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A Cahn–Hilliard system with forward-backward dynamic boundary condition and non-smooth potentials

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Abstract. A system with equation and dynamic boundary condition of Cahn–Hilliard type is considered. This system comes from a derivation performed in Liu–Wu (Arch. Ration. Mech. Anal., 233:167–247, 2019) via an energetic variational approach. Actually, the related problem can be seen as a transmission problem for the phase variable in the bulk and the corresponding variable on the boundary. The asymptotic behavior as the coefficient of the surface diffusion acting on the boundary phase variable goes to 0 is investigated. By this analysis we obtain a forward-backward dynamic boundary condition at the limit. We can deal with a general class of potentials having a double-well structure, including the non-smooth double-obstacle potential. We illustrate that the limit problem is well-posed by also proving a continuous dependence estimate. Moreover, in the case when the two graphs, in the bulk and on the boundary, exhibit the same growth, we show that the solution of the limit problem is more regular and we prove an error estimate for a suitable order of the diffusion parameter.

1. Introduction

Let $T > 0$ be some finite time and let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded smooth domain. Consider the heat equation: for a given initial data $u_0 := u_0(x)$, $x \in \Omega$, and heat source $f := f(t, x)$, find $u := u(t, x)$, $(t, x) \in Q := (0, T) \times \Omega$, satisfying

$$\partial_t u - \Delta u = f \quad \text{in } Q, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{1.1}$$

besides some suitable boundary condition. If instead the sign in front of the Laplace term Δu appearing in the heat equation is positive, that is,

$$\partial_t u + \Delta u = f \quad \text{in } Q, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{1.2}$$

the resultant is known to be an ill-posed problem. Indeed, (1.2) is backward-in-time and can be interpreted as a determination problem of the history of heat diffusion as follows: by the change of variable $U(t) := u(T - t)$, $t \in (0, T)$, we obtain

$$\partial_t U - \Delta U = -f \quad \text{in } Q, \quad U(T) = u_0 \quad \text{in } \Omega, \tag{1.3}$$

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where the initial condition is changed as a terminal condition at time T. From the general theory of partial differential equations, it is known that the forward heat equation (1.1) has the special property of the smoothing effect. More precisely, you can gain the smoothness of the solution at any short time even if the initial datum is not so smooth. Therefore, this consideration suggests us that some small noise in the terminal data may come from pathological deviations on intermediate states for the backward heat equation (1.3). In this sense, the continuous dependence is a delicate problem and we can say that the backward heat equation is ill posed, in general. The issue of the existence of solutions is also delicate. In order to discuss it, one needs some additional settings (see, e.g., [37]).

About this class of problems, let us raise the question: what can happen when the backward problem is set on the boundary as a dynamic boundary condition?

In this paper, we are concerned with a (possible) backward heat equation on the boundary $\Gamma := \partial\Omega$ of some smooth bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$); namely, we address a backward equation as a dynamic boundary condition of a problem which consists in finding $v : \Sigma \to \mathbb{R}$ that satisfy

$$\begin{aligned} \partial_t v + \Delta_\Gamma v &= Gu \quad \text{on } \Sigma := (0, T) \times \Gamma, \\ v(0) &= v_0 \quad \text{on } \Gamma, \end{aligned}$$

where ∂_t and Δ_Γ stand for the time derivative and Laplace–Beltrami operator (see, e.g., [26]), respectively. Moreover, $v_0 : \Gamma \to \mathbb{R}$ is prescribed. The backward nature of the boundary problem is due to the fact that the sign of the Laplace–Beltrami term appearing in the dynamic boundary condition is positive. The detail about the right-hand side Gu is given later: indeed, the variable $u : Q := (0, T) \times \Omega \to \mathbb{R}$ is also unknown and runs in the bulk, being related by a transmission condition to the unknown $v : \Sigma \to \mathbb{R}$ on the boundary.

In order to give rigorous sense to the backward dynamics on the boundary, first we artificially provide the problem with a suitable equation in the bulk with a fourth-order boundary condition in such a way that the respective bulk-boundary problem is well-posed. Then, by performing a vanishing diffusion on the boundary, in particular we recover the second-order backward heat equation on the boundary. The equation considered in the bulk is of Cahn–Hilliard type (see [8]), that refers to a celebrated model describing the spinodal decomposition in a simple framework of fourth-order partial differential equations. Some historical and mathematical description of Cahn–Hilliard systems can be found in the papers [7,16,30,38,39], to mention only a few. On the boundary, we consider the following dynamic condition of Cahn–Hilliard type (see, e.g., [14,23,36]): for $\delta \in (0, 1]$ we look for $v : \Sigma \to \mathbb{R}$ fulfilling

$$\partial_t v - \Delta_\Gamma w = 0 \quad \text{on } \Sigma, \tag{1.4}$$

$$w = -\delta\Delta_\Gamma v + \beta_\Gamma(v) + \pi_\Gamma(v) - g + \partial_{\boldsymbol{\nu}} u \quad \text{on } \Sigma, \tag{1.5}$$

$$v(0) = v_0 \quad \text{on } \Gamma, \tag{1.6}$$

where β_Γ is a monotone function (it may be also a graph), π_Γ is an anti-monotone Lipschitz continuous function, $\partial_{\boldsymbol{\nu}}$ stands for the normal derivative, $g : \Sigma \to \mathbb{R}$ is a given datum. In the last term of (1.5) the normal derivative of another unknown function $u : Q \to \mathbb{R}$ appears, and correspondingly u has to satisfy

$$\partial_t u - \Delta\mu = 0 \quad \text{in } Q, \tag{1.7}$$

$$\mu = -\Delta u + \beta(u) + \pi(u) - f \quad \text{in } Q, \tag{1.8}$$

$$\partial_{\boldsymbol{\nu}}\mu = 0 \quad \text{on } \Sigma, \tag{1.9}$$

$$u_{|\Gamma} = v \quad \text{on } \Sigma, \tag{1.10}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{1.11}$$

where the symbol Δ stands for the Laplacian, $u_{|\Gamma}$ represents the trace of u on Γ, β and π play the same role in the bulk as β_Γ and π_Γ on the boundary, $f : Q \to \mathbb{R}$ is another datum. Of course, in (1.4)–(1.11) two auxiliary variables $w : \Sigma \to \mathbb{R}$ and $\mu : Q \to \mathbb{R}$, which have the physical meaning of chemical potentials, are also outlined.

Here, we intentionally construct the system from the equations on the boundary with side conditions on the bulk. This implies that the system presents the main equations on the boundary with the equations in the bulk interpreted as auxiliary conditions (same procedure as, e.g., in [11,17,18] and references therein). Note that if we simply take $\beta_\Gamma(r) = 0$, $\pi_\Gamma(r) = -r$ for $r \in \mathbb{R}$, and let $\delta \to 0$ in (1.4)–(1.5), then the target equation on the boundary reads

$$\partial_t v + \Delta_\Gamma v = Gu := \Delta_\Gamma(\partial_{\boldsymbol{\nu}} u - g) \quad \text{on } \Sigma \tag{1.12}$$

and actually makes sense as a backward equation. On the other hand, the complementary system (1.7)–(1.11) is ready to help in order to gain solvability of the full problem despite the backward equation on the boundary.

The main topic of this paper is related to the rigorous discussion of the limiting procedure as $\delta \to 0$ for the complete system (1.4)–(1.11) and the novelty is the treatment of wider classes for β and β_Γ. Indeed, we can postulate that β and β_Γ are maximal monotone graphs, that may be multivalued, with suitable growth properties. In this respect, the equations (1.5) and (1.8) should be rewritten for suitable selections η of $\beta_\Gamma(v)$ and ξ of $\beta(u)$, respectively. In fact, in our approach β and β_Γ are the subdifferentials of proper convex lower semicontinuous functions $\widehat{\beta}$, $\widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]$ such that $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$, and the growth of β is dominated by the one of β_Γ, in the sense of assumption (A1) below with condition (2.24). In this framework, we can prove that the solution to (1.4)–(1.11), whose determination is ensured by the results in [14], suitably converges as $\delta \to 0$ to the solution of the limit problem in which (1.5) is replaced by the analogous condition with $\delta = 0$. Actually, it occurs that in the limiting process the solution of the problem with $\delta \in (0, 1]$ looses some regularity at the limit, and the limit boundary equation $w = \partial_{\boldsymbol{\nu}} u - g + \beta_\Gamma(v) + \pi_\Gamma(v)$ has to be properly interpreted in the sense of a subdifferential inclusion in dual spaces. However, the limit problem turns out to exhibit a well-posedness property since the continuous dependence of the solution with respect to the initial data and the source terms f and g can be proved. In addition to these results, in the special situation when the two graphs β and β_Γ have a comparable growth (cf. assumption (2.50) later on), we show that the solution enjoys more regularity and the limit boundary equation makes sense also almost everywhere. Moreover, we examine the refined convergence and arrive at an error estimate, for the difference of solutions, of order $\delta^{1/2}$.

Let us now mention some related work. Recently the equation and dynamic boundary condition of Cahn–Hilliard type have been studied in several papers from various viewpoints. In particular, the Cahn–Hilliard system coupled with the dynamic boundary condition of Cahn–Hilliard type as (1.4)–(1.11) has been introduced and examined by Liu–Wu in [36] for smooth or singular potentials. Then, it is important to quote the article [23] where the same problem is treated with a gradient flow approach. After that, the well-posedness problem for non-smooth potentials has been discussed in [14]. Among other contributions for this model, we point out [40] for the long time behavior and [42] for the numerical analysis. As a remark, there is a similar system of equation and dynamic boundary condition of Cahn–Hilliard type, which has been analysed, earlier than the one in [36], by Gal [22] or Goldstein–Miranville–Schimperna [25]. For this similar model, which however does not postulate a transmission condition like (1.10), the same authors of this paper investigated the problem with forward-backward boundary condition in [13]. A sort of intermediate problem between Goldstein–Miranville–Schimperna [25] and Liu–Wu [36] has been considered (see, e.g., [1,31]). About the vanishing diffusion on the dynamic boundary condition, the reader may also see the treatments in [12,44] for other Cahn–Hilliard systems, as well as [3,4,15] for vanishing diffusion in the bulk and convergence to regularised forward-backward problems. In the light of vanishing diffusion, let us additionally mention the contributions [10,19], in which the asymptotic limit of a Cahn–Hilliard system converging to a nonlinear diffusion equation is considered: the approach of [10,19] consists in taking, for $\delta \in (0, 1]$, the Cahn–Hilliard system

$$
\begin{aligned}
\partial_t u - \Delta\mu &= 0 \quad \text{in } Q,\\
\mu = -\delta\Delta u + \beta(u) + \delta\pi(u) - f &\quad \text{in } Q,
\end{aligned}
$$

with Neumann boundary conditions, where the functions β and $\delta\pi$ are the monotone and anti-monotone parts of the derivative of a double well potential. Letting $\delta \to 0$, the target problem is based on the nonlinear diffusion equation $\partial_t u - \Delta(\beta(u) - f) = 0$ in Q. Similar asymptotic limits have been applied also in other contexts (see, e.g., [20,21,23,29,32–34,45,48]).

We present a brief outline of the paper which is structured as follows. In Sect. 2, the reader can find the notation and the basic tools for a precise interpretation of the problem, which is clearly stated in terms of variational equations and regularity of solutions. After that, the main theorems are precisely stated. Section 3 is devoted to the proof of the uniform estimates, independent of the coefficient δ, for the solution to a viscous approximation of the system (1.4)–(1.11), this viscous approximation having already been used in [14]. Finally, in Sect. 4 the main theorems are finally proved, with the proofs presented in this order: we start with proving the passage to the limit as $\delta \to 0$ on the basis of the uniform estimates; next, we deal with the continuous dependence estimate, of the solution with respect to the data; then, we examine the refined convergence and show the error estimate of order $\delta^{1/2}$ in the case when the two graphs exhibit the same growth.

2. Main theorems

In this section, we present the main theorems. To this aim, we set up the target problem and its fundamental settings.

2.1. Notation and useful tools

Let $T > 0$ be a finite time and let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. Moreover, we define the sets $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \Gamma$. We use the following notation for the function spaces: $H := L^2(\Omega)$, $V := H^1(\Omega)$, and $W := H^2(\Omega)$. Norms and inner products will be denoted by $|\cdot|_X$ and $(\cdot, \cdot)_X$, respectively, where X is the corresponding Banach or Hilbert space. Analogously, let $H_\Gamma := L^2(\Gamma)$, $V_\Gamma := H^1(\Gamma)$, $W_\Gamma := H^2(\Gamma)$, and set $Z_\Gamma := H^{1/2}(\Gamma)$ as well. Next, we define the bilinear forms $a : V \times V \to \mathbb{R}$ and $a_\Gamma : V_\Gamma \times V_\Gamma \to \mathbb{R}$ by

$$a(z, \tilde{z}) := \int_\Omega \nabla z \cdot \nabla \tilde{z} \,\mathrm{d}x \quad \text{for } z, \tilde{z} \in V,$$
$$a_\Gamma(z_\Gamma, \tilde{z}_\Gamma) := \int_\Gamma \nabla_\Gamma z_\Gamma \cdot \nabla_\Gamma \tilde{z}_\Gamma \,\mathrm{d}\Gamma \quad \text{for } z_\Gamma, \tilde{z}_\Gamma \in V_\Gamma,$$

where the symbol ∇_Γ stands for the surface gradient. Moreover, we define two functions $m : V^* \to \mathbb{R}$ and $m_\Gamma : V_\Gamma^* \to \mathbb{R}$ by

$$m(z^*) := \frac{1}{|\Omega|}\langle z^*, 1\rangle_{V^*,V} \quad \text{for } z^* \in V^*,$$
$$m_\Gamma(z_\Gamma^*) := \frac{1}{|\Gamma|}\langle z_\Gamma^*, 1\rangle_{V_\Gamma^*,V_\Gamma} \quad \text{for } z_\Gamma^* \in V_\Gamma^*,$$

where the symbol X^* stands for the dual spaces of the corresponding Banach space X, $|\Omega| := \int_\Omega 1 \,\mathrm{d}x$, and $|\Gamma| := \int_\Gamma 1 \,\mathrm{d}\Gamma$. If $z^* \in H$, then $m(z^*)$ is the mean value of z^*. Analogously, $m_\Gamma(z_\Gamma^*)$ has the same meaning for $z_\Gamma^* \in H_\Gamma$. Using them, we define $H_0 := H \cap \ker(m) = \{z \in H : m(z) = 0\}$, $H_{\Gamma,0} := H_\Gamma \cap \ker(m_\Gamma)$, $V_0 := V \cap H_0$, and $V_{\Gamma,0} := V_\Gamma \cap H_{\Gamma,0}$ with the following inner products

$$(z, \tilde{z})_{H_0} := (z, \tilde{z})_H \qquad \text{for } z, \tilde{z} \in H_0,$$
$$(z, \tilde{z})_{V_0} := a(z, \tilde{z}) \qquad \text{for } z, \tilde{z} \in V_0,$$
$$(z_\Gamma, \tilde{z}_\Gamma)_{H_{\Gamma,0}} := (z_\Gamma, \tilde{z}_\Gamma)_{H_\Gamma} \quad \text{for } z_\Gamma, \tilde{z}_\Gamma \in H_{\Gamma,0},$$

$$(z_\Gamma, \tilde{z}_\Gamma)_{V_{\Gamma,0}} := a_\Gamma(z_\Gamma, \tilde{z}_\Gamma) \quad \text{for } z_\Gamma, \tilde{z}_\Gamma \in V_{\Gamma,0}.$$

