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# AN EXACT SOLUTION FOR THE MACROSCOPIC APPROACH TO EXTENDED THERMODYNAMICS OF DENSE GASES WITH MANY MOMENTS

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**Abstract:** Extended Thermodynamics of Dense Gases with an arbitrary but fixed number of moments has been recently studied in literature; the arbitrariness of the number of moments is linked to a number $N$ and the resulting model is called an $(N)-Model$. As usual in Extended Thermodynamics, in the field equations some unknown functions appear; restriction on their generalities are obtained by imposing the entropy principle, the Galilean relativity principle and some symmetry conditions.

The solution of these conditions is called the "closure problem" and it has not been written explicitly because an hard notation is necessary, but it has been shown how the theory is self-generating in the sense that, if we know the closure of the $(N)-Model$, than we will be able to find that of the $(N+1)-Model$. Instead of this, we find here an exact solution which holds for every number $N$.

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## 1. Introduction

Extended Thermodynamics of Dense Gases (ETDG) with many moments proposes for every number $N$ the following balance equations

$$\begin{aligned} &\partial_t F^{i_1\cdots i_n} + \partial_k F^{ki_1\cdots i_n} = P^{i_1\cdots i_n} \quad \text{for}\, n = 0, \cdots N+2. \\ &\partial_t G^{i_1\cdots i_m} + \partial_k G^{ki_1\cdots i_m} = Q^{i_1\cdots i_m} \quad \text{for}\, m = 0, \cdots N\,. \end{aligned} \tag{1}$$

and the resulting model is called an $N - Model$.

We recall that the earlier versions of Extended Thermodynamics (ET) used only eqs. $(1)_1$ obtaining the important result to have a symmetric hyperbolic system of partial differential equations, with finite speed of propagation of shock waves and other important properties (See refs. [1]-[6] ). These properties were not satisfied by the models of Ordinary Thermodynamics except for some simple cases. But the structure of ET implied a restriction on the state function relating the pressure $p$ to the mass density $\rho$ and energy density $\epsilon$, that is $p = \frac{2}{3}\rho\epsilon$. This problem have been overcome in ETDG by considering all the balance equations (1) of which the first one is called the "Mass-Block" of equations and the second is called the "Energy-Block" of equations (See refs. [7]-[14] as examples).

In (1) the independent variables are $F^{i_1\cdots i_n}$ and $G^{i_1\cdots i_m}$; the quantities $F^{ki_1\cdots i_n}$ and $G^{ki_1\cdots i_m}$ are their corresponding fluxes. We see that each flux is equal to the independent variable of the subsequent equation, except for the flux in the last equation of the Mass-Block and for that in the last equation of the Energy-Block; for these last fluxes we know only that they are symmetric tensors. We will refer to this property as "the symmetry conditions". The problem will be closed when we know the expressions of $F^{ki_1\cdots i_{N+2}}$ and $G^{ki_1\cdots i_N}$ as functions of the independent variables. Restrictions on their generality are obtained by imposing the entropy principle, the Galilean relativity principle and the symmetry conditions.

The Entropy Principle can be exploited through Liu' s Theorem [15] and by using a bright idea conceived by Ruggeri [16]; so it becomes equivalent to assuming the existence of Lagrange Multipliers $\mu_A$ and $\lambda_B$ which can be taken as independent variables and, after that, we have

$$\begin{aligned} F^{i_1\cdots i_n} &= \frac{\partial h'}{\partial \mu_{i_1\cdots i_n}} \quad , \quad G^{i_1\cdots i_m} = \frac{\partial h'}{\partial \lambda_{i_1\cdots i_m}} \\ F^{ki_1\cdots i_n} &= \frac{\partial h'^k}{\partial \mu_{i_1\cdots i_n}} \quad , \quad G^{ki_1\cdots i_m} = \frac{\partial h'^k}{\partial \lambda_{i_1\cdots i_m}}\,. \end{aligned} \tag{2}$$

which expresses all the moments in terms of only two unknown functions, the 4-potentials $h'$, $h'^k$. A nice consequence of eqs. (2) is that the field equations assume the symmetric form. Another consequence of (2) is that the above mentioned symmetry conditions and the Galilean Relativity Principle can be expressed as

