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A not so common boundary problem related to the membrane equilibrium equations

Giuseppe Viglialoro[1] **and Juan Murcia**[2]

[1] *Department of Mathematics, University of Cadiz*

[2] *Instituto de Ciencias de la Construcción Eduardo Torroja, CSIC*

emails: `giuseppe.viglialoro@uca.es`, `murcia@ietcc.csic.es`

Abstract

This paper presents a mathematical problem related to the equilibrium analysis of a membrane with rigid and cable boundary for the so called *prestressing phase*. The membrane and its boundary are respectively identified with a regular surface and a set of regular curves. The equilibrium is directly expressed by means of a boundary (differential) problem, in terms of the shape of the membrane and its stress tensor. The membrane-cable interface equilibrium leads to take into account a singular condition; as a consequence, an unusual elliptic problem will appear.

Key words: membrane, footbridge, equilibrium, rigid boundary, cable boundary, elliptic problem.

1 Introduction

This work studies the bi-dimensional and continuous equilibrium of a membrane for the *prestressing phase*, when the prestressed membrane is ready to the serviceability phase.

More exactly, a new membrane technology for footbridges has been developed in Spain and a membrane footbridge prototype has been built near Barcelona (see [2] for the main technological aspects and the Figure 1(a) for a real picture). The prestressing is introduced and obtained by means of the membrane's boundaries, that are one-dimensional elements defined by spatial curves, that may or may not have curvature (straight elements).

In the paper the membrane tensions are identified with a positive second order tensor and its shape with a surface with negative gaussian curvature. So, the equilibrium is directly defined by partial differential equations, one of them involving the product between the

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variables associated to the membrane and to its stress tensor. Hence, we will discuss the following problem: *once the stress tensor is given find the shape of the membrane that achieves the equilibrium.*

Along with the equilibrium equations the boundary conditions must be defined; these conditions depend on the used boundary elements; herein both rigid and cable boundary will be considered. Contrary to what happens with the rigid elements, the shapes of a cable and the corresponding membrane are linked by a compatibility relation (see [3] and [4]), making that the form of the membrane (i.e. the unknown) has to verify a not *ad hoc* boundary condition for a second order problem.

2 Equilibrium equations of a membrane

Let us identify the membrane with a surface $\mathcal{S}$ with a negative gaussian curvature (Figure 1(c)); $\mathcal{S}$ is parameterized by

$$\mathcal{S} \to \boldsymbol{\varphi}(x,y) := (x, y, z(x,y)) \ \forall \ (x,y) \in D,$$

where $z = z(x,y)$ is a regular function defined in a domain $D \subset \mathbb{R}^2$.

If $\boldsymbol{\sigma} := N_{\alpha\beta} = N_{\alpha\beta}(x,y)$ $(\alpha,\ \beta = 1,2)$ is the projected stress tensor of the membrane (in fact, force per unit length), the membrane equilibrium equations in terms of $N_{\alpha\beta}$, neglecting its weight and considering no external load, are expressed by (see [5] for the details):

$$\begin{cases} N_{xx,x} + N_{xy,y} = 0 \quad \text{in } D, \\ N_{xy,x} + N_{yy,y} = 0 \quad \text{in } D, \\ N_{xx} z_{,xx} + 2N_{xy} z_{,xy} + N_{yy} z_{,yy} = 0 \quad \text{in } D. \end{cases} \tag{1}$$

The system (1) shows that if a positive tensor $\boldsymbol{\sigma}$ is fixed the function z has to solve an elliptic equation; then, the problem we will consider consists of *fixing the stress tensor of the membrane* and *finding its shape.*

3 Boundary equilibrium equations

Once the membrane equilibrium equations are given (system (1)), the problem has to be completed by defining the corresponding boundary conditions on $\Gamma = \partial D$. We will put Γ^{r} the subset of Γ corresponding to the rigid boundary and Γ^{c} the one corresponding to the cable boundary; of course $\Gamma = \Gamma^{\mathrm{r}} \cup \Gamma^{\mathrm{c}}$ (see Figure 1(b)).