We point out that, owing to the Poincaré–Wirtinger inequality, there exists a constant $C_\mathrm{P} > 0$ such that

$$|z|_V^2 \le C_\mathrm{P}\Big(\big|z - m(z)\big|_{V_0}^2 + \big|m(z)\big|^2\Big) \qquad \text{for all } z \in V, \tag{2.1}$$

$$|z|_V^2 \le C_\mathrm{P}|z|_{V_0}^2 \qquad \text{for all } z \in V_0, \tag{2.2}$$

$$|z_\Gamma|_{V_\Gamma}^2 \le C_\mathrm{P}\Big(\big|z_\Gamma - m_\Gamma(z_\Gamma)\big|_{V_{\Gamma,0}}^2 + \big|m_\Gamma(z_\Gamma)\big|^2\Big) \qquad \text{for all } z \in V_\Gamma, \tag{2.3}$$

$$|z_\Gamma|_{V_\Gamma}^2 \le C_\mathrm{P}|z_\Gamma|_{V_{\Gamma,0}}^2 \qquad \text{for all } z_\Gamma \in V_{\Gamma,0}. \tag{2.4}$$

Therefore, we can define the bounded linear operators $F : V_0 \to V_0^*$ and $F_\Gamma : V_{\Gamma,0} \to V_{\Gamma,0}^*$ as follows:

$$\langle Fz, \tilde{z}\rangle_{V_0^*, V_0} := a(z, \tilde{z}) \qquad \text{for } z, \tilde{z} \in V_0,$$
$$\langle F_\Gamma z_\Gamma, \tilde{z}_\Gamma\rangle_{V_{\Gamma,0}^*, V_{\Gamma,0}} := a_\Gamma(z_\Gamma, \tilde{z}_\Gamma) \qquad \text{for } z_\Gamma, \tilde{z}_\Gamma \in V_{\Gamma,0},$$

and observe that F and F_Γ are duality mappings. Moreover, $Fz = 0$ in V_0^* if and only if $z = 0$ in V_0, that is, F is invertible. Analogously, F_Γ is also invertible. Therefore, we can define the inner products

$$(z^*, \tilde{z}^*)_{V_0^*} := \langle z^*, F^{-1}\tilde{z}^*\rangle_{V_0^*, V_0} \qquad \text{for } z^*, \tilde{z}^* \in V_0^*,$$
$$(z_\Gamma^*, \tilde{z}_\Gamma^*)_{V_{\Gamma,0}^*} := \langle z_\Gamma^*, F_\Gamma^{-1}\tilde{z}_\Gamma^*\rangle_{V_{\Gamma,0}^*, V_{\Gamma,0}} \qquad \text{for } z_\Gamma^*, \tilde{z}_\Gamma^* \in V_{\Gamma,0}^*,$$

which give the related norms

$$|z^*|_{V_0^*} = \left\{\int_\Omega |\nabla F^{-1}z^*|\,\mathrm{d}x\right\}^{1/2} \qquad \text{for } z^* \in V_0^*,$$
$$|z_\Gamma^*|_{V_{\Gamma,0}^*} = \left\{\int_\Gamma |\nabla_\Gamma F_\Gamma^{-1}z_\Gamma^*|\,\mathrm{d}\Gamma\right\}^{1/2} \qquad \text{for } z_\Gamma^* \in V_{\Gamma,0}^*.$$

Finally, we introduce the following norms in V^* and V_Γ^*,

$$|z^*|_* = \Big\{\big|z^* - m(z^*)\big|_{V_0^*}^2 + \big|m(z^*)\big|^2\Big\}^{1/2} \qquad \text{for } z^* \in V^*, \tag{2.5}$$

$$|z_\Gamma^*|_{\Gamma,*} = \Big\{\big|z_\Gamma^* - m_\Gamma(z_\Gamma^*)\big|_{V_{\Gamma,0}^*}^2 + \big|m_\Gamma(z_\Gamma^*)\big|^2\Big\}^{1/2} \qquad \text{for } z_\Gamma^* \in V_\Gamma^*, \tag{2.6}$$

and observe that they are equivalent to the standard induced norms $|\cdot|_{V^*}$ of V^* and $|\cdot|_{V_\Gamma^*}$ of V_Γ^*, respectively. Then we obtain the following dense and compact embeddings:

$$V \hookrightarrow\hookrightarrow H \hookrightarrow V^*, \quad V_0 \hookrightarrow\hookrightarrow H_0 \hookrightarrow V_0^*,$$
$$V_\Gamma \hookrightarrow\hookrightarrow H_\Gamma \hookrightarrow V_\Gamma^*, \quad Z_\Gamma \hookrightarrow\hookrightarrow H_\Gamma \hookrightarrow V_\Gamma^*, \quad V_{\Gamma,0} \hookrightarrow\hookrightarrow H_{\Gamma,0} \hookrightarrow V_{\Gamma,0}^*,$$

where "$\hookrightarrow\hookrightarrow$" stands for the dense and compact embedding.

For the reader's convenience, we recall useful tools in functional analysis. The first tool is related to the trace theorem (see, e.g., [5, Theorem 2.24], [43, Chapter 2, Theorem 5.7]), which states that there exist unique continuous linear operators $\gamma_0 : V \to Z_\Gamma$ and $\gamma_1 : W \to Z_\Gamma$ such that

$$\gamma_0 z = z_{|\Gamma} \quad \text{for all } z \in C^\infty(\overline{\Omega}) \cap V,$$
$$\gamma_1 z = \partial_{\boldsymbol{\nu}} z \quad \text{for all } z \in C^\infty(\overline{\Omega}) \cap W.$$

Moreover, there exists a positive constant C_{tr} such that

$$|\gamma_0 z|_{Z_\Gamma} \le C_{\mathrm{tr}} |z|_V \quad \text{for all } z \in V. \tag{2.7}$$

2.2. Target problem

Now we set up our target problem of the forward-backward dynamic boundary equation along with the bulk condition of Cahn–Hilliard type and considering non-smooth potentials. Find $v, w, \eta : \Sigma \to \mathbb{R}$ and $u, \mu, \xi : Q \to \mathbb{R}$ satisfying

$$\partial_t v - \Delta_\Gamma w = 0 \quad \text{a.e. on } \Sigma, \tag{2.8}$$
$$w = \partial_{\boldsymbol{\nu}} u + \eta + \pi_\Gamma(v) - g, \quad \eta \in \beta_\Gamma(v) \quad \text{a.e. on } \Sigma, \tag{2.9}$$
$$\partial_t u - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{2.10}$$
$$\mu = -\Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \tag{2.11}$$
$$\partial_{\boldsymbol{\nu}} \mu = 0 \quad \text{a.e. on } \Sigma, \tag{2.12}$$
$$u_{|\Gamma} = v \quad \text{a.e. on } \Sigma, \tag{2.13}$$
$$v(0) = v_0 \quad \text{a.e. on } \Gamma, \tag{2.14}$$
$$u(0) = u_0 \quad \text{a.e. in } \Omega, \tag{2.15}$$

where β_Γ and β are maximal monotone graphs on $\mathbb{R} \times \mathbb{R}$, π_Γ and π are Lipschitz continuous functions, $g : \Sigma \to \mathbb{R}$, $f : Q \to \mathbb{R}$, $v_0 : \Gamma \to \mathbb{R}$, and $u_0 : \Omega \to \mathbb{R}$ are given functions. Combining (2.8) and (2.9), we find a structure of second-order partial differential equation of forward-backward type on the boundary equation. Indeed, in general the sum $\beta_\Gamma + \pi_\Gamma$ is not monotonically increasing on the whole domain. As prototypes, we can choose

- ▹ $\beta_\Gamma(r) := r^3$, $\pi_\Gamma(r) := -r$ for $r \in \mathbb{R}$ (corresponding to the smooth double well potential);
- ▹ $\beta_\Gamma(r) := \ln((1+r)/(1-r))$, $\pi_\Gamma(r) := -2cr$ for $r \in (-1, 1)$ (derived from the singular potential of logarithmic type, where $c > 0$ is a large constant which breaks monotonicity);
- ▹ $\beta_\Gamma(r) := \partial I_{[-1,1]}(r)$, $\pi_\Gamma(r) := -r$ for $r \in [-1, 1]$ (for the non-smooth potential, where the symbol ∂ stands for the subdifferential in $\mathbb{R}$);
- ▹ $\beta_\Gamma(r) := 0$, $\pi_\Gamma(r) := -r$ for $r \in \mathbb{R}$ (for the backward-like heat equation on the boundary).

In our approach, according to previous contributions (cf., e.g., [9,12–14]), about β we prescribe a condition on the growth, that sets a control by the growth of β_Γ, see the later assumption (A1) and condition (2.24). Instead, we can choose any Lipschitz continuous function for π, independent of π_Γ.

2.3. Main theorems

We recall an auxiliary Cahn–Hilliard system approaching our target problem: for $\delta \in (0, 1]$, find $u_\delta, \mu_\delta, \xi_\delta : Q \to \mathbb{R}$ and $v_\delta, w_\delta, \eta_\delta : \Sigma \to \mathbb{R}$ satisfying

$$\partial_t u_\delta - \Delta \mu_\delta = 0 \quad \text{a.e. in } Q, \tag{2.16}$$
$$\mu_\delta = -\Delta u_\delta + \xi_\delta + \pi(u_\delta) - f, \quad \xi_\delta \in \beta(u_\delta) \quad \text{a.e. in } Q, \tag{2.17}$$
$$\partial_{\boldsymbol{\nu}} \mu_\delta = 0 \quad \text{a.e. on } \Sigma, \tag{2.18}$$
$$(u_\delta)_{|\Gamma} = v_\delta \quad \text{a.e. on } \Sigma, \tag{2.19}$$
$$\partial_t v_\delta - \Delta_\Gamma w_\delta = 0 \quad \text{a.e. on } \Sigma, \tag{2.20}$$
$$w_\delta = \partial_{\boldsymbol{\nu}} u_\delta - \delta \Delta_\Gamma v_\delta + \eta_\delta + \pi_\Gamma(v_\delta) - g, \quad \eta_\delta \in \beta_\Gamma(v_\delta) \quad \text{a.e. on } \Sigma, \tag{2.21}$$
$$u_\delta(0) = u_0 \quad \text{a.e. in } \Omega, \tag{2.22}$$
$$v_\delta(0) = v_0 \quad \text{a.e. on } \Gamma. \tag{2.23}$$

This system of equation and dynamic boundary condition of Cahn–Hilliard type has been introduced by Liu–Wu in [36] and its solvability is discussed in the papers [23,36] under some restrictions for β and β_Γ, while in the case $\delta > 0$ the well-posedness issue is examined in [14, Theorems 2.3, 2.4, and 4.1] under our general conditions on the graphs β and β_Γ (cf. the assumption (A1) below). The aim of the present paper is the extension of the results in [14] to the limiting situation $\delta = 0$. In particular, in our analysis we are able to avoid the geometric conditions of Liu–Wu (cf. [36, Theorem 3.2], see also [23]).

In this paper, we assume:

(A1) β and β_Γ are maximal monotone graphs on $\mathbb{R} \times \mathbb{R}$, and there exist proper, lower semicontinuous, and convex functions $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ such that

$$\beta = \partial \widehat{\beta}, \quad \beta_\Gamma = \partial \widehat{\beta}_\Gamma.$$

Moreover, we assume that $D(\beta_\Gamma) \subset D(\beta)$ and there exists positive constants $\varrho_1, c_1 > 0$ such that

$$\left|\beta^\circ(r)\right| \le \varrho_1 \left|\beta_\Gamma^\circ(r)\right| + c_1 \quad \text{for all } r \in D(\beta_\Gamma). \tag{2.24}$$

(A2) $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, with their constants L and L_Γ, respectively. Moreover, we set $\widehat{\pi}(\rho) := \int_0^\rho \pi(r)\, dr$ and $\widehat{\pi}_\Gamma(\rho) := \int_0^\rho \pi_\Gamma(r)\, dr$, $\rho \in \mathbb{R}$;

(A3) $f \in L^2(0, T; V)$ and $g \in L^2(0, T; V_\Gamma)$;

(A4) $u_0 \in V$, $v_0 \in V_\Gamma$ satisfy $\gamma_0 u_0 = v_0$ in Z_Γ. Moreover, $u_0 \in L^\infty(\Omega)$, so that $v_0 \in L^\infty(\Gamma)$ as well, and

$$\left[\operatorname*{ess\,inf}_{x\in\Omega} u_0(x),\ \operatorname*{ess\,sup}_{x\in\Omega} u_0(x)\right] \subset \operatorname{int} D(\beta),$$
$$\left[\operatorname*{ess\,inf}_{x\in\Gamma} v_0(x),\ \operatorname*{ess\,sup}_{x\in\Gamma} v_0(x)\right] \subset \operatorname{int} D(\beta_\Gamma).$$

Note that this implies that $\widehat{\beta}(u_0) \in L^1(\Omega)$, $\widehat{\beta}_\Gamma(v_0) \in L^1(\Gamma)$, $m_0 := m(u_0) \in \operatorname{int} D(\beta)$, and $m_{\Gamma 0} := m_\Gamma(v_0) \in \operatorname{int} D(\beta_\Gamma)$.

We notice that in (A1) the symbol β° stands for the minimal section defined by

$$\beta^\circ(r) := \left\{r^* \in \beta(r) : |r^*| = \min_{s\in\beta(r)} |s|\right\},$$

and same definition holds for β_Γ°. Of course we can choose $\beta(r) = \beta_\Gamma(r) = 0$ for $r \in D(\beta_\Gamma) := \mathbb{R}$.