$$\begin{aligned}
&\frac{\partial h'^k}{\partial \mu_{i_1\cdots i_n}} = \frac{\partial h'}{\partial \mu_{k i_1\cdots i_n}} \quad \text{for}\, n=0,\cdots,N+1\,; \ \frac{\partial h'^{[k}}{\partial \mu_{i_1]i_2\cdots i_{N+2}}} = 0 \\
&\frac{\partial h'^k}{\partial \lambda_{i_1\cdots i_m}} = \frac{\partial h'}{\partial \lambda_{k i_1\cdots i_m}} \quad \text{for}\, m=0,\cdots,N-1\,; \ \frac{\partial h'^{[k}}{\partial \lambda_{i_1]\cdots i_N}} = 0\,. \\
&\frac{\partial h'}{\partial \mu_k}\mu_i + \sum_{n=1}^{N+1} \frac{\partial h'}{\partial \mu_{k j_1\cdots j_n}}\left[(n+1)\mu_{i j_1\cdots j_n} + 2\lambda_{j_1\cdots j_{n-1}}\delta_{j_n i}\right] + \\
&+ \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{k h_1\cdots h_s}}(s+1)\lambda_{i h_1\cdots h_s} + h'\delta^{ki} = 0\,. 
\end{aligned} \tag{3}$$

We don' t go in the details on how the the Galilean Relativity Principle, thanks also to the other conditions, is equivalent to $(3)_5$; this equivalence can be already found in literature as [17]-[18] and others. In the next section, eqs. (4) and (5), we will exhibit a particular but significative solution of the conditions (3), which is expressed through a Taylor' s expansion around equilibrium; this is defined as the state where $\mu_{i_1\cdots i_n} = 0$ for $n = 1,\cdots,N+2$, $\lambda_{i_1\cdots i_m} = 0$ for $m = 1,\cdots,N$, so that the only variables which are not zero at equilibrium are $\mu$ and $\lambda$. The first of these is the chemical potential, while $\lambda = \frac{1}{2T}$ with $T$ absolute temperature.

In sect. 3 we will report a part of the proof that eqs. (4) and (5) give a solution of (3). The second and final part of the proof is described in sect. 4.

## 2. An Exact Solution of Conditions (3)

Before writing our solution we need to describe an hard but straightforward

**Notation:** To do the derivatives with respect to $\mu_{i_1\cdots i_n}$ a number $p_n$ of times, we use the compact form $A_{n,1}$ to indicate a set of $n$ indexes, $A_{n,2}$ to indicate another set of $n$ indexes, and so on up to $A_{n,p_n}$. For example,

$$\frac{\partial^{p_2} h'^k}{\partial \mu_{h_1k_1}\partial \mu_{h_2k_2}\cdots\partial \mu_{h_{p_2}k_{p_2}}} \quad \text{in the compact notation becomes}$$

$$\frac{\partial^{p_2} h'^k}{\partial \mu_{A_{2,1}}\partial \mu_{A_{2,2}}\cdots\partial \mu_{A_{2,p_2}}}$$

because the first index of $A_{\ldots,\ldots}$ indicates how many indexes has the corresponding $\mu_{\ldots}$, while the second index of $A_{\ldots,\ldots}$ indicates how many derivatives we are taking with respect to it.

$$\text{Similarly,}\quad \frac{\partial^{p_3} h'^k}{\partial \mu_{h_1 k_1 j_1} \partial \mu_{h_2 k_2 j_2} \cdots \partial \mu_{h_{p_3} k_{p_3} j_{p_3}}}$$

$$\text{becomes}\quad \frac{\partial^{p_3} h'^k}{\partial \mu_{A_{3,1}} \partial \mu_{A_{3,2}} \cdots \partial \mu_{A_{3,p_3}}}$$

and so on. By using this notation, we can now write our solution and it is

$$h' = \sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i + \\ +\sum_{j=1}^{N} jr_j \; even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot \tag{4}$$

$$\cdot \frac{\left[1+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^{N} jr_j\right]!!}{1+\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot \frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^{N} r_j}}{\partial \lambda^{\sum_{j=1}^{N} r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot \psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot \delta^{(A_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}} B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\cdot$$

$$\cdot \mu_{A_{1,1}}\cdots \mu_{A_{1,p_1}}\cdots \mu_{A_{N+2,1}}\cdots \mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\cdot \lambda_{B_{1,1}}\cdots \lambda_{B_{1,r_1}}\cdots \lambda_{B_{N,1}}\cdots \lambda_{B_{N,r_N}},$$

$$h'^k = \sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i + \\ +\sum_{j=1}^{N} jr_j \; odd}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot \tag{5}$$

$$\cdot\left[\sum_{i=1}^{N+2} ip_i + \sum_{j=1}^{N} jr_j\right]!!\cdot$$

$$\cdot\frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{1+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[1+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{(kA_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\cdot\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,,$$

where $\psi_n$ is a family of functions depending only on $\mu$ and $\lambda$ and constrained only by

