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Non-locality and entropic uncertainty relations in neutrino oscillations

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Abