On prospective teachers' evaluations of freshman students' written arguments

Cristina Coppola[1], Pier Luigi Ferrari2 and Annamaria Miranda[1]

[1]Università degli Studi di Salerno, Italia; ccoppola@unisa.it, amiranda@unisa.it

[2]Università del Piemonte Orientale, Italia; pierluigi.ferrari@uniupo.it

In this paper we present a qualitative study in which we analyse how a group of prospective mathematics teachers evaluates written arguments produced by freshman students solving a problem involving graphs. The prospective teachers were attending a course of Mathematics Education for the second year of a master degree in Mathematics. In particular we focus on how prospective teachers in their evaluations look at mathematical content, language, and argumentative structure. The outcomes show, on the one hand, a certain attention to some linguistic aspects, on the other hand, a greater attention to the content than to the argumentative structure of the texts. We suggest that prospective teachers' models for linguistic education are often the standard ones, based on conformity to grammar or style rather than on adequacy with respect to goals.

Keywords: Evaluations, Language, Argument, Prospective teachers, Mathematical content.

Introduction.

Argumentation processes are growing more and more important in both research and curricula. In Italian curricula for the primary and secondary school level, argumentation is regarded as an interdisciplinary practice, related to mathematics but also to language, history, science and philosophy. Furthermore, it is given the function not only of representing what has already been learned but also of promoting learning. Its interdisciplinary character can be an obstacle to the development of appropriate teaching practices, as some teachers may stick to their subject and disregard the other aspects. Albano et al. (2019) found in an experiment that high school teachers evaluating students' arguments focused on the correctness of mathematical content only, and preferred concise though incomplete rather than more detailed and complete arguments. The goal of our study is to investigate more systematically how mathematics teachers (in this case, prospective teachers) evaluate students' written arguments. In particular, we want to highlight the aspects to which teachers give greater importance, what weight they give to material errors related to mathematical contents, to the lack of explicitation of the premises and logical connections, to the improper use of symbolic notations or verbal language. We expected attention to the argumentative structure, in accordance with school curricula, since prospective teachers have had some opportunities to reflect on argumentation in mathematics education as a teaching goal. The issue addressed in the paper can be seen as part of the more general problem of how teachers can promote students' argumentative competence.

Theoretical framework.

Argumentation in mathematics education has been widely studied in the last years, and a number of theories and models have been proposed, related both to purposes and form of argument. In this regard we assume that the main goal of argumentation in school is making the links between mathematical properties explicit, rather than convincing or warranting. Contrary to some current perspectives, which relate arguments to the application of external rules or constraints (e.g., logical rules), we assume that arguments in mathematics are based not on form but on meaning, and validity is a semantic property, according to Tarski's definition of *logical consequence* (Tarski, 1944). A statement P is a logical consequence of a set X of statements if P is true in all the interpretations that make all the statements of X true. A deep and insightful discussion on these topics has been carried out by Catarina Dutilh Novaes (2020). The idea of logical consequence is the core of classical mathematical logic and thus of mathematics. A good example of the role of meaning in mathematics is given by the way D. Hilbert formulates and comments the first axiom of the book in his Foundations of Geometry (Hilbert, 1909).

> I, 1. Two distinct points A and B always completely determine a straight line a. We write AB = a or BA = a.
>
> Instead of "determine," we may also employ other forms of expression; for example, we may say A "lies upon" a, A "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b "have the point A in common," etc. (Hilbert, 1909, p.2)

Here Hilbert is not proposing the application of logical rules or other constraints, but a set of synonymous, interchangeable expressions. Expressions like "A and B lie upon A" and "a joins A and B" have different syntactical structure but the same meaning. So, what relates one expression to another is not form but meaning.

We adopt a semantic approach to argumentation for two main reasons. First, it is much nearer to mathematical practice, as it is impossible to describe any non-trivial argument according to its conformity to logical rules or other. Second, it is more useful in mathematics teaching, as the progressive achievement and refinement of meaning is a common feature of development of arguments at any school level, that any teacher can promote, recognise and evaluate.

Furthermore, an argument is, first of all, a piece of text, as argued by Bermejo-Luque (2011). So, language must be regarded as a relevant factor in the analysis of arguments. In this regard, we adopt a functional-linguistic perspective according to Halliday (1985). In this frame the focus is on the functions of a piece of text rather than on its form. In particular, we regard linguistic competence as the ability to critically and effectively use resources in different contexts rather than as conformity to grammatical or stylistic models. So, in our perspective an argument is the verbal description of the relationship of logical consequence between a set of assumptions and a conclusion. This means showing that in any interpretation where the assumptions are true, the conclusion is also true. Of course, a verbal description cannot be complete, as it is based on the meaning of words, that cannot be all defined. So, the degree of explicitation is never absolute but has to be related to the teaching goals and, possibly, negotiated in the class.