$$\frac{\partial\psi_{n+1}}{\partial\mu}=\psi_n\,. \tag{6}$$

Despite the appearance, these expressions are not complicated. In fact, the factorial $\frac{1}{p_1!}$ appears also in the Taylor' s expansions of functions depending on a single variable; we have here one of these for every variable, both of the mass block than of the energy block. Moreover, in these expressions appears the sum of these numbers $p_1+p_2+\cdots+p_{N+2}$ of the mass block and $r_1+r_2+\cdots+r_N$ of the energy block. Moreover, these numbers appear also through $p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N$ where these numbers are multiplied by the order of the Lagrange multiplier which they represent; after that, their sum is taken. The condition "$p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N$ odd" in the expression of $h'^k$ is necessary because the following $\delta^{\cdots}$ must have an even number of indexes; similarly, for the expression of $h'$. Here we find only a letter $\delta^{\cdots}$ but it is understood that it is a shortened symbol denoting the product of some $\delta^{\cdots}$ each one with 2 indexes, and with a final symmetrization over all these indexes. Finally, in the last line there is the product of the variables with respect to which we have done the Taylor' s expansions.

## 3. Proof of the Solution (4), (5) – Part I

We prove now that (4), (5) is a solution of $(3)_{1-4}$, while $(3)_5$ will be considered in the next section.

- Let us begin with $(3)_1$ in the case $n \geq 1$.

  To verify it we have to take into account that $\mu_{i_1 \cdots i_n}$ here is denoted with $\mu_{A_{n,1}}$, or with $\mu_{A_{n,2}}$, and so on up to $\mu_{A_{n,p_n}}$. Similarly, $\mu_{k i_1 \cdots i_n}$ is denoted with $\mu_{A_{n+1,1}}$, or with $\mu_{A_{n+1,2}}$, and so on up to $\mu_{A_{n+1,p_{n+1}}}$. Consequently, the left hand side of $(3)_1$ with $n \geq 1$ causes, with respect to the expression (5), a rise of one unity of the index $p_n$; similarly, the right hand side of $(3)_1$ with $n \geq 1$ causes, with respect to the expression (4), a rise of one unity of the index $p_{n+1}$. More precisely, we obtain for both sides the following expression

$$\sum_{\substack{p_1, \cdots, p_{N+2}, r_1, \cdots, r_N \\ n + \sum_{i=1}^{N+2} i p_i + \\ + \sum_{j=1}^{N} j r_j \, odd}}^{0 \cdots \infty} \frac{1}{p_1!} \frac{1}{p_2!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot$$

$$\cdot \left[ n + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j \right]!! \cdot$$

$$\cdot \frac{\partial^{2 + \sum_{i=1}^{N+2} p_i + \sum_{j=1}^{N} r_j}}{\partial \lambda^{\sum_{j=1}^{N} r_j} \partial \mu^{2 + \sum_{i=1}^{N+2} p_i}} \left[ \left( \frac{-1}{2\lambda} \right)^{\frac{1}{2}\left[ n + 1 + \sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j \right]} \cdot \right.$$

$$\left. \cdot \psi_{\frac{1}{2}[n+1+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+N r_N]} \right] \cdot$$

$$\cdot \delta^{(k i_1 \cdots i_n A_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} \cdot$$

$$\cdot \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} \, .$$

- Regarding $(3)_1$ in the case $n = 0$, we have the same situation of the above case for its right hand side, while for its left hand side we have to take simply the derivative of (5) with respect to $\mu$. More precisely, we obtain for both sides the following expression

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j\, odd}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!\cdot$$

$$\cdot\frac{\partial^{2+\sum_{i=1}^{N+2}p_i+\sum_{j=1}^{N}r_j}}{\partial\lambda^{\sum_{j=1}^{N}r_j}\partial\mu^{2+\sum_{i=1}^{N+2}p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[1+\sum_{i=1}^{N+2}ip_i+\sum_{j=1}^{N}jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[1+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{(kA_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,.$$

- It is easy to verify $(3)_{2,4}$.
- Let us continue verifying $(3)_3$ in the case $m \geq 1$.

