Environmental sustainability, nonlinear dynamics and chaos reloaded: 0 matters!

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Abstract

In this paper, we reconsider the overlapping generations (OLG) environmental model introduced in John and Pecchenino (1994) and Zhang (1999) by adopting the specification of the environmental dynamics proposed by Naimzada and Sodini (2010). The model two different regimes that may alternate: one in which the economy and the environment co-evolve in the same direction; the other in which the environmental problem is not internalized by the agents, that is, the agents do not devolve any private resource to the environmental good, leading to a possible trade-off between environment and economic growth. The analysis of the equilibrium dynamics, described by a two dimensional piecewise smooth map, shows that starting from a parametric configuration in which the dynamics are definitely driven by a unique regime, the increase of the negative effect of the agents' consumption activity ends up in scenarios where the two regimes alternate, determining the arise of stable cycles or the occurrence of chaotic regimes. It is interesting to notice that because of the nonsmoothness of the map, the rise in environmental harm produced by economic activity may induce a sudden transition to chaotic regime.

Keywords: Nonlinear Dynamics, Bifurcations, Complex Dynamics, Overlapping Generations, Sustainability.

JEL Codes: O13, O41, C62.

1 Introduction

In recent years, the literature focused on dynamic economic models has increasingly recognized the importance of (exogenously or endogenously determined) constraints for investigating the evolution of economic variables. From a mathematical point of view, in discrete time, this implies the study of the so-called piecewise smooth dynamic systems whose state space can be divided into regions where the specification of the map changes (see Mosekilde and Zhusubaliyev, 2003; Di Bernardo et al., 2008).

Several examples in this line of research can be found in the study of oligopoly dynamics. Among others, we recall Tramontana et al. (2010) where the authors, reconsidering the duopoly model in Puu (1991), introduce constraints to avoid negative replies by the rivals and focus on the role of possible lower bounds of the production level showing how such bounds can cause very interesting dynamic phenomena, different with respect to the original model of Puu (1991). Also Bischi and Lamantia (2012) and Bischi et. al (2012) study oligopoly models whose dynamics are characterized by a piecewise smooth dynamic system. The authors focus on the occurrence

of particular dynamic phenomena such as border collision and homoclinic bifurcations.
More recently, Sushko et al. (2016) reconsider the works of Matsuyama (2013) and Matsuyama et al. (2016) to study a growth model where, in an overlapping generations framework, firms arrive sequentially and, upon arrival, sell their endowment of inputs to acquire a certain net worth successively used to finance two types of investment projects, *good* and *bad* (where these have a different effect on the net worths of subsequent generations and thus on future levels of capital stocks). The analysis of the one-dimensional nonlinear and homogeneous piecewise map, generated by the two different types of investments, allows to observe the arise of border collision bifurcations and the direct transition towards a chaotic regime from an attractive equilibrium (or cycle), when exogenous parameters of the model are changed. Still in a overlapping generations context, Agliari and Vachadze (2011) study a model with forward looking young agents[1] who make endogenous labor supply decisions on the basis of two possible plans for the subsequent generation: run a project (and thus access an imperfect credit market) or become a depositor. These two possible choices determine a piecewise (differentiable) map. Concerning the dynamic analysis, the authors show several phenomena, as the existence of a heteroclinic connection or the occurrence of a homoclinic bifurcation that may be associated with global indeterminacy, where given the initial condition of the state variable, the system may converge to different invariant sets according to the initial value of the control variable.
In the present paper, in order to investigate the evolution of the economic-environmental variables we consider a variant of the model presented in Zhang (1999) where, differently from the author, the equilibrium dynamics are described by a piecewise map, whose definition is driven by the allocative choice made by the economic agents.
More specifically regarding the original article, the author studies the equilibrium dynamics in an overlapping generations framework in which agents internalize the environmental problem and contribute (with their own private resources) to environmental maintenance. Indeed, in Zhang (1999), in contrast to John and Pecchenino (1994), some assumptions made by the author prevent having, as the optimal solution, that agents do not contribute to environmental defense. The current article aims to investigate a slightly different specification of the environmental dynamics, as introduced in Naimzada and Sodini (2010) and Caravaggio and Sodini (2022), that allows to have two different regimes that may alternate: one in which the economy and the environment co-evolve in the same direction; the other in which the environmental problem is not internalized by the agents and where, therefore, they do not devolve any private resource in favor of the environmental good, leading to a possible trade-off between environment and economic growth. Specifically, the study of the local and global properties of the piecewise defined map allows to notice that, either starting from a parametric configuration in which the dynamics definitely evolve in the regime with positive contribution, or starting from a parametric configuration in which the agents definitely do not invest for environmental maintenance, the increase of the negative magnitude of the agents' consumption activity ends up generating dynamics in which the two regimes alternate. From a mathematical point of view, the onset of border collision bifurcations, able of generating sudden transitions (as the reference parameter varies) from a stable cycle to a chaotic attractor, can be observed. The remainder of the paper is organized as follows: Section 2 describes the structure of the model; Section 3 provides the local analysis of the piecewise defined dynamic system; Section 4 shows, through numerical exercises, the global dynamic properties of the model; Section 5 concludes.

[1]In this context, forward looking agents are defined as economic agents who do not consider the variables to be estimated as setted at their past value, but they *perfectly foresee* them.

2 The model

We consider a modification of the Overlapping Generations (OLG, hereafter) model proposed in John and Pecchenino (1994) and Zhang (1999). Specifically, we assume that each representative individual born at time t has preferences defined over consumption and environmental quality at $t+1$ (old age), C_{t+1} and E_{t+1} respectively. In particular, such preferences are described by the utility function

$$U(c_{t+1}, E_{t+1}) = \ln c_{t+1} + \eta \ln E_{t+1} \tag{1}$$

where η measures the relative weight individuals give to environmental quality rather than consumption. During the youth, the agent supplies inelastically his time endowment (which is normalized to 1) to the productive sector receiving a wage w_t that he will divide between saving, s_t, for consumption when old, and investment in environmental maintenance, m_t, for improving the environmental quality at $t+1$.

Differently by John and Pecchenino (1994) and Zhang (1999), we assume that the environmental quality index evolves according to

$$E_{t+1} = (1-b)E_t + b\overline{E} - \beta\, c_t + \gamma\, m_t; \tag{2}$$

where the parameter $\overline{E} > 0$ represents the value toward the index tends when consumption and environmental expenditures are null, $b \in (0,1)$ measures the speed of reversion of the environmental quality to $\overline{E}$ and $E_t > 0$ is the current level of the index. As in Zhang (1999), the term $\beta\, c_t$ is the consumption degradation of the environment by the old, while $\gamma\, m_t$ measures environmental improvement.

Agent supplies his saving s_t to firms and earns the gross return $(1 + r_{t+1} - \delta)$ where r_{t+1} is the real interest rate and $\delta \in (0,1)$ is the depreciation rate of capital. Then, the individual faces the following life-cycle budget constraints:

$$c_{t+1} = (1 + r_{t+1} - \delta)s_t; \tag{3}$$

$$w_t = s_t + m_t; \tag{4}$$

$$c_{t+1}, m_t, s_t \geq 0. \tag{5}$$

and the expression in (2).