In this frame, we wonder how prospective mathematics teachers evaluate students' written arguments, in particular how much weight they give to the correctness of the contents or to the level of detail in the description of the logical connections.

Methodology.

The experiment.

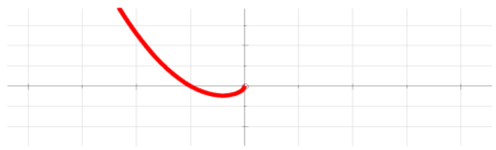
The first step of our study consisted in collecting (and analysing) a number of written explanations produced by about 150 freshman students enrolled in a 3-year BSc degree in Biology (hereinafter, "students"). They were asked to solve the task in Table 1 and explain their answer. Then we selected some of the answers and we asked a group of prospective mathematics teachers (hereinafter, PTs) to evaluate these arguments. The PTs were attending a course of Mathematics Education for the second year of a master degree in Mathematics. The focus of this paper is on how prospective teachers evaluate the written arguments produced by students, while solving the task in Table 1.

Let *f* be defined by: $f(x) = -x^2 + 2\sqrt{-x}$

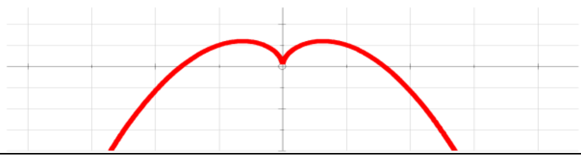
a) Find the domain of *f*. b) Find *f'(x)*

c) One of the following graphs corresponds to *f*. Which one? Explain.

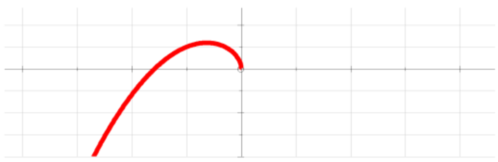
A)



B)



C)



D)

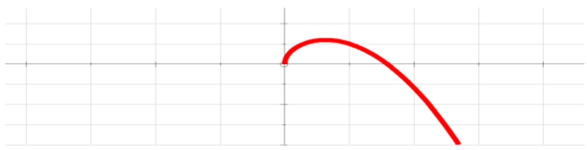


Table 1: The task for the students

A priori analysis.

Problems of this kind have been presented by some of the authors in other occasions (see for example (Albano et al., 2019). They have proved suitable for triggering arguments because they involve different modes of representation (verbal, figural, symbolic), they require some basic knowledge only but cannot be solved just through the perfunctory application of methods learned at school. The crucial information to solve the problem is in the statement "One of the following graphs corresponds to f". Without it there is no way to give an answer and in any case reasoning by exclusion is necessary, since without further information it is never possible to conclude that a visualization of any graph corresponds to some equation. In some cases (such as here) it is possible to conclude that a graph does not correspond to an equation.

Freshman students' explanations.

Here is the English translation of a selection of five explanations provided by freshman Biology students. We are aware that some features of the text may be lost or distorted in the translation.

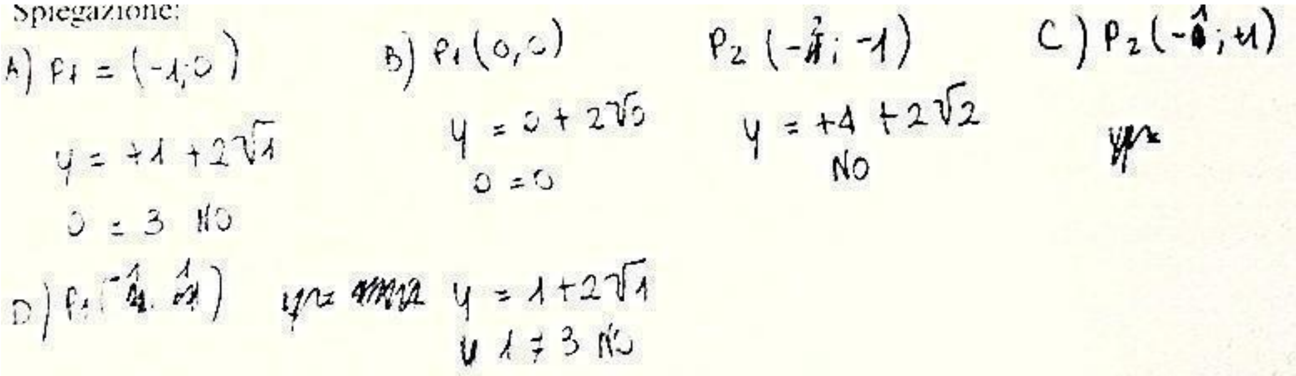
The selection of the arguments was made by taking those that were representative of the most frequent patterns. In particular we chose a detailed one (E1), a couple of concise ones (E2, E3), one without words (E4) and an incoherent one (E5).