  To verify it we have to take into account that $\lambda_{i_1\cdots i_m}$ here is denoted with $\lambda_{B_{m,1}}$, or with $\lambda_{B_{m,2}}$, and so on up to $\lambda_{B_{m,r_m}}$. Similarly, $\lambda_{ki_1\cdots i_m}$ is denoted with $\lambda_{B_{m+1,1}}$, or with $\lambda_{B_{m+1,2}}$, and so on up to $\lambda_{B_{m+1,r_{m+1}}}$. Consequently, the left hand side of $(3)_3$ with $m \geq 1$ causes, with respect to the expression (5), a rise of one unity of the index $r_m$; similarly, the right hand side of $(3)_3$ with $m \geq 1$ causes, with respect to the expression (4), a rise of one unity of the index $r_{m+1}$. More precisely, we obtain for both sides the following expression

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ m+\sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \, odd}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\left[m+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!\cdot$$

$$\cdot\frac{\partial^{2+\sum_{i=1}^{N+2}p_i+\sum_{j=1}^{N}r_j}}{\partial\lambda^{1+\sum_{j=1}^{N}r_j}\partial\mu^{1+\sum_{i=1}^{N+2}p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[m+1+\sum_{i=1}^{N+2}ip_i+\sum_{j=1}^{N}jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[m+1+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{(ki_1\cdots i_m A_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,.$$

- Let us conclude verifying $(3)_3$ in the case $m=0$.

  We have the same situation of the above case for its right hand side, while for its left hand side we have to take simply the derivative of (5) with respect to $\lambda$. More precisely, we obtain for both sides the following expression

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \, odd}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!\cdot$$

$$\cdot \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^{N} r_j}}{\partial \lambda^{1+\sum_{j=1}^{N} r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[ \left( \frac{-1}{2\lambda} \right)^{\frac{1}{2}\left[1+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j\right]} \cdot \right.$$

$$\left. \cdot \psi_{\frac{1}{2}[1+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+N r_N]} \right] \cdot$$

$$\cdot \delta^{(k A_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} \cdot$$

$$\cdot \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} \cdot$$

So we have finished to verify that (4), (5) is a solution of $(3)_{1-4}$ and there remains to prove that it satisfies also $(3)_5$.

## 4. Proof of the Solution (4), (5) – Part II

We prove now that (4), (5) is a solution of $(3)_5$. To this regard we note firstly that in this condition we can put under an unique summation the coefficients of $\mu_{\dots}$, from the first coefficient of a $\lambda_{\dots}$ we can isolate the term with $n = 1$ and in the other ones we can change the index of the summation according to the law $n = 2 + s$. In this way $(3)_5$ can be rewritten as

$$\sum_{n=0}^{N+1} \frac{\partial h'}{\partial \mu_{kj_1\cdots j_n}} (n+1)\mu_{ij_1\cdots j_n} + \sum_{s=0}^{N-1} 2\lambda_{j_1\cdots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1\cdots j_{s+1}i}} + \tag{7}$$
$$+ \quad \frac{\partial h'}{\partial \mu_{ki}} 2\,\lambda + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{kh_1\cdots h_s}} (s+1)\lambda_{ih_1\cdots h_s} + h'\delta^{ki} = 0\,.$$

- Now, for the first term of this relation we can use (4) and the derivation causes a presence of the factor $p_{n+1}$, the substitution of $\mu_{A_{n+1,1}} \cdots \mu_{A_{n+1,p_{n+1}}}$ with $\mu_{A_{n+1,1}} \cdots \mu_{A_{n+1,p_{n+1}-1}}$ and the new free indexes $kj_1 \cdots j_n$ in the expression of $\delta^{\cdots}$ ; more precisely, that term becomes

$$\sum_{n=0}^{N+1} (n+1)\mu_{ij_1\cdots j_n} \frac{\partial h'}{\partial \mu_{kj_1\cdots j_n}} = \sum_{n=0}^{N+1} (n+1) p_{n+1} \cdot \tag{8}$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \; even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\frac{\left[1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot\frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{1+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{j_{n+1}i}\delta^{(kj_1\cdots j_nA_{1,1}\cdots A_{1,p_1}\cdots A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}}$$

$$^{B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{n+1,1}}\cdots\mu_{A_{n+1,p_{n+1}-1}}$$

$$\mu_{j_1\cdots j_{n+1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\cdot\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}},$$

where we have also substituted $\mu_{ij_1\cdots j_n}$ with $\mu_{j_1\cdots j_{n+1}}\delta^{j_{n+1}i}$. Now in this expression we can insert a symmetrization over all the indexes of the set

$$j_1\cdots j_{n+1}A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}$$

because that expression remains the same if we exchange two of these indexes. This fact is evident if the two indexes are taken between $j_1\cdots j_{n+1}$; for the proof in the other cases, let us consider the shortened expression

$$\delta^{j_{n+1}i}\delta^{(kj_1\cdots j_nk_1\cdots k_nk_{n+1}\cdots)}\mu_{j_1\cdots j_{n+1}}\mu_{k_1\cdots k_{n+1}};$$

here we can exchange the nomes of the indexes $j.$ with those of the $k.$, so that the above shortened expression becomes

$$\delta^{k_{n+1}i}\delta^{(kk_1\cdots k_nj_1\cdots j_nj_{n+1}\cdots)}\mu_{k_1\cdots k_{n+1}}\mu_{j_1\cdots j_{n+1}}.$$

Now we can exchange the indexes $k_1\cdots k_n$ with the indexes $j_1\cdots j_n$ in the expression of $\delta^{(kk_1\cdots k_nj_1\cdots j_nj_{n+1}\cdots)}$ because this is a symmetric tensor.