On the production side, a consumption good is produced by N competitive firms where N is normalized to 1. Then, the output y at time t is produced according to the following Cobb-Douglas technology

$$y_t = h(k_t) = Ak_t^{\alpha} \tag{6}$$

where A is a positive parameter measuring the technological progress, assumed as given, and $\alpha \in (0,1)$ represents the elasticity of substitution of capital. Therefore, at the equilibrium we have the following prices:

$$w_t = (1-\alpha)Ak_t^{\alpha}, \tag{7}$$
$$r_t = \alpha Ak_t^{\alpha-1}. \tag{8}$$

We can notice that the specification in (2) differs from John and Pecchenino (1994)'s one by the introduction of the extra term $b\overline{E}$. Before going any further, we point out some properties of the model in Zhang (1999) where $\overline{E} = 0$. Compared with the model in John and Pecchenino (1994), the Zhang's model makes a strong simplification. Indeed, it assumes the marginal rate of substitution between c and E as constant, that is

$$\frac{U'_E(c,E)E}{U'_c(c,E)c} = \eta. \tag{9}$$

From one hand it can be proved that this assumption avoids the possibility of having solutions with $m_t = 0$. From the other hand, depending on the parameter set considered, the assumption in (9) generates several problems regarding the well-defined trajectories. Specifically, by solving the partial differential equation

$$U'_E(c,E)E - \eta U'_c(c,E)c = 0 \tag{10}$$

equivalent to (9), we can note that the unique functions having this property are functions depending on the quantity $yx^{\frac{1}{\eta}}$.

At this point, considering the economic assumption of monotonicity of U with respect to its arguments, this last fact prevents having utility functions defined for both positive and negative values of the environmental index. Thus, parameter configurations and initial conditions for which the environmental index takes negative values make it impossible to define successive iterations of the map. As we will show, the introduction of an appropriate term $\overline{E}$ actually leads to results more in line with what was suggested, at least as a possibility, in John and Pecchenino's original article (see their footnote 21). In particular, we will show cases in which, or in transient or definitively the case $m_t = 0$ occurs.

2.1 Well-defined dynamics

From the budget constraint (3) and the market clearing condition $k_{t+1} = s_t$, we have that

$$c_t = (1-\delta)k_t + A\alpha k_t^\alpha. \tag{11}$$

If all resources were devoted to saving and thus to consumption by the old, capital accumulation is governed by the equation

$$k_{t+1} = A(1-\alpha)k_t^\alpha. \tag{12}$$

Therefore, because the right hand side of the expression (11) is monotonically increasing with respect k_t, called $\overline{k} = [A(1-\alpha)]^{\frac{1}{1-\alpha}}$ the unique positive stationary point of (12), we have that for any initial condition $k_0 \in [0, \overline{k}]$, k assumes values lower that $\overline{k}$ for every t. Then, $c_t \leq c^{\max} = (1-\delta)\overline{k} + A\alpha\overline{k}^\alpha$. Therefore, from the inequality $E_{t+1} \geq (1-b)E_t + b\overline{E} - \beta\, c^{\max}$, we get the necessary and sufficient condition

$$\overline{E} > \frac{\beta\, c^{\max}}{b} \tag{13}$$

such that every trajectory starting from the set of initial conditions[2]

$$(k_0, E_0) \in D = \left\{(k,E) \in \Re^2 : k \in [0, \overline{k}],\ E > 0\right\} \tag{14}$$

[2] We note that the assumption of restricting the set of initial conditions to pairs (k_0, E_0) with $k_0 \leq \overline{k}$ does not represent a strong limitation in the analysis of the model, as it imposes the condition that k_0 is no greater than the maximum level of accumulation allowed for the economy, given the technology.

is well defined, that is it lies in D, for every t.

2.2 Consumer's choices

The agent's problem is maximizing (1) with respect to c_{t+1}, s_t m_t, given the constraints in (3)-(4)-(5). By substituting in the utility function the expressions of c_{t+1} and s_t in terms of m_t as can be obtained by (3)-(4), the optimization problem can be rewritten as

$$\max_{m_t \in [0,\, w_t)} \ln(w_t - m_t) + \eta \ln\left((1-b)E_t + b\overline{E} - \beta\, c_t + \gamma\, m_t\right). \tag{15}$$

By considering the concavity property of the objective function with respect to m_t, we obtain that the optimal solution for m_t is given by

$$m_t^* = \begin{cases} \frac{\eta\gamma w_t + \beta c_t - (\overline{E}b + E_t(1-b))}{\gamma(\eta+1)} & \text{if } E_t(1-b) + b\overline{E} - \beta c_t < \eta\gamma w_t \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

The optimal values of s_t and c_{t+1} are obtained by direct substitution of (16) into (3)-(4). Therefore, the economic agents young in t will contribute to the environmental good if and only if the environmental quality they will experience in absence of environmental expenditures, $E_t(1-b) + b\overline{E} - \beta c_t$, is too low with respect to their financial resources, w_t, weighted by environmental awareness η and the efficacy of environmental expenditures, γ.
Since we have

$$\frac{U'_E(c, E)E}{U'_c(c, E)c} = \eta,$$

we can notice that if the interior solution ($m_t^* > 0$) solves (15) at t, then the relationship

$$E_{t+1} = \eta\gamma k_{t+1} \tag{17}$$

holds at $t+1$.

3 Dynamics

By recalling the optimal solution in (16), the constraints in (3)-(4), the prices defined in (7)-(8), the law of motion for the environment in (2) and the market clearing condition $k_{t+1} = s_t$, we have that, given the function

$$Q(k, E) = A(\eta(1-\alpha)\gamma + \beta\alpha)k^\alpha + k(1-\delta)\beta - (1-b)E - b\overline{E}$$

the equilibrium dynamics of the model are defined by the two-dimensional piecewise-smooth map M which reads as

$$M_{zm} : \begin{cases} k_{t+1} = w_t = A(1-\alpha)k_t^\alpha \\ E_{t+1} = (1-b)E_t + b\overline{E} - \beta(1 + \alpha\, A k_t^{\alpha-1} - \delta)k_t, \end{cases} \tag{18}$$

if

$$Q(k_t, E_t) \leq 0 \tag{19}$$

and

$$M_{pm}:\begin{cases} k_{t+1} = \frac{1}{\gamma(1+\eta)}\left[(1-b)E_t + b\overline{E} + ((\gamma(1-\alpha)-\beta\alpha)Ak_t^{\alpha} - k_t(1-\delta)\beta\right] \\ E_{t+1} = \frac{\eta}{1+\eta}\left[(1-b)E_t + b\overline{E} + ((\gamma(1-\alpha)-\beta\alpha)Ak_t^{\alpha} - k_t(1-\delta)\beta\right] \end{cases} \tag{20}$$

if

$$Q(k_t, E_t) > 0. \tag{21}$$

The curve $Q(k_t, E_t) = 0$, obtained by considering the inequality in (16), divides the set D in the two regions in which $m_t^* = 0$ and $m_t^* > 0$, respectively. By solving $Q(k_t, E_t) = 0$ in terms of E_t, we have that such a curve can be defined as follows:

$$E_t = v(k_t) := \frac{A(\eta(1-\alpha)\gamma + \beta\alpha)k_t^{\alpha} + (1-\delta)\beta k_t - b\overline{E}}{1-b}. \tag{22}$$

By straightforward calculations, we deduce that $v' > 0$ and $v'' < 0$ $\forall k > 0$ (see Figure 1).

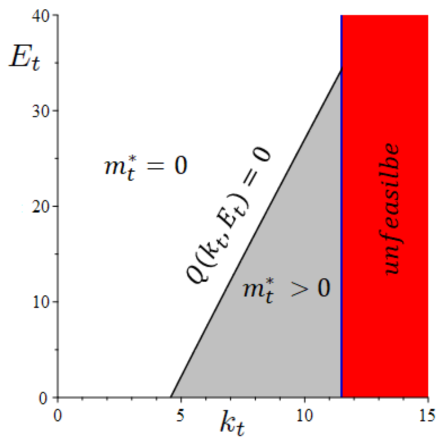


Figure 1: Parameter set: $\alpha = 0.1$, $\beta = 2.21$, $\gamma = 0.9$, $\delta = 0.03$, $\eta = 0.75$, $A = 10$, $\overline{E} = 36$, $b = 0.54$. The set D partitioned by the curve $Q(k_t, E_t) = 0$ (depicted in black) in the region in which $m_t^* = 0$, (the white part in the positive orthant $\Re^2_{++}$, included the boundary in black), and the one in which $m_t^* > 0$ (in gray included the boundary in blue). The red region describes the complement of D on $\Re^2_{++}$, that is the set of points with a value of $k > \overline{k}$.

