Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 51, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SECOND-ORDER BOUNDARY ESTIMATES FOR SOLUTIONS TO SINGULAR ELLIPTIC EQUATIONS IN BORDERLINE CASES

CLAUDIA ANEDDA, GIOVANNI PORRU

Abstract. Let $\Omega \subset R^N$ be a bounded smooth domain. We investigate the effect of the mean curvature of the boundary $\partial\Omega$ on the behaviour of the solution to the homogeneous Dirichlet boundary value problem for the equation $\Delta u + f(u) = 0$. Under appropriate growth conditions on $f(t)$ as t approaches zero, we find asymptotic expansions up to the second order of the solution in terms of the distance from x to the boundary $\partial\Omega$.

1. Introduction

In this paper we study the Dirichlet problem

$$
\begin{gathered}
\Delta u + f(u) = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $f(t)$ is a decreasing and positive smooth function in $(0, \infty)$, which approaches infinity as $t \to 0$. Equation (1.1) arises in problems of heat conduction and in fluid mechanics.

Problems of this kind are discussed in many papers; see, for instance, [5, 6, 8, 9, 11, 12] and references therein. For $f(t) = t^{-\gamma}$, $\gamma > 0$, in [4] it is shown that there exists a positive solution continuous up to the boundary $\partial\Omega$. For $f(t) = t^{-\gamma}$, $\gamma > 1$, in [3] it is shown that there exists a constant $B > 0$ such that

$$
\Big| u(x) - \Big(\frac{\gamma + 1}{\sqrt{2(\gamma - 1)}} \delta \Big)^{\frac{2}{1+\gamma}} \Big| < B \delta^{\frac{2\gamma}{\gamma+1}},
$$

where $\delta = \delta(x)$ denotes the distance from x to the boundary $\partial\Omega$. For $f(t) = t^{-\gamma}$, $\gamma > 3$, in [2] it is proved that

$$
u(x) = \Big(\frac{\gamma + 1}{\sqrt{2(\gamma - 1)}} \delta \Big)^{\frac{2}{1+\gamma}} \Big[1 + \frac{1}{3 - \gamma} H\delta + o(\delta) \Big],
$$

where $H = H(x)$ is related with the mean curvature of $\partial\Omega$ at the nearest point to x.

2000 *Mathematics Subject Classification.* 35B40, 35J67.
Key words and phrases. Elliptic problems; singular equations;
second order boundary approximation.

Submitted January 10, 2011. Published April 13, 2011.

In [1], more general nonlinearities are discussed. More precisely, let

$$F(t)=\int_t^1 f(\tau)d\tau, \quad \lim_{t\to 0^+} F(t)=\infty, \quad \frac{f'(t)F(t)}{(f(t))^2}=\frac{\gamma}{1-\gamma}+O(1)t^{\beta}, \tag{1.2}$$

where $\gamma \geq 3$, $\beta > 0$ and $O(1)$ denotes a bounded quantity as $t \to 0$. In addition, we suppose there is M finite such that for all $\theta \in (1/2, 2)$ and for $t \in (0,1)$ we have

$$\frac{|f''(\theta t)|t^2}{f(t)} \leq M. \tag{1.3}$$

An example which satisfies these conditions is $f(t) = t^{-\gamma} + t^{-\nu}$ with $0 < \nu < \gamma$; here $\beta = \min[\gamma - \nu, \gamma - 1]$.

Let $\phi(\delta)$ be defined as

$$\int_0^{\phi(\delta)} \frac{1}{(2F(t))^{1/2}} dt = \delta. \tag{1.4}$$

For $3 < \gamma < \infty$, in [1] it is proved that

$$u(x) = \phi(\delta)\Big[1 + \frac{1}{3-\gamma} H\delta + O(1)\delta^{\sigma+1}\Big], \tag{1.5}$$

where σ is any number such that $0 < \sigma < \min[\frac{\gamma-3}{\gamma+1}, \frac{2\beta}{\gamma+1}]$. Note that ϕ satisfies the one dimensional problem

$$\phi'' + f(\phi) = 0, \quad \phi(0) = 0.$$

The estimate (1.5) shows that the expansion of $u(x)$ in terms of δ has the first part which is independent of the geometry of the domain, and the second part which depends on the mean curvature of the boundary as well as on γ.

In the present paper we investigate the borderline cases $\gamma = 3$ and $\gamma = \infty$. In the case of $\gamma = 3$ we find the expansion

$$u(x) = \phi(\delta)\Big[1 + \frac{1}{4} H\delta \log\delta + O(1)\delta(-\log\delta)^{\sigma}\Big], \tag{1.6}$$

where $0 < \sigma < 1$ and $O(1)$ is bounded as $\delta \to 0$.

To discuss the case $\gamma = \infty$, we make the following assumption

$$f(t) > 0, \quad \frac{f'(t)}{f(t)} = -\frac{\ell}{t^{\beta+1}}\big(1 + O(1)t^{\beta}\big), \tag{1.7}$$

with $\ell > 0$ and $\beta > 0$. Note that the above condition implies

$$\frac{F(t)}{f(t)} = \frac{t^{\beta+1}}{\ell}\big(1 + O(1)t^{\beta}\big), \quad F(t) = \int_t^1 f(\tau)d\tau. \tag{1.8}$$

Furthermore, (1.7) together with (1.8) imply (1.2) with $\gamma = \infty$; that is,

$$\frac{f'(t)F(t)}{(f(t))^2} = -1 + O(1)t^{\beta}. \tag{1.9}$$

Instead of (1.3), now we suppose that for some $m > 2$ and some $\epsilon \in (0,1)$, there is $M > 0$ such that

$$\frac{|f''(\theta t)|t^2}{f(t)} \leq M \frac{1}{t^{2\beta}} (F(t))^{1/m}, \quad \forall t \in (0,1/2),\ \forall \theta \in (1-\epsilon, 1+\epsilon). \tag{1.10}$$

The function $f(t) = e^{\frac{\ell}{\beta t^{\beta}}}$ satisfies all these conditions.

Under assumptions (1.7) and (1.10), we find the estimate

$$u(x) = \phi(\delta)\Big[1 - \frac{1}{\ell} H\delta\big(\phi(\delta)\big)^{\beta} + O(1)\delta\big(\phi(\delta)\big)^{2\beta}\Big],$$

where ϕ is defined as in (1.4).

Throughout this paper, the boundary $\partial\Omega$ is smooth in the sense that it belongs to C^4.

2. Preliminary results

Lemma 2.1. *Let $A(\rho, R) \subset \mathbb{R}^N$, $N \ge 2$, be the annulus with radii ρ and R centered at the origin. Let $f(t) > 0$ smooth, decreasing for $t > 0$, and such that $\int_t^1 (F(\tau))^{1/2} d\tau \to \infty$ as $t \to 0^+$, where $F(t) = \int_t^1 f(\tau)d\tau$. We also suppose that the function $s \mapsto (F(s))^{-1} \int_s^1 (F(t))^{1/2} dt$ is increasing for s close to 0. If $u(x)$ is a solution to problem (1.1) in $\Omega = A(\rho, R)$ and $v(r) = u(x)$ for $r = |x|$, then*

$$v(r) > \phi(R - r) - C\frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}}(R - r), \quad \tilde{r} < r < R, \tag{2.1}$$

and

$$v(r) < \phi(r - \rho) + C\phi'(r - \rho)\frac{\int_v^1 (F(t))^{1/2} dt}{F(v)}(r - \rho), \quad \rho < r < \overline{r}, \tag{2.2}$$

where ϕ is defined as in (1.4), $\rho < \overline{r} \le \tilde{r} < R$ and C is a suitable positive constant.