We obtain $\delta^{k_{n+1}i}\delta^{(kj_1\cdots j_n k_1\cdots k_n j_{n+1}\cdots)}\mu_{k_1\cdots k_{n+1}}\mu_{j_1\cdots j_{n+1}}$. By comparing this result with the expression which we started from, it is the same if he had exchanged the indexes $j_{n+1}$ and $k_{n+1}$. This completes the proof of the fact that the expression (8) remains the same if we exchange two indexes of the set $j_1\cdots j_{n+1}A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}$; so we can insert there a symmetrization over those indexes and (8) becomes

$$\sum_{n=0}^{N+1}(n+1)\mu_{ij_1\cdots j_n}\frac{\partial h'}{\partial \mu_{kj_1\cdots j_n}} = \sum_{n=0}^{N+1}(n+1)p_{n+1}\cdot \tag{9}$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N\\ \sum_{i=1}^{N+2} ip_i+\\ +\sum_{j=1}^{N} jr_j\ even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\frac{\left[1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot\frac{\partial^{1+\sum_{i=1}^{N+2}p_i+\sum_{j=1}^{N}r_j}}{\partial\lambda^{\sum_{j=1}^{N}r_j}\partial\mu^{1+\sum_{i=1}^{N+2}p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{\underline{j_{n+1}}i}\delta^{(k\underline{j_1\cdots j_n}A_{1,1}\cdots A_{1,p_1}\cdots\underline{A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}}$$

$$^{B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{n+1,1}}\cdots\mu_{A_{n+1,p_{n+1}-1}}$$

$$\mu_{j_1\cdots j_{n+1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\cdot\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}},$$

where underlined indexes denote symmetrization over these indexes. Now we observe that $(n+1)p_{n+1}$ is exactly the number of the indexes of the set $j_1\cdots j_{n+1}A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}$ and that, thanks to the summation $\sum_{n=0}^{N+1}$ the index near $i$ in $\delta^{j_{n+1}i}$ can be every index of the set

$$A_{1,1}\cdots A_{1,p_1}\cdots A_{n+1,1}\cdots A_{n+1,p_{n+1}}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}.$$

These facts allow to rewrite (9) as

$$
\sum_{n=0}^{N+1}(n+1)\mu_{ij_1\cdots j_n}\frac{\partial h'}{\partial\mu_{kj_1\cdots j_n}} = \left[\sum_{n=0}^{N+1}(n+1)p_{n+1}\right]\cdot \tag{10}
$$

$$
\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N\\ \sum_{i=1}^{N+2} ip_i+\\ +\sum_{j=1}^{N} jr_j\ even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot
$$

$$
\cdot\frac{\left[1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot
$$

$$
\cdot\frac{\partial^{1+\sum_{i=1}^{N+2}p_i+\sum_{j=1}^{N}r_j}}{\partial\lambda^{\sum_{j=1}^{N}r_j}\partial\mu^{1+\sum_{i=1}^{N+2}p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2}ip_i+\sum_{j=1}^{N}jr_j\right]}\cdot\right.
$$

$$
\left.\cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot
$$

$$
\cdot\delta^{\underline{j_{n+1}}i}\delta^{(kj_1\cdots j_n\underline{A_{1,1}\cdots A_{1,p_1}\cdots A_{n+1,1}\cdots A_{n+1,p_{n+1}-1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}}}
$$

$$
{}^{B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{n+1,1}}\cdots\mu_{A_{n+1,p_{n+1}-1}}
$$

$$
\mu_{j_1\cdots j_{n+1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot
$$

$$
\cdot\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\ ,
$$

- For the fourth term of (7) we can do similar passages (the difference is that we have the $\lambda_{...}$ instead of the $\mu_{...}$, $N-2$ instead of $N$ and $s$ instead of $n$); in this way that term becomes

$$
\sum_{s=0}^{N-1}(s+1)\lambda_{ih_1\cdots h_s}\frac{\partial h'}{\partial\lambda_{kh_1\cdots h_s}} = \left[\sum_{s=0}^{N-1}(s+1)r_{s+1}\right]\cdot \tag{11}
$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \; even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\frac{\left[1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{1+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot\frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{1+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{\underline{h_{s+1}}i}\delta^{(k\underline{h_1\cdots h_s}A_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}}$$

$$\underline{{}^{B_{1,1}\cdots B_{1,r_1}\cdots B_{s+1,1}\cdots B_{s+1,r_{s+1}-1}\cdots B_{N,1}\cdots B_{N,r_N})}}\cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\cdot\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{s+1,1}}\cdots\lambda_{B_{s+1,r_{s+1}-1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,.$$