We notice that in the expression (18) wage is entirely saved and the environmental quality experiences the negative effects produced by consumption, without any defensive action. In this case, in order to contrast the low quality of environment, agents react by increasing the consumption level, inducing a further decrease of environmental quality. This stage of growth resembles the characteristics of interaction between environment and economic activity described in Antoci et al. (2016) where the concept of maladaptation to pollution is introduced. In other terms, because the non coordination among the generations, a non optimal allocation of the resources happens, producing the so called tragedy of commons, that is a scenario where even if the Pareto optimal allocation can require a positive amount of resources for environmental quality, no one positively invest in environmental maintenance. In this last case, the dynamics of the system are determined by the two-dimensional *triangular* dynamical system M_{zm}. By imposing stationarity conditions for M_{zm} we get the following unique solution

$$k = k^*_{zm} := [A(1-\alpha)]^{1/(1-\alpha)} \qquad E = E^*_{zm} := \overline{E} - \frac{\beta(1+\frac{\alpha}{1-\alpha}-\delta)[A(1-\alpha)]^{1/(1-\alpha)}}{b}. \tag{23}$$

where $E^*_{zm} > 0$ because of the assumption in (13). In terms of existence and stability of the fixed point, the following Proposition holds:

Proposition 1 *The map M_{zm} admits a fixed point if and only if $Q(k^*_{zm}, E^*_{zm}) \leq 0$. In this case the fixed point is unique and given by (k^*_{zm}, E^*_{zm}). Moreover, whenever the fixed point exists, it is always locally asymptotically stable.*

Proof. From the previous analysis we have that the only possible fixed point for the dynamical system in (18) is given by the expression (k^*_{zm}, E^*_{zm}) in (23). This is a feasible fixed point if and only if its coordinates satisfy the inequality in (19). By evaluating the Jacobian matrix

$$J(k_t, E_t) = \begin{bmatrix} \alpha A(1-\alpha)k_t^{\alpha-1} & 0 \\ -\beta(1+\alpha^2 A k_t^{\alpha-1} - \delta) & 1-b \end{bmatrix},$$

at the fixed point (k^*_{zm}, E^*_{zm}) we get that the two eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = 1-b$, with $0 < \lambda_1, \lambda_2 < 1$. Then, the result follows. □

When this regime applies, because of the simple specification of the map, given the initial conditions we can obtain the solution of the system of difference equations. Indeed, given the map (18) and an initial condition (k_0, E_0) satisfying $Q(k_0, E_0) \leq 0$, the dynamics of k and E are defined by the expressions

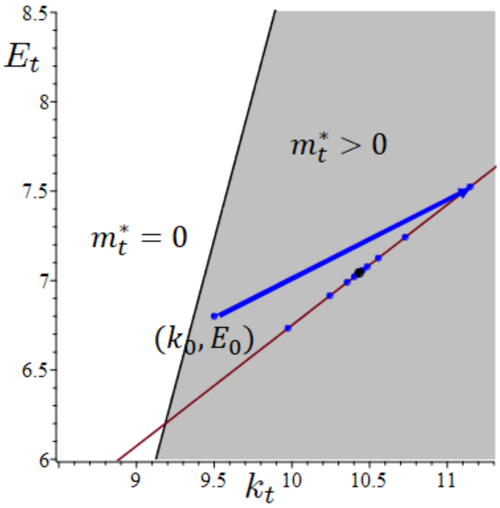
$$\begin{aligned} k_t &= [A(1-\alpha)]^{\frac{\alpha^t-1}{\alpha-1}} k_0^{\alpha^t} \\ E_t &= \beta \sum_{l=1}^{t} \left\{ (1-b)^{t-l} (A(1-\alpha))^{\frac{\alpha^l-\alpha}{(\alpha-1)\alpha}} k_0^{\alpha^{l-1}} \left[\left(A(1-\alpha))^{\frac{\alpha^l-\alpha}{(\alpha-1)\alpha}} k_0^{\alpha^{l-1}} \right)^{\alpha} + 1 - \delta \right] \right\} + \\ &\quad + (E_0 - \overline{E})(1-b)^t + \overline{E}. \end{aligned}$$

When economic agents contribute for the environmental maintenance, the dynamics of the system are determined by the two-dimensional map M_{pm}. Notice that within this regime even if the agents of a specific generation start from a couple of states (k, E) outside the (17) because the optimization process, the subsequent iterate will lie on (17), as highlighted in Figure 2.a.

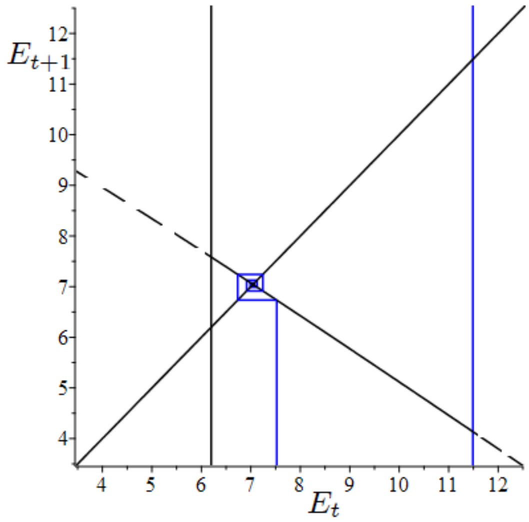
Remark 1 *If all the iterates satisfy the condition (21), then the dynamics of (20) can be analyzed by studying the dynamics generated by the unidimensional map*

$$\widetilde{M}_{pm} : E_{t+1} = f(E_t) = \frac{1}{\gamma(1+\eta)} \left[(\gamma(1-\alpha) - \alpha\beta)\gamma\eta\, A \left(\frac{E_t}{\gamma\eta} \right)^{\alpha} + ((1-b)E_t + b\overline{E})\gamma\eta - \beta(1-\delta)E_t \right]. \tag{24}$$

The property in Remark 1 is exemplified in Figure 2.b. In light of this Remark, the following results can be easily obtained:



(a)



(b)

Figure 2: Parameter set: $\alpha = 0.1$, $\beta = 1.488$, $\gamma = 0.9$, $\delta = 0.03$, $\eta = 0.75$, $A = 10$, $\overline{E} = 36$, $b = 0.54$. (a) Although the agents start from a couple of states $(k_0, E_0) = (9.5, 6.8)$ outside the line defined by (17), depicted in red, the subsequent iterates (see the arrow that highlights the jump on the red line), represented by the blue points, lie on the red line and the sequence converge to the fixed point $(k^*_{pm}, E^*_{pm}) \simeq (10.43, 7.04)$ (the black point). (b) Cobweb diagram representing the evolution of the environmental dynamics for $t \geq 1$, in the plane (E_t, E_{t+1}). The black vertical line defines the boundary of the regime with $m^*_t > 0$ obtained by considering (22); the blue vertical line identifies the boundary for E such that, starting from E_t, the next iterate satisfies the condition $k_t < \overline{k}$, given the relationship between k and E in (22).

