if the number n of the equations is not too large and the kernels and the right-hand sides are sufficiently smooth, the dimensions $n2j$ of the systems (15) are kept down. Then, they can be well solved using the Matlab function "\" or the Mathematica function `LinearSolve` with a computational cost that is of the order of $(n2j)^3$.

In Examples 2–4, the exact solutions of the systems are unknown but exist and are unique in the proper spaces $\mathbf{C_u}$ since we have verified that

$$\|\mathbf{K}\|_{\mathbf{C_u}} < 1.$$

Moreover, we will think as exact their approximate solutions obtained for $m = 512$.

Example 1. The exact solution of the system

$$f_r(x) - \frac{1}{8}\sum_{s=1}^{2}\int_{-\infty}^{+\infty} h^{r,s}(x,y)f_s(y)w^{\alpha_s,\beta_s}(y)dy = g_r(x), \quad r = 1,2,$$

with

$$w^{\alpha_1,\beta_1}(y) = e^{-y^2}, \quad w^{\alpha_2,\beta_2}(y) = e^{-|y|^3},$$

$$(h^{r,s}(x,y))_{r,s=1,2} = \begin{pmatrix} ye^x & y^2\sin(x) \\ x^2\cos(y) & yx^3 \end{pmatrix}$$

and

$$\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} e^x\left(x - \frac{3}{32}e^{\frac{1}{4}}\sqrt{\pi}\right) \\ \left(1 - \frac{1}{12}\Gamma\left(\frac{5}{3}\right)\right)x^3 - \frac{1}{16}\sqrt{\pi}\left(\cos\left(\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right)\right)x^2 \end{pmatrix},$$

is

$$\mathbf{f}^*(x) = \begin{pmatrix} xe^x \\ x^3 \end{pmatrix}$$

and it lives in the space $C_{u_1} \times C_{u_2}$, where $u_1(x) = (1+|x|)e^{-\frac{x^2}{2}}$ and $u_2(x) = (1+|x|)e^{-\frac{|x|^3}{2}}$.

Setting

$$e_{2j,1} = \max_{k=1,\dots,10000}\left\{(1+|y_k|)e^{-\frac{y_k^2}{2}}|f^*_{2j,1}(y_k) - y_ke^{y_k}|\right\},$$

$$e_{2j,2} = \max_{k=1,\dots,10000}\left\{(1+|y_k|)e^{-\frac{|y_k|^3}{2}}|f^*_{2j,2}(y_k) - y_k^3|\right\},$$

where y_k are 10000 equally spaced points in the interval $[-15,15]$, in Table 1 we show that, by solving a linear system of order 24, we get an approximation of the solution with an error of the order of the machine precision. This very fast convergence is due to the fact that both the kernels and the known terms are very smooth. The condition numbers of the matrices of the systems (15) are, for increasing values of j, less than 2.5.

TABLE 1. Absolute errors $e_{2j,i}, i = 1, 2$

$2j$	$e_{2j,1}$	$e_{2j,2}$
8 (m =8)	$2.397e-9$	$5.758e-9$
12 (m =12)	$3.679e-16$	$3.370e-15$

Example 2. Now we consider the system

$$f_r(x) - \frac{1}{7\sqrt{2\pi}} \sum_{s=1}^{2} \int_{-\infty}^{+\infty} h^{r,s}(x,y) f_s(y) w^{\alpha_s,\beta_s}(y)dy = g_r(x), \quad r = 1, 2,$$

with

$$w^{\alpha_1,\beta_1}(y) = |y|^{\frac{1}{4}} e^{-|y|^{\frac{5}{2}}}, \quad w^{\alpha_2,\beta_2}(y) = e^{-|y|^3},$$

$$(h^{r,s}(x,y))_{r,s=1,2} = \begin{pmatrix} \cos(x^2+y^2) & \sin(x+y) \\ e^{-\frac{x+y}{8}} & xy^3 e^{-\frac{xy}{5}} \end{pmatrix}$$

and

$$\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} x^3 \cos(x) \\ xe^x \end{pmatrix},$$