- If we look at the last term of (7), we see that it can be written together with (10) and (11) and they become

$$\sum_{n=0}^{N+1}\frac{\partial h'}{\partial\mu_{kj_1\cdots j_n}}(n+1)\mu_{ij_1\cdots j_n}+\sum_{s=0}^{N-1}\frac{\partial h'}{\partial\lambda_{kh_1\cdots h_s}}(s+1)\lambda_{ih_1\cdots h_s}+$$

$$+h'\delta^{ki}=\left[1+\sum_{n=0}^{N+1}(n+1)p_{n+1}+\sum_{s=0}^{N-1}(s+1)r_{s+1}\right]\cdot$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \; even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$
\cdot \frac{\left[1+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j\right]!!}{1+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j} \cdot
$$

$$
\cdot \frac{\partial^{1+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^{N} r_j}}{\partial \lambda^{\sum_{j=1}^{N} r_j} \partial \mu^{1+\sum_{i=1}^{N+2} p_i}} \left[ \left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j\right]} \cdot \right.
$$

$$
\left. \cdot \psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+N r_N]} \right] \cdot
$$

$$
\cdot \delta^{\underline{k}i} \delta^{\underline{(A_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}} B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}} \cdot
$$

$$
\cdot \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot
$$

$$
\lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} \, , \tag{12}
$$

where also the index $k$ has been put under the symmetrization and we recall for the sequel is that a property of symmetrization is $\delta^{i\underline{k}}\delta^{\underline{(\cdots)}} = \delta^{i\underline{k}}\delta^{\underline{\cdots}} = \delta^{(ik\cdots)}$. It is interesting to see that the coefficient in square bracket at the beginning of the right hand side of eq. (12) has become equal to the denominator of the half factorial of that right hand side!

- For the second term of (7), we can use (4) and the derivation causes someway the rising of one unity of the index $p_2$; more precisely we have

$$
\frac{\partial h'}{\partial \mu_{ki}} 2\,\lambda = 2\,\lambda \sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ \sum_{i=1}^{N+2} i p_i + \\ +\sum_{j=1}^{N} j r_j \; even}}^{0\cdots\infty} \frac{1}{p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \cdots \frac{1}{r_N!} \cdot
$$

$$
\cdot \left[1+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j\right]!! \cdot
$$

$$
\cdot \frac{\partial^{2+\sum_{i=1}^{N+2} p_i + \sum_{j=1}^{N} r_j}}{\partial \lambda^{\sum_{j=1}^{N} r_j} \partial \mu^{2+\sum_{i=1}^{N+2} p_i}} \left[ \left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[2+\sum_{i=1}^{N+2} i p_i + \sum_{j=1}^{N} j r_j\right]} \cdot \right.
$$

$$\left.\cdot\psi_{\frac{1}{2}[2+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right] \cdot$$

$$\cdot\delta^{(kiA_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})} \cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,, \tag{13}$$

- There remains to consider the third term of (7); for it we can use (4) and the derivation causes someway the rising of one unity of the index $p_{s+3}$; more precisely we have

$$\sum_{s=0}^{N-1} 2\lambda_{j_1\cdots j_{s+1}} \frac{\partial h'}{\partial\mu_{kj_1\cdots j_{s+1}i}} = \sum_{s=0}^{N-1} 2\lambda_{j_1\cdots j_{s+1}} \cdot \tag{14}$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ s+3+\sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j\ even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\frac{\left[s+4+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{s+4+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot\frac{\partial^{2+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{2+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[s+3+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[s+3+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{(kj_1\cdots j_{s+1}iA_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}.$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,.$$