Recalling the known result in [14] we obtain the following proposition for $\delta \in (0, 1]$.

Proposition 2.1. [14, Theorems 2.3, 2.4] *Under the assumptions* (A1)–(A4)*, there exists a sextuplet* $(u_\delta, \mu_\delta, \xi_\delta, v_\delta, w_\delta, \eta_\delta)$*, where* u_δ *and* v_δ *are uniquely determined, so that*

$$\begin{aligned}
&u_\delta \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\mu_\delta \in L^2(0,T;V), \quad \xi_\delta \in L^2(0,T;H),\\
&v_\delta \in H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;W_\Gamma),\\
&w_\delta \in L^2(0,T;V_\Gamma), \quad \eta_\delta \in L^2(0,T;H_\Gamma)
\end{aligned}$$

and they satisfy

$$\langle \partial_t u_\delta, z\rangle_{V^*,V} + \int_\Omega \nabla\mu_\delta \cdot \nabla z \,\mathrm{d}x = 0 \quad \textit{for all } z \in V, \ \textit{a.e. in } (0,T), \tag{2.25}$$

$$\mu_\delta = -\Delta u_\delta + \xi_\delta + \pi(u_\delta) - f, \quad \xi_\delta \in \beta(u_\delta) \quad \textit{a.e. in } Q, \tag{2.26}$$

$$(u_\delta)_{|\Gamma} = v_\delta \quad \textit{a.e. on } \Sigma, \tag{2.27}$$

$$\langle \partial_t v_\delta, z_\Gamma\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Gamma \nabla_\Gamma w_\delta \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma = 0 \quad \textit{for all } z_\Gamma \in V_\Gamma, \ \textit{a.e. in } (0,T), \tag{2.28}$$

$$w_\delta = \partial_{\boldsymbol{\nu}} u_\delta - \delta\Delta_\Gamma v_\delta + \eta_\delta + \pi_\Gamma(v_\delta) - g, \quad \eta_\delta \in \beta_\Gamma(v_\delta) \quad \textit{a.e. on } \Sigma, \tag{2.29}$$

$$u_\delta(0) = u_0 \quad \textit{a.e. in } \Omega, \tag{2.30}$$

$$v_\delta(0) = v_0 \quad \textit{a.e. on } \Gamma. \tag{2.31}$$

We note that, due to the lack of the regularities of time derivatives, Eqs. (2.16) and (2.20) are replaced by the variational formulations (2.25) and (2.28), respectively. Moreover, the boundary condition (2.18) is hidden in the weak form (2.25). Here and hereafter we frequently use the notations $z_{|\Gamma}$ and $\partial_{\boldsymbol{\nu}} z$ in place of $\gamma_0 z$ and $\gamma_1 z$, respectively.

Our main theorem is stated here:

Theorem 2.2. *Under the assumptions* (A1)–(A4)*, there exists at least one sextuplet* $(u, \mu, \xi, v, w, \eta)$ *fulfilling*

$$
\begin{aligned}
& u \in H^1(0,T;V^*) \cap L^\infty(0,T;V), \quad \Delta u \in L^2(0,T;H) \\
& \mu \in L^2(0,T;V), \quad \xi \in L^2(0,T;H), \\
& v \in H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;Z_\Gamma), \\
& w \in L^2(0,T;V_\Gamma), \quad \eta \in L^2(0,T;Z_\Gamma^*)
\end{aligned}
$$

and satisfying (2.8)–(2.15) *in the following sense:*

$$\langle \partial_t u, z\rangle_{V^*,V} + \int_\Omega \nabla\mu \cdot \nabla z \, dx = 0 \ \ \textit{for all } z \in V, \ \textit{a.e. in } (0,T), \tag{2.32}$$

$$\mu = -\Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \ \ \textit{a.e. in } Q, \tag{2.33}$$

$$u_{|\Gamma} = v \ \ \textit{a.e. on } \Sigma, \tag{2.34}$$

$$\langle \partial_t v, z_\Gamma\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma z_\Gamma \, d\Gamma = 0$$
$$\textit{for all } z_\Gamma \in V_\Gamma, \ \textit{a.e. in } (0,T), \tag{2.35}$$

$$(w, z_\Gamma)_{H_\Gamma} = \langle \partial_{\boldsymbol{\nu}} u + \eta, z_\Gamma\rangle_{Z_\Gamma^*,Z_\Gamma} + \big(\pi_\Gamma(v) - g, z_\Gamma\big)_{H_\Gamma}$$
$$\textit{for all } z_\Gamma \in Z_\Gamma, \ \textit{a.e. in } (0,T), \tag{2.36}$$

$$\langle \eta, z_\Gamma - v\rangle_{Z_\Gamma^*,Z_\Gamma} \le \int_\Gamma \widehat{\beta}_\Gamma(z_\Gamma)\, d\Gamma - \int_\Gamma \widehat{\beta}_\Gamma(v)\, d\Gamma$$
$$\textit{for all } z_\Gamma \in Z_\Gamma, \ \textit{a.e. in } (0,T), \tag{2.37}$$

$$u(0) = u_0 \ \ \textit{a.e. in } \Omega, \tag{2.38}$$

$$v(0) = v_0 \ \ \textit{a.e. on } \Gamma. \tag{2.39}$$

Moreover, $(u, \mu, \xi, v, w, \eta)$ *is obtained as limit of the family* $\{(u_\delta, \mu_\delta, \xi_\delta, v_\delta, w_\delta, \eta_\delta)\}_{0<\delta\le 1}$ *of the sextuplet solutions given by Proposition 2.1, in the sense that there is a subsequence* $\{\delta_k\}_{k\in\mathbb{N}}$ *such that, as* $k \to +\infty$*,*

$$u_{\delta_k} \to u \quad \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V), \tag{2.40}$$

$$\Delta u_{\delta_k} \to \Delta u \quad \textit{weakly in } L^2(0,T;H), \tag{2.41}$$

$$\partial_{\boldsymbol{\nu}} u_{\delta_k} \to \partial_{\boldsymbol{\nu}} u \quad \textit{weakly in } L^2(0,T;Z_\Gamma^*), \tag{2.42}$$

$$\mu_{\delta_k} \to \mu \quad \textit{weakly in } L^2(0,T;V), \tag{2.43}$$

$$\xi_{\delta_k} \to \xi \quad \textit{weakly in } L^2(0,T;H), \tag{2.44}$$

$$v_{\delta_k} \to v \quad \textit{weakly star in } H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;Z_\Gamma), \tag{2.45}$$

$$\delta_k v_{\delta_k} \to 0 \quad \textit{strongly in } L^\infty(0,T;V_\Gamma), \tag{2.46}$$

$$w_{\delta_k} \to w \quad \textit{weakly in } L^2(0,T;V_\Gamma), \tag{2.47}$$

$$\eta_{\delta_k} \to \eta \quad \textit{weakly in } L^2(0,T;V_\Gamma^*), \tag{2.48}$$

$$(-\delta_k \Delta_\Gamma v_{\delta_k} + \eta_{\delta_k}) \to \eta \quad \textit{weakly in } L^2(0,T;Z_\Gamma^*). \tag{2.49}$$

Remark 2.3. About the inequality (2.37), we point out that whenever $\eta \in L^2(0,T;H_\Gamma)$ then (2.37) is actually equivalent to the inclusion $\eta \in \beta_\Gamma(v)$ a.e. on Σ, or equivalently

$$\eta \in \partial I_\Sigma(v),$$

where

$$I_\Sigma : L^2(0,T;H_\Gamma) \to [0,+\infty],\ I_\Sigma(z_\Gamma) := \begin{cases} \displaystyle\int_\Sigma \widehat{\beta}_\Gamma(z_\Gamma)\,\mathrm{d}\Gamma\mathrm{d}t & \text{if } \widehat{\beta}_\Gamma(z_\Gamma) \in L^1(\Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

On the other hand, if we only have $\eta \in L^2(0,T;Z_\Gamma^*)$, then (2.37) means that $\eta \in \partial J_\Sigma(v)$, where

$$J_\Sigma : L^2(0,T;Z_\Gamma) \to [0,+\infty],\ J_\Sigma(z_\Gamma) := \begin{cases} \displaystyle\int_\Sigma \widehat{\beta}_\Gamma(z_\Gamma)\,\mathrm{d}\Gamma\mathrm{d}t & \text{if } \widehat{\beta}_\Gamma(z_\Gamma) \in L^1(\Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

Here, the main point is that, since we are identifying H_Γ to its dual, the subdifferential ∂I_Σ is intended as a multivalued operator

$$\partial I_\Sigma \ \text{ from } \ L^2(0,T;H_\Gamma) \ \text{ to } \ L^2(0,T;H_\Gamma),$$

while ∂J_Σ is seen as an operator, multivalued as well,

$$\partial J_\Sigma \ \text{ from } \ L^2(0,T;Z_\Gamma) \ \text{ to } \ L^2(0,T;Z_\Gamma^*).$$

For further details we refer to [2,6].

Remark 2.4. Take, for instance, the case $\beta_\Gamma \equiv 0$, which yields that β should be at most bounded due to (A1) and (2.24). In this case, it is compulsory to have $\eta = 0$ and, therefore, by a comparison of term in (2.36) we deduce that $\partial_{\boldsymbol{\nu}} u \in L^2(0,T;Z_\Gamma)$, being in fact $w = \partial_{\boldsymbol{\nu}} u + \pi_\Gamma(v) - g$ an element of $L^2(0,T;V_\Gamma)$. Then we interpret the backward equation (2.35) on the boundary as

$$\begin{aligned} &\langle \partial_t v, z_\Gamma \rangle_{V_\Gamma^*, V_\Gamma} + \int_\Gamma \nabla_\Gamma \big(\partial_{\boldsymbol{\nu}} u + \pi_\Gamma(v)\big) \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma \\ &\quad = \int_\Gamma \nabla_\Gamma g \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma \quad \text{for all } z_\Gamma \in V_\Gamma, \end{aligned}$$

a.e. in $(0,T)$, where thanks to (A3) we can move the term containing g to the right-hand side, but we cannot split $\partial_{\boldsymbol{\nu}} u + \pi_\Gamma(v) \in L^2(0,T;V_\Gamma)$.

Next theorem is related to the continuous dependence on the given data:

Theorem 2.5. *For any data* $\{(f^{(i)}, g^{(i)}, u_0^{(i)}, v_0^{(i)})\}_{i=1,2}$ *satisfying* (A3), (A4) *and such that* $m(u_0^{(1)}) = m(u_0^{(2)})$, $m_\Gamma(v_0^{(1)}) = m_\Gamma(v_0^{(2)})$, *let* $(u^{(i)}, \mu^{(i)}, \xi^{(i)}, v^{(i)}, w^{(i)}, \eta^{(i)})$ *be some respective solutions obtained by Theorem* 2.2. *Then there exists a positive constant* $C > 0$ *such that*

$$\begin{aligned}
&\left|u^{(1)}(t) - u^{(2)}(t)\right|_*^2 + \left|v^{(1)}(t) - v^{(2)}(t)\right|_{\Gamma,*}^2 \\
&\quad + \int_0^t \left|u^{(1)}(s) - u^{(2)}(s)\right|_V^2 \mathrm{d}s + \int_0^t \left|v^{(1)}(s) - v^{(2)}(s)\right|_{Z_\Gamma}^2 \mathrm{d}s \\
&\le C\biggl(\left|u_0^{(1)} - u_0^{(2)}\right|_*^2 + \left|v_0^{(1)} - v_0^{(2)}\right|_{\Gamma,*}^2 \\
&\quad + \int_0^t \left|f^{(1)}(s) - f^{(2)}(s)\right|_H^2 \mathrm{d}s + \int_0^t \left|g^{(1)}(s) - g^{(2)}(s)\right|_{H_\Gamma}^2 \mathrm{d}s\biggr)
\end{aligned}$$

for all $t \in [0, T]$.

Of course, this theorem entails the uniqueness property for u and v. If β and β_Γ are single-valued functions, then the whole sextuplet $(u, \mu, \xi, v, w, \eta)$ obtained by Theorem 2.2 is unique as well.

As a remark, the discussion of the continuous dependence is delicate for backward problems in general. In such a problem, under the assumption of the existence of bounded solutions, the conditional stability is discussed in some sense in [28] (see references therein) and in [46] for the Cahn–Hilliard equation.

The results that follow are inspired by the analogous ones in [13].

Theorem 2.6. *Under the assumptions* (A1)–(A4)*, suppose also that*

$$\begin{aligned}
&D(\beta) = D(\beta_\Gamma), \quad \textit{there exists a constant } M \ge 1 \textit{ such that} \\
&\frac{1}{M}\left|\beta_\Gamma^\circ(r)\right| - M \le \left|\beta^\circ(r)\right| \le M\left(\left|\beta_\Gamma^\circ(r)\right| + 1\right) \quad \textit{for all } r \in D(\beta).
\end{aligned} \tag{2.50}$$

Then, the limiting sextuplet $(u, \mu, \xi, v, w, \eta)$ *obtained in Theorem* 2.2 *also satisfies*

$$\begin{aligned}
&u \in L^2\bigl(0, T; H^{3/2}(\Omega)\bigr), \quad \partial_\nu u \in L^2(0, T; H_\Gamma), \quad v \in L^2(0, T; V_\Gamma), \\
&\eta \in L^2(0, T; H_\Gamma), \quad \eta \in \beta_\Gamma(v) \quad \textit{a.e. on } \Sigma.
\end{aligned}$$

Moreover, in addition to (2.40)–(2.49)*, the following convergences hold, as* $k \to +\infty$*,*

$$\eta_{\delta_k} \to \eta \quad \textit{weakly in } L^2(0, T; H_\Gamma), \tag{2.51}$$

$$\delta_k v_{\delta_k} \to 0 \quad \textit{weakly in } L^2\bigl(0, T; H^{3/2}(\Gamma)\bigr), \tag{2.52}$$

$$\partial_{\boldsymbol{\nu}} u_{\delta_k} - \delta_k \Delta_\Gamma v_{\delta_k} \to \partial_{\boldsymbol{\nu}} u \quad \textit{weakly in } L^2(0, T; H_\Gamma). \tag{2.53}$$

In particular, (2.36) *can be rewritten as*

$$w = \partial_\nu u + \eta + \pi_\Gamma(v) - g \qquad \textit{a.e. on } \Sigma. \tag{2.54}$$

Remark 2.7. We note that the additional assumption (2.50) is a reinforcement of (A1) and (2.24), for some constant $M \geq \max\{\varrho_1, c_1\}$. In fact, (2.50) implies that the two graphs β and β_Γ have the same growth properties.