Proof. If $\Omega = A(\rho, R)$, the corresponding solution $u(x)$ to problem (1.1) is radially symmetric (by uniqueness) and positive (by the maximum principle). With $v(r) = u(x)$ for $r = |x|$ we have

$$v'' + \frac{N-1}{r} v' + f(v) = 0, \quad v(\rho) = v(R) = 0. \tag{2.3}$$

The latter equation can be rewritten as

$$\big(r^{N-1} v'\big)' + r^{N-1} f(v) = 0.$$

Since $v(\rho) = v(R)$, we must have $v'(r_0) = 0$ for some $r_0 \in (\rho, R)$. Integrating over (r_0, r) we obtain

$$r^{N-1} v' + \int_{r_0}^r t^{N-1} f(v) dt = 0.$$

Hence, $v(r)$ is increasing for $\rho < r < r_0$ and decreasing for $r_0 < r < R$. Multiplying (2.3) by v' and integrating over (r_0, r) we find

$$\frac{(v')^2}{2} + (N-1)\int_{r_0}^r \frac{(v')^2}{s} ds = F(v) - F(v_0), \quad v_0 = v(r_0). \tag{2.4}$$

Since $\int_t^1 (F(\tau))^{1/2} d\tau \to \infty$ as $t \to 0$, we have $F(t) \to \infty$ as $t \to 0$. Therefore, $F(v(r)) \to \infty$ as $r \to R$, and (2.4) implies

$$|v'| < 2(F(v))^{1/2}, \quad r \in (r_1, R), \quad r_0 \le r_1 < R. \tag{2.5}$$

As a consequence, with $v_1 = v(r_1)$ we have

$$\int_{r_1}^r \frac{(v')^2}{s} ds \le \frac{2}{r_1}\int_{r_1}^r (F(v))^{1/2}(-v')ds = \frac{2}{r_1}\int_v^{v_1} (F(t))^{1/2} dt. \tag{2.6}$$

Since
$$\int_v^{v_1} (F(t))^{1/2} dt \le (F(v))^{1/2} v_1,$$
using (2.6) we find
$$\lim_{r\to R} \frac{\int_{r_1}^r \frac{(v')^2}{s} ds}{F(v)} = \lim_{r\to R} \frac{\int_v^{v_1} (F(t))^{1/2} dt}{F(v)} = 0. \tag{2.7}$$
Now, by (2.4) we have
$$\frac{(v')^2}{2F(v)} = 1 - \frac{(N-1)\int_{r_0}^r \frac{(v')^2}{s} ds + F(v_0)}{F(v)}. \tag{2.8}$$
Note that, if $v_0 > 1$ then $F(v_0) < 0$. We claim that
$$(N-1)\int_{r_0}^r \frac{(v')^2}{s} ds + F(v_0) > 0$$
for r close to R. Indeed, by (2.7) and (2.8) it follows that $|v'| > (F(v))^{1/2}$ for $r \in (r_2, R)$. Hence,
$$\int_{r_2}^r \frac{(v')^2}{s} ds > \frac{1}{R}\int_{r_2}^r (F(v))^{1/2}(-v') ds = \frac{1}{R}\int_{v(r)}^{v(r_2)} (F(\tau))^{1/2} d\tau.$$
By using the assumption $\int_t^1 (F(\tau))^{1/2} d\tau \to \infty$ as $t \to 0$, the latter inequality implies that $\int_{r_2}^r \frac{(v')^2}{s} ds \to \infty$ as $r \to R$, and the claim follows.

Equation (2.8) yields
$$\frac{-v'}{(2F(v))^{1/2}} = 1 - \Gamma(r), \tag{2.9}$$
where
$$\Gamma(r) = 1 - \Big[1 - \frac{(N-1)\int_{r_0}^r \frac{(v')^2}{s} ds + F(v_0)}{F(v)}\Big]^{1/2}.$$
Since
$$1 - [1-\epsilon]^{1/2} < \epsilon, \quad \forall \epsilon \in (0,1),$$
using (2.6) we find a constant M such that, for r close to R,
$$0 \le \Gamma(r) \le \frac{(N-1)\int_{r_0}^r \frac{(v')^2}{s} ds + F(v_0)}{F(v)} \le M \frac{\int_v^{v_0} (F(t))^{1/2} dt}{F(v)}. \tag{2.10}$$
Note that, by (2.10) and (2.7) we have $\Gamma(r) \to 0$ as $r \to R$.

The inverse function of ϕ is
$$\psi(s) = \int_0^s \frac{1}{(2F(t))^{1/2}} dt.$$
Integration of (2.9) over (r, R) yields
$$\psi(v) = R - r - \int_r^R \Gamma(s) ds,$$
from which we find
$$v(r) = \phi\big(R - r - \int_r^R \Gamma(s) ds\big). \tag{2.11}$$

By (2.11), we have

$$v(r) = \phi(R-r) - \phi'(\omega)\int_r^R \Gamma(s)ds, \tag{2.12}$$

with

$$R - r - \int_r^R \Gamma(s)ds < \omega < R - r.$$

Since $\phi'(\omega) = (2F(\phi(\omega)))^{1/2}$, and since the function $t \to F(\phi(t))$ is decreasing we have

$$\phi'(\omega) < \Big(2F\big(\phi(R-r-\int_r^R \Gamma(s)ds)\big)\Big)^{1/2} = (2F(v))^{1/2},$$

where (2.11) has been used in the last step. Hence, by (2.12) we have

$$v(r) > \phi(R-r) - (2F(v))^{1/2}\int_r^R \Gamma(s)ds.$$

Using (2.10), we find

$$v(r) > \phi(R-r) - (2F(v))^{1/2} M \int_r^R \frac{\int_{v(s)}^{v_0}(F(\tau))^{1/2}d\tau}{F(v(s))}ds. \tag{2.13}$$

Since $(F(t))^{-1}\int_t^1 (F(\tau))^{1/2}d\tau$ is increasing and since $v(s)$ is decreasing, for s close to R the function

$$s \mapsto \frac{\int_{v(s)}^{v_0}(F(\tau))^{1/2}d\tau}{F(v(s))}$$

is decreasing. Using the monotonicity of this function, inequality (2.1) follows from (2.13).

To prove (2.2), we observe that (2.4) also holds for $\rho < r < r_0$. Let us write equation (2.4) as

$$\frac{(v')^2}{2} = F(v) - F(v_0) + (N-1)\int_r^{r_0}\frac{(v')^2}{s}ds, \tag{2.14}$$

with $\rho < r < r_0$. By (2.14), $(v'(r))^2 \to \infty$ as $r \to \rho$. Moreover, since $v'(r) > 0$ for $r \in (\rho, r_0)$, by (2.3) we have $v''(r) < 0$. Hence, by [10, Lemma 2.1], we have

$$\lim_{r\to\rho}\frac{\int_r^{r_0}\frac{(v')^2}{t}dt}{(v'(r))^2} = 0.$$

Using this result and (2.14) we find $0 < v' < 2(F(v))^{1/2}$ for $r \in (\rho, r_3)$, $r_3 \le r_0$. As a consequence we have, with $v(r_3) = v_3$,

$$\int_r^{r_3}\frac{(v')^2}{s}ds \le \frac{2}{\rho}\int_r^{r_3}(F(v))^{1/2}v'ds = \frac{2}{\rho}\int_v^{v_3}(F(t))^{1/2}dt. \tag{2.15}$$

Since $\int_v^{v_3}(F(t))^{1/2}dt \le (F(v))^{1/2}v_3$, (2.15) implies

$$\lim_{r\to\rho}\frac{\int_r^{r_0}\frac{(v')^2}{s}ds}{F(v)} = 0. \tag{2.16}$$

By (2.14), we find

$$\frac{(v')^2}{2F(v)} = 1 + \frac{(N-1)\int_r^{r_0}\frac{(v')^2}{s}ds - F(v_0)}{F(v)}. \tag{2.17}$$

Using (2.16) and (2.17) and arguing as in the previous case one finds that

$$(N-1)\int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0) > 0$$

for r close to ρ. Equation (2.17) yields

$$\frac{v'}{(2F(v))^{1/2}} = 1 + \tilde{\Gamma}(r), \tag{2.18}$$

where

$$\tilde{\Gamma}(r) = \Big[1 + \frac{(N-1)\int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0)}{F(v)}\Big]^{1/2} - 1.$$