3.1 The rigid boundary: equilibrium equations

Let us consider the equilibrium on Γ^{r}; as you can check in [5], Γ^{r} can assume any shape and moreover it is not necessary to impose any kind of restriction between its own shape

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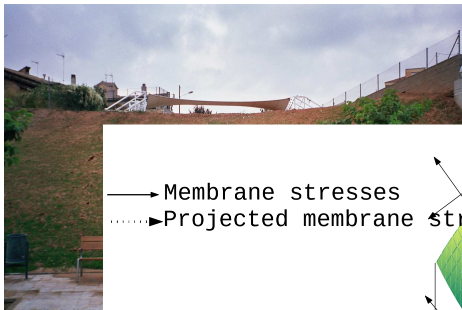
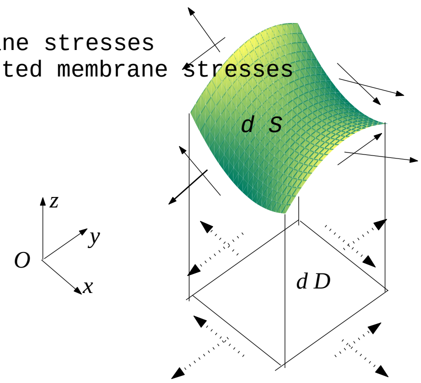
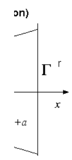


(a) Pic





(c) Membrane element and stress tensors.

Figure 1: Characterization of a membrane footbridge.

and the shape of the membrane (the unknown z). In this case, the corresponding boundary condition is the usual Dirichel condition

$$z = g \text{ on } \Gamma^{\mathrm{r}},$$

being g the value of z on the same Γ^{r}, i.e. the 3−D shape of the rigid boundary of membrane.

3.2 The cable boundary: equilibrium equations

Let us consider the equilibrium on Γ^{c}; the cable tensions are tangent to the cable and they belong to the *osculator plan* of the curve $\mathcal{C}$, that represents the 3−D shape of the cable. On

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the other hand, the stress tensor of the membrane belong to the *tangent plan* of the surface $\mathcal{S}$; of course, in order to reach the equilibrium these two plans have to coincide.

As a consequence (you can find the proof in [5]), the boundaries equilibrium returns the following *cable-membrane compatibility equation*:

$$z_{,xx} + 2z_{,xy}y' + z_{,yy}y'^2 = 0 \quad \text{on } \Gamma^{\mathrm{c}}. \tag{2}$$

4 Definition and properties of the mathematical problem

4.1 Mathematical problem

With reference to the equilibrium expressed by the system (1) *and the above notation, let* $N_{\alpha\beta}$ *be a positive and symmetric second order tensor such that* $\sum_{\beta=1}^{2} N_{\alpha\beta,\beta} = 0$ $(\alpha = 1, 2)$, *in a bounded domain* D *of the plan* xOy.

Find the surface z*, defined in* D*, such that*

$$\begin{cases} z_{,xx}N_{xx} + 2z_{,xy}N_{xy} + z_{,yy}N_{yy} = \operatorname{div}(\boldsymbol{\sigma} \cdot \nabla z) = 0 \ \textit{in } D, \\ z = g \ \textit{on } \Gamma^{\mathrm{r}}, \\ z_{,xx} + 2z_{,xy}y' + z_{,yy}y'^2 = 0 \ \textit{on } \Gamma^{\mathrm{c}}. \end{cases} \tag{3}$$

Due to the second boundary condition, the previous **is not** a typical elliptic problem with usual Dirichlet or Dirichlet-Neumann boundary conditions (see [1]). By means of *Hopf's Lemma*, one can prove the uniqueness of the solution of the system (3); on the other hand we have to underline that the corresponding existence problem is still an open question.

In the next section, we will propose a numerical method to solve the system (3).

4.2 Solving the problem by a numerical method

In [5] is shown that the problem (3) is equivalent to the following one, that is more direct.

With the same notations of the previous problem (system (3)*), find* z *and* h*, such that*

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \cdot \nabla z) = 0 \quad \textit{in } D, \\ z = g \ \textit{on } \Gamma^{\mathrm{r}}, \\ z = h \ \ \textit{on} \ \ \Gamma^{\mathrm{c}}, \\ z_{,y} = \dfrac{h''}{y''} \ \ \textit{on} \ \ \Gamma^{\mathrm{c}}. \end{cases} \tag{4}$$

Let us consider the following system

$$\begin{cases} -\operatorname{div}(\boldsymbol{\sigma} \cdot \nabla z) = 0 \quad \text{in } D, \\ z = g \ \ \text{on} \ \ \Gamma^{\mathrm{r}}, \\ z = h \ \ \text{on} \ \ \Gamma^{\mathrm{c}}, \end{cases} \tag{5}$$