Theorem 2.8. *In the setting of Theorem 2.6, let $(u, \mu, \xi, v, w, \eta)$ denote the sextuplet solution of the problem (2.32)–(2.39) given by Theorem 2.2 and, for $0 < \delta \leq 1$, let $(u_\delta, \mu_\delta, \xi_\delta, v_\delta, w_\delta, \eta_\delta)$ be the sextuplet solution of the problem (2.25)–(2.31) given by Proposition 2.1. Then, there exists a constant $C > 0$, independent of δ, such that*

$$|u_\delta - u|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + |v_\delta - v|_{L^\infty(0,T;V_\Gamma^*)\cap L^2(0,T;Z_\Gamma)} \leq C\delta^{1/2} \tag{2.55}$$

for every $\delta \in (0, 1]$ and, as $\delta \to 0$,

$$v_\delta \to v \quad \textit{weakly in } L^2(0, T; V_\Gamma). \tag{2.56}$$

3. Uniform estimates

In this section, we will obtain uniform estimates independent of the parameter $0 < \delta \leq 1$. To do so, we recall another suitable approximation to the auxiliary problem. Then, taking care of the previous known results, we will obtain uniform estimates that are useful for the limiting procedure.

3.1. Yosida approximation and viscous Cahn–Hilliard system

In the approach of [14], Proposition 2.1 has been proved by considering the following viscous Cahn–Hilliard system: for $\delta, \lambda \in (0, 1]$

$$\partial_t u_{\delta,\lambda} - \Delta\mu_{\delta,\lambda} = 0 \quad \text{a.e. in } Q, \tag{3.1}$$

$$\mu_{\delta,\lambda} = \lambda\partial_t u_{\delta,\lambda} - \Delta u_{\delta,\lambda} + \beta_\lambda(u_{\delta,\lambda}) + \pi(u_{\delta,\lambda}) - f \quad \text{a.e. in } Q, \tag{3.2}$$

$$\partial_{\boldsymbol{\nu}}\mu_{\delta,\lambda} = 0 \quad \text{a.e. on } \Sigma, \tag{3.3}$$

$$(u_{\delta,\lambda})_{|\Gamma} = v_{\delta,\lambda} \quad \text{a.e. on } \Sigma, \tag{3.4}$$

$$\partial_t v_{\delta,\lambda} - \Delta_\Gamma w_{\delta,\lambda} = 0 \quad \text{a.e. on } \Sigma, \tag{3.5}$$

$$\begin{aligned} w_{\delta,\lambda} &= \lambda\partial_t v_{\delta,\lambda} + \partial_{\boldsymbol{\nu}} u_{\delta,\lambda} - \delta\Delta_\Gamma v_{\delta,\lambda} \\ &\quad + \beta_{\Gamma,\lambda}(v_{\delta,\lambda}) + \pi_\Gamma(v_{\delta,\lambda}) - g \quad \text{a.e. on } \Sigma, \end{aligned} \tag{3.6}$$

$$u_{\delta,\lambda}(0) = u_0 \quad \text{a.e. in } \Omega, \tag{3.7}$$

$$v_{\delta,\lambda}(0) = v_0 \quad \text{a.e. on } \Gamma, \tag{3.8}$$

where β_λ and $\beta_{\Gamma,\lambda}$ are the Yosida approximations of β and β_Γ, respectively, defined by

$$\beta_\lambda(r) := \frac{1}{\lambda}\bigl(r - J_\lambda(r)\bigr) := \frac{1}{\lambda}\bigl(r - (I + \lambda\beta)^{-1}(r)\bigr),$$
$$\beta_{\Gamma,\lambda}(r) := \frac{1}{\lambda}\bigl(r - J_{\Gamma,\lambda}(r)\bigr) := \frac{1}{\lambda}\bigl(r - (I + \lambda\beta_\Gamma)^{-1}(r)\bigr) \quad \text{for } r \in \mathbb{R}.$$

From the well-known theory of maximal monotone operators (see, e.g., [2]), we see that β_λ and $\beta_{\Gamma,\lambda}$ are Lipschitz continuous functions with Lipschitz constant $1/\lambda$. Moreover, it holds that

$$\big|\beta_\lambda(r)\big| \le \big|\beta^\circ(r)\big|, \quad 0 \le \widehat{\beta}_\lambda(r) \le \widehat{\beta}(r), \qquad \text{for all } r \in D(\beta), \tag{3.9}$$
$$\big|\beta_{\Gamma,\lambda}(r)\big| \le \big|\beta_\Gamma^\circ(r)\big|, \quad 0 \le \widehat{\beta}_{\Gamma,\lambda}(r) \le \widehat{\beta}_\Gamma(r) \quad \text{for all } r \in D(\beta_\Gamma). \tag{3.10}$$

The approximating problem (3.1)–(3.8) is well posed [14], namely, there exists a unique quadruplet $(u_{\delta,\lambda}, \mu_{\delta,\lambda}, v_{\delta,\lambda}, w_{\delta,\lambda})$, with

$$\begin{aligned}
&u_{\delta,\lambda} \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\mu_{\delta,\lambda} \in L^2(0,T;W),\\
&v_{\delta,\lambda} \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;W_\Gamma),\\
&w_{\delta,\lambda} \in L^2(0,T;W_\Gamma),
\end{aligned}$$

satisfying (3.1)–(3.8). Moreover, $(u_{\delta,\lambda}, \mu_{\delta,\lambda}, v_{\delta,\lambda}, w_{\delta,\lambda})$ converges to the sextuplet

$$(u_\delta, \mu_\delta, \xi_\delta, v_\delta, w_\delta, \eta_\delta)$$

given by Proposition 2.1 in a suitable sense, where ξ_δ and η_δ are the limits of $\beta_\lambda(u_{\delta,\lambda})$ and $\beta_{\Gamma,\lambda}(v_{\delta,\lambda})$ as $\lambda \to 0$, respectively (see, [14, Theorem 2.3]). Therefore, we omit the details of the limiting procedure $\lambda \to 0$ in this paper.

From the next subsection, we will obtain the uniform estimates for the approximating problem (3.1)–(3.8), whereas we will discuss the limiting procedure $\delta \to 0$ in the next section.

3.2. 1st estimate (related to the volume conservation).

Integrating (3.1) over $\Omega \times (0,t)$, multiplying by $1/|\Omega|$, and using (3.3), (3.7) we obtain

$$m\big(u_{\delta,\lambda}(t)\big) = m(u_0) = m_0 \tag{3.11}$$

for all $t \in [0,T]$. On the other hand, integrating (3.5) over $\Gamma \times (0,t)$ and multiplying by $1/|\Gamma|$, from (3.8) we have that

$$m_\Gamma\big(v_{\delta,\lambda}(t)\big) = m_\Gamma(v_0) = m_{\Gamma 0}$$

for all $t \in [0,T]$. Also, we observe that

$$\big\langle \partial_t\big(u_{\delta,\lambda}(t) - m_0\big), 1\big\rangle_{V^*,V} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{\delta,\lambda}(t)\,\mathrm{d}x = 0,$$

which yields that $\partial_t(u_{\delta,\lambda}(t) - m_0) \in V_0^*$, and analogously $\partial_t(v_{\delta,\lambda}(t) - m_{\Gamma 0}) \in V_{\Gamma,0}^*$ for a.a. $t \in (0,T)$. Moreover, there exists a positive constant $M_1 > 0$ such that

$$\big|m(u_{\delta,\lambda})\big|_{L^\infty(0,T)} + \big|m_\Gamma(v_{\delta,\lambda})\big|_{L^\infty(0,T)} \le M_1. \tag{3.12}$$

3.3. 2nd estimate

Multiply (3.1) by $F^{-1}(u_{\delta,\lambda}(t)-u_0)$ and (3.5) by $F_\Gamma^{-1}(v_{\delta,\lambda}(t)-v_0)$. Then, using (3.3) we obtain

$$\begin{aligned}&\big\langle \partial_t\big(u_{\delta,\lambda}(t)-u_0\big), F^{-1}\big(u_{\delta,\lambda}(t)-u_0\big)\big\rangle_{V_0^*,V_0}\\&\quad+\int_\Omega \nabla\mu_{\delta,\lambda}(t)\cdot\nabla F^{-1}\big(u_{\delta,\lambda}(t)-u_0\big)\,\mathrm{d}x=0,\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&\big\langle \partial_t\big(v_{\delta,\lambda}(t)-v_0\big), F_\Gamma^{-1}\big(v_{\delta,\lambda}(t)-v_0\big)\big\rangle_{V_{\Gamma,0}^*,V_{\Gamma,0}}\\&\quad+\int_\Gamma \nabla_\Gamma w_{\delta,\lambda}(t)\cdot\nabla_\Gamma F_\Gamma^{-1}\big(v_{\delta,\lambda}(t)-v_0\big)\,\mathrm{d}\Gamma=0\end{aligned}\tag{3.14}$$

for a.a. $t\in(0,T)$. Next, multiplying (3.2) by $u_{\delta,\lambda}(t)-u_0$ and using (3.4) we infer that

$$\begin{aligned}&\big(\mu_{\delta,\lambda}(t),u_{\delta,\lambda}(t)-u_0\big)_H\\&=\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|u_{\delta,\lambda}(t)-u_0\big|_{H_0}^2+\int_\Omega\nabla u_{\delta,\lambda}(t)\cdot\nabla\big(u_{\delta,\lambda}(t)-u_0\big)\,\mathrm{d}x\\&\quad-\big(\partial_{\boldsymbol{\nu}}u_{\delta,\lambda}(t),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}\\&\quad+\big(\beta_\lambda\big(u_{\delta,\lambda}(t)\big),u_{\delta,\lambda}(t)-u_0\big)_H+\big(\pi\big(u_{\delta,\lambda}(t)\big)-f(t),u_{\delta,\lambda}(t)-u_0\big)_H.\end{aligned}\tag{3.15}$$

Analogously, multiplying (3.6) by $v_{\delta,\lambda}(t)-v_0$ we have that

$$\begin{aligned}&\big(w_{\delta,\lambda}(t),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}\\&=\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|v_{\delta,\lambda}(t)-v_0\big|_{H_{\Gamma,0}}^2+\big(\partial_{\boldsymbol{\nu}}u_{\delta,\lambda}(t),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}\\&\quad+\delta\int_\Gamma\nabla_\Gamma v_{\delta,\lambda}(t)\cdot\nabla_\Gamma\big(v_{\delta,\lambda}(t)-v_0\big)\,\mathrm{d}\Gamma\\&\quad+\big(\beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}+\big(\pi_\Gamma\big(v_{\delta,\lambda}(t)\big)-g(t),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}.\end{aligned}\tag{3.16}$$

By merging (3.13), (3.14), (3.15), and (3.16), and then adding $|u_{\delta,\lambda}(t)|_H^2$ to both sides of the resultant we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|u_{\delta,\lambda}(t)-u_0\big|_{V_0^*}^2+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|u_{\delta,\lambda}(t)-v_0\big|_{H_0}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|v_{\delta,\lambda}(t)-v_0\big|_{V_{\Gamma,0}^*}^2\\&\quad+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|v_{\delta,\lambda}(t)-v_0\big|_{H_{\Gamma,0}}^2+\big|u_{\delta,\lambda}(t)\big|_V^2+\delta\int_\Gamma\big|\nabla_\Gamma v_{\delta,\lambda}(t)\big|^2\,\mathrm{d}\Gamma\\&\quad+\big(\beta_\lambda\big(u_{\delta,\lambda}(t)\big),u_{\delta,\lambda}(t)-u_0\big)_H+\big(\beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big),v_{\delta,\lambda}(t)-v_0\big)_{H_\Gamma}\\&\le\big|u_{\delta,\lambda}(t)\big|_H^2+\int_\Omega\nabla u_{\delta,\lambda}(t)\cdot\nabla u_0\,\mathrm{d}x+\delta\int_\Gamma\nabla_\Gamma v_{\delta,\lambda}(t)\cdot\nabla_\Gamma v_0\,\mathrm{d}\Gamma\end{aligned}$$

$$- \big(\pi\big(u_{\delta,\lambda}(t)\big) - f(t), u_{\delta,\lambda}(t) - u_0\big)_H - \big(\pi_\Gamma\big(v_{\delta,\lambda}(t)\big) - g(t), v_{\delta,\lambda}(t) - v_0\big)_{H_\Gamma} \tag{3.17}$$

for a.a. $t \in (0, T)$. Now, on the left-hand side, by the convexity of $\widehat{\beta}_\lambda$ and $\widehat{\beta}_{\Gamma,\lambda}$, as well as (3.9)–(3.10), we deduce that

$$\begin{aligned}&\big(\beta_\lambda\big(u_{\delta,\lambda}(t)\big), u_{\delta,\lambda}(t) - u_0\big)_H + \big(\beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big), v_{\delta,\lambda}(t) - v_0\big)_{H_\Gamma}\\&\quad\geq \int_\Omega \widehat{\beta}_\lambda\big(u_{\delta,\lambda}(t)\big)\,\mathrm{d}x - \int_\Omega \widehat{\beta}(u_0)\,\mathrm{d}x + \int_\Gamma \widehat{\beta}_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big)\,\mathrm{d}\Gamma - \int_\Gamma \widehat{\beta}_\Gamma(v_0)\,\mathrm{d}\Gamma.\end{aligned}$$

On the right-hand side, by the Young inequality we have

$$\begin{aligned}&\int_\Omega \nabla u_{\delta,\lambda}(t)\cdot\nabla u_0\,\mathrm{d}x + \delta\int_\Gamma \nabla_\Gamma v_{\delta,\lambda}(t)\cdot\nabla_\Gamma v_0\,\mathrm{d}\Gamma\\&\quad\leq \frac{1}{2}|u_{\delta,\lambda}(t)|_V^2 + \frac{\delta}{2}\int_\Gamma \big|\nabla_\Gamma v_{\delta,\lambda}(t)\big|^2\,\mathrm{d}\Gamma + |u_0|_V^2 + \delta|v_0|_{V_\Gamma}^2.\end{aligned}$$