Since

$$[1+\epsilon]^{1/2} - 1 < \epsilon, \quad \forall \epsilon > 0,$$

using (2.15) one finds, for r close to ρ,

$$0 \le \tilde{\Gamma}(r) \le \frac{(N-1)\int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0)}{F(v)} \le \tilde{M} \frac{\int_v^{v_0} (F(t))^{1/2} dt}{F(v)}. \tag{2.19}$$

Integration of (2.18) over (ρ, r) yields

$$\psi(v) = r - \rho + \int_\rho^r \tilde{\Gamma}(s) ds,$$

from which we find

$$v(r) = \phi(r-\rho) + \phi'(\omega_1) \int_\rho^r \tilde{\Gamma}(s) ds, \tag{2.20}$$

with

$$r - \rho < \omega_1 < r - \rho + \int_\rho^r \tilde{\Gamma}(s) ds.$$

Since $\phi'(s)$ is decreasing we have

$$\phi'(\omega_1) < \phi'(r-\rho).$$

The latter estimate, (2.20) and (2.19) imply

$$v(r) < \phi(r-\rho) + \phi'(r-\rho) \int_\rho^r \tilde{M} \frac{\int_v^{v_0} (F(\tau))^{1/2} d\tau}{F(v)} ds. \tag{2.21}$$

Since $v(s)$ is increasing for s close to ρ, the function

$$s \mapsto \frac{\int_{v(s)}^{v_0} (F(\tau))^{1/2} d\tau}{F(v(s))}$$

is increasing. Hence, inequality (2.2) follows from (2.21). The lemma is proved. □

Corollary 2.2. *Assume the same notation and assumptions as in Lemma 2.1. Given $\epsilon > 0$ there are r_ϵ and $\tilde{r}_\epsilon$ such that*

$$\phi(R-r) > v(r) > (1-\epsilon)\phi(R-r), \quad r_\epsilon < r < R, \tag{2.22}$$
$$\phi(r-\rho) < v(r) < (1+\epsilon)\phi(r-\rho), \quad \rho < r < \tilde{r}_\epsilon. \tag{2.23}$$

Proof. By (2.9) we have
$$\frac{-v'}{(2F(v))^{1/2}} < 1.$$
Integrating over (r, R) we find $\psi(v) < R-r$, from which the left hand side of (2.22) follows. By (2.1) we have
$$v(r) > \Big[1 - C\frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} \frac{R-r}{\phi(R-r)}\Big]\phi(R-r).$$
Since $F(t)$ is decreasing we find
$$\frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} \le 1.$$
Moreover, putting $R - r = \psi(s)$ we have
$$0 \le \lim_{r\to R} \frac{R-r}{\phi(R-r)} = \lim_{s\to 0} \frac{\psi(s)}{s} \le \lim_{s\to 0} \frac{1}{(2F(s))^{1/2}} = 0.$$
The right hand side of (2.22) follows from these estimates.

By (2.18) we have
$$\frac{v'}{(2F(v))^{1/2}} > 1.$$
Integrating over (ρ, r), we find $\psi(v) > r-\rho$, from which the left hand side of (2.23) follows. By (2.2) we have
$$v(r) < \Big[1 + C\phi'(r-\rho)\frac{\int_v^1 (F(t))^{1/2} dt}{F(v)} \frac{r-\rho}{\phi(r-\rho)}\Big]\phi(r-\rho).$$
We find
$$0 \le \lim_{r\to\rho} \frac{\int_v^1 (F(t))^{1/2} dt}{F(v)} \le \lim_{r\to\rho} \frac{1}{(F(v))^{1/2}} = 0.$$
Moreover, putting $r - \rho = \psi(s)$, we have
$$\frac{(r-\rho)\phi'(r-\rho)}{\phi(r-\rho)} = \frac{\psi(s)(2F(s))^{1/2}}{s} \le 1.$$
The right hand side of (2.23) follows from these estimates. The proof is complete. □

3. The case $\gamma = 3$

Let $f(t)$ be a smooth, decreasing and positive function in $(0, \infty)$. Assume (1.2) with $\gamma = 3$; that is,
$$F(t) = \int_t^1 f(\tau)d\tau, \quad \lim_{t\to 0^+} F(t) = \infty, \quad \frac{f'(t)F(t)}{(f(t))^2} = -\frac{3}{2} + O(1)t^{\beta}, \tag{3.1}$$
where $\beta > 0$ and $O(1)$ denotes a bounded quantity as $t \to 0$. This condition implies, for t small,
$$-\frac{f'(t)}{f(t)} = \Big(\frac{3}{2} + O(1)t^{\beta}\Big)\frac{f(t)}{F(t)} > \frac{5}{4}\frac{f(t)}{F(t)}.$$
Integration over (t, t_0), t_0 small, yields
$$\log\frac{f(t)}{f(t_0)} > \frac{5}{4}\log\frac{F(t)}{F(t_0)}, \quad \frac{f(t)}{F(t)} > \frac{f(t_0)}{(F(t_0))^{5/4}}(F(t))^{1/4}.$$

It follows that
$$\lim_{t\to 0}\frac{F(t)}{f(t)}=0. \tag{3.2}$$
Let us rewrite (3.1) as
$$(F(t))^{-1/2}\Big(\frac{(F(t))^{3/2}}{f(t)}\Big)'=O(1)t^{\beta}. \tag{3.3}$$
Integrating by parts over $(0,t)$ and using (3.2) we find
$$\frac{F(t)}{tf(t)}=\frac{1}{2}+O(1)t^{\beta}. \tag{3.4}$$
Using the latter estimate and (3.1) again we find
$$\frac{tf'(t)}{f(t)}=-3+O(1)t^{\beta}. \tag{3.5}$$
Let us write (3.5) as
$$\frac{f'(t)}{f(t)}=-\frac{3}{t}+O(1)t^{\beta-1}.$$
Integration over $(t,1)$ yields
$$\log\frac{f(1)}{f(t)}=\log t^3+O(1).$$
Therefore, we can find two positive constants C_1, C_2 such that
$$C_1t^{-3}<f(t)<C_2t^{-3},\quad \forall t\in(0,1). \tag{3.6}$$
Since $F(t)=\int_t^1 f(\tau)d\tau$, using (3.6) we find two positive constants C_3, C_4 such that
$$C_3t^{-2}<F(t)<C_4t^{-2},\quad \forall t\in(0,1/2). \tag{3.7}$$

Lemma 3.1. *If* (3.1) *holds and if* $\phi(\delta)$ *is defined as in* (1.4) *then we have*
$$\frac{\phi'(\delta)}{\delta f(\phi(\delta))}=2+O(1)(\phi(\delta))^{\beta}, \tag{3.8}$$
$$\frac{\phi(\delta)}{\delta\phi'(\delta)}=2+O(1)(\phi(\delta))^{\beta}, \tag{3.9}$$
$$\frac{\phi(\delta)}{\delta^2 f(\phi(\delta))}=4+O(1)(\phi(\delta))^{\beta}, \tag{3.10}$$
$$\phi(\delta)=O(1)\delta^{1/2}. \tag{3.11}$$

For a proof of the above lemma, see [1, Lemma 2.3].

Lemma 3.2. *Let* $\Omega\subset\mathbb{R}^N$, $N\geq 2$, *be a bounded smooth domain, and let* $f(t)>0$ *be smooth, decreasing and satisfy* (3.1) *with* $\beta>0$. *If* $u(x)$ *is a solution to problem* (1.1) *then*
$$\phi(\delta)\big[1-C\delta(-\log\delta)\big]<u(x)<\phi(\delta)\big[1+C\delta(-\log\delta)\big], \tag{3.12}$$
where ϕ *is defined as in* (1.4), δ *denotes the distance from* x *to* $\partial\Omega$, *and* C *is a suitable constant.*

Proof. If $P \in \partial\Omega$ we can consider a suitable annulus of radii ρ and R contained in Ω and such that its external boundary is tangent to $\partial\Omega$ in P. If $v(x)$ is the solution of problem (1.1) in this annulus, by using the comparison principle for elliptic equations ([7], Theorem 10.1) we have $u(x) \geq v(x)$ for x belonging to the annulus. Choose the origin in the center of the annulus and put $v(x) = v(r)$ for $r = |x|$.