Proposition 2 *Let assume $\beta < \frac{\gamma(1-\alpha)}{\alpha}$. The map M_{pm} admits a unique fixed point if and only if $Q\left(\frac{E^*_{pm}}{\gamma\eta}, E^*_{pm}\right) > 0$, where E^*_{pm} is the unique fixed point of $\widetilde{M}_{pm}$.*

Proof. By Remark 1, the existence of fixed points for M_{pm} can be performed by studying the map $\widetilde{M}_{pm}$. Because we have $f(0) = \frac{b\eta\overline{E}}{1+\eta} > 0$ and the concavity of $f(E_t)$ ensured by the assumption $\beta < \frac{\gamma(1-\alpha)}{\alpha}$, the graph of the map $\widetilde{M}_{pm}$ crosses the 45-degree line only one time and then the fixed point, denoted by E^*_{pm}, is unique. From (17) it follows that the k-coordinate of the fixed point for M_{pm} is $k^*_{pm} = \frac{E^*_{pm}}{\gamma\eta}$. Then, the condition $Q\left(\frac{E^*_{pm}}{\gamma\eta}, E^*_{pm}\right) > 0$ guarantees the feasibility of the stationary solution defined by the equation $f(E) = E$. □

Although the expression of the fixed point for M_{pm} cannot be found analytically, the following Proposition classifies its stability:

Proposition 3 *Let M_{pm} be the map defined in (20).*
(a) If $\gamma\frac{1+\eta(2-b)}{1-\delta} < \beta < \min\left\{\gamma\frac{\eta[2-b(1-\alpha)]+1+\alpha}{(1-\delta)(1-\alpha)}, \frac{\gamma(1-\alpha)}{\alpha}\right\}$, then there exists a threshold value $\overline{E}_{th}$ such that for $\overline{E} < \overline{E}_{th}$ $\left(k^*_{pm}, E^*_{pm}\right)$ is locally asymptotically stable for M_{pm}, while for $\overline{E} > \overline{E}_{th}$, $\left(k^*_{pm}, E^*_{pm}\right)$ does not exist or is unstable.
(b) If $\gamma\frac{\eta[2-b(1-\alpha)]+1+\alpha}{(1-\delta)(1-\alpha)} < \beta < \frac{\gamma(1-\alpha)}{\alpha}$, and $Q\left(k^*_{pm}, E^*_{pm}\right) > 0$ then $\left(k^*_{pm}, E^*_{pm}\right)$ is unstable.

Proof. By Remark 1, the stability of the fixed point for M_{pm} can be performed by studying the map $\widetilde{M}_{pm}$. By evaluating the first derivative of f at E^*_{pm}, we get

$$f'(E^*_{pm}) = \frac{(\gamma(1-\alpha) - \alpha\beta)\alpha A \left(\frac{E^*_{pm}}{\gamma\eta}\right)^{\alpha-1} + \eta(1-b)\gamma - (1-\delta)\beta}{\gamma(\eta+1)}.$$

It is straightforward to prove that, under the assumption $\beta < \frac{\gamma(1-\alpha)}{\alpha}$, there is a positive monotonic relationship between $\overline{E}$ and E^*_{pm} and, as $\overline{E}$ tends to $+\infty$, E^*_{pm} tends to $+\infty$. Furthermore, because $\beta < \frac{\gamma(1-\alpha)}{\alpha}$, $f'(E)$ is monotonically decreasing with respect to E and then f is concave. (a) For the assumption on β, we get $f'(E^*_{pm})|_{\overline{E}=0} \in (-1,1)$ and $\lim_{\overline{E}\to+\infty} f'(E^*_{pm}) < -1$. This implies that there exists a threshold value $\overline{E}_{th}$ such that $f'(E^*_{pm})|_{\overline{E}=\overline{E}_{th}} = -1$. By taking into account that $\left(k^*_{pm}, E^*_{pm}\right)|_{\overline{E}=\overline{E}_{th}}$ may be feasible or not, depending on the sign of Q, the result follows. (b) In this case, for $\overline{E} = 0$ we have that $f'(E^*_{pm}) < -1$ and due to the monotonic behavior of $f'(E^*_{pm})$, the result trivially follows. □

Proposition 4 *If $\beta > \frac{\gamma(1-\alpha)}{\alpha}$, then the function f is always decreasing and a fixed point exists. If $f'(E^*_{pm}) > -1$ then the fixed point E^*_{pm} is the Ω-limit set for the dynamics starting on the diagonal with an initial condition (k_0, E_0) such that $Q(k_0, E_0) \leq 0$. For sufficiently high value of $\overline{E}$, the dynamics are captured by a 2-cycle or they exit the regime.*

The previous results clarify the dynamic properties of the map M_{pm} from initial conditions (k_0, E_0) close to fixed point (k^*_{pm}, E^*_{pm}). We also note that from (13), in order to have well defined dynamics in the set D, the condition $\beta < \frac{b\overline{E}}{c^{max}}$ has to hold, too. Regarding the map M, we can observe that the fixed points (k^*_{zm}, E^*_{zm}) and $\left(k^*_{pm}, E^*_{pm}\right)$ are mutually exclusive. Indeed, the following result holds:

Proposition 5 *The fixed point (k^*_{zm}, E^*_{zm}) exists if and only if $Q\left(k^*_{pm}, E^*_{pm}\right) \leq 0$.*

Proof. Suppose by contradiction that (k^*_{pm}, E^*_{pm}) exists. Therefore, we have $Q\left(k^*_{pm}, E^*_{pm}\right) > 0$. A stationary point with $m^* = 0$ would identify a point (k', E') with $k' > k^*_{pm}$ and $E' < E^*_{pm}$, because a lower level in m^* induces both an higher level of k and a lower level of E. This represents a contradiction, because the point (k', E') would belong to the region characterized by $Q(k', E') > 0$ and then $m^* > 0$.
Suppose now by contradiction that (k^*_{zm}, E^*_{zm}) exists. Therefore, we have $Q(k^*_{zm}, E^*_{zm}) \leq 0$. A stationary point with $m^* > 0$ would identify a point (k'', E'') with $k'' < k^*_{zm}$ and $E'' > E^*_{zm}$, because a higher level in m^* induces both a lower level of k and a higher level of E. This represents a contradiction, because the point (k'', E'') would belong to the region characterized by $Q(k'', E'') \leq 0$ and then $m^* = 0$. □

We note that, in general, the regions defined by the curve (22) may be non-invariant, as shown in Figure 3. In this case even if there is an area where $Q(k, E) > 0$, then no fixed point exists for this regime. Figure 3.a and the enlargement in Figure 3.b show a trajectory generated by an initial condition that verifies $Q(k_0, E_0) > 0$. The dynamics synchronize on the half-line (17) for the first three iterates, the last of which is, however, outside the region $Q(k, E) > 0$. From the successive the dynamics develop in the region $Q(k_0, E_0) \leq 0$, converging to equilibrium

(k_{zm}, E_{zm}). Figure 3.c highlights that also the region where $Q(k, E) \leq 0$ is non-invariant. Indeed, even if starting from an initial condition (k_0, E_0) that satisfies $Q(k_0, E_0) \leq 0$ and converging to the fixed point (k^*_{zm}, E^*_{zm}), the trajectory enters, at least for one iterate, the region with $Q(k_t, E_t) > 0$ before exiting again. We note that the cases in Figure 3.a and Figure 3.c were obtained with the condition $\gamma(1-\alpha) - \beta\alpha > 0$. Figure 3.d shows that the non-invariancy of $Q(k, E) > 0$ can also occur with $\gamma(1-\alpha) - \beta\alpha < 0$.