whose exact solution is unknown. Here $\alpha_1 = \frac{1}{4}$ and $\alpha_2 = 0$, and therefore, taking into account (16), the system is unisolvent in the space $C_{u_1} \times C_{u_2}$, where $u_1(x) = (1+|x|)|x|^{\frac{1}{2}} e^{-\frac{|x|^{\frac{5}{2}}}{2}}$ and $u_2(x) = (1+|x|)|x|^{\frac{1}{4}} e^{-\frac{|x|^3}{2}}$. Also in this case the kernels and the known terms are smooth and, as one can see in Tables 2 and 3, the convergence is fast. In Figure 1 we show the graph of the approximate solutions $u_i f^*_{52,i}, i = 1, 2$. The condition numbers of the matrices of the solved linear systems are, for increasing values of j, less than 1.3.

TABLE 2. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1, 2$, at the point $x = 0.5$

$2j$	$(u_1 f^*_{2j,1})(0.5)$	$(u_2 f^*_{2j,2})(0.5)$
14 (m =16)	0.1373949	0.99558
26 (m =32)	0.13739498774	0.995584068
52 (m =64)	0.137394987743170	0.995584068525489

Unlike the previous two examples, in the next ones not all the kernels and the right-hand sides are very smooth. As one can see, in these cases the "truncation" has more effect.

TABLE 3. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1, 2$, at the point $x = 2.5$

$2j$	$(u_1 f^*_{2j,1})(2.5)$	$(u_2 f^*_{2j,2})(2.5)$
14 (m =16)	−0.49560642	0.05434295
26 (m =32)	−0.4956064289	0.054342952404
52 (m =64)	−0.495606428980036	0.054342952404796425

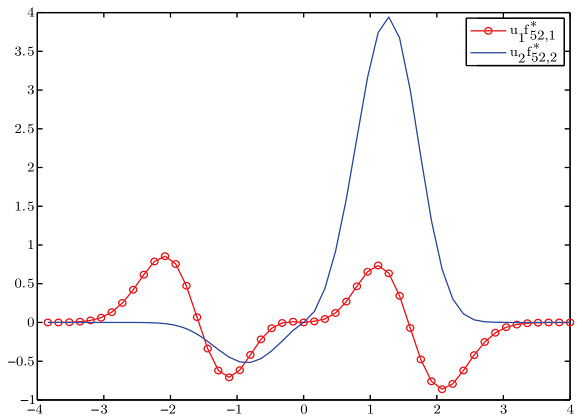

FIGURE 1. Graph of $u_1 f^*_{52,1}$ and $u_2 f^*_{52,2}$

Example 3. The solution of the system

$$f_r(x) - \frac{1}{10\pi} \sum_{s=1}^{3} \int_{-\infty}^{+\infty} h^{r,s}(x,y) f_s(y) w^{\alpha_s,\beta_s}(y) dy = g_r(x), \quad r = 1, 2, 3,$$

with

$$w^{\alpha_1,\beta_1}(y) = |y|^{\frac{1}{2}} e^{-|y|^3}, \quad w^{\alpha_2,\beta_2}(y) = e^{-|y|^{\frac{5}{2}}}, \quad w^{\alpha_3,\beta_3}(y) = |y|^{\frac{1}{4}} e^{-y^2},$$

$$(h^{r,s}(x,y))_{r,s=1,2,3} = \begin{pmatrix} \frac{e^x}{(1+x^2+y^2)^2} & \sin(x+y) & \frac{|x-y|^{\frac{11}{2}}}{(1+x^2+y^2)^4} \\ xy^2 \cos(x+y) & x^2 y \sin(x+y) & xy \\ |\sin(x-y)|^7 & \sin(2+x^2+y^3) & y^2 \cos(x) \end{pmatrix}$$

and

$$\begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} \frac{|x|^{\frac{15}{2}}}{(1+x^2)^2} \\ x\cos(x) \\ xe^x \end{pmatrix},$$