We note that our assumptions imply those of Lemma 2.1. Indeed, the condition $\int_t^1 (F(\tau))^{1/2} d\tau \to \infty$ as $t \to 0$, follows from (3.7). Furthermore, using (3.7) again and (3.6), for s close to 0 we have

$$\frac{d}{ds}\Big[(F(s))^{-1}\int_s^1 (F(t))^{1/2} dt\Big] = (F(s))^{-1/2}\Big[\frac{f(s)\int_s^1 (F(\tau))^{1/2} d\tau}{(F(s))^{3/2}} - 1\Big] > 0.$$

Therefore, we can use Lemma 2.1 and Corollary 2.2. By (2.1), we have

$$v(r) > \phi(R-r) - C_1 \frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}}(R-r), \quad \tilde{r} < r < R. \tag{3.13}$$

By using (3.7) we find that

$$\lim_{r\to R}\int_{v(r)}^1 (F(t))^{1/2} dt = \infty = \lim_{r\to R} v(r)\big(F(v(r))\big)^{1/2}\log(R-r)^{-1}.$$

Using de l'Hôpital rule and (3.4) we find

$$\begin{aligned}
&\lim_{r\to R}\frac{\int_v^1 (F(t))^{1/2} dt}{v(F(v))^{1/2}\log(R-r)^{-1}} \\
&= \lim_{r\to R}\frac{-(F(v))^{1/2} v'}{v'\Big((F(v))^{1/2} - \frac{vf(v)}{2(F(v))^{1/2}}\Big)\log(R-r)^{-1} + \frac{v(F(v))^{1/2}}{R-r}} \\
&= \lim_{r\to R}\frac{1}{\Big(-1 + \frac{vf(v)}{2F(v)}\Big)\log(R-r)^{-1} - \frac{v}{v'(R-r)}} \\
&= \lim_{r\to R}\frac{1}{O(1)v^\beta\log(R-r)^{-1} - \frac{v}{v'(R-r)}}.
\end{aligned}$$

By (2.22) we have $v(r) < \phi(R-r)$. Using this inequality and (3.11) with $\delta = R - r$ we obtain

$$\lim_{r\to R} v^\beta \log(R-r)^{-1} = 0.$$

Moreover, using (2.9), de l'Hôpital rule and (3.4) we find

$$\begin{aligned}
\lim_{r\to R}\frac{v}{-v'(R-r)} &= \lim_{r\to R}\frac{v(2F(v))^{-1/2}}{R-r} \\
&= \lim_{r\to R}(-v')\Big((2F(v))^{-1/2} + v(2F(v))^{-\frac{3}{2}} f(v)\Big) \\
&= \lim_{r\to R}\Big(1 + \frac{vf(v)}{2F(v)}\Big) = 2.
\end{aligned}$$

Hence,

$$\lim_{r\to R}\frac{\int_v^1 (F(\tau))^{1/2} d\tau}{v(F(v))^{1/2}\log(R-r)^{-1}} = \frac{1}{2}. \tag{3.14}$$

From (3.13) and (3.14) we find
$$v(r) > \phi(R-r) - C_2 v(r)(R-r)\log(R-r)^{-1}.$$
By (2.22), $v(r) < \phi(R-r)$, hence
$$v(r) > \phi(R-r)\big(1 - C_2(R-r)\log(R-r)^{-1}\big). \tag{3.15}$$
For x near to P we have $\delta = R-r$; therefore, (3.15) and the inequality $u(x) \geq v(x)$ yield the left hand side of (3.12).

Consider a new annulus of radii ρ and R containing Ω and such that its internal boundary is tangent to $\partial\Omega$ in P. If $w(x)$ is the solution of problem (1.1) in this annulus, by using the comparison principle for elliptic equations we have $u(x) \leq w(x)$ for x belonging to Ω. Choose the origin in the center of the annulus and put $w(x) = w(r)$ for $r = |x|$. By (2.2) of Lemma 2.1 (with w in place of v) we have
$$w(r) < \phi(r-\rho) + C_3(r-\rho)\phi'(r-\rho)\frac{\int_w^1 (F(t)^{1/2}dt}{F(w)}, \quad \rho < r < \overline{r}. \tag{3.16}$$
The same proof used to get (3.14) yields
$$\lim_{r\to\rho} \frac{\int_w^1 (F(t))^{1/2}dt}{w(F(w))^{1/2}\log(r-\rho)^{-1}} = \frac{1}{2}.$$
Hence, for r near ρ,
$$\frac{\int_w^1 (F(t))^{1/2}dt}{F(w)} \leq C_4(F(w))^{-1/2} w\log(r-\rho)^{-1}. \tag{3.17}$$
Since $\phi' = (2F(\phi))^{1/2}$, (3.16) and (3.17) imply
$$w(r) < \phi(r-\rho) + C_5(r-\rho)\Big(\frac{F(\phi)}{F(w)}\Big)^{1/2} w\log(r-\rho)^{-1}.$$
By (3.7) and (2.23) (with w instead of v) we have
$$\Big(\frac{F(\phi)}{F(w)}\Big)^{1/2} w \leq C_6\phi.$$
Hence,
$$w(r) < \phi(r-\rho)\big(1 + C_7(r-\rho)\log(r-\rho)^{-1}\big).$$
For x near to P, this estimate and the inequality $u(x) \leq w(x)$ yield the right hand side of (3.12). The lemma is proved. $\square$

To state the next theorem, we define
$$H(x) = \sum_{i=1}^{N-1} \frac{-k_i}{1-k_i\delta}, \tag{3.18}$$
where $\delta = \delta(x)$ denotes the distance from x to $\partial\Omega$, and $k_i = k_i(\overline{x})$ denote the principal curvatures of $\partial\Omega$ at $\overline{x}$, the nearest point to x. We note that in several papers, instead of $H(x)$, the function $\frac{1}{N-1}H(x)$ is considered.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain, and let $f(t) > 0$ be smooth, decreasing and satisfy* (3.1), *as well as* (1.3). *If $u(x)$ is a solution to problem* (1.1), *then*
$$\phi(\delta)\Big[1 + \frac{1}{4}H\delta\log\delta - C\delta(-\log\delta)^{\sigma}\Big] < u(x) < \phi(\delta)\Big[1 + \frac{1}{4}H\delta\log\delta + C\delta(-\log\delta)^{\sigma}\Big],$$

where ϕ is defined as in (1.4), *$H = H(x)$ is defined as in* (3.18), *$0 < \sigma < 1$ and C is a suitable constant.*

Proof. We look for a super-solutions of the kind

$$w(x) = \phi(\delta)\Big[1 + A\delta\log\delta + \alpha\delta(-\log\delta)^{\sigma}\Big], \quad A = \frac{H}{4},$$

where α is a positive constant to be determined. We have

$$\begin{aligned} w_{x_i} = \phi'\delta_{x_i}\Big[1 + A\delta\log\delta + \alpha\delta(-\log\delta)^{\sigma}\Big] + \phi\Big[A_{x_i}\delta\log\delta \\ + A\log(e\delta)\delta_{x_i} + \alpha\delta_{x_i}(-\log\delta)^{\sigma} - \alpha\sigma\delta_{x_i}(-\log\delta)^{\sigma-1}\Big]. \end{aligned}$$

We know that (see for example [7, page 355])

$$\sum_{i=1}^{N}\delta_{x_i}\delta_{x_i} = 1, \quad \sum_{i=1}^{N}\delta_{x_ix_i} = -H. \tag{3.19}$$