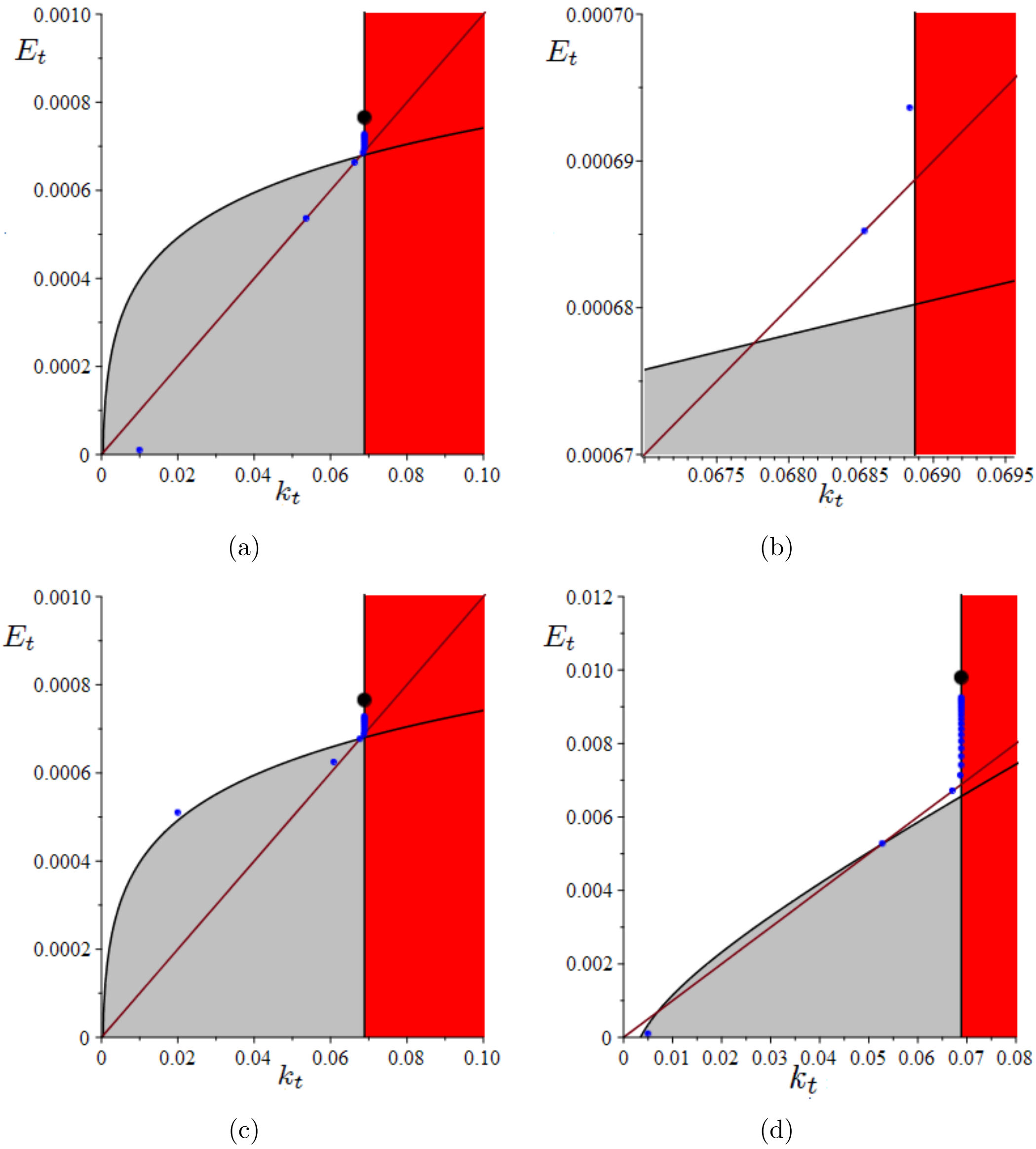
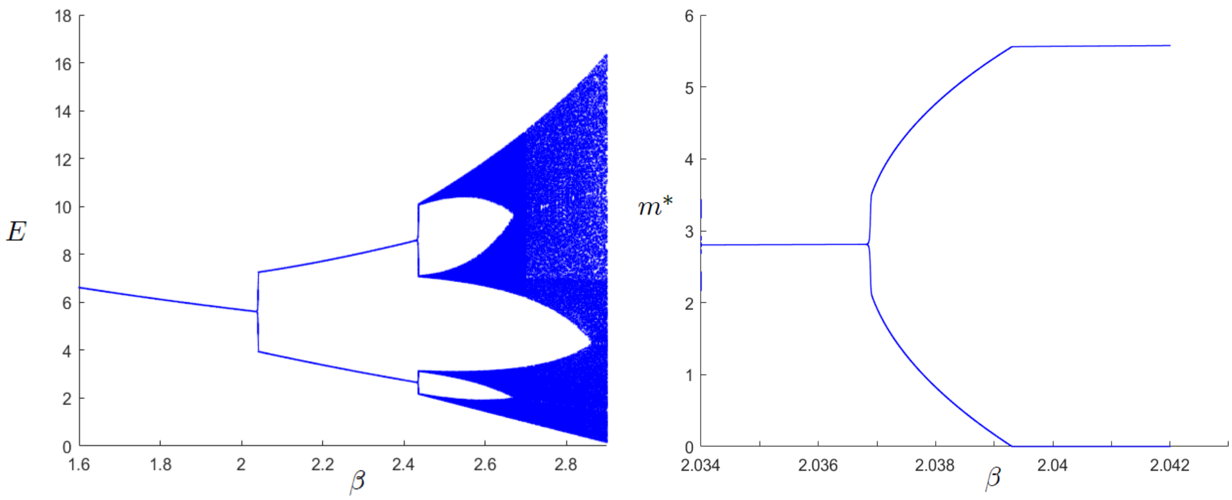


Figure 3: Panels (a) and (b) show the non-invariancy of the set $Q(k, E) > 0$ when $\gamma(1-\alpha) - \beta\alpha > 0$, starting from the initial condition $(k_0, E_0) = (0.01, 0.00001)$. Panel (c) displays the non-invariancy of the set $Q(k, E) \leq 0$ when $\gamma(1-\alpha) - \beta\alpha > 0$, starting from the initial condition $(k_0, E_0) = (0.02, 0.00051)$. The parameter set applied is $\alpha = \beta = \gamma = \eta = A = b = 0.1,\ \delta = 1,\ \overline{E} = 0.0084$. Panel (d) shows the non-invariancy of the set $Q(k, E) > 0$ when $\gamma(1-\alpha) - \beta\alpha < 0$, starting from the initial condition $(k_0, E_0) = (0.005, 0.0001)$. The parameter set applied is $\alpha = \beta = \eta = A = b = 0.1,\ \gamma = 1,\ \delta = 0.4,\ \overline{E} = 0.0588$.

The next section will be devoted to the study of more complex configurations where infinite switches between the two regimes occur.

4 Global analysis

In the previous section, we dealt with the definition of the piecewise map M and the local analysis of the two possible regimes that may arise, M_{pm} and M_{zm}, depending on the value assumed by m_t (null or positive). In this section we will explore, by means of numerical exercises, the global dynamics of the map M. Specifically, focusing on the role of β and $\overline{E}$, we will show that the dynamics (i) may exhibit an involvement (in the long run) of both regimes, and how (ii) the emergence of *peculiar* dynamic phenomena unexplored in both John and Pecchenino (1994) and Zhang (1999) can be observed. One of the features of the map is that, since in the regime where $Q(k, E) > 0$ the dynamics are constrained on a half-line, even complex attractors that will occupy the region where $Q(k, E) < 0$ will be strongly affected by this shape. This means that for such a parametric configuration, regardless of the initial condition (k_0, E_0), the dynamics will be described from a certain t onwards by the map M_{pm}.
Consider the parameter set: $\alpha = 0.1,\ \gamma = 0.9,\ \delta = 0.03,\ \eta = 0.75,\ A = 10,\ \overline{E} = 36,\ b = 0.54$. The bifurcation diagram in Figure 4.a shows the phenomena that can be observed as β increases.



(a) (b)

Figure 4: (a) Bifurcation diagram of E as β increases. (b) Bifurcation diagram of m^* for β close to the first border collision bifurcation. From $\beta \simeq 2.0393$ one of the poles of the 2-cycle involves null environmental investments.