is unknown. In this case the system is unisolvent in $C_{u_1} \times C_{u_2} \times C_{u_3}$, where $u_1(x) = (1 + |x|)|x|^{\frac{1}{2}} e^{-\frac{|x|^3}{2}}$, $u_2(x) = (1 + |x|)|x|^{\frac{1}{4}} e^{-\frac{|x|^{\frac{5}{2}}}{2}}$ and $u_3(x) = (1 + |x|)|x|^{\frac{1}{2}} e^{-\frac{x^2}{2}}$. Since any of the kernels and right-hand sides are not very smooth (for example, $h^{1,3}(x, y) = \frac{|x-y|^{\frac{11}{2}}}{(1+x^2+y^2)^4} \in W_5(u_1)$ w.r.t. both the variables), according to (22), we need to take $2j = 104$ to get approximations of the solutions with at least 10 exact decimal digits. In Tables 4 and 5 we show the values of the weighted approximations of the components of the solution in two different points, while in Figure 2 we show the graph of $u_i f^*_{104,i}$, $i = 1, 2, 3$. In this case the condition numbers of the matrices of the systems (15) are, for increasing values of j, less than 1.3.

TABLE 4. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1, 2, 3$, at the point $x = 0.3$

$2j$	$(u_1 f^*_{2j,1})(0.3)$	$(u_2 f^*_{2j,2})(0.3)$	$(u_3 f^*_{2j,3})(0.3)$
26 (m=50)	0.00937	0.28601	0.3218
52 (m=100)	0.00937207	0.28601295	0.3218693
104 (m=200)	0.0093720773	0.286012951748789	0.32186936407973

TABLE 5. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1, 2, 3$, at the point $x = 2.5$

$2j$	$(u_1 f^*_{2j,1})(2.5)$	$(u_2 f^*_{2j,2})(2.5)$	$(u_3 f^*_{2j,3})(2.5)$
26 (m=50)	0.04109223	−0.05749	7.3912
52 (m=100)	0.041092238457	−0.05749129	7.39124664
104 (m=200)	0.0410922384573266	−0.0574912958587280	7.391246647118550

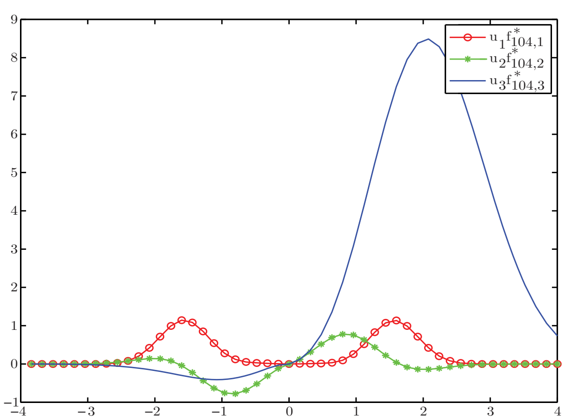

FIGURE 2. Graph of $u_i f^*_{104,i}, i = 1, 2, 3$

Example 4. As the last example we take the system

$$f_r(x) - \frac{1}{7\pi}\sum_{s=1}^{3}\int_{-\infty}^{+\infty} h^{r,s}(x,y) f_s(y) w^{\alpha_s,\beta_s}(y)dy = g_r(x), \quad r = 1,2,3,$$

with

$$w^{\alpha_1,\beta_1}(y) = |y|^{\frac{1}{2}} e^{-y^2}, \quad w^{\alpha_2,\beta_2}(y) = e^{-|y|^{\frac{5}{2}}}, \quad w^{\alpha_3,\beta_3}(y) = e^{-|y|^3},$$

$$(h^{r,s}(x,y))_{r,s=1,2,3} = \begin{pmatrix} y\sin(x) & xy^2\cos(x-y) & \sin(1+x+y^2) \\ xy|\sin(x-y)|^{\frac{3}{2}} & \frac{x^2 y}{4}+1 & \frac{5x^2+2y^3}{10} \\ x\cos(y) & \cos(1+x^2+y^2) & y^3 x\cos(xy) \end{pmatrix}$$

and

$$\begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} (1+x^2)^3 \\ x^2\sin(x) \\ x^3 \end{pmatrix}.$$