Using (3.19) we find

$$\begin{aligned} \Delta w = \phi''\Big[1 + A\delta\log\delta + \alpha\delta(-\log\delta)^{\sigma}\Big] - \phi' H\Big[1 + A\delta\log\delta + \alpha\delta(-\log\delta)^{\sigma}\Big] \\ + 2\phi'\Big[\nabla A\cdot\nabla\delta\ \delta\log\delta + A + A\log\delta + \alpha(-\log\delta)^{\sigma} - \alpha\sigma(-\log\delta)^{\sigma-1}\Big] \\ + \phi\Big[\Delta A\ \delta\log\delta + 2\nabla A\cdot\nabla\delta\log(e\delta) + A\delta^{-1} - AH\log(e\delta) - \alpha H(-\log\delta)^{\sigma} \\ - \alpha\sigma(-\log\delta)^{\sigma-1}\delta^{-1} + \alpha\sigma H(-\log\delta)^{\sigma-1} + \alpha\sigma(\sigma-1)(-\log\delta)^{\sigma-2}\delta^{-1}\Big]. \end{aligned}$$

By using the equation $\phi'' = -f(\phi)$, as well as (3.8) and (3.10), we find

$$\begin{aligned} \Delta w = f(\phi)\Big\{-1 - A\delta\log\delta - \alpha\delta(-\log\delta)^{\sigma} - \big(2 + O(1)\phi^{\beta}\big)\delta H\Big[1 + A\delta\log\delta \\ + \alpha\delta(-\log\delta)^{\sigma}\Big] + 2\big(2 + O(1)\phi^{\beta}\big)\delta\Big[\nabla A\cdot\nabla\delta\ \delta\log\delta + A + A\log\delta \\ + \alpha(-\log\delta)^{\sigma} - \alpha\sigma(-\log\delta)^{\sigma-1}\Big] + \big(4 + O(1)\phi^{\beta}\big)\delta^2\Big[\Delta A\ \delta\log\delta + A\delta^{-1} \\ + 2\nabla A\cdot\nabla\delta\log(e\delta) - AH\log(e\delta) - \alpha H(-\log\delta)^{\sigma} - \alpha\sigma(-\log\delta)^{\sigma-1}\delta^{-1} \\ + \alpha\sigma H(-\log\delta)^{\sigma-1} + \alpha\sigma(\sigma-1)(-\log\delta)^{\sigma-2}\delta^{-1}\Big]\Big\}. \end{aligned}$$

After some simplification,

$$\begin{aligned} \Delta w = f(\phi)\Big\{-1 + 3A\delta\log\delta + 3\alpha\delta(-\log\delta)^{\sigma} - 2H\delta + O(1)\delta^2\log\delta + O(1)\phi^{\beta}\delta\log\delta \\ + 8A\delta - 8\alpha\sigma\delta(-\log\delta)^{\sigma-1} + \alpha O(1)\phi^{\beta}\delta(-\log\delta)^{\sigma} + \alpha O(1)\delta(-\log\delta)^{\sigma-2}\Big\}. \end{aligned}$$

Hence, since $-2H + 8A = 0$, for some positive constants C_1, C_2 and C_3 we have

$$\begin{aligned} \Delta w < f(\phi)\Big\{-1 + 3A\delta\log\delta + C_1\delta^2(-\log\delta) + C_2\phi^{\beta}\delta(-\log\delta) \\ + \alpha\delta(-\log\delta)^{\sigma}\Big[3 - 8\sigma(-\log\delta)^{-1} + C_3(-\log\delta)^{-2}\Big]\Big\}. \end{aligned} \tag{3.20}$$

Note that (3.11) has been used to compare $\phi^{\beta}\delta(-\log\delta)^{\sigma}$ with $\delta(-\log\delta)^{\sigma-2}$.

On the other hand, using Taylor's expansion we have

$$\begin{aligned} f(w) = f(\phi)\Big\{ 1 + \phi \frac{f'(\phi)}{f(\phi)} \big[A\delta \log\delta + \alpha\delta(-\log\delta)^\sigma \big] \\ + \phi^2 \frac{f''(\overline{\phi})}{2f(\phi)} \big[A\delta\log\delta + \alpha\delta(-\log\delta)^\sigma \big]^2 \Big\}, \end{aligned} \tag{3.21}$$

with $\overline{\phi}$ between ϕ and $\phi\big(1 + A\delta\log\delta + \alpha\delta(-\log\delta)^\sigma\big)$. We consider points $x \in \Omega$ such that

$$-\frac{1}{2} < A\delta\log\delta + \alpha\delta(-\log\delta)^\sigma < 1. \tag{3.22}$$

This means that $1/2 < 1 + A\delta\log\delta + \alpha\delta(-\log\delta)^\sigma < 2$; therefore, the term $\overline{\phi}$ which appears in (3.21) satisfies $\overline{\phi} = \theta\phi$, with $1/2 < \theta < 2$. Using the estimates (3.5) and (1.3), by (3.21) we find

$$\begin{aligned} f(w) = f(\phi)\Big\{ 1 + \big(-3 + O(1)\phi^\beta\big) A\delta\log\delta + O(1)(\delta\log\delta)^2 \\ + \alpha\delta(-\log\delta)^\sigma \Big[-3 + O(1)\phi^\beta + O(1)\alpha\delta(-\log\delta)^\sigma \Big] \Big\}. \end{aligned} \tag{3.23}$$

By (3.23), we can take suitable positive constants C_4, C_5, C_6 and C_7 such that

$$\begin{aligned} f(w) < f(\phi)\Big\{ 1 - 3A\delta\log\delta + C_4\phi^\beta\delta(-\log\delta) + C_5(\delta\log\delta)^2 \\ + \alpha\delta(-\log\delta)^\sigma \Big[-3 + C_6\phi^\beta + C_7\alpha\delta(-\log\delta)^\sigma \Big] \Big\}. \end{aligned} \tag{3.24}$$

By (3.20) and (3.24) we have

$$\Delta w + f(w) < 0 \tag{3.25}$$

whenever

$$\begin{aligned} & C_1\delta^2(-\log\delta) + C_2\phi^\beta\delta(-\log\delta) + \alpha\delta(-\log\delta)^\sigma \Big[-8\sigma(-\log\delta)^{-1} + C_3(-\log\delta)^{-2} \Big] \\ & + C_4\phi^\beta\delta(-\log\delta) + C_5(\delta\log\delta)^2 + \alpha\delta(-\log\delta)^\sigma \Big[C_6\phi^\beta + C_7\alpha\delta(-\log\delta)^\sigma \Big] < 0. \end{aligned}$$

Rearranging we find

$$\begin{aligned} & C_1\delta(-\log\delta)^{2-\sigma} + (C_2 + C_4)\phi^\beta(-\log\delta)^{2-\sigma} + C_5\delta(-\log\delta)^{3-\sigma} \\ & < \alpha\Big[8\sigma - C_3(-\log\delta)^{-1} - C_6\phi^\beta(-\log\delta) - C_7\alpha\delta(-\log\delta)^{1+\sigma} \Big]. \end{aligned} \tag{3.26}$$

Since, by (3.11), $\phi^\beta \le C\delta^{\frac{\beta}{2}}$, and since $\sigma > 0$, (3.26) holds for α fixed and δ small enough.