Indeed, as the value of β raises, we can notice that the fixed point loses its stability as a result of a flip bifurcation (which occurs for $\beta \simeq 2.0367$) and the system convergences to a 2-cycle whose poles are both within the regime M_{pm}. By considering larger values of β, we can observe some interesting phenomena. At $\beta \simeq 2.0393$ the dynamics of the system end up involving both regimes (persistence border collision bifurcation), as also highlighted by the bifurcation diagram of m^* in Figure 4.b. Therefore, the iterations are no longer governed by only the map M_{pm} (see Figure 5.a) and this means that for value of β higher than 2.0393, the 2-cycle has one pole inside the blue region (with $m_t^* > 0$) and the other outside it (the white region with $m_t^* = 0$), as shown by the phase plane (k, E) in Figure 5.b. This phenomenon reveals to be interesting also because it shows that, once both regimes are involved, the dynamics that at a certain t are outside the curve given by (17), coordinate on it at the next iterate, but outside the region with $m_t^* > 0$.

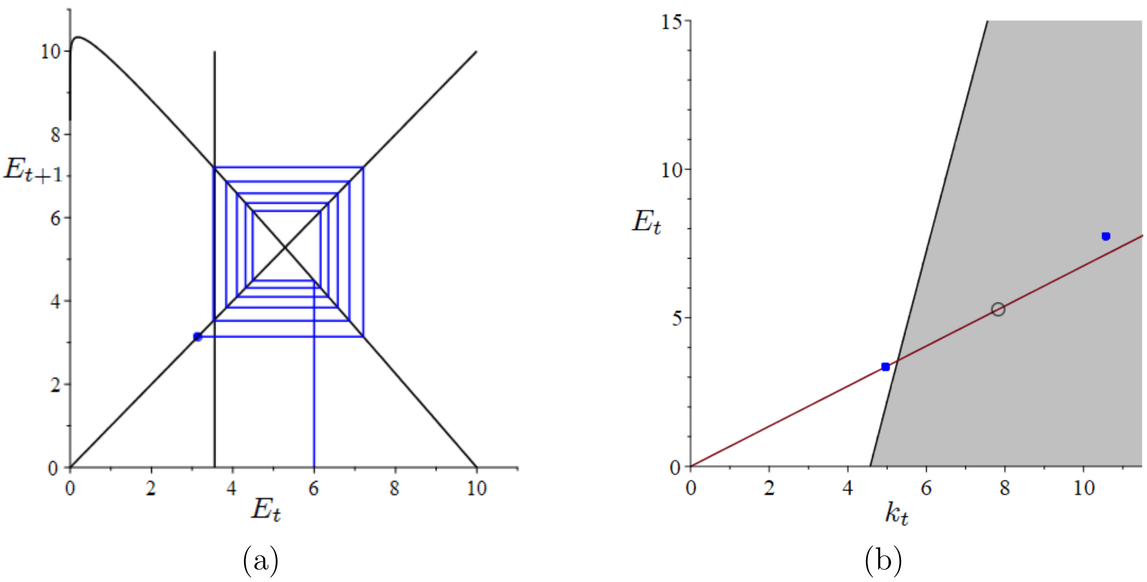


Figure 5: (a) The Cobweb diagram showing the iterates governed by the map M_{pm}, starting from the initial value $E_0 = 6.255$. The black vertical line represents the value of E from which the dynamics are governed by both M_{pm} and M_{zm}. (b) The phase plane depicted for $\beta = 2.21$. The poles of the 2-cycles lie on the two different regions of the plane, respectively. The empty circle represents the unstable fixed point (k^*_{pm}, E^*_{pm}).

From an economic point of view, we notice that a strong negative effect of consumption activity, in line with what has been shown in Zhang (1999), destabilizes the system and favors at first (i) the onset of a cycle in which, in an alternating manner, agents first contribute more (observing an insufficient level of E) and at the next time contribute, but with less intensity, then it ends up producing (ii) reactions in the contribution of agents such that there will be alternating periods in which m is very high in response to an insufficient level of E and periods in which the level of environmental quality is so high that agents are encouraged to decide not to contribute.
In the bifurcation diagram in Figure 4.a, one can notice a property of piecewise-smooth dynamical systems. Differently from what occurs in smooth systems, here the border collision bifurcation at $\beta \simeq 2.43$ causes a sudden transition from a stable 2-cycle to a chaotic attractor divided into 4 pieces (or intervals), involving both regimes (see Figure 6.a).

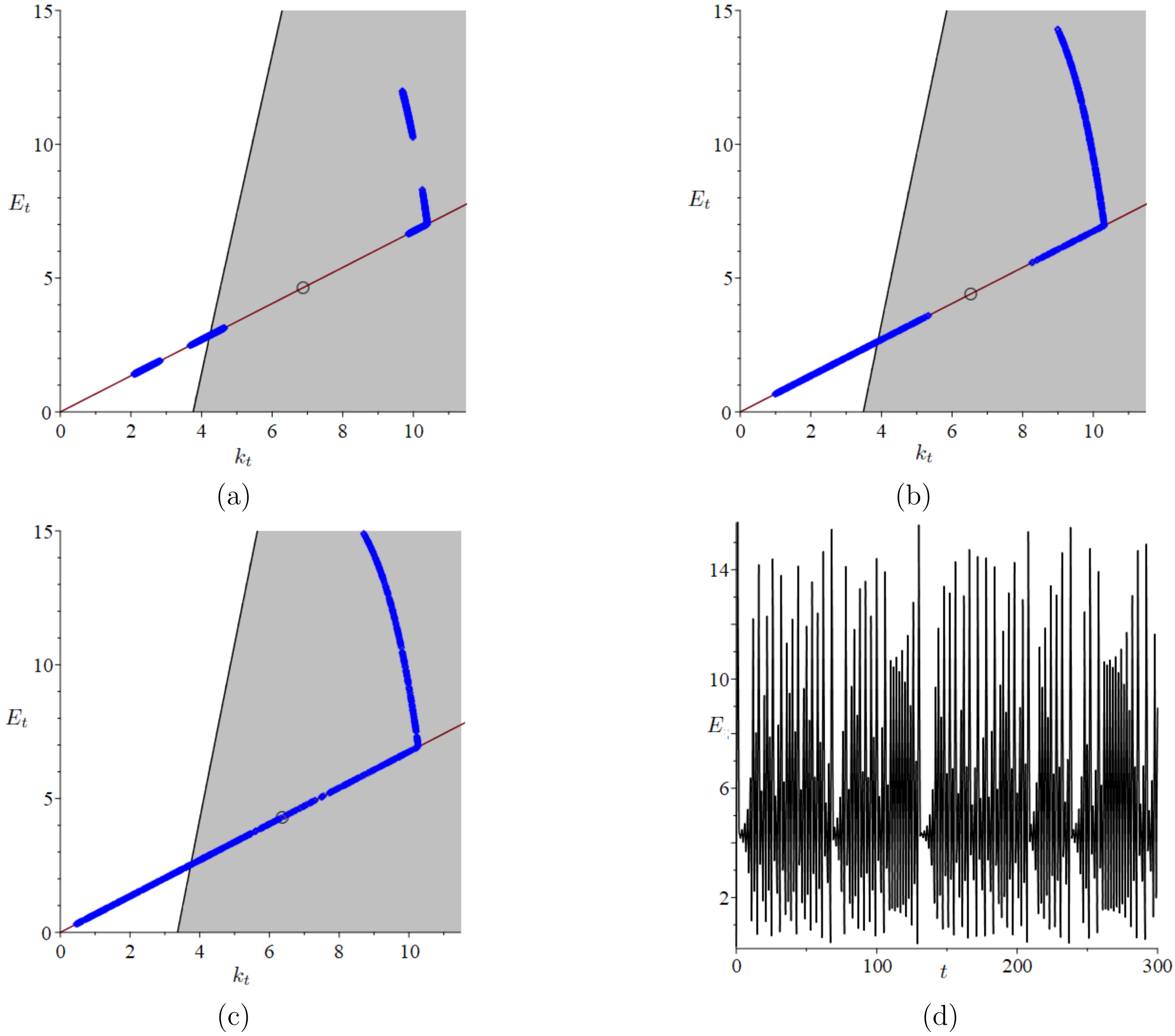
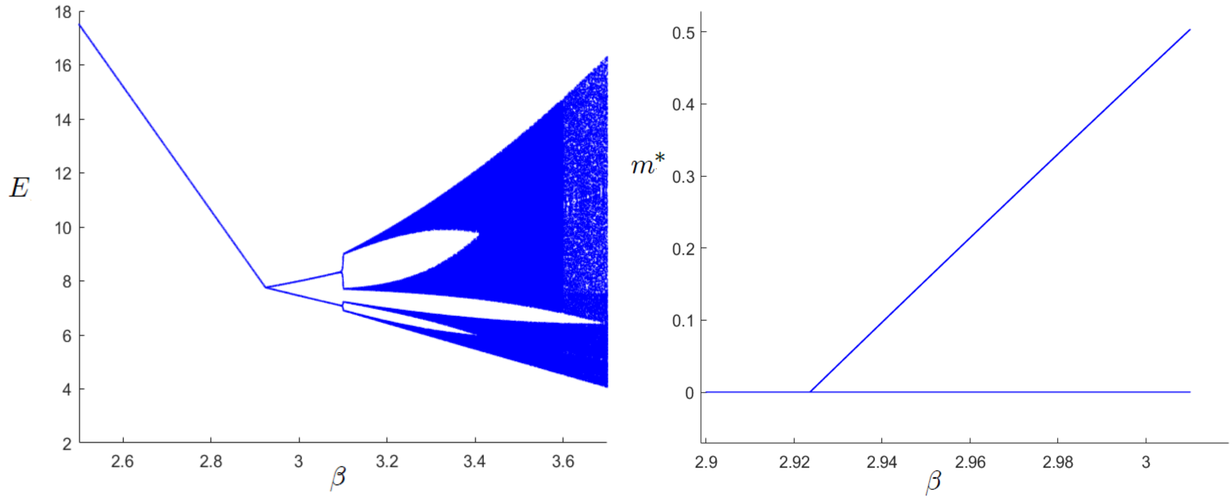