E1: From the graphs given, observing the domain of the function f (x) (-∞, 0] we know that the function exists only for values from −∞ to 0, so we do not expect to have the function for positive values from 0, to + ∞; from this observation we can exclude graphs B and D, as they prove that the function exists for positive values (0, + ∞). Substituting in the function f (x) −1, then f (−1) ⇒ = −1, therefore values of x = −1, we obtain that y is equal to 1. The graph A can be excluded because for x = -1 the function in the graph has values y = 0. So by exclusion the graph corresponding to f (x) is graph C, as it satisfies the conditions.

E2: [Checks graph C on the diagram] I exclude the other graphs since B has domain in all R. Graph D has domain in x ≥ 0 and graph A has domain x ≤ 0 but for x = −1 y = 1 so I choose graph C.

E3: Graphs B and D are excluded because the domain is x ≤ 0. Graph A is excluded because f (-1) = 1. By exclusion, the graph corresponding to f(x) is C.

E4: [There is no verbal text, beyond three occurrences of "NO", so we present it as an image]



E5: Graphs B and D can be excluded for the domain. Studying the 1st derivative which is always negative because the domain, the function is always increasing so it is C. [1]

The sample.

The sample considered for our study consists of 50 prospective secondary school mathematics teachers, working in groups of 3 to 5. They were given copies of students' explanations and asked to evaluate them, in accordance with some guidelines, which were intended to make explicit the aspects we wanted to focus on: the weight given to mathematical content, to language, including symbolic notations, and to the argumentative structure. To this end, the guidelines drew the attention of PTs on the clarity of students' explanation, on their length (if they could be shortened or expanded in their opinion), on the mathematical correctness, on the linguistic appropriateness (possibly related to school level) and on the overall acceptability. Moreover, PTs were asked to indicate the critical points and add any comments they believed appropriate. It is worth mentioning that in the course of mathematics education PTs have had the opportunity to reflect on language, and on argumentation in mathematics education and on the importance of developing linguistic and argumentative competencies as a teaching goal.

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[1] This is a good example of the drawbacks of the translations of students' excerpts from one language to another. The english translation could be mended by simply adding 'of' after 'because', whereas the Italian original text ("perché il dominio") requires the addition of some verb.

To analyse the written evaluations provided by the PTs, we observed which elements they took into consideration. More in detail, we searched for finding if there were more references to mathematical content, e.g. material errors, or to the explicitation of the links between the parts of the arguments or to linguistic inaccuracies. We looked at which linguistic aspects the PTs gave weight and how they assessed, for example, improper use of symbols, or of language. For example, comments discussing the mathematical content of poorly formulated explanations without any remarks on language would show a strong focus on the content and a weaker interest in the other aspects. Furthermore, we also found it very interesting to look at the extent to which their attention is paid to other aspects such as completeness and coherence. Finally, we observed also the overall evaluation of each argument and also the models emerging from teachers' comments, in particular if there were expressions showing that they were more concerned about the conformity of arguments to some pattern or about their appropriateness related to some goals.

Data analysis.

We are focusing on the three main aspects outlined in the description of the task:

(1) Mathematical content
(2) Language (including symbolic notations)
(3) Argument (explanation of the steps, coherence, …)

What seems to emerge from the analysis of the protocols is that in most cases PTs, when dealing with the arguments produced by students, evaluate the content rather than the text with which it is communicated. In other words, the texts seem to be considered as carriers of mathematical content rather than as a component of students' productions to be evaluated. Some of them, for example, ascribe the incoherent organization of the text of the 2nd statement of E5 to the wrong calculation of the derivative. An extreme example is given by explanation E4 above: all the groups of the sample gave some interpretation of the procedure adopted by the student and, although there is almost no written text beyond formulas, about half of them describe the (presumed) argument as inadequate or lacking in rigor. According to a couple of groups this procedure shows some understanding of the mathematical ideas involved. Another group claims that this explanation could be appropriate if the student were at level 11. The reference to some school level as a term of comparison is another aspect emerging very often in teachers' evaluations, in most cases related to language. Arguments are considered more or less acceptable according to the (real or hypothetical) level of the student. The following comment, related to argument E1 is an example of this viewpoint.