Using Lemma 3.2 we find

$$w(x) - u(x) \ge \phi(\delta)\big(-\log\delta\big)^{-1} \big[-A\delta(\log\delta)^2 + \alpha\delta(-\log\delta)^{1+\sigma} - C\delta(\log\delta)^2 \big].$$

If α and δ are such that (3.22) and (3.26) hold, define $q = \alpha\delta(-\log\delta)^{1+\sigma}$ and decrease δ (increasing α) so that $\alpha\delta(-\log\delta)^{1+\sigma} = q$ until

$$-A\delta(\log\delta)^2 + q - C\delta(\log\delta)^2 > 0$$

for $\delta(x) = \delta_1$. Then, applying the comparison principle to (3.25) and (1.1) we find

$$w(x) \ge u(x), \quad x \in \Omega : \delta(x) < \delta_1.$$

By a similar argument one finds a sub-solution of the kind

$$w(x) = \phi(\delta)\Big(1 + A\delta\log\delta - \alpha\delta(-\log\delta)^\sigma \Big),$$

where A and σ are the same as before and α is a suitable positive constant. The theorem follows. $\square$

4. The case $\gamma = \infty$

Let $f(t)$ be a smooth, decreasing and positive function in $(0,\infty)$. In this section we assume conditions (1.7) and (1.10). By (1.7) one finds positive constants c_1, c_2, ℓ_1 and ℓ_2 such that

$$c_1 e^{\ell_1/t^\beta} < f(t) < c_2 e^{\ell_2/t^\beta}, \quad t > 0. \tag{4.1}$$

Similarly, by (1.8) (which follows from (1.7)), one finds

$$c_3 e^{\ell_1/t^\beta} < F(t) < c_4 e^{\ell_2/t^\beta}, \quad t \in \big(0, \frac{1}{2}\big). \tag{4.2}$$

By (4.2), for $m > \ell_2 2^{\beta+1}/\ell_1$, we find

$$\sup_{0<t<1/2} \frac{(F(t))^{\frac{2}{m}}}{F(2t)} < \infty. \tag{4.3}$$

Lemma 4.1. *If* (1.7) *holds, we have*

$$\frac{\phi'(\delta)}{f(\phi(\delta))} = \delta + O(1)\delta(\phi(\delta))^\beta, \tag{4.4}$$

where $\phi(\delta)$ is defined as in (1.4).

Proof. Recall that (1.7) implies (1.9). Using (1.9) and the relation

$$-1 - 2\big[-1 + O(1)t^\beta\big] = 1 + O(1)t^\beta,$$

we have

$$-1 - 2F(t)f'(t)(f(t))^{-2} = 1 + O(1)t^\beta.$$

Multiplying by $(2F(t))^{-1/2}$ we find

$$-(2F(t))^{-1/2} - (2F(t))^{1/2} f'(t)(f(t))^{-2} = (2F(t))^{-1/2} + O(1)t^\beta(2F(t))^{-1/2},$$

and

$$\big((2F(t))^{1/2}(f(t))^{-1}\big)' = (2F(t))^{-1/2} + O(1)t^\beta(2F(t))^{-1/2}. \tag{4.5}$$

By (1.8) we have

$$\frac{(F(t))^{1/2}}{f(t)} = \frac{1}{(F(t))^{1/2}}\frac{F(t)}{f(t)} = \frac{1}{(F(t))^{1/2}}\frac{t^{\beta+1}}{\ell}\big(1 + O(1)t^\beta\big).$$

The latter estimate yields

$$\lim_{t\to 0}(F(t))^{1/2}(f(t))^{-1} = 0.$$

Hence, integrating (4.5) on $(0,s)$ we obtain

$$(2F(s))^{1/2}(f(s))^{-1} = \int_0^s \big(2F(t)\big)^{-1/2} dt + O(1)\int_0^s t^\beta (2F(t))^{-1/2} dt. \tag{4.6}$$

Since t^β is increasing we have

$$0 \le \int_0^s t^\beta (2F(t))^{-1/2} dt \le s^\beta \int_0^s (2F(t))^{-1/2} dt,$$

and equation (4.6) implies

$$\frac{(2F(s))^{1/2}}{f(s)} = \int_0^s \big(2F(t)\big)^{-1/2} dt + O(1)s^\beta \int_0^s (2F(t))^{-1/2} dt.$$

Putting $s = \phi(\delta)$ and recalling that $\phi'(\delta) = (2F(\phi(\delta)))^{1/2}$, (4.4) follows and the lemma is proved. □

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain, let $f(t) > 0$ be smooth, decreasing and satisfying* (1.7)*.$If u(x) is a solution to problem$*(1.1) *then*

$$\phi\big[1 - C\delta\phi^{\beta}\big] < u(x) < \phi\Big[1 + C\delta\Big(\frac{F(\phi)}{F(2\phi)}\Big)^{1/2}\phi^{\beta}\Big], \tag{4.7}$$

where $\phi = \phi(\delta)$ is defined as in (1.4)*, C is a suitable constant and $\delta = \delta(x)$ denotes the distance from x to $\partial\Omega$.*

Proof. We proceed as in the proof of Lemma 3.2 using the same notation. We prove first that our assumptions imply those of Lemma 2.1. Indeed, estimate (4.2) implies

$$\lim_{t\to 0}\int_t^1 (F(\tau))^{1/2}d\tau = \infty.$$

To prove the monotonicity of the function $s \mapsto (F(s))^{-1}\int_s^1 (F(t))^{1/2}dt$ for s close to 0, we claim that

$$\frac{d}{ds}\Big[(F(s))^{-1}\int_s^1 (F(t))^{1/2}dt\Big] = (F(s))^{-1/2}\Big[\frac{\int_s^1 (F(\tau))^{1/2}d\tau}{(F(s))^{3/2}(f(s))^{-1}} - 1\Big] > 0.$$

Indeed, using (1.9), for s close to 0 we have

$$\begin{aligned}
(F(s))^{3/2}(f(s))^{-1} &= -\int_s^1 \Big((F(t))^{3/2}(f(t))^{-1}\Big)'dt \\
&= \int_s^1 (F(t))^{1/2}\Big(\frac{3}{2} + F(t)f'(t)(f(t))^{-2}\Big)dt \\
&> \frac{1}{4}\int_s^1 (F(t))^{1/2}dt.
\end{aligned}$$

The above estimate and (4.2) yield

$$\lim_{s\to 0}(F(s))^{3/2}(f(s))^{-1} = +\infty.$$

Using de l'Hôpital rule and (1.9) we find

$$\lim_{s\to 0}\frac{\int_s^1 (F(\tau))^{1/2}d\tau}{(F(s))^{3/2}(f(s))^{-1}} = \lim_{s\to 0}\frac{1}{\frac{3}{2} + F(s)(f(s))^{-2}f'(s)} = 2.$$

It follows that

$$\frac{d}{ds}\Big[(F(s))^{-1}\int_s^1 (F(t))^{1/2}dt\Big] > 0,$$

as claimed.

Now we can use Lemma 2.1 and its Corollary. By (2.1),

$$v(r) > \phi(R-r) - C\frac{\int_v^1 (F(t))^{1/2}dt}{(F(v))^{1/2}}(R-r), \quad \tilde{r} < r < R. \tag{4.8}$$

By (4.2) we have

$$\lim_{t\to 0} t^{\beta+1}(F(t))^{1/2} = +\infty.$$

Using de l'Hôpital rule and (1.8) we find

$$\lim_{t\to 0}\frac{\int_t^1 (F(\tau))^{1/2}d\tau}{t^{\beta+1}(F(t))^{1/2}} = \lim_{t\to 0}\frac{1}{-(\beta+1)t^{\beta}+\frac{t^{\beta+1}f(t)}{2F(t)}} = \frac{2}{\ell}. \tag{4.9}$$

Equations (4.8) and (4.9) imply

$$v(r) > \phi(R-r) - C_1(v(r))^{\beta+1}(R-r).$$

By (2.22), $v(r) < \phi(R-r)$. Hence,

$$v(r) > \phi(R-r)\big[1 - C_1(\phi(R-r))^{\beta}(R-r)\big]. \tag{4.10}$$

Arguing as in the proof of Lemma 3.2, one proves that (4.10) implies the left hand side of (4.7).