Figure 6: (a) 4-pieces chaotic attractor for $\beta = 2.61$. (b) 2-pieces chaotic attractor for $\beta = 2.78$. (c) The merged chaotic attractor for $\beta = 2.865$. The empty circle in the phase plane represents the unstable fixed point (k^*_{pm}, E^*_{pm}). (d) The chaotic oscillations of E for $\beta = 2.865$.

From a dynamical point of view, for such values of β, we have that alternating trajectories will visit pairs (k, E) inside and outside the blue region, but in different intervals of the chaotic attractor. Further increases of β suggest before the merging of the intervals first into two intervals of greater amplitude (see Figure 6.b), and finally into a single chaotic area (see Figure 6.c). According to the latter phase plane, the time series for E, displayed in Figure 6.d, exhibits chaotic oscillations.

Differently, consider the parameter set: $\alpha = 0.1,\ \gamma = 0.9,\ \delta = 0.03,\ \eta = 0.75,\ A = 10,\ \overline{E} = 75,\ b = 0.54$. We notice that, for a sufficient value of β, the fixed point (k^*_{zm}, E^*_{zm}) exists and it is stable. As β increases, we can observe several phenomena, summarized by the bifurcation diagram in Figure 7.a.

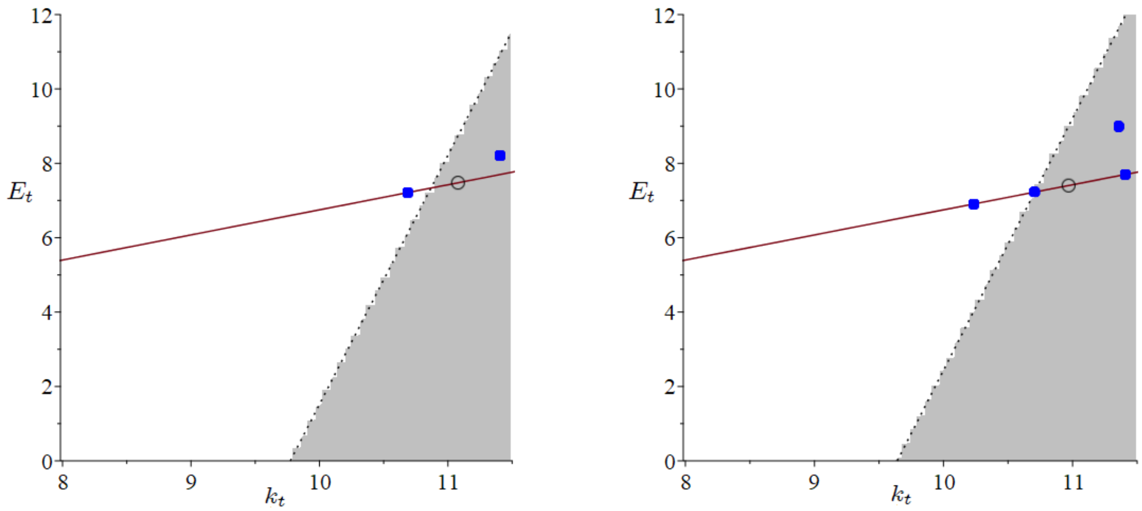
By increasing the value of β, the fixed point (k^*_{zm}, E^*_{zm}) no longer exists and there exists the



(a) (b)

Figure 7: (a) Bifurcation diagram of E as β increases. (b) An enlarged bifurcation diagram of m^* highlighting that from $\beta \simeq 2.927$ the dynamics are attracted by a stable 2-cycle involving both the regimes.

fixed point (k^*_{pm}, E^*_{pm}), which is unstable. Indeed, at $\beta = 2.927$, the system undergoes a border collision bifurcation and a stable 2-cycle, involving both the regimes, becomes the attractor (see also the bifurcation diagram of m^* in Figure 7.b). A further increase in β induces the raise of a attractive 4-cycle, as shown in Figure 8.



(a) (b)

Figure 8: (a) The phase plane (k, E) for $\beta = 3.0616$. The two poles of the 2-cycle lies in two different regions of the plane. The empty circle represents the unstable fixed point (k^*_{pm}, E^*_{pm}). (b) The phase plane (k, E) for $\beta = 3.10$. The poles of the 4-cycle lies in two different regions of the plane.

At $\beta \simeq 3.12$, the bifurcation diagram in Figure 7.a shows the emergence of a further border collision bifurcation that induces a sudden transition to a chaotic regime. As in the previous numerical example, the chaotic attractor is first composed of four intervals (see Figure 9.a), and for higher values of the parameter the intervals reduce to two (see Figure 9.b) and then to a single chaotic area (see Figure 9.c). The final chaotic regime in which dynamics are trapped for high values of β are also described by the time series for E in Figure 9.d.