In this expression, we can substitute $\lambda_{j_1\cdots j_{s+1}}$ with $\lambda_{B_{s+1,r_{s+1}+1}}$ and $j_1\cdots j_{s+1}$ with $B_{s+1,r_{s+1}+1}$; moreover, we substitute $\frac{1}{r_{s+1}!}$ with $\frac{r_{s+1}+1}{(r_{s+1}+1)!}$ and, after that, decrease $r_{s+1}$ of one unity (which is equivalent to a change of index). So this expression becomes

$$\sum_{s=0}^{N-1} 2\lambda_{j_1\cdots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1\cdots j_{s+1}i}} = \sum_{s=0}^{N-1} 2r_{s+1} \cdot \tag{15}$$

$$\sum_{\substack{p_1,\cdots,p_{N+2},r_1,\cdots,r_N \\ 2+\sum_{i=1}^{N+2} ip_i+ \\ +\sum_{j=1}^{N} jr_j \; even}}^{0\cdots\infty} \frac{1}{p_1!}\frac{1}{p_2!}\cdots\frac{1}{p_{N+2}!}\frac{1}{r_1!}\frac{1}{r_2!}\cdots\frac{1}{r_N!}\cdot$$

$$\cdot\frac{\left[3+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]!!}{3+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j}\cdot$$

$$\cdot\frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{-1+\sum_{j=1}^{N} r_j}\partial\mu^{2+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[2+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[2+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]\cdot$$

$$\cdot\delta^{(kiA_{1,1}\cdots A_{1,p_1}\cdots A_{N+2,1}\cdots A_{N+2,p_{N+2}}B_{1,1}\cdots B_{1,r_1}\cdots B_{N,1}\cdots B_{N,r_N})}\cdot$$

$$\cdot\mu_{A_{1,1}}\cdots\mu_{A_{1,p_1}}\cdots\mu_{A_{N+2,1}}\cdots\mu_{A_{N+2,p_{N+2}}}\cdot$$

$$\lambda_{B_{1,1}}\cdots\lambda_{B_{1,r_1}}\cdots\lambda_{B_{N,1}}\cdots\lambda_{B_{N,r_N}}\,.$$

It is true that with change of index we have in the summation the extra term with $r_{s+1}=0$; but it doesn't effect the result for the presence of the coefficient $r_{s+1}$.

If we look at the expressions (12), (13) and (15) we conclude that to prove (7) it is sufficient that the following relation holds

$$\frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{1+\sum_{i=1}^{N+2} p_i}}\left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]}\cdot\right.$$

$$\left.\cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]+$$

$$+2\lambda \frac{\partial^{2+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{\sum_{j=1}^{N} r_j}\partial\mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[2+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]} \cdot \right.$$

$$\left. \cdot\psi_{\frac{1}{2}[2+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right] +$$

$$+\sum_{s=0}^{N-1} 2r_{s+1} \frac{\partial^{1+\sum_{i=1}^{N+2} p_i+\sum_{j=1}^{N} r_j}}{\partial\lambda^{-1+\sum_{j=1}^{N} r_j}\partial\mu^{2+\sum_{i=1}^{N+2} p_i}} \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}\left[2+\sum_{i=1}^{N+2} ip_i+\sum_{j=1}^{N} jr_j\right]} \cdot \right.$$

$$\left. \cdot\psi_{\frac{1}{2}[2+p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right] = 0\,.$$

Now, for the last two terms of this expression we can use (6), after that only the function $\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}$ will be present there and to prove that relation it will be sufficient to prove that

$$\frac{\partial^r}{\partial\lambda^r}\vartheta + 2\lambda\frac{\partial^r}{\partial\lambda^r}\left(\frac{-1}{2\lambda}\vartheta\right) + 2r\frac{\partial^{-1+r}}{\partial\lambda^{-1+r}}\left(\frac{-1}{2\lambda}\vartheta\right) = 0\,, \tag{16}$$

where we have put $r = r_1 + r_2 + \cdots + r_N$ and $\vartheta = \left[\left(\frac{-1}{2\lambda}\right)^{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]} \cdot \right.$
$\left. \cdot\psi_{\frac{1}{2}[p_1+2p_2+\cdots+(N+2)p_{N+2}+r_1+2r_2+\cdots+Nr_N]}\right]$.

Obviously, eq. (16) is an identity because we have

$$\frac{\partial^r}{\partial\lambda^r}\vartheta = \frac{\partial^r}{\partial\lambda^r}\left(-2\lambda\,\frac{-1}{2\lambda}\vartheta\right) = -2\lambda\frac{\partial^r}{\partial\lambda^r}\left(\frac{-1}{2\lambda}\vartheta\right) - 2r\frac{\partial^{-1+r}}{\partial\lambda^{-1+r}}\left(\frac{-1}{2\lambda}\vartheta\right)$$

so we have finished all the arguments which we had to prove.

## Conclusions

We observe that in the particular case $N = 0$, we obtain the 11 moments model and the solution here indicated is exactly that already obtained in literature with the macroscopic approach. In the case $N = 1$ we have that the present one is a particular solution of those already known in literature. Aim of a future research is to find the explicit expression of the general solution.