> Although we reasoned exactly as [the student] did to proceed more quickly, we find this a sufficient but simplistic explanation of the problem for a student starting university. The answer would have been good if the student had been at a lower school level.

As far as language is concerned, most of the teachers who take it into consideration seemingly appreciate the conciseness rather than the completeness of the explanations. Moreover, they are much more focused on the use of mathematical terms or notations rather than on the overall construction of the argument. For example, although some groups appreciated E1 (which is probably the most detailed of the lot), some others claimed it is redundant. Most of the evaluations point out that expressions like 'even graph' are wrong, as (they claim) evenness is a property of functions and not of graphs, but on the other side a large part of the teachers was very cooperative (in the sense of Grice, 1975) when interpreting more ambiguous statements.

In general, most groups mainly focused on mathematical content and interpreted students' productions in a very cooperative way, filling the gaps and neglecting some linguistic inaccuracy. In spite of their sensitivity to content errors, most groups interpreted them as occasional slips, ascribed to distraction or forgetfulness.

We report an evaluation of E2 where special attention is paid to the linguistic aspect.

> The explanation is extremely concise, but by no means incomplete. […] The calculation of the derivative is correct except for one multiplicative factor. We can reasonably conclude that […] the student did not arrive at the correct solution due to a distraction. The explanation […] reveals a fair knowledge of the topics covered […] Expressions such as 'having domain in' are not officially recognized nor in common use. The use of mathematical symbology is also poor, as can be seen in 'for $x = -1$ $y = 1$'. The student omits the use of commas due either to sloppiness or to some difficulty in linking different propositions.

This comment on E2 is a bit different.

> Synthetic and not rigorous explanation, the student has the basic notions and can roughly solve the problem, but cannot adequately motivate it. The description of the domain is informally, but not formally, correct. The calculation of the derivative is incorrect. […] The language is not mathematical, and it is not appropriate for a student who should have mathematics maturity.

Also another group criticizes the use of language in E2 (but their symbolic, 'meticulous' representation of the domain is wrong).

> […] For example, a statement like "the graph X has domain Y" is a common error. We are sure that the student has understood the underlying concept but it is not the graph that has that particular domain but the function represented by the graph. There is also an error in the derivative, as a factor '2' is omitted. We assume however, that this is just a simple, not trivial, slip, since the theorem of derivation of compound functions has been correctly applied, which in other answers has not been done. To be meticulous, the domain of the function could have been written more precisely, for example: $\{x \in R \vee x \leq 0\}$, however we believe it is quite a passable thing. As regards the use of Italian, there is a doubt about the phrase "I exclude the other graphs" placed at the beginning of the answer, since it is not clear whether it begins directly in this way (and in this case it would be unjustifiable, especially for one fifth high school or first year university student) or, having marked answer C as correct, she begins by directly motivating why the other graphs cannot be exact (in this case it is justifiable), precisely according to the answer already marked in the diagram. Not only that, we also add that in explaining why graph A could not be the exact one, one could be a little more precise by saying who the $x = -1$ $y = 1$ was referring to and not throw it there without adding any details. We believe that the answer is correct and suggests that the student has quite clear the basic mathematical concepts.

This group points out the improper use of mathematical notation as well.

> Apart from an error in the calculation of the derivative, the answer is mathematically correct and the thinking process is easily interpretable […] we consider it acceptable. The language could be improved, for example the student could have formalized the link between '$x = -1$' and '$y = 1$', however on the whole it is concise and clear and therefore adequate for his school level.

Some groups give a better evaluation to E2. However, all the PTs approved the reasoning in E2.

A fairly large number of PTs criticize reasoning by exclusion as in the next two evaluations.

> In this case an argument stronger than the simple exclusion would have been preferable, even a simple condition of belonging of particular points to the function would have been more adequate and appreciated.

> However, since the student has solved the question by exclusion, no certain confirmation can be drawn from it regarding the understanding of the “alternative” concepts which could be referred to for the direct resolution of the problem.

The following evaluation, even bolder than others, contains mathematically wrong suggestions.

> The derivative is wrong and as regards the choice of the graph, even if correct, it goes to the exclusion when it was enough to make the limit to $-\infty$ to identify the correct graph.