By (2.2) of Lemma 2.1 (with w in place of v) we have

$$w(r) < \phi(r-\rho) + C\phi'(r-\rho)\frac{\int_w^1 (F(t))^{1/2}dt}{F(w)}(r-\rho), \quad \rho < r < \tilde{r}. \tag{4.11}$$

By (4.9) we can find a constant C_2 such that

$$\frac{\int_w^1 (F(t))^{1/2}dt}{F(w)} \le C_2\frac{1}{(F(w))^{1/2}}w^{\beta+1}.$$

By using this estimate and the equation $\phi' = (2F(\phi))^{1/2}$, from (4.11) we find

$$w(r) < \phi + C_3(r-\rho)\Big(\frac{F(\phi)}{F(w)}\Big)^{1/2}w^{\beta+1}. \tag{4.12}$$

By (2.23) (with w in place of v and with $\epsilon = 1$), for r close to ρ we have $w(r) < 2\phi(r-\rho)$. Hence, from (4.12) we find

$$w(r) < \phi\Big[1 + C_4(r-\rho)\Big(\frac{F(\phi)}{F(2\phi)}\Big)^{1/2}\phi^{\beta}\Big].$$

Proceeding as in the proof of Lemma 3.2, we obtain the right hand side of (4.7). The proof is complete. $\square$

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded smooth domain, let $f(t)$ be smooth, decreasing and satisfying* (1.7) *and* (1.10)*. If $u(x)$ is a solution to problem* (1.1) *then*

$$\phi\Big[1 - \frac{1}{\ell}H\delta\phi^{\beta} - C\delta\phi^{2\beta}\Big] \le u(x) \le \phi\Big[1 - \frac{1}{\ell}H\delta\phi^{\beta} + C\delta\phi^{2\beta}\Big],$$

where $\phi = \phi(\delta)$ is defined as in (1.4)*, $H = H(x)$ is defined as in* (3.18)*, and C is a suitable positive constant.*

Proof. We look for a super-solution of the form

$$w(x) = \phi(\delta) - A\delta\phi^{\beta+1} + \alpha\delta\phi^{2\beta+1}, \quad A = \frac{1}{\ell}H,$$

where α is a positive constant to be determined. We have

$$w_{x_i} = \phi'\delta_{x_i} - A_{x_i}\delta\phi^{\beta+1} - A\big[\phi^{\beta+1} + (\beta+1)\delta\phi^{\beta}\phi'\big]\delta_{x_i} + \alpha\big[\phi^{2\beta+1} + (2\beta+1)\delta\phi^{2\beta}\phi'\big]\delta_{x_i}.$$

Recalling (3.19) we find

$$\begin{aligned}\Delta w &= \phi'' - \phi' H - \Delta A \delta \phi^{\beta+1} - 2\nabla A\cdot\nabla\delta\big(\phi^{\beta+1} + (\beta+1)\delta\phi^\beta\phi'\big)\\ &\quad - A\big[2(\beta+1)\phi^\beta\phi' + (\beta+1)\beta\delta\phi^{\beta-1}(\phi')^2 + (\beta+1)\delta\phi^\beta\phi''\big]\\ &\quad + AH\big[\phi^{\beta+1} + (\beta+1)\delta\phi^\beta\phi'\big] + \alpha\big[2(2\beta+1)\phi^{2\beta}\phi' + (2\beta+1)2\beta\delta\\ &\quad - \phi^{2\beta-1}(\phi')^2 + (2\beta+1)\delta\phi^{2\beta}\phi'' - \big(\phi^{2\beta+1} + (2\beta+1)\delta\phi^{2\beta}\phi'\big)H\big].\end{aligned}\tag{4.13}$$

Equation (4.4) yields

$$\phi' = \big[1 + O(1)\phi^\beta\big]\delta f(\phi).\tag{4.14}$$

Since $\phi'' = -f(\phi)$, by (4.13) and (4.14) we find

$$\begin{aligned}\Delta w = f(\phi)\Big[&-1 - H\delta + O(1)\delta\phi^\beta + O(1)\frac{\phi^{\beta+1}}{f(\phi)} + O(1)\delta^3\phi^{\beta-1}f(\phi)\\ &+ \alpha O(1)\delta\phi^{2\beta} + \alpha O(1)\frac{\phi^{2\beta+1}}{f(\phi)} + \alpha O(1)\delta^3\phi^{2\beta-1}f(\phi)\Big].\end{aligned}\tag{4.15}$$

We claim that, for δ small,

$$\frac{\phi^{\beta+1}}{f(\phi)} \le \delta\phi^\beta.\tag{4.16}$$

Rewrite (4.16) as

$$\frac{\phi}{\delta f(\phi)} \le 1.$$

The latter inequality follows by the statement

$$\begin{aligned}\lim_{\delta\to 0}\frac{\phi}{\delta f(\phi)} &= \lim_{t\to 0}\frac{t(f(t))^{-1}}{\psi(t)} = \lim_{t\to 0}\frac{(f(t))^{-1} - t(f(t))^{-2}f'(t)}{(2F(t))^{-1/2}}\\ &= \lim_{t\to 0}\Big[\Big(\frac{2F(t)}{f(t)}\Big)^{1/2}\frac{1}{(f(t))^{1/2}} - \frac{t}{(2F(t))^{1/2}}\frac{2F(t)f'(t)}{(f(t))^2}\Big] = 0.\end{aligned}$$

In the last step we have used (1.8), (1.9), (4.1) and (4.2).

Now we claim that, for δ small,

$$\delta^3\phi^{\beta-1}f(\phi) \le \delta\phi^\beta.\tag{4.17}$$

Rewrite (4.17) as

$$\frac{\delta^2 f(\phi)}{\phi} \le 1.$$

The latter inequality follows by the statement

$$\begin{aligned}\lim_{\delta\to 0}\frac{\delta}{\phi^{1/2}(f(\phi))^{-1/2}} &= \lim_{t\to 0}\frac{\psi(t)}{t^{1/2}(f(t))^{-1/2}}\\ &= \lim_{t\to 0}\frac{2(2F(t))^{-1/2}}{(tf(t))^{-1/2} - t^{1/2}(f(t))^{-\frac{3}{2}}f'(t)}\\ &= \lim_{t\to 0}\frac{\sqrt{2}\Big(\frac{F(t)}{tf(t)}\Big)^{1/2}}{\frac{F(t)}{tf(t)} - \frac{F(t)f'(t)}{(f(t))^2}} = 0,\end{aligned}$$

where (1.8) and (1.9) have been used.

Let us consider now the terms containing α. By (4.16), for δ small we have

$$\frac{\phi^{2\beta+1}}{f(\phi)} \le \delta\phi^{2\beta}.\tag{4.18}$$

Finally, by (4.17) we find

$$\delta^3\phi^{2\beta-1}f(\phi) \le \delta\phi^{2\beta}. \tag{4.19}$$

Therefore, by (4.15) and estimates (4.16)-(4.19), we find suitable positive constants M_1, M_2, such that

$$\Delta w < f(\phi)\big[-1 - H\delta + M_1\delta\phi^\beta + \alpha M_2\delta\phi^{2\beta}\big]. \tag{4.20}$$

On the other hand, by Taylor's formula we have

$$f(t+\omega t) = f(t)\Big[1 + \frac{tf'(t)}{f(t)}\omega + \frac{1}{2}\frac{t^2f''(\theta t)}{f(t)}\omega^2\Big], \tag{4.21}$$

where θ is between 1 and $1+\omega$. If $-\epsilon < \omega < \epsilon$ we can use (1.10); using also (1.7), from (4.21) we find

$$f(t+\omega t) = f(t)\Big[1 - \frac{\ell}{t^\beta}\big(1 + O(1)t^\beta\big)\omega + O(1)\frac{1}{t^{2\beta}}(F(t))^{1/m}\omega^2\Big].$$

Here m is so large that (1.10) and (4.3) hold. Let

$$\omega = -A\delta\phi^\beta + \alpha\delta\phi^{2\beta},$$

and take α and δ_0 so that, for $\{x \in \Omega : \delta(x) < \delta_0\}$

$$-\epsilon < -A\delta\phi^\beta + \alpha\delta\phi^{2\beta} < \epsilon. \tag{4.22}$$