Its exact solution is unknown. By virtue of (16), the system admits a unique solution in the space $C_{u_1}\times C_{u_2}\times C_{u_3}$, where $u_1(x) = (1+|x|)|x|^{\frac{1}{2}} e^{-\frac{x^2}{2}}$, $u_2(x) = (1+|x|)|x|^{\frac{1}{4}} e^{-\frac{|x|^{\frac{5}{2}}}{2}}$ and $u_3(x) = (1+|x|)|x|^{\frac{1}{4}} e^{-\frac{|x|^3}{2}}$. Since the kernel $h^{2,1}(x,y) = xy|\sin(x-y)|^{\frac{3}{2}}$ belongs to $W_1(u_1)$ w.r.t. both the variables, as one can see in Tables 6 and 7, it is necessary to increase m to obtain approximations of the solutions with exact decimal digits. In figures 3 and 4 we show the graphs of the approximate solutions $u_i f^*_{296,i}$, $i = 1,2,3$. In this case the condition numbers of the matrices of the solved linear systems are, for increasing values of j, less than 1.5.

TABLE 6. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1,2,3$, at the point $x = -2.5$

$2j$	$(u_1 f^*_{2j,1})(-2.5)$	$(u_2 f^*_{2j,2})(-2.5)$	$(u_3 f^*_{2j,3})(-2.5)$
26 (m =50)	92.66	−0.1220	−0.02779
52 (m =100)	92.6689751	−0.1220	−0.027798797
104 (m =200)	92.6689751	−0.12201	−0.0277987977
158 (m =300)	92.66897515	−0.12201	−0.0277987977
210 (m =400)	92.66897515	−0.12201	−0.02779879772

TABLE 7. Weighted solutions $(u_i f^*_{2j,i})(x), i = 1, 2, 3$, at the point $x = -0.7$

$2j$	$(u_1 f^*_{2j,1})(-0.7)$	$(u_2 f^*_{2j,2})(-0.7)$	$(u_3 f^*_{2j,3})(-0.7)$
26 (m =50)	3.696	−0.31	−0.4929
52 (m =100)	3.696358	−0.311	−0.492911
104 (m =200)	3.6963584	−0.311	−0.4929115
158 (m =300)	3.6963584	−0.3113	−0.4929115
210 (m =400)	3.69635844	−0.3113	−0.49291154

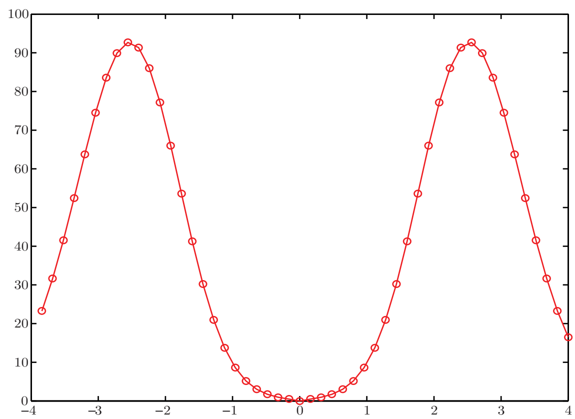

FIGURE 3. Graph of $u_1 f^*_{296,1}$

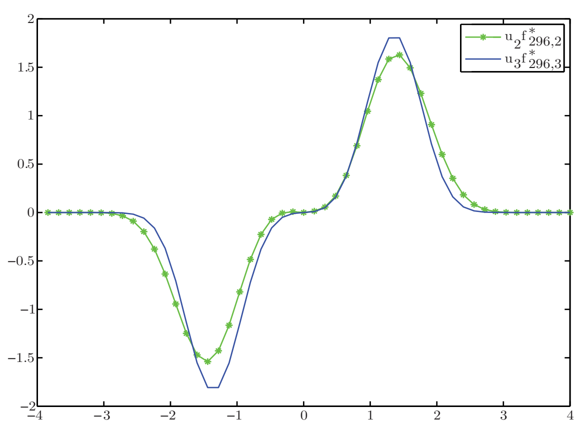

FIGURE 4. Graph of $u_i f^*_{296,i}, i = 2, 3$

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