Furthermore, applying the Ehrling lemma (see, e.g., [35, Chapter 1, Lemma 5.1]) for $V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*$, we see that for any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon > 0$ such that

$$\big|u_{\delta,\lambda}(t)\big|_H^2 \leq \varepsilon\big|u_{\delta,\lambda}(t)\big|_V^2 + C_\varepsilon\Big(1 + \big|u_{\delta,\lambda}(t) - u_0\big|_{V_0^*}^2\Big), \tag{3.18}$$

where we have added and subtracted u_0 in the second term on the right-hand side and used the equivalence of $|\cdot|_{V^*}$ and $|\cdot|_{V_0^*}$ on V_0^*. Moreover, thanks to (A2) and (A3), using the Young inequality and again the Ehrling lemma it turns out that there exists a positive constant $C > 0$ depending on π, $|u_0|_H$, and $|\Omega|$ such that

$$\begin{aligned}&- \big(\pi\big(u_{\delta,\lambda}(t)\big) - f(t), u_{\delta,\lambda}(t) - u_0\big)_H\\&\leq \big(L\big|u_{\delta,\lambda}(t)\big|_H + \big|\pi(0)\big|_H + \big|f(t)\big|_H\big)\big(\big|u_{\delta,\lambda}(t)\big|_H + |u_0|_H\big)\\&\leq \varepsilon\big|u_{\delta,\lambda}(t)\big|_V^2 + C_\varepsilon\Big(1 + \big|u_{\delta,\lambda}(t) - u_0\big|_{V_0^*}^2\Big) + C\Big(1 + \big|f(t)\big|_H^2\Big)\end{aligned}$$

for a.a. $t \in (0, T)$. Analogously, using the Young inequality, the Ehrling lemma with respect to the inclusions $Z_\Gamma \hookrightarrow\hookrightarrow H_\Gamma \hookrightarrow V_\Gamma^*$, and the estimate (2.7) for the trace γ_0, we deduce that

$$\begin{aligned}&- \big(\pi_\Gamma\big(v_{\delta,\lambda}(t)\big) - g(t), v_{\delta,\lambda}(t) - v_0\big)_{H_\Gamma}\\&\leq \big(L_\Gamma\big|v_{\delta,\lambda}(t)\big|_{H_\Gamma} + \big|\pi_\Gamma(0)\big|_{H_\Gamma} + \big|g(t)\big|_{H_\Gamma}\big)\big(\big|v_{\delta,\lambda}(t)\big|_{H_\Gamma} + |v_0|_{H_\Gamma}\big)\\&\leq \varepsilon\big|u_{\delta,\lambda}(t)\big|_V^2 + C_\varepsilon\big|v_{\delta,\lambda}(t) - v_0\big|_{V_{\Gamma,0}^*}^2 + C\Big(1 + \big|g(t)\big|_{H_\Gamma}^2\Big)\end{aligned}$$

for a.a. $t \in (0, T)$, where we exploited the equivalence of $|\cdot|_{V_\Gamma^*}$ and $|\cdot|_{V_{\Gamma,0}^*}$ on $V_{\Gamma,0}^*$ and we let the updated value of C depend also on π_Γ, $|v_0|_{H_\Gamma}$, and $|\Gamma|$. Therefore, going back to (3.17) we choose ε small enough and apply the Gronwall inequality, obtaining

$$\sup_{t\in[0,T]}\big|u_{\delta,\lambda}(t) - u_0\big|_{V_0^*}^2 + \sup_{t\in[0,T]}\lambda\big|u_{\delta,\lambda}(t) - u_0\big|_{H_0}^2$$

$$
\begin{aligned}
&+ \sup_{t\in[0,T]} \bigl|v_{\delta,\lambda}(t) - v_0\bigr|_{V_{\Gamma,0}^*}^2 + \sup_{t\in[0,T]} \lambda \bigl|v_{\delta,\lambda}(t) - v_0\bigr|_{H_{\Gamma,0}}^2 \\
&+ \int_0^T \bigl|u_{\delta,\lambda}(s)\bigr|_V^2 \, \mathrm{d}s + \delta \int_0^T \bigl|\nabla_\Gamma v_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2 \, \mathrm{d}s \\
&+ \int_0^T \bigl|\widehat{\beta}_\lambda\bigl(u_{\delta,\lambda}(s)\bigr)\bigr|_{L^1(\Omega)} \, \mathrm{d}s + \int_0^T \bigl|\widehat{\beta}_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(s)\bigr)\bigr|_{L^1(\Gamma)} \, \mathrm{d}s \le M_2, \qquad (3.19)
\end{aligned}
$$

where the constant M_2 depends on T, $|f|_{L^2(0,T;H)}$, $|g|_{L^2(0,T;H_\Gamma)}$, $|u_0|_V$, and $\delta^{1/2}|v_0|_{V_\Gamma}$.

3.4. 3rd estimates

Firstly, multiplying (3.1) by $\mu_{\delta,\lambda}(t) + f(t)$ and using (3.3) we obtain

$$
\begin{aligned}
&\bigl\langle \partial_t u_{\delta,\lambda}(t), \mu_{\delta,\lambda}(t) + f(t) \bigr\rangle_{V^*,V} + \int_\Omega \bigl|\nabla \mu_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}x \\
&= -\int_\Omega \nabla \mu_{\delta,\lambda}(t) \cdot \nabla f(t) \, \mathrm{d}x \le \frac{1}{2} \int_\Omega \bigl|\nabla \mu_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega \bigl|\nabla f(t)\bigr|^2 \, \mathrm{d}x \qquad (3.20)
\end{aligned}
$$

for a.a. $t \in (0,T)$. Secondly, multiplying (3.2) by $\partial_t u_{\delta,\lambda}(t)$ leads to

$$
\begin{aligned}
&\bigl\langle \partial_t u_{\delta,\lambda}(t), \mu_{\delta,\lambda}(t) + f(t) \bigr\rangle_{V^*,V} \\
&= \lambda \bigl|\partial_t u_{\delta,\lambda}(t)\bigr|_H^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \bigl|\nabla u_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}x - \bigl(\partial_{\boldsymbol{\nu}} u_{\delta,\lambda}(t), \partial_t v_{\delta,\lambda}(t)\bigr)_{H_\Gamma} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_\Omega \widehat{\beta}_\lambda\bigl(u_{\delta,\lambda}(t)\bigr) \, \mathrm{d}x + \int_\Omega \widehat{\pi}\bigl(u_{\delta,\lambda}(t)\bigr) \, \mathrm{d}x \right\}. \qquad (3.21)
\end{aligned}
$$

Analogously, multiplying (3.6) by $\partial_t v_{\delta,\lambda}(t)$ we infer that

$$
\begin{aligned}
&\bigl\langle \partial_t v_{\delta,\lambda}(t), w_{\delta,\lambda}(t) + g(t) \bigr\rangle_{V_\Gamma^*,V_\Gamma} \\
&= \lambda \bigl|\partial_t v_{\delta,\lambda}(t)\bigr|_{H_\Gamma}^2 + \bigl(\partial_{\boldsymbol{\nu}} u_{\delta,\lambda}(t), \partial_t v_{\delta,\lambda}(t)\bigr)_{H_\Gamma} + \frac{\delta}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \bigl|\nabla_\Gamma v_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}\Gamma \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_\Gamma \widehat{\beta}_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(t)\bigr) \, \mathrm{d}\Gamma + \int_\Gamma \widehat{\pi}_\Gamma\bigl(v_{\delta,\lambda}(t)\bigr) \, \mathrm{d}\Gamma \right\}, \qquad (3.22)
\end{aligned}
$$

while multiplying (3.5) by $w_{\delta,\lambda}(t) + g(t)$ gives

$$
\begin{aligned}
&\bigl\langle \partial_t v_{\delta,\lambda}(t), w_{\delta,\lambda}(t) + g(t) \bigr\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Gamma \bigl|\nabla_\Gamma w_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}\Gamma \\
&\le \frac{1}{2} \int_\Gamma \bigl|\nabla_\Gamma w_{\delta,\lambda}(t)\bigr|^2 \, \mathrm{d}\Gamma + \frac{1}{2} \int_\Gamma \bigl|\nabla_\Gamma g(t)\bigr|^2 \, \mathrm{d}\Gamma. \qquad (3.23)
\end{aligned}
$$

Combining (3.20)–(3.23), integrating the resulting inequality from 0 to t, adding the term $(1/2)|u_{\delta,\lambda}(t)|_H^2$ to both sides, and using (3.7)–(3.10), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \bigl|\nabla \mu_{\delta,\lambda}(s)\bigr|_H^2\,ds + \frac{1}{2}\int_0^t \bigl|\nabla_\Gamma w_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2\,ds \\
&\quad + \lambda \int_0^t \bigl|\partial_t u_{\delta,\lambda}(s)\bigr|_H^2\,ds + \lambda \int_0^t \bigl|\partial_t v_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2\,ds + \frac{1}{2}\bigl|u_{\delta,\lambda}(t)\bigr|_V^2 \\
&\quad + \frac{\delta}{2}\bigl|\nabla_\Gamma v_{\delta,\lambda}(t)\bigr|_{H_\Gamma}^2 + \int_\Omega \widehat{\beta}_\lambda\bigl(u_{\delta,\lambda}(t)\bigr)\,dx + \int_\Gamma \widehat{\beta}_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(t)\bigr)\,d\Gamma \\
&\quad \le \frac{1}{2}|\nabla u_0|_H^2 + \frac{\delta}{2}|\nabla_\Gamma v_0|_{H_\Gamma}^2 + \int_\Omega \widehat{\beta}(u_0)\,dx + \int_\Gamma \widehat{\beta}_\Gamma(v_0)\,d\Gamma \\
&\quad + \frac{1}{2}\int_0^T \bigl|f(s)\bigr|_V^2\,ds + \frac{1}{2}\int_0^T \bigl|g(s)\bigr|_{V_\Gamma}^2\,ds + \frac{1}{2}\bigl|u_{\delta,\lambda}(t)\bigr|_H^2 \\
&\quad + \int_\Omega \bigl|\widehat{\pi}\bigl(u_{\delta,\lambda}(t)\bigr)\bigr|\,dx + \int_\Omega \bigl|\widehat{\pi}(u_0)\bigr|\,dx + \int_\Gamma \bigl|\widehat{\pi}_\Gamma\bigl(v_{\delta,\lambda}(t)\bigr)\bigr|\,d\Gamma + \int_\Gamma \bigl|\widehat{\pi}_\Gamma(v_0)\bigr|\,d\Gamma
\end{aligned} \tag{3.24}
$$

for all $t \in [0, T]$. Here, from (A2) we see that

$$
\bigl|\widehat{\pi}(r)\bigr| \le L|r|^2 + \frac{1}{2L}\bigl|\pi(0)\bigr|^2, \qquad \bigl|\widehat{\pi}_\Gamma(r)\bigr| \le L_\Gamma|r|^2 + \frac{1}{2L_\Gamma}\bigl|\pi_\Gamma(0)\bigr|^2
$$

for all $r \in \mathbb{R}$, therefore

$$
\begin{aligned}
\int_\Omega \bigl|\widehat{\pi}\bigl(u_{\delta,\lambda}(t)\bigr)\bigr|\,dx + \int_\Omega \bigl|\widehat{\pi}(u_0)\bigr|\,dx &\le L\bigl|u_{\delta,\lambda}(t)\bigr|_H^2 + L|u_0|_H^2 + \frac{1}{L}\bigl|\pi(0)\bigr|^2, \\
\int_\Gamma \bigl|\widehat{\pi}_\Gamma\bigl(v_{\delta,\lambda}(t)\bigr)\bigr|\,d\Gamma + \int_\Gamma \bigl|\widehat{\pi}_\Gamma(v_0)\bigr|\,d\Gamma &\le L_\Gamma\bigl|v_{\delta,\lambda}(t)\bigr|_{H_\Gamma}^2 + L_\Gamma|v_0|_{H_\Gamma}^2 + \frac{1}{L_\Gamma}\bigl|\pi_\Gamma(0)\bigr|^2.
\end{aligned}
$$

Now, applying again the compactness inequalities and the estimate (2.7) for the trace, we see that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that (3.18) and

$$
\bigl|v_{\delta,\lambda}(t)\bigr|_{H_\Gamma}^2 \le \varepsilon\bigl|u_{\delta,\lambda}(t)\bigr|_V^2 + C_\varepsilon\Bigl(1 + \bigl|v_{\delta,\lambda}(t) - v_0\bigr|_{V_{\Gamma,0}^*}^2\Bigr) \tag{3.25}
$$

hold, where C_ε depends on $|u_0|_H$, $|v_0|_{H_\Gamma}$, $|\Omega|$, and $|\Gamma|$. Thus, using (3.19), we deduce that there exists a positive constant $M_3 > 0$ such that

$$
\begin{aligned}
&\int_0^T \bigl|\nabla \mu_{\delta,\lambda}(s)\bigr|_H^2\,ds + \int_0^T \bigl|\nabla_\Gamma w_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2\,ds \\
&\quad + \lambda \int_0^T \bigl|\partial_t u_{\delta,\lambda}(s)\bigr|_H^2\,ds + \lambda \int_0^T \bigl|\partial_t v_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2\,ds \\
&\quad + \sup_{t\in[0,T]} \bigl|u_{\delta,\lambda}(t)\bigr|_V^2 + \sup_{t\in[0,T]} \delta\bigl|\nabla_\Gamma v_{\delta,\lambda}(t)\bigr|_{H_\Gamma}^2 \\
&\quad + \sup_{t\in[0,T]} \int_\Omega \widehat{\beta}_\lambda\bigl(u_{\delta,\lambda}(t)\bigr)\,dx + \sup_{t\in[0,T]} \int_\Gamma \widehat{\beta}_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(t)\bigr)\,d\Gamma \le M_3.
\end{aligned} \tag{3.26}
$$

From (3.1), (3.3), and (3.5), it straightforward to infer that

$$
\bigl|\partial_t u_{\delta,\lambda}(s)\bigr|_{V_0^*}^2 \le \bigl|\nabla \mu_{\delta,\lambda}(s)\bigr|_H^2, \quad \bigl|\partial_t v_{\delta,\lambda}(s)\bigr|_{V_{\Gamma,0}^*}^2 \le \bigl|\nabla_\Gamma w_{\delta,\lambda}(s)\bigr|_{H_\Gamma}^2,
$$

for a.a. $s \in (0, T)$. Thus, the estimate (3.26) also implies that

$$\int_0^T \left|\partial_t u_{\delta,\lambda}(s)\right|_{V_0^*}^2 \mathrm{d}s + \int_0^T \left|\partial_t v_{\delta,\lambda}(s)\right|_{V_{\Gamma,0}^*}^2 \mathrm{d}s \le M_3. \tag{3.27}$$

3.5. 4th estimate.