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and the equation

$$h'' = z_{,y} y'' \quad \text{on} \quad \Gamma^{\mathrm{c}}. \tag{6}$$

By means of the *Finite Element Method* (see [6]), let us fix a mesh for D; if n_{t} is the total number of nodes of D, n_{r} those of Γ^{r} and n_{c} those of Γ^{c}, putting $z \simeq \sum_{j=1}^{n_{\mathrm{t}}} z_j N_j$ and $h \simeq \sum_{j=1}^{n_{\mathrm{c}}} h_j N_j$ and replacing them into the system (5), we obtain

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{h} + \boldsymbol{G}\boldsymbol{g}, \tag{7}$$

being $\boldsymbol{z} = (z_1, \dots, z_{\mathrm{t}})$, $\boldsymbol{g} = (g_1, \dots, g_{\mathrm{r}})$ and $\boldsymbol{h} = (h_1, \dots, h_{\mathrm{c}})$ the nodal values vectors of z in D, g on Γ^{r} and h on Γ^{c}, and $\boldsymbol{H} \in M_{n_{\mathrm{t}} \times n_{\mathrm{c}}}(\mathbb{R})$ and $\boldsymbol{G} \in M_{n_{\mathrm{t}} \times n_{\mathrm{r}}}(\mathbb{R})$.

With reference to the equation (6), let us define the $n_{\mathrm{c}}-$dimensional ***residual vector*** $\boldsymbol{R}$, whose components are

$$(\boldsymbol{R})_i = R_i(h) := \int_{\Gamma^{\mathrm{c}}} (z_{,y} y'' - h'') N_i \, d\,\Gamma^{\mathrm{c}}, \quad i = 1, \dots, n_{\mathrm{c}},$$

and let us make it zero:

$$R_i(h) = \int_{\Gamma^{\mathrm{c}}} (z_{,y} y'' - h'') N_i \, d\,\Gamma^{\mathrm{c}} = 0, \quad i = 1, \dots, n_{\mathrm{c}}. \tag{8}$$

Putting $z \simeq \sum_{j=1}^{n_{\mathrm{t}}} z_j N_j$ and $h \simeq \sum_{j=1}^{n_{\mathrm{c}}} h_j N_j$, replacing them into the relation (8) and integrating by parts, we obtain

$$\sum_{j=1}^{n_{\mathrm{t}}} z_j \int_{\Gamma^{\mathrm{c}}} y'' N_{j,y} N_i \, d\,\Gamma^{\mathrm{c}} + \sum_{j=1}^{n_{\mathrm{c}}} h_j \int_{\Gamma^{\mathrm{c}}} N'_j N'_i \, d\,\Gamma^{\mathrm{c}} = 0, \quad i = 1, \dots, n_{\mathrm{c}}.$$

Defining

$$M_{ij} := \int_{\Gamma^{\mathrm{c}}} y'' N_{j,y} N_i \, d\,\Gamma^{\mathrm{c}} \quad \text{and} \quad W_{ij} = \int_{\Gamma^{\mathrm{c}}} N'_j N'_i \, d\,\Gamma^{\mathrm{c}},$$

we conclude

$$\boldsymbol{M}\boldsymbol{z} + \boldsymbol{W}\boldsymbol{h} = \boldsymbol{0}, \tag{9}$$

with $\boldsymbol{M} \in M_{n_{\mathrm{c}}, n_{\mathrm{t}}}(\mathbb{R})$ and $\boldsymbol{W} \in M_{n_{\mathrm{c}}}(\mathbb{R})$.

Comparing the systems (9) and (7) we have the solutions $\boldsymbol{h}$ and $\boldsymbol{z}$ given by

$$\boldsymbol{h} = -(\boldsymbol{M}\boldsymbol{H} + \boldsymbol{W})^{-1} \boldsymbol{M}\boldsymbol{G}\boldsymbol{g} \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{H}\boldsymbol{h} + \boldsymbol{G}\boldsymbol{g},$$

that are the nodal vector corresponding to the shape of the surface z on Γ^{c} (i.e., the shape of the cable) and, the nodal vector corresponding to the shape of the surface z all over the domain D (i.e., the shape of the total membrane).

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5 Conclusion and future work

In this paper we have analyzed the membrane equilibrium equations, for the prestressing phase. The mathematical formulation has been written down and the corresponding problem leads one to consider an elliptic problem with a singular boundary condition for the unknown (the shape of the membrane). More exactly, it is possible to prove the uniqueness of the solution but it is not known if the problem has or not always a solution; moreover the numerical method herein proposed is generally not stable (see [4]).

Finally, the authors are working on another resolution strategy for the same problem based on a *fix point* procedure, being the final goal the proof that the corresponding boundary equilibrium problem is well posed, in terms of existence, uniqueness and continuous dependence on data.

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