Discussion.

Mathematical content.

Prospective teachers were able to somehow interpret the mathematical content of the explanations, even when they were vague, inconsistent or even missing. In most cases they ascribed mathematical errors to slips or distractions. This is an interpretive pattern rather common among secondary school mathematics teachers and seems to depend on the interpretation of knowledge as separate from performance. The cooperative attitude of teachers is a known phenomenon that affects the attitude of students as well, who are aware of it and sometimes attribute failure in communication to the lack of cooperation of the teacher rather than the inadequacy of their product.

Language.

Teachers in some cases did not consider language at all, in others they seemingly evaluated students' productions according to their conformity to some (generally unspecified) pattern rather than to their effectiveness in order to communicate their answers. A clear example of separation between effectiveness and conformity is the comment reported as the first indented citation of the section ‘Data analysis’: teachers rate the explanation as inappropriate, but admit they had followed the same pattern to proceed faster. Also the comparison between the text produced and the school level suggests a view of language as an (optional) tool to appropriately express already developed concepts rather than a fundamental one to promote learning. The other linguistic remarks are concerned with mathematical vocabulary: most teachers criticize the use of expressions like ‘domain of the graph’ or ‘even graph’ in place of ‘domain of the function’, ‘even function’. Also in this case the use of ‘graph’ may not conform to mathematical vocabulary but is not a great danger for the clarity of the text.

Argument.

In general, there have been very few remarks focusing on the structure and the effectiveness of arguments. Teachers have discussed the length of the explanations or the use of particular expressions but have not shown much interest in the explicitation of the relationships between the statements. E1, which is by far the most explicit, has been criticized for being redundant, whereas many groups have praised explanation E2 or E3. Actually, explanations like E3 (but also some of the others) are incomplete and do not give much information about the level of understanding of the student. Some teachers claimed that reasoning by exclusion is not the most appropriate way to solve the problem. We have remarked in the a priori analysis that there is no other way to answer the question. Reasoning by exclusion is based on a well known deductive rule (from ‘P or Q’ and ‘not P’ one can infer ‘Q’) and is perfectly correct from the viewpoint of logic. Most likely some teacher interpreted the explanations as occasions for students to display their mathematical knowledge. Again, there is some separation between knowledge and goals.

Summing up, the PTs involved in this experiment showed some more attention to language compared to the experienced, in-service teachers involved in the experiment discussed by Albano et al. (2019). Trouble is that their models for linguistic education are the standard ones, based on conformity to grammar or style rather than on adequacy related to goals. This suggests that language's evaluation by mathematics teachers is based on different criteria from those adopted by language teachers who follow the national curricula. Also mathematical notations have been considered as warrants for some unspecified rigour, to be applied in accordance with norms rather than semiotic tools capable of expressing meaning in relation to different purposes. This suggests that the semiotic systems of mathematics (verbal language, symbolic notations, figural representations) are to be regarded as such rather than as highways to get some already existing mathematical content not depending on them. So the use of such systems and in particular the conversions between them should be a topic for reflection for prospective teachers. Similar considerations hold for the argumentative aspects: the explicitation of the links among the statements has not been regarded as a relevant factor for evaluation. Apparently, the PTs expected students to write explanations to display their mathematical knowledge rather than to make their thinking process explicit and motivated.

References

Albano G., Coppola C., & Ferrari P.L. (2019). High school teachers' evaluation of argumentative texts in mathematics. In Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). *Proceedings of CERME11*, 1648–1655.

Bermejo Luque, Lilian (2011). *Giving Reasons. A Linguistic-Pragmatic Approach to Argumentation Theory*. Dordrecht: Springer.

Dutilh Novaes, C. (2020). *The Dialogical Roots of Deduction: Historical, Cognitive, and Philosophical Perspectives on Reasoning*. Cambridge University Press.

Grice, H.P. (1975). 'Logic and conversation', in Cole, P. & Morgan, J.L. (Eds.), *Syntax and semantics: Vol.3. Speech acts* (pp.41–58), New York: Academic Press.

Halliday, M.A.K. (1985). *An introduction to functional grammar*. London: Arnold.

Hilbert, D. (1909). *Grundlagen der Geometrie*. Leipzig-Berlin: Teubner. (English translation: *The Foundations of Geometry*. La Salle (Il): The Open Court, 1950).

Tarski, A. (1944). The Semantic Concept of Truth and the Foundations of Semantics. *Philosophy and Phenomenological Research*, *4*(3).