Thanks to (A1) and (A4), we can use the following useful inequality (see [41, Appendix, Prop. A.1] and/or the detailed proof given in [24, p. 908]): there exist two positive constants $c_2, c_3 > 0$ such that

$$\left(\beta_\lambda\left(u_{\delta,\lambda}(t)\right), u_{\delta,\lambda}(t) - u_0\right)_H \ge c_2 \int_\Omega \left|\beta_\lambda\left(u_{\delta,\lambda}(t)\right)\right| \mathrm{d}x - c_3|\Omega|,$$
$$\left(\beta_{\Gamma,\lambda}\left(v_{\delta,\lambda}(t)\right), v_{\delta,\lambda}(t) - v_0\right)_{H_\Gamma} \ge c_2 \int_\Gamma \left|\beta_{\Gamma,\lambda}\left(v_{\delta,\lambda}(t)\right)\right| \mathrm{d}\Gamma - c_3|\Gamma|$$

for a.a. $t \in (0, T)$. Therefore, merging (3.13)–(3.16) again and recalling the definition of inner products of V_0^* and $V_{\Gamma,0}^*$, we have

$$\begin{aligned}
&c_2 \left\{ \int_\Omega \left|\beta_\lambda\left(u_{\delta,\lambda}(t)\right)\right| \mathrm{d}x + \int_\Gamma \left|\beta_{\Gamma,\lambda}\left(v_{\delta,\lambda}(t)\right)\right| \mathrm{d}\Gamma \right\} - c_3\left(|\Omega| + |\Gamma|\right) \\
&\le \left(f(t) - \pi\left(u_{\delta,\lambda}(t)\right) - \lambda\partial_t u_{\delta,\lambda}(t), u_{\delta,\lambda}(t) - u_0\right)_H - \left(\partial_t u_{\delta,\lambda}(t), u_{\delta,\lambda}(t) - u_0\right)_{V_0^*} \\
&\quad + \left(g(t) - \pi_\Gamma\left(v_{\delta,\lambda}(t)\right) - \lambda\partial_t v_{\delta,\lambda}(t), v_{\delta,\lambda}(t) - v_0\right)_{H_\Gamma} - \left(\partial_t v_{\delta,\lambda}(t), v_{\delta,\lambda}(t) - v_0\right)_{V_{\Gamma,0}^*} \\
&\le \left\{ \left|f(t)\right|_H + \left|\pi\left(u_{\delta,\lambda}(t)\right)\right|_H + \lambda\left|\partial_t u_{\delta,\lambda}(t)\right|_H \right\} \left|u_{\delta,\lambda}(t) - u_0\right|_H \\
&\quad + \left\{ \left|g(t)\right|_{H_\Gamma} + \left|\pi_\Gamma\left(v_{\delta,\lambda}(t)\right)\right|_{H_\Gamma} + \lambda\left|\partial_t v_{\delta,\lambda}(t)\right|_{H_\Gamma} \right\} \left|v_{\delta,\lambda}(t) - v_0\right|_{H_\Gamma} \\
&\quad + \left|\partial_t u_{\delta,\lambda}(t)\right|_{V_0^*} \left|u_{\delta,\lambda}(t) - u_0\right|_{V_0^*} + \left|\partial_t v_{\delta,\lambda}(t)\right|_{V_{\Gamma,0}^*} \left|v_{\delta,\lambda}(t) - v_0\right|_{V_{\Gamma,0}^*}
\end{aligned} \tag{3.28}$$

for a.a. $t \in (0, T)$. Here, from (A2) and (3.26)–(3.27) it follows that the right-hand side of (3.28) is uniformly bounded in $L^2(0, T)$: hence, there exists a positive constant $M_4 > 0$ such that

$$\int_0^T \left|\beta_\lambda\left(u_{\delta,\lambda}(s)\right)\right|_{L^1(\Omega)}^2 \mathrm{d}s + \int_0^T \left|\beta_{\Gamma,\lambda}\left(v_{\delta,\lambda}(s)\right)\right|_{L^1(\Gamma)}^2 \mathrm{d}s \le M_4. \tag{3.29}$$

3.6. 5th estimate

Setting $W_0 := H^2(\Omega) \cap H_0^1(\Omega)$, we multiply (3.2) by an arbitrary $\zeta \in L^2(0, T; W_0)$ and integrate by parts. Recalling the continuous embedding $W_0 \hookrightarrow L^\infty(\Omega)$, we obtain that

$$\begin{aligned}
&\int_0^T \left(\mu_{\delta,\lambda}(s), \zeta(s)\right)_H \mathrm{d}s \\
&\le \int_0^T \left|\lambda\partial_t u_{\delta,\lambda}(s) + \pi\left(u_{\delta,\lambda}(s)\right) - f(s)\right|_H \left|\zeta(s)\right|_H \mathrm{d}s
\end{aligned}$$

$$+\int_0^T \left|\nabla u_{\delta,\lambda}(s)\right|_H \left|\nabla\zeta(s)\right|_H \,\mathrm{d}s + C\int_0^T \left|\beta_\lambda\big(u_{\delta,\lambda}(s)\big)\right|_{L^1(\Omega)} \left|\zeta(s)\right|_{W_0} \mathrm{d}s,$$

where the positive constant C only depends on Ω. Therefore, exploiting the estimates (3.26) and (3.29) we infer that

$$\int_0^T \left|\mu_{\delta,\lambda}(s)\right|_{W_0^*}^2 \mathrm{d}s \le M_5. \tag{3.30}$$

Now, we apply the Ehrling lemma for the spaces $V \hookrightarrow\hookrightarrow H \hookrightarrow W_0^*$ to deduce that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\left|\mu_{\delta,\lambda}(s)\right|_H^2 \le \varepsilon\left|\nabla\mu_{\delta,\lambda}(s)\right|_H^2 + C_\varepsilon\left|\mu_{\delta,\lambda}(s)\right|_{W_0^*}^2 \quad \text{for a.a. } s \in (0,T).$$

Consequently, the estimates (3.26) and (3.30) yield, possibly updating M_5,

$$\int_0^T \left|\mu_{\delta,\lambda}(s)\right|_V^2 \mathrm{d}s \le M_5. \tag{3.31}$$

Next, we test (3.2) by 1 and integrate by parts using the boundary equations (3.4) and (3.6). Recalling that $\partial_t u_{\delta,\lambda}(t) \in V_0^*$ and $\partial_t v_{\delta,\lambda}(t) \in V_{\Gamma,0}^*$, it easily follows that

$$\begin{aligned}
&\int_\Omega \mu_{\delta,\lambda}(t)\,\mathrm{d}x + \int_\Gamma w_{\delta,\lambda}(t)\,\mathrm{d}\Gamma \\
&= \int_\Omega \beta_\lambda\big(u_{\delta,\lambda}(t)\big)\,\mathrm{d}x + \big(\pi\big(u_{\delta,\lambda}(t)\big) - f(t), 1\big)_H \\
&\quad + \int_\Gamma \beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big)\,\mathrm{d}\Gamma + \big(\pi_\Gamma\big(v_{\delta,\lambda}(t)\big) - g(t), 1\big)_{H_\Gamma}
\end{aligned} \tag{3.32}$$

for a.a. $t \in (0,T)$. Then, by virtue of (3.19), (3.29), (3.31) and assumptions (A2) and (A3), comparing the terms in (3.32) yields that

$$\text{the function } t \mapsto m_\Gamma\big(w_{\delta,\lambda}(t)\big) = \frac{1}{|\Gamma|}\int_\Gamma w_{\delta,\lambda}(t)\,\mathrm{d}\Gamma \text{ is uniformly bounded in } L^2(0,T),$$

whence the estimate (3.26) and the Poincaré–Wirtinger inequality allow us to infer that

$$\int_0^T \left|w_{\delta,\lambda}(s)\right|_{V_\Gamma}^2 \mathrm{d}s \le M_5. \tag{3.33}$$

3.7. 6th and 7th estimates

We test now equation (3.2) by $\beta_\lambda(u_{\delta,\lambda})$ and equation (3.6) by $\beta_\lambda(v_{\delta,\lambda})$, then we combine them obtaining

$$\begin{aligned}
&\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \widehat{\beta}_\lambda(u_{\delta,\lambda})\,\mathrm{d}x + \int_\Omega \beta_\lambda'(u_{\delta,\lambda})|\nabla u_{\delta,\lambda}|^2\,\mathrm{d}x + \int_\Omega \left|\beta_\lambda(u_{\delta,\lambda})\right|^2\mathrm{d}x \\
&\quad + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \widehat{\beta}_\lambda(v_{\delta,\lambda})\,\mathrm{d}\Gamma + \delta\int_\Gamma \beta_\lambda'(v_{\delta,\lambda})|\nabla_\Gamma v_{\delta,\lambda}|^2\,\mathrm{d}\Gamma + \int_\Gamma \beta_\lambda(v_{\delta,\lambda})\beta_{\Gamma,\lambda}(v_{\delta,\lambda})\,\mathrm{d}\Gamma
\end{aligned}$$

$$= \int_\Omega \big(\mu_{\delta,\lambda} + f - \pi(u_{\delta,\lambda})\big)\, \beta_\lambda(u_{\delta,\lambda})\, \mathrm{d}x + \int_\Gamma \big(w_{\delta,\lambda} + g - \pi_\Gamma(v_{\delta,\lambda})\big)\, \beta_\lambda(v_{\delta,\lambda})\, \mathrm{d}\Gamma \tag{3.34}$$

Now, we recall assumption (A1) and point out that (2.24) entails that the same inequality holds for the Yosida approximations β_λ and $\beta_{\Gamma,\lambda}$ (see, e.g., [12, Appendix]). Hence, for the coupling term above we have the control

$$\int_\Gamma \beta_\lambda(v_{\delta,\lambda})\beta_{\Gamma,\lambda}(v_{\delta,\lambda})\, \mathrm{d}\Gamma \geq \frac{1}{2\varrho_1} \int_\Gamma \big|\beta_\lambda(v_{\delta,\lambda})\big|^2 \mathrm{d}\Gamma - C$$

for some constant C. Then, integrating (3.34) over $(0, T)$ and applying the Young inequality, from (A1)–(A4) and the estimates (3.26), (3.31), (3.33), it is standard matter to deduce that

$$\begin{aligned} &\lambda \int_\Omega \widehat{\beta}_\lambda\big(u_{\delta,\lambda}(T)\big)\, \mathrm{d}x + \lambda \int_\Gamma \widehat{\beta}_\lambda\big(v_{\delta,\lambda}(T)\big)\, \mathrm{d}\Gamma \\ &\quad + \int_0^T \big|\beta_\lambda\big(u_{\delta,\lambda}(s)\big)\big|_H^2\, \mathrm{d}s + \int_0^T \big|\beta_\lambda\big(v_{\delta,\lambda}(s)\big)\big|_{H_\Gamma}^2\, \mathrm{d}s \leq M_6 \end{aligned} \tag{3.35}$$

for some positive constant M_6. Next, by comparing the terms in equation (3.2) we have that

$$\big|\Delta u_{\delta,\lambda}(t)\big|_H \leq \big|\mu_{\delta,\lambda}(t)\big|_H + \lambda\big|\partial_t u_{\delta,\lambda}(t)\big|_H + \big|\beta_\lambda\big(u_{\delta,\lambda}(t)\big)\big|_H + \big|\pi\big(u_{\delta,\lambda}(t)\big)\big|_H + \big|f(t)\big|_H$$

for a.a. $t \in (0, T)$, whence

$$\int_0^T \big|\Delta u_{\delta,\lambda}(s)\big|_H^2\, \mathrm{d}s \leq M_6. \tag{3.36}$$

We proceed now by exploiting the idea of [13]. Together with the trace theorems for the normal derivative (see, e.g., [27, Lemma 5.1.1, p. 209]), estimates (3.26) and (3.36) yield that

$$\int_0^T \big|\partial_{\boldsymbol{\nu}} u_{\delta,\lambda}(s)\big|_{Z_\Gamma^*}^2\, \mathrm{d}s \leq M_6. \tag{3.37}$$

Analogously, recalling the estimate for $\delta^{1/2}\nabla_\Gamma v_{\delta,\lambda}$ in $L^\infty(0, T; H_\Gamma)$ in (3.26), by (3.4) the trace of $\delta^{1/2}u_{\delta,\lambda}$ is uniformly bounded in $L^2(0, T; V_\Gamma)$. Therefore, by virtue of the elliptic regularity (see, e.g., [5, Theorem 3.2, p. 1.79]) and again the trace theorems for the normal derivative, we obtain that

$$\delta \int_0^T \big|\partial_{\boldsymbol{\nu}} u_{\delta,\lambda}(s)\big|_{H_\Gamma}^2\, \mathrm{d}s \leq M_7. \tag{3.38}$$

Consequently, by comparing the terms in equation (3.6) one deduces that

$$\big|-\delta\Delta_\Gamma v_{\delta,\lambda}(t) + \beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big)\big|_{Z_\Gamma^*}$$

$$\leq \left|\partial_{\boldsymbol{\nu}} u_{\delta,\lambda}(t)\right|_{Z_\Gamma^*} + C\Bigl(\left|w_{\delta,\lambda}(t)\right|_{H_\Gamma} + \lambda\left|\partial_t v_{\delta,\lambda}(t)\right|_{H_\Gamma} + \left|\pi_\Gamma\bigl(v_{\delta,\lambda}(t)\bigr)\right|_{H_\Gamma} + \left|g(t)\right|_{H_\Gamma}\Bigr)$$

for a.a. $t \in (0, T)$, hence that

$$\int_0^T \left|-\delta\Delta_\Gamma v_{\delta,\lambda}(s) + \beta_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(s)\bigr)\right|_{Z_\Gamma^*}^2 \mathrm{d}s \leq M_7. \tag{3.39}$$

Since $\delta^{1/2}\Delta_\Gamma v_{\delta,\lambda}$ is bounded in $L^\infty(0, T; V_\Gamma^*)$ by the estimate (3.26), a direct comparison in (3.39) yields also

$$\delta \int_0^T \left|\Delta_\Gamma v_{\delta,\lambda}(s)\right|_{V_\Gamma^*}^2 \mathrm{d}s + \int_0^T \left|\beta_{\Gamma,\lambda}\bigl(v_{\delta,\lambda}(s)\bigr)\right|_{V_\Gamma^*}^2 \mathrm{d}s \leq M_7. \tag{3.40}$$