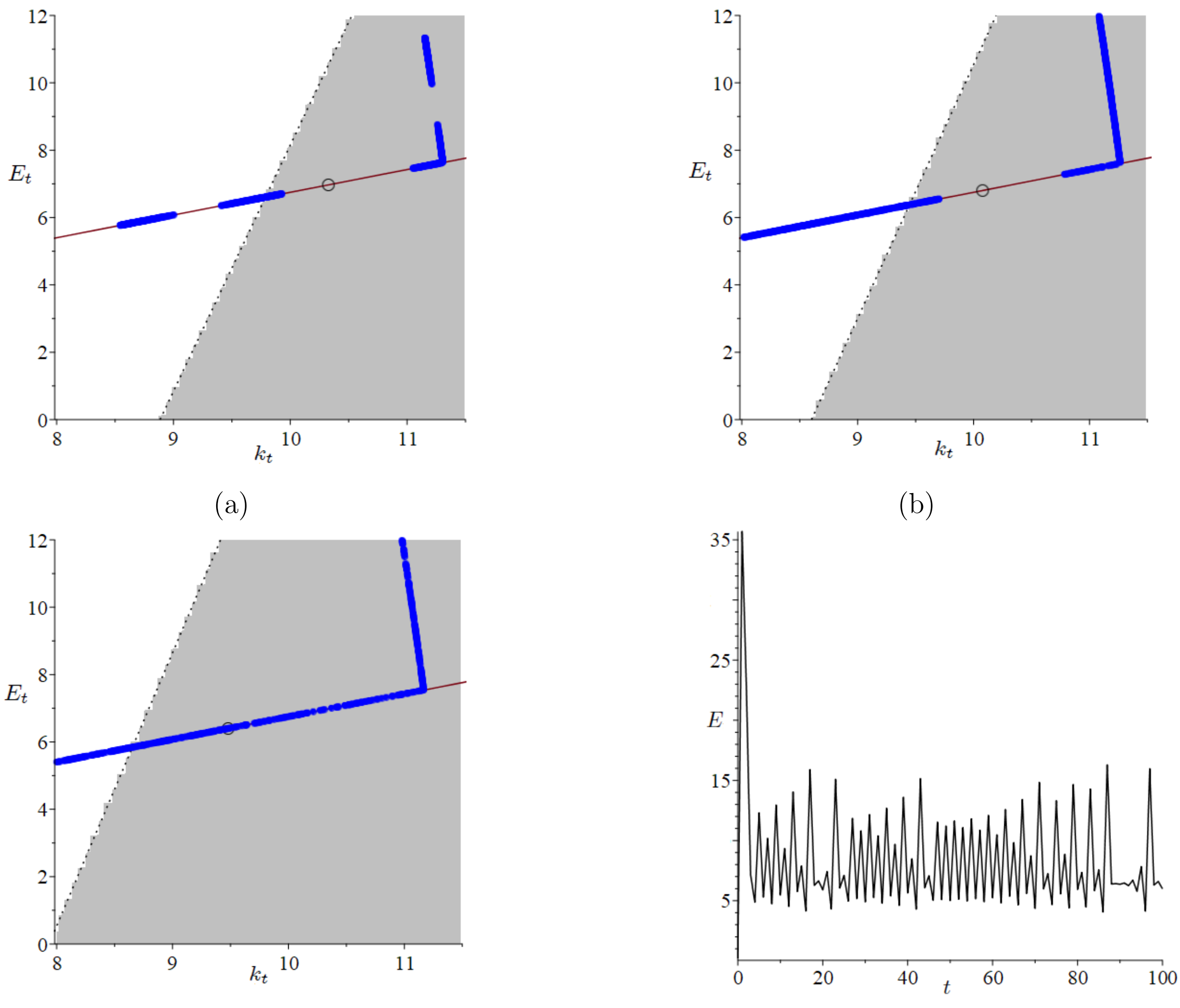


Figure 9: (a) 4-pieces chaotic attractor for $\beta = 3.34$. (b) 2-pieces chaotic attractor for $\beta = 3.44$. (c) the merged chaotic attractor for $\beta = 3.7$. The empty circle in the phase plane represents the unstable fixed point (k^*_{pm}, E^*_{pm}). (d) The chaotic oscillations of E for $\beta = 3.7$.

At the economic-environmental level, this means that, considering an initial parametric configuration such that the system is attracted to a state (k, E) in which there is no contribution to environmental maintenance (therefore, E will converge to $E_{zm} < E_{pm}$), higher values of environmental damage stimulate a reaction of agents on their contribution's decisions. However, this reaction ends up generating dynamics that do not converge to an equilibrium in which (in the long run) agents always contribute but rather to a state of the system where cyclically (or chaotically) agents first observe levels of E on which to intervene with their private resources and then an E considered sufficiently high to incentivize non-contribution actions. Comparing the dynamic phenomena of two numerical exercises proposed, we can additionally notice that in this second scenario the emergence of an attractive 2-cycle is not caused by a flip bifurcation, but it is a consequence of the border collision bifurcation involving the fixed point (k^*_{zm}, E^*_{zm}) which becomes *virtual*.

5 Conclusions

In this paper, we have reconsidered the overlapping generations environmental model introduced in John and Pecchenino (1994) and further studied in Zhang (1999), and adopted a specification of the environmental dynamics in which the possible convergence to a natural value of environmental quality in the absence of anthropogenic activity is taken into account (as in Naimzada and Sodini, 2010 and Caravaggio and Sodini, 2022). This specification allows to have agents who decide to do not devolve any private resource to the environmental issue. Related to this point, we show that two different regimes may alternate: one in which the economy and the environment co-evolve in the same direction; the other in which the environmental problem is not internalized by the agents and where, therefore, they do not devolve any private resource to the environmental good, leading to a possible trade-off between environment and economic growth. The analysis of the local and global properties of the resulting piecewise defined dynamic system makes it possible to highlight that, either starting from a parametric configuration in which the dynamics evolve in the regime with positive contribution, or starting from a parametric configuration such as to determine dynamics with zero contribution, the increase of the negative magnitude of the agents' consumption activity ends up generating dynamics in which the two regimes alternate, first in a cyclic manner, then in a chaotic one. From a mathematical point of view, the occurrence of border collision bifurcations, typical of piecewise maps and able of generating sudden transitions (as the reference parameter varies) from a stable cycle to a chaotic attractor, may be observed.

References

[1] Agliari, A., Vachadze, G., 2011. Homoclinic and heteroclinic bifurcations in an overlapping generations model with credit market imperfection. Computational Economics 38 (3), 241–260.

[2] Antoci, A., Gori, L., Sodini, M., 2016. Nonlinear dynamics and global indeterminacy in an overlapping generations model with environmental resources. Communications in Nonlinear Science and Numerical Simulation 38, 59–71.

[3] Bischi, G.I., Lamantia, F., 2012. Routes to complexity induced by constraints in Cournot oligopoly games with linear reaction functions. Studies in Nonlinear Dynamics & Econometrics 16 (2).

[4] Bischi, G.I., Sushko, I., Lamantia, F., 2012. Border collision bifurcation in a simple oligopoly model with constraints. International Journal of Applied Mathematics & Statistics 26 (2).

[5] Caravaggio, A., Sodini, M., 2022. Local environmental quality and heterogeneity in an OLG agent-based model with spatial externalities. Journal of Economic Interaction and Coordination 17 (1), 287–317.

[6] Di Bernardo, M., Budd, C., Champneys, A. Kowalczyk, P., 2008. Piecewise-smooth dynamical systems, Springer Verlag.

[7] John, A., Pecchenino, R., 1994. An overlapping generations model of growth and the environment. The Economic Journal 104, 1393–1410.

[8] Matsuyama, K., 2013. The Good, the Bad, and the Ugly: An inquiry into the causes and nature of credit cycles. Theoretical Economics (8), 623–651.

[9] Matsuyama, K., Sushko, I., Gardini, L., 2016. Revisiting the model of credit cycles with good and bad projects. Journal of Economic Theory (163), 525–556.

[10] Naimzada, A., Sodini, M., 2010. Multiple attractors and nonlinear dynamics in an overlapping generations model with environment. Discrete Dynamics in Nature and Society.

[11] Puu, T., 1991. Chaos in duopoly pricing. Chaos, Solitons & Fractals 1 (6), 573–581.

[12] Sushko, I., Gardini, L., Matsuyama, K., 2016. Robust chaos in a credit cycle model defined by a one-dimensional piecewise smooth map. Chaos, Solitons & Fractals 91, 299–309.

[13] Tramontana, F., Gardini, L., Puu, T., 2010. Global bifurcations in a piecewise-smooth Cournot duopoly game. Chaos, Solitons & Fractals 43(1-12), 15–24.

[14] Zhang, J., 1999. Environmental sustainability, nonlinear dynamics and chaos. Economic Theory 14, 489–500.

[15] Zhusubaliyev, Z.T., Mosekilde, E., 2003. Bifurcations and chaos in piecewise-smooth dynamical systems. Vol. 44. World Scientific.