## References


[1] I-S. Liu, I. Müller, Extended Thermodynamics of Classical and Degenerate Ideal Gases, *Arch. Rat. Mech. Anal.*, **83**, (1983), 285-332. DOI 10.1007/BF00963838.

[2] I.S. Liu, I. Müller, T. Ruggeri, Relativistic thermodynamics of gases, *Ann. Phys. (N.Y.)*, **169**, (1986), 191-219. DOI 10.1016/0003-4916(86)90164-8.

[3] I. Müller, T. Ruggeri, *Rational Extended Thermodynamics 2nd edn.*, Springer Tracts in Natural Philosophy, Springer, New York (1998). DOI 10.1007/978-1-4612-2210-1.

[4] I. Müller, Entropy and Energy,-a Universal Competition, *Entropy*, **10**, (2008), 462-476. DOI 10.3390/entropy-e10040462.

[5] M.C. Carrisi, A. Farci, M. Obounou, S. Pennisi, Relativistic Extended Thermodynamics from the Lagrangian view-point, *Ricerche di Matematica*, **64**, (2015), 357-376. DOI 10.1007/s11587-015-0244-x.

[6] A. Atangana, B.S.T. Alkahatani, Analysys of the Keller-Segel Model with a Fractional Derivative without Singular Kernel, *Entropy*, **17**, (2015), 4439-4453. DOI 10.3390/e17064439.

[7] T. Arima, S. Taniguchi, T. Ruggeri, M. Sugiyama, Extended Thermodynamics of dense gases, *Continuum Mech. Thermodyn.*, **24**, (2012), 271-292. DOI 10.1007/s00161-011-0213-x.

[8] T. Arima, S. Taniguchi, T. Ruggeri, M. Sugiyama, Extended Thermodynamics of real gases with dynamic pressure: An extension of Meixner's theory, *Physics Letters A.*, **376**, (2012), 2799-2803. DOI 10.1016/j.physleta.2012.08.030.

[9] T. Arima, A. Mentrelli, T. Ruggeri, Molecular Extended Thermodynamics of rarefied polyatomic gases and wave velocities for increasing number of moments, *Ann. Phys.*, **345**, (2014), 111-132. DOI 10.1016/j.aop.2014.03.011.

[10] T. Ruggeri, M. Sugiyama, *Rational Extended Thermodynamics beyond the Monatomic Gas*, Springer Verlag, (2015). DOI 10.1007/978-3-319-13341-6.

[11] M.C. Carrisi, S. Pennisi, An 18 Moments Model for Dense Gases: Entropy and Galilean Relativity Principles without Expansions, *Entropy.*, **17**, (2015), 214-230. DOI 10.3390/e17010214.

[12] M.C. Carrisi, S. Pennisi, Extended thermodynamics for dense gases up to whatever order, *Int. Journal of Non-Linear Mech.*, **77**, (2015), 74-84. DOI 10.1016/ijnonlinmec.2015.07.011.

[13] M.C. Carrisi, R. E. Tchame, M.Obounou, S. Pennisi, Extended Thermodynamics for dense gases up to whatever order and with only some symmetries, *Entropy.*, **17**, (2015), 7052-7075. DOI 10.3390/e17107052.

[14] M.C. Carrisi, S. Pennisi, T. Ruggeri, M. Sugiyama, Extended thermodynamics of dense gases in the presence of dynamic pressure, *Ricerche di Matematica*, **64**, (2015), 403-419. DOI 10.1007/s11587-015-0247-7.

[15] I-S. Liu, Method of Lagrange Multipliers for Exploitation of the Entropy Principle, *Arch. Rat. Mech. Anal.*, **46**, (1972), 131-148. DOI 10.1007/BF00250688.

[16] T. Ruggeri, A. Strumia, Main Field and Convex Covariant Density for Quasi-Linear Hyperbolic Systems. Relativistic Fluid Dynamics, *Ann. Inst. H. Poincaré*, **34**, (1981), 65-84. DOI not available.

[17] T. Ruggeri, Galilean Invariance and Entropy Principle for Systems of Balance Laws. The Structure of the Extended Thermodynamics, *Continuum Mech. Thermodyn.*, **1**, (1989), 3-20. DOI 10.1007/BF01125883.

[18] S. Pennisi, T. Ruggeri, A new method to exploit the Entropy Principle and Galilean Invariance in the macroscopic approach of Extended Thermodynamics, *Ricerche di Matematica*, **55**, (2006), 319-339. DOI 10.1007/s11587-006-0019-5.