4. Proofs of main theorems

We start by discussing the limiting procedure. The main issue concerns the passage to the limit as $\delta \to 0$. Indeed, it is known from [14, Theorem 2.3] that letting $\lambda \to 0$ with weak and weak star convergences, we can prove Proposition 2.1. Moreover, the limit functions u_δ, μ_δ, ξ_δ, v_δ, w_δ, and η_δ satisfy (2.25)–(2.31) and same kind of uniform estimates obtained in the previous section, that is, the estimates

$$\sup_{t\in[0,T]} \left|u_\delta(t)\right|_V^2 + \sup_{t\in[0,T]} \delta\left|\nabla_\Gamma v_\delta(t)\right|_{H_\Gamma}^2 \leq M_3, \tag{4.1}$$

$$\int_0^T \left|\partial_t u_\delta(s)\right|_{V_0^*}^2 \mathrm{d}s + \int_0^T \left|\partial_t v_\delta(s)\right|_{V_{\Gamma,0}^*}^2 \mathrm{d}s \leq M_3, \tag{4.2}$$

$$\int_0^T \left|\mu_\delta(s)\right|_V^2 \mathrm{d}s + \int_0^T \left|w_\delta(s)\right|_{V_\Gamma}^2 \mathrm{d}s \leq 2M_5, \tag{4.3}$$

$$\int_0^T \left|\xi_\delta(s)\right|_H^2 \mathrm{d}s + \int_0^t \left|\Delta u_\delta(s)\right|_H^2 \mathrm{d}s + \int_0^t \left|\partial_{\boldsymbol{\nu}} u_\delta(s)\right|_{Z_\Gamma^*}^2 \mathrm{d}s \leq 3M_6, \tag{4.4}$$

$$\int_0^T \left|-\delta\Delta_\Gamma v_\delta(s) + \eta_\delta(s)\right|_{Z_\Gamma^*}^2 \mathrm{d}s \leq M_7, \tag{4.5}$$

$$\delta \int_0^T \left|\partial_{\boldsymbol{\nu}} u_\delta(s)\right|_{H_\Gamma}^2 \mathrm{d}s + \delta \int_0^T \left|\Delta_\Gamma v_\delta(s)\right|_{V_\Gamma^*}^2 \mathrm{d}s + \int_0^T \left|\eta_\delta(s)\right|_{V_\Gamma^*}^2 \mathrm{d}s \leq 2M_7. \tag{4.6}$$

Moreover, we have that

$$m\bigl(\partial_t u_\delta(t)\bigr) = 0, \quad m_\Gamma\bigl(\partial_t v_{\delta,\lambda}(t)\bigr) = 0 \tag{4.7}$$

for a.a. $t \in (0, T)$. As a remark, using (4.2) and recalling the definition of norms in (2.5)–(2.6), we deduce similar uniform estimates for $\{\partial_t u_\delta\}_{\delta\in(0,1]}$ in $L^2(0, T; V^*)$ and $\{\partial_t v_\delta\}_{\delta\in(0,1]}$ in $L^2(0, T; V_\Gamma^*)$, respectively.

4.1. Proof of Theorem 2.2.

From (4.1)–(4.6) it follows that there exist a sextuplet $(u, \mu, \xi, v, w, \eta)$, with

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;V), \quad \Delta u \in L^2(0,T;H),$$
$$\mu \in L^2(0,T;V), \quad \xi \in L^2(0,T;H),$$
$$v \in H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;Z_\Gamma),$$
$$w \in L^2(0,T;V_\Gamma), \quad \eta \in L^2(0,T;Z_\Gamma^*),$$

and a subsequence $\{\delta_k\}_{k\in\mathbb{N}}$ such that, as $k \to +\infty$, the convergences (2.40)–(2.49) hold. Moreover, by virtue of the Aubin–Lions compactness results [47] and the compact embeddings $V \hookrightarrow\hookrightarrow H$ and $Z_\Gamma \hookrightarrow\hookrightarrow H_\Gamma$, the following strong convergence properties

$$u_{\delta_k} \to u \quad \text{in } C\big([0,T];H\big), \tag{4.8}$$
$$v_{\delta_k} \to v \quad \text{in } C\big([0,T];H_\Gamma\big) \tag{4.9}$$

hold as well. The Lipschitz continuities of π and π_Γ give us then, as $k \to \infty$,

$$\pi(u_{\delta_k}) \to \pi(u) \quad \text{in } C\big([0,T];H\big), \tag{4.10}$$
$$\pi_\Gamma(v_{\delta_k}) \to \pi_\Gamma(v) \quad \text{in } C\big([0,T];H_\Gamma\big). \tag{4.11}$$

Therefore, taking the limit in (2.25) and (2.28) we can obtain the variational formulations (2.32) and (2.35). The conditions (2.38) and (2.39) are also inferred from (2.30)–(2.31) on account of (4.8)–(4.9). Thanks to (2.40) and (2.45), the boundary condition (2.34) follows from (2.27) and the continuity of the linear trace operator γ_0 from V to Z_Γ.

The first equation in (2.33) is coming from the one in (2.26) owing to the convergences (2.41), (2.43), (2.44), and (4.10). The second condition in (2.33) is proved by the demi-closedness of the maximal monotone operator induced by β, by virtue of the strong convergence (4.8) and the weak convergence (2.44). The variational formulation (2.36) is also obtained from the first equation in (2.29), due to the convergences (2.47), (2.42), (2.49), and (4.11).

To conclude the proof of Theorem 2.2, it remains to prove (2.37). To this aim, we multiply the equality in (2.26) by u_{δ_k} and integrate the resultant over Q with respect to time and space variables. Using (2.27), we have that

$$\int_Q |\nabla u_{\delta_k}|^2 \,\mathrm{d}x\mathrm{d}t - \int_\Sigma \partial_{\boldsymbol{\nu}} u_{\delta_k} v_{\delta_k} \,\mathrm{d}\Gamma\mathrm{d}t + \int_Q \xi_{\delta_k} u_{\delta_k} \,\mathrm{d}x\mathrm{d}t = \int_Q \big(\mu_{\delta_k} - \pi(u_{\delta_k}) + f\big) u_{\delta_k} \,\mathrm{d}x\mathrm{d}t. \tag{4.12}$$

On the other hand, multiplying the equality in (2.29) by v_{δ_k} and integrating then over Σ, we find out that

$$\int_\Sigma \partial_{\boldsymbol{\nu}} u_{\delta_k} v_{\delta_k} \,\mathrm{d}\Gamma\mathrm{d}t + \delta_k \int_\Sigma |\nabla_\Gamma v_{\delta_k}|^2 \,\mathrm{d}\Gamma\mathrm{d}t + \int_\Sigma \eta_{\delta_k} v_{\delta_k} \,\mathrm{d}\Gamma\mathrm{d}t = \int_\Sigma \big(w_{\delta_k} - \pi_\Gamma(v_{\delta_k}) + g\big) v_{\delta_k} \,\mathrm{d}\Gamma\mathrm{d}t. \tag{4.13}$$

Summing (4.12) and (4.13), using lower semicontinuity and weak-strong convergences, we obtain that

$$
\begin{aligned}
&\limsup_{k\to+\infty}\int_\Sigma \eta_{\delta_k} v_{\delta_k}\,\mathrm{d}\Gamma\mathrm{d}t\\
&\le \limsup_{k\to+\infty}\int_Q \big(\mu_{\delta_k}-\pi(u_{\delta_k})+f\big)u_{\delta_k}\,\mathrm{d}x\mathrm{d}t+\limsup_{k\to+\infty}\int_\Sigma \big(w_{\delta_k}-\pi_\Gamma(v_{\delta_k})+g\big)v_{\delta_k}\,\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\liminf_{k\to+\infty}\int_Q |\nabla u_{\delta_k}|^2\,\mathrm{d}x\mathrm{d}t-\liminf_{k\to+\infty}\int_Q \xi_{\delta_k}u_{\delta_k}\,\mathrm{d}x\mathrm{d}t-\liminf_{k\to+\infty}\delta_k\int_\Sigma |\nabla_\Gamma v_{\delta_k}|^2\,\mathrm{d}\Gamma\mathrm{d}t\\
&\le \int_Q \big(\mu-\pi(u)+f\big)u\,\mathrm{d}x\mathrm{d}t+\int_\Sigma \big(w-\pi_\Gamma(v)+g\big)v\,\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\int_Q |\nabla u|^2\,\mathrm{d}x\mathrm{d}t-\int_Q \xi u\,\mathrm{d}x\mathrm{d}t\;=\int_0^T \langle \eta, v\rangle_{Z_\Gamma^*,Z_\Gamma}\,\mathrm{d}t
\end{aligned}
\tag{4.14}
$$

and the last equality can be recovered combining the equation in (2.33) tested by u and (2.36) with $z_\Gamma = v$ (cf. also (2.34)). Now, using the definition of subdifferential for β_Γ in $L^2(\Sigma)$, from the second inclusion in (2.29) we have that

$$
\int_\Sigma \eta_{\delta_k}(\zeta_\Gamma - v_{\delta_k})\,\mathrm{d}\Gamma\mathrm{d}t+\int_\Sigma \widehat{\beta}_\Gamma(v_{\delta_k})\,\mathrm{d}\Gamma\mathrm{d}t\le\int_\Sigma \widehat{\beta}_\Gamma(\zeta_\Gamma)\,\mathrm{d}\Gamma\mathrm{d}t \tag{4.15}
$$

for all $\zeta_\Gamma \in L^2(0,T;H_\Gamma)$. If $\zeta_\Gamma \in L^2(0,T;V_\Gamma)$, then by virtue of the weak convergence (2.48), the strong convergence (4.9), the weak lower semicontinuity of $\widehat{\beta}_\Gamma$, and (4.14), we obtain

$$
\begin{aligned}
&\lim_{k\to+\infty}\int_\Sigma \eta_{\delta_k}\zeta_\Gamma\,\mathrm{d}\Gamma\mathrm{d}t=\int_0^T\langle \eta,\zeta_\Gamma\rangle_{Z_\Gamma^*,Z_\Gamma}\,\mathrm{d}t,\\
&\liminf_{k\to+\infty}\left(-\int_\Sigma \eta_{\delta_k}v_{\delta_k}\,\mathrm{d}\Gamma\mathrm{d}t\right)=-\limsup_{k\to+\infty}\int_\Sigma \eta_{\delta_k}v_{\delta_k}\,\mathrm{d}\Gamma\mathrm{d}t\ge-\int_0^T\langle \eta, v\rangle_{Z_\Gamma^*,Z_\Gamma}\,\mathrm{d}t,\\
&\liminf_{k\to+\infty}\int_\Sigma \widehat{\beta}_\Gamma(v_{\delta_k})\,\mathrm{d}\Gamma\mathrm{d}t\ge\int_\Sigma \widehat{\beta}_\Gamma(v)\,\mathrm{d}\Gamma\mathrm{d}t.
\end{aligned}
$$

Therefore, taking the infimum limit in (4.15), we deduce that

$$
\int_0^T\langle \eta,\zeta_\Gamma - v\rangle_{Z_\Gamma^*,Z_\Gamma}\,\mathrm{d}t\le\int_\Sigma \widehat{\beta}_\Gamma(\zeta_\Gamma)\,\mathrm{d}\Gamma\mathrm{d}t-\int_\Sigma \widehat{\beta}_\Gamma(v)\,\mathrm{d}\Gamma\mathrm{d}t \tag{4.16}
$$

for all $\zeta_\Gamma \in L^2(0,T;V_\Gamma)$. As $\eta \in L^2(0,T;Z_\Gamma^*)$, by a density argument we can prove that (4.16) holds also for all $\zeta_\Gamma \in L^2(0,T;Z_\Gamma)$. Indeed, for a given arbitrary $\zeta_\Gamma \in L^2(0,T;Z_\Gamma)$ and $\varepsilon > 0$, we can take the approximations $\{\zeta_{\Gamma,\varepsilon}\}_{\varepsilon>0} \subset L^2(0,T;V_\Gamma)$ defined as the solutions to

$$
\zeta_{\Gamma,\varepsilon}-\varepsilon\Delta_\Gamma\zeta_{\Gamma,\varepsilon}=\zeta_\Gamma\quad\text{a.e. on }\Sigma.
$$

In fact, thanks to [10, Lemma A.1] we have that

$$
\zeta_{\Gamma,\varepsilon}\to\zeta_\Gamma\quad\text{in }L^2(0,T;Z_\Gamma)\quad\text{as }\varepsilon\to 0,
$$

$$\widehat{\beta}_\Gamma(\zeta_{\Gamma,\varepsilon}) \leq \widehat{\beta}_\Gamma(\zeta_\Gamma) \quad \text{a.e. on } \Sigma, \text{ for all } \varepsilon > 0.$$

Thus, replacing ζ_Γ by $\zeta_{\Gamma,\varepsilon}$ in (4.16) and letting $\varepsilon \to 0$, we easily obtain the validity of (4.16) for all $\zeta_\Gamma \in L^2(0, T; Z_\Gamma)$, which is an equivalent formulation of (2.37). $\square$

4.2. Proof of Theorem 2.5.

Next, we prove the continuous dependence result stated in Theorem 2.5. Assume that $f^{(1)}, f^{(2)}, g^{(1)}, g^{(2)}$ satisfy (A3), $u_0^{(1)}, u_0^{(2)}, v_0^{(1)}, v_0^{(2)}$ satisfy (A4) and

$$m\big(u_0^{(1)}\big) = m\big(u_0^{(2)}\big), \quad m_\Gamma\big(v_0^{(1)}\big) = m_\Gamma\big(v_0^{(2)}\big). \tag{4.17}$$