With $t = \phi(\delta)$ we have $t + t\omega = w$, and

$$\begin{aligned}
f(w) &= f(\phi)\Big[1 - \ell\big(1 + O(1)\phi^\beta\big)\big(-A\delta + \alpha\delta\phi^\beta\big) + O(1)\Big(-A\delta + \alpha\delta\phi^\beta\Big)^2(F(\phi))^{1/m}\Big] \\
&= f(\phi)\Big[1 + \ell A\delta - \alpha\ell\delta\phi^\beta + O(1)\delta\phi^\beta + \alpha O(1)\delta\phi^{2\beta} + O(1)\delta^2(F(\phi))^{1/m} \\
&\quad + \alpha^2 O(1)\delta^2\phi^{2\beta}(F(\phi))^{1/m}\Big].
\end{aligned}$$

Note that, using (1.8), (4.2), and recalling that $m > 2$ we find

$$\begin{aligned}
0 \le \lim_{\delta\to 0}\frac{\delta^2(F(\phi))^{1/m}}{\delta\phi^\beta} &= \lim_{\delta\to 0}\frac{\delta}{\phi^\beta(F(\phi))^{-1/m}} = \lim_{t\to 0}\frac{\psi(t)}{t^\beta(F(t))^{-1/m}} \\
&= \lim_{t\to 0}\frac{(2F(t))^{-1/2}}{\beta t^{\beta-1}(F(t))^{-1/m} + \frac{1}{m}t^\beta(F(t))^{-\frac{1}{m}-1}f(t)} \\
&\le \frac{m}{\sqrt{2}}\lim_{t\to 0}\frac{F(t)}{f(t)}\frac{1}{t^\beta(F(t))^{\frac{1}{2}-\frac{1}{m}}} = 0.
\end{aligned}$$

Hence, we can find positive constants M_3, M_4, M_5 such that

$$f(w) < f(\phi)\big[1 + \ell A\delta - \alpha\ell\delta\phi^\beta + M_3\delta\phi^\beta + \alpha M_4\delta\phi^{2\beta} + \alpha^2 M_5\delta^2\phi^{2\beta}(F(\phi))^{1/m}\big].$$

Recalling that $H = \ell A$, by (4.20) and the latter inequality we have

$$\Delta w + f(w) < 0 \tag{4.23}$$

provided

$$M_1\delta\phi^\beta + \alpha M_2\delta\phi^{2\beta} - \alpha\ell\delta\phi^\beta + M_3\delta\phi^\beta + \alpha M_4\delta\phi^{2\beta} + \alpha^2 M_5\delta^2\phi^{2\beta}(F(\phi))^{1/m} < 0.$$

Rearranging we find

$$M_1 + M_3 < \alpha\big[\ell - (M_2 + M_4)\phi^\beta - \alpha M_5\delta\phi^\beta(F(\phi))^{1/m}\big]. \tag{4.24}$$

Since

$$\lim_{\delta\to 0}\delta(F(\phi))^{1/m}=\lim_{t\to 0}\psi(t)(F(t))^{1/m}\le\lim_{t\to 0}t(F(t))^{\frac{1}{m}-\frac{1}{2}}=0,$$

it follows that (4.24) holds for δ small and α large.

Using the right hand side of (4.7) we have

$$w-u>\phi^{\beta+1}(F(\phi))^{-1/m}\Big[-A\delta(F(\phi))^{1/m}+\alpha\delta\phi^{\beta}(F(\phi))^{1/m}-C\delta\frac{(F(\phi))^{\frac{1}{2}+\frac{1}{m}}}{(F(2\phi))^{1/2}}\Big].$$

Take α_1 large and δ_1 small so that (4.22) and (4.24) hold for $\{x\in\Omega:\delta(x)<\delta_1\}$, and define

$$q=\alpha_1\delta_1\phi^{\beta}(F(\phi))^{1/m}.$$

Let us show that we can decrease δ increasing α according to $\alpha\delta\phi^{\beta}(F(\phi))^{1/m}=q$ until

$$-A\delta(F(\phi))^{1/m}+q-C\delta\frac{(F(\phi))^{\frac{1}{2}+\frac{1}{m}}}{(F(2\phi))^{1/2}}>0 \tag{4.25}$$

for $\{x\in\Omega:\delta(x)=\delta_2\}$. Indeed, we have

$$0\le\lim_{\delta\to 0}\delta(F(\phi))^{1/m}=\lim_{t\to 0}\psi(t)(F(t))^{1/m}\le\lim_{t\to 0}(F(t))^{-\frac{1}{2}+\frac{1}{m}}=0.$$

Furthermore, using (4.3) we find

$$0\le\lim_{\delta\to 0}\delta\frac{(F(\phi))^{\frac{1}{2}+\frac{1}{m}}}{(F(2\phi))^{1/2}}=\lim_{t\to 0}\frac{\psi(t)(F(t))^{\frac{1}{2}+\frac{1}{m}}}{(F(2t))^{1/2}}\le\lim_{t\to 0}\frac{t(F(t))^{1/m}}{(F(2t))^{1/2}}=0.$$

If (4.25) holds, then $w-u>0$ for $\delta(x)=\delta_2$. Since $w-u=0$ on $\partial\Omega$, by (4.23) and (1.1) we have $w-u\ge 0$ on $\{x\in\Omega:\delta(x)<\delta_2\}$. We have proved that, for C large,

$$u(x)<\phi\Big[1-\frac{1}{\ell}H\delta\phi^{\beta}+C\delta\phi^{2\beta}\Big].$$

In a very similar manner, using the left hand side of (4.7), one finds that

$$v=\phi-\frac{1}{\ell}H\delta\phi^{\beta+1}-\alpha\delta\phi^{2\beta+1},$$

satisfies $v-u\le 0$ in a neighborhood of $\partial\Omega$ provided α is large enough. The proof is complete. $\square$

References

[1] C. Anedda; *Second order boundary estimates for solutions to singular elliptic equations*, Electronic Journal of Differential Equations, Vol 2011 (2011), no. 90, 1-15.

[2] C. Anedda, F. Cuccu and G. Porru; *Boundary estimates for solutions to singular elliptic equations*, Le Matematiche, LX (2005), 339-352.

[3] S. Berhanu, F. Cuccu and G. Porru; *On the boundary behaviour, including second order effects, of solutions to singular elliptic problems*, Acta Math. Sin. (Engl. Ser.), 23 (2007), 479-486.

[4] M. G. Crandall, P. H. Rabinowitz and L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Part. Diff. Eq., 2 (1997), 193-222.

[5] F. Cuccu, E. Giarrusso and G. Porru; *Boundary behaviour for solutions of elliptic singular equations with a gradient term*, Nonlinear Analysis 69 (2008), 4550-4566.

[6] M. Ghergu and V. Rădulescu; *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl., 311 (2005), 635-646.

[7] D. Gilbarg and N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin (1977).

[8] B. Kawohl; *On a class of singular elliptic equations*, Progress in PDE: elliptic and parabolic problems, Pitman Res. Notes Math. Series, 266, Longman (1992), 156-163.

[9] A. Lazer and P. J. McKenna; *On a singular nonlinear elliptic boundary value problem*, Proc. American. Math. Soc., 111 (1991), 721-730.

[10] A. Lazer and P. J. McKenna; *Asymptotic behaviour of solutions of boundary blow-up problems*, Differential and Integral Equations, 7 (1994), 1001-1019.

[11] Z. Zhang and J. Cheng; *Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems*, Nonlinear Analysis, 57 (2004), 473-484.

[12] Z. Zhang and J. Yu; *On a singular nonlinear Dirichlet problem with a convection term*, SIAM J. Math. Anal. 32 (2000), 916-927.

Claudia Anedda

Dipartimento di Matematica e Informatica, Universitá di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

E-mail address: `canedda@unica.it`

Giovanni Porru

Dipartimento di Matematica e Informatica, Universitá di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

E-mail address: `porru@unica.it`