For these data let $(u^{(i)}, \mu^{(i)}, \xi^{(i)}, v^{(i)}, w^{(i)}, \eta^{(i)})$, $i = 1, 2$, be respective solutions obtained by Theorem 2.2 Put $\bar{u} := u^{(1)} - u^{(2)}$ and analogously use the same notation for the differences of functions. Taking the difference of (2.32), (2.33), (2.35), and (2.36), we have

$$\langle \partial_t \bar{u}, z \rangle_{V^*,V} + \int_\Omega \nabla \bar{\mu} \cdot \nabla z \,\mathrm{d}x = 0, \tag{4.18}$$

$$(\bar{\mu}, z)_H = (\nabla \bar{u}, \nabla z)_H - \langle \partial_{\boldsymbol{\nu}} \bar{u}, z_{|\Gamma} \rangle_{Z_\Gamma^*, Z_\Gamma} + (\bar{\xi}, z)_H + \big(\pi(u_1) - \pi(u_2) - \bar{f}, z\big)_H \tag{4.19}$$

for all $z \in V$ and a.e. in $(0, T)$,

$$\langle \partial_t \bar{v}, z_\Gamma \rangle_{V_\Gamma^*, V_\Gamma} + \int_\Gamma \nabla_\Gamma \bar{w} \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma = 0, \tag{4.20}$$

for all $z_\Gamma \in V_\Gamma$ and a.e. in $(0, T)$,

$$(\bar{w}, z_\Gamma)_{H_\Gamma} = \langle \partial_{\boldsymbol{\nu}} \bar{u} + \bar{\eta}, z_\Gamma \rangle_{Z_\Gamma^*, Z_\Gamma} + \big(\pi_\Gamma(v_1) - \pi(v_2) - \bar{g}, z_\Gamma\big)_{H_\Gamma}, \tag{4.21}$$

for all $z_\Gamma \in Z_\Gamma$ and a.e. in $(0, T)$. Moreover, using (4.17) we have

$$m\big(\bar{u}(t)\big) = 0, \quad m_\Gamma\big(\bar{v}(t)\big) = 0$$

for all $t \in [0, T]$. Take $z = F^{-1}\bar{u}$ in (4.18), $z = \bar{u}$ in (4.19), $z_\Gamma = F_\Gamma^{-1}\bar{v}$ in (4.20), and $z_\Gamma = \bar{v}$ in (4.21), respectively. Then, combining the obtained equalities and integrating over $(0, t)$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\big|\bar{u}(t)\big|_*^2 + \frac{1}{2}\big|\bar{v}(t)\big|_{\Gamma,*}^2 + \int_0^t \big|\bar{u}(s)\big|_{V_0}^2 \,\mathrm{d}s + \int_0^t \big(\bar{\xi}(s), \bar{u}(s)\big)_H \,\mathrm{d}s + \int_0^t \big\langle \bar{\eta}(s), \bar{v}(s)\big\rangle_{Z_\Gamma^*, Z_\Gamma} \,\mathrm{d}s \\
&= \frac{1}{2}|\bar{u}_0|_*^2 + \frac{1}{2}|\bar{v}_0|_{\Gamma,*}^2 + \int_0^t \big(\bar{f}(s) + \pi\big(u^{(2)}(s)\big) - \pi\big(u^{(1)}(s)\big), \bar{u}(s)\big)_H \,\mathrm{d}s \\
&\quad + \int_0^t \big(\bar{g}(s) + \pi_\Gamma\big(v^{(2)}(s)\big) - \pi_\Gamma\big(v^{(1)}(s)\big), \bar{v}(s)\big)_{H_\Gamma} \,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, T]$. Now, we invoke the monotonicity of the maximal monotone operators induced by β and β_Γ (cf. Remark 2.3) to see that the last two terms on the left-hand side are nonnegative. We also use the following estimate

$$\left|\bar{v}(s)\right|_{Z_\Gamma}^2 \le C_{\mathrm{tr}}^2 \left|\bar{u}(s)\right|_V^2 \le C_{\mathrm{tr}}^2 C_{\mathrm{P}} \left|\bar{u}(s)\right|_{V_0}^2,$$

which comes from (2.7) and (2.2). Then, on account of the Lipschitz continuity of π and π_Γ, by applying twice the Ehrling lemma we can conclude that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} &\left|\bar{u}(t)\right|_*^2 + \left|\bar{v}(t)\right|_{\Gamma,*}^2 + \int_0^t \left|\bar{u}(s)\right|_{V_0}^2 \mathrm{d}s + \frac{1}{C_{\mathrm{tr}}^2 C_{\mathrm{P}}} \int_0^t \left|\bar{v}(s)\right|_{Z_\Gamma}^2 \mathrm{d}s \\ &\le |\bar{u}_0|_*^2 + |\bar{v}_0|_{\Gamma,*}^2 + \int_0^t \left|\bar{f}(s)\right|_H^2 \mathrm{d}s + \varepsilon \int_0^t \left|\bar{u}(s)\right|_{V_0}^2 \mathrm{d}s + C_\varepsilon \int_0^t \left|\bar{u}(s)\right|_*^2 \mathrm{d}s \\ &\quad + \int_0^t \left|\bar{g}(s)\right|_{H_\Gamma}^2 \mathrm{d}s + \varepsilon \int_0^t \left|\bar{v}(s)\right|_{Z_\Gamma}^2 \mathrm{d}s + C_\varepsilon \int_0^t \left|\bar{v}(s)\right|_{\Gamma,*}^2 \mathrm{d}s \end{aligned}$$

for all $t \in [0, T]$. Thus, choosing $\varepsilon > 0$ sufficiently small and applying the Gronwall lemma, by the Poincaré–Wirtiger inequality (2.2) we complete the proof of Theorem 2.5. $\square$

4.3. Proof of Theorem 2.6.

We point out that the further assumption (2.50) on the graphs yields additional estimates on the solutions. Indeed, since assumption (2.50) induces the same inequalities on the respective Yosida approximations (details are given in [11, Appendix]) and, in particular, (2.50) implies that

$$\frac{1}{2M^2} \int_\Gamma \left|\beta_{\Gamma,\lambda}\big(v_{\delta,\lambda}(t)\big)\right|^2 \mathrm{d}\Gamma \le \int_\Gamma \left(\left|\beta_\lambda\big(v_{\delta,\lambda}(t)\big)\right|^2 + M^2\right) \mathrm{d}\Gamma$$

for a.a. $t \in (0, T)$, the estimate (3.35) entails that

$$\left|\beta_\lambda(u_{\delta,\lambda})\right|_{L^2(0,T;H)} + \left|\beta_{\Gamma,\lambda}(v_{\delta,\lambda})\right|_{L^2(0,T;H_\Gamma)} \le C$$

for some positive constant C. Hence, recalling the estimates (3.26), (3.33) and (3.37), by comparison of terms in (3.6) we find out that

$$\left|\partial_{\boldsymbol{\nu}} u_{\delta,\lambda} - \delta \Delta_\Gamma v_{\delta,\lambda}\right|_{L^2(0,T;H_\Gamma)} + \delta \left|\Delta_\Gamma v_{\delta,\lambda}\right|_{L^2(0,T;Z_\Gamma^*)} \le C \tag{4.22}$$

and consequently, by elliptic regularity,

$$\left|\delta v_{\delta,\lambda}\right|_{L^2(0,T;H^{3/2}(\Gamma))} \le C. \tag{4.23}$$

Then, we can take the limit as $\lambda \to 0$ and infer that

$$\int_0^T \big|\eta_\delta(s)\big|_{H_\Gamma}^2 \,\mathrm{d}s + \int_0^T \big|\partial_{\boldsymbol{\nu}} u_\delta(s) - \delta\Delta_\Gamma v_\delta(s)\big|_{H_\Gamma}^2 \,\mathrm{d}s + \int_0^T \big|\delta v_\delta(s)\big|_{H^{3/2}(\Gamma)}^2 \,\mathrm{d}s \le C \tag{4.24}$$

in addition to (4.1)–(4.6). Thus, in view of (2.40)–(2.49), when passing to the limit on a subsequence δ_k we also deduce (2.51)–(2.53) and the boundary equation (2.54) at the limit. At this point, as $u \in L^2(0,T;V)$, $\Delta u \in L^2(0,T;H)$ and $\partial_{\boldsymbol{\nu}} u \in L^2(0,T;H_\Gamma)$, by elliptic regularity (see [5, Thm. 3.2]) it follows that

$$u \in L^2\big(0,T;H^{3/2}(\Omega)\big),$$

whence, from (2.34) and the trace theory,

$$v \in L^2(0,T;V_\Gamma).$$

Eventually, the pointwise inclusion $\xi_\Gamma \in \beta_\Gamma(u_\Gamma)$ a.e. on Σ is ensured in this framework, as explained in Remark 2.3. This ends the proof of Theorem 2.6. □

4.4. Proof of Theorem 2.8.

For $\delta \in (0,1]$ let $(u_\delta, \mu_\delta, \xi_\delta, v_\delta, w_\delta, \eta_\delta)$ be the sextuplet, solution of the problem (2.25)–(2.31), obtained in the passage to the limit as $\lambda \to 0$ and let $(u, \mu, \xi, v, w, \eta)$ denote the solution of the problem (2.32)–(2.39) arising from the above proof of Theorem 2.6 (cf. Theorem 2.2 as well).

Now, we argue similarly as in the proof of Theorem 2.5 and use the notations $\bar u_\delta := u_\delta - u$, $\bar\mu_\delta := \mu_\delta - \mu$, $\bar\xi_\delta := \xi_\delta - \xi$, $\bar v_\delta := v_\delta - v$, $\bar w_\delta := w_\delta - w$, $\bar\eta_\delta := \eta_\delta - \eta$. Here, in place of (4.18)–(4.21) we have the equalities

$$\langle \partial_t \bar u_\delta, z\rangle_{V^*,V} + \int_\Omega \nabla\bar\mu_\delta \cdot \nabla z \,\mathrm{d}x = 0, \tag{4.25}$$

$$(\bar\mu_\delta, z)_H = (\nabla\bar u_\delta, \nabla z)_H - (\partial_{\boldsymbol{\nu}}\bar u_\delta, z_{|\Gamma})_{H_\Gamma} + (\bar\xi_\delta, z)_H + \big(\pi(u_\delta) - \pi(u), z\big)_H \tag{4.26}$$

for all $z \in V$ and a.e. in $(0,T)$;

$$\langle \partial_t \bar v_\delta, z_\Gamma\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Gamma \nabla_\Gamma \bar w_\delta \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma = 0, \tag{4.27}$$

$$\begin{aligned}(\bar w_\delta, z_\Gamma)_{H_\Gamma} &= \delta \int_\Gamma \nabla_\Gamma v_\delta \cdot \nabla_\Gamma z_\Gamma \,\mathrm{d}\Gamma + (\partial_{\boldsymbol{\nu}}\bar u_\delta + \bar\eta_\delta, z_\Gamma)_{H_\Gamma} \\ &\quad + \big(\pi_\Gamma(v_\delta) - \pi(v), z_\Gamma\big)_{H_\Gamma}\end{aligned} \tag{4.28}$$

for all $z_\Gamma \in V_\Gamma$ and a.e. in $(0,T)$. As

$$m\big(\bar u_\delta(s)\big) = 0, \quad m_\Gamma\big(\bar u_\delta(s)\big) = 0$$

for all $s \in [0,T]$, we can take $z = F^{-1}\bar u_\delta(s)$ in (4.25), $z = -\bar u_\delta(s)$ in (4.26), and add them with a cancellation; then, we choose $z_\Gamma = F_\Gamma^{-1}\bar v_\delta(s)$ in (4.27), and $z_\Gamma = -\bar v_\delta(s)$

in (4.28), and add the two resultants with another cancellation. Finally, we can take the sum and integrate over $(0, t)$, obtaining

$$\begin{aligned}
&\frac{1}{2}\left|\bar{u}_\delta(t)\right|_*^2 + \frac{1}{2}\left|\bar{v}_\delta(t)\right|_{\Gamma,*}^2 + \int_0^t \left|\bar{u}_\delta(s)\right|_{V_0}^2 \mathrm{d}s + \delta \int_0^t \int_\Gamma \left|\nabla_\Gamma v_\delta(s)\right|^2 \mathrm{d}\Gamma \mathrm{d}s \\
&+ \int_0^t \left(\bar{\xi}_\gamma(s), \bar{u}_\gamma(s)\right)_H \mathrm{d}s + \int_0^t \left(\bar{\eta}_\gamma(s), \bar{v}(s)\right)_{H_\Gamma} \mathrm{d}s \\
&= \delta \int_0^t \int_\Gamma \nabla_\Gamma v_\delta(s) \cdot \nabla_\Gamma v(s) \,\mathrm{d}\Gamma \mathrm{d}s + \int_0^t \left(\pi\left(u_\delta(s)\right) - \pi\left(u(s)\right), \bar{u}_\delta(s)\right)_H \mathrm{d}s \\
&\quad + \int_0^t \left(\pi_\Gamma\left(v_\delta(s)\right) - \pi_\Gamma\left(v(s)\right), \bar{v}_\delta(s)\right)_{H_\Gamma} \mathrm{d}s
\end{aligned}$$

for all $t \in [0, T]$. Next, we observe that

$$\int_0^t \left(\bar{\xi}_\gamma(s), \bar{u}_\gamma(s)\right)_H \mathrm{d}s \geq 0, \quad \int_0^t \left(\bar{\eta}_\gamma(s), \bar{v}(s)\right)_{H_\Gamma} \mathrm{d}s \geq 0$$

due to the monotonicity of β and β_Γ;

$$\delta \int_0^t \int_\Gamma \nabla_\Gamma v_\delta(s) \cdot \nabla_\Gamma v(s) \,\mathrm{d}\Gamma \mathrm{d}s \leq \frac{\delta}{2} \int_0^t \int_\Gamma \left|\nabla_\Gamma v_\delta(s)\right|^2 \mathrm{d}\Gamma \mathrm{d}s + \frac{\delta}{2} \int_0^t \int_\Gamma \left|\nabla_\Gamma v(s)\right|^2 \mathrm{d}\Gamma \mathrm{d}s$$

by the Young inequality; moreover, we can treat the terms containing the differences $\pi(u_\delta(s)) - \pi(u(s))$ and $\pi_\Gamma(v_\delta(s)) - \pi_\Gamma(v(s))$ exactly in the same way as in the proof of Theorem 2.5, using Lipschitz continuity and the Ehrling lemma. Then, with the help of the Gronwall lemma and the Poincaré–Wirtiger inequality (2.2) we arrive at

$$\begin{aligned}
&|\bar{u}_\delta|_{L^\infty(0,T;V^*)}^2 + |\bar{v}_\delta|_{L^\infty(0,T;V_\Gamma^*)}^2 + |\bar{u}_\delta|_{L^2(0,T;V)}^2 \\
&+ |\bar{v}_\delta|_{L^2(0,T;Z_\Gamma)}^2 + \delta \int_0^T \int_\Gamma \left|\nabla_\Gamma v_\delta(t)\right|^2 \mathrm{d}\Gamma \mathrm{d}t \leq C\delta \int_0^T \int_\Gamma \left|\nabla_\Gamma v(t)\right|^2 \mathrm{d}\Gamma \mathrm{d}t
\end{aligned}$$

for some positive constant C depending only on data. Then, as v belongs to $L^2(0, T; V_\Gamma)$, it is straightforward to deduce both the error estimate (2.55) and the additional convergence (2.56), which is a consequence of the boundedness of $\int_0^T \int_\Gamma |\nabla_\Gamma v_\delta(t)|^2 \mathrm{d}\Gamma \, \mathrm{d}t$ independent of δ and the strong convergence $v_\delta \to v$ in $L^2(0, T; Z_\Gamma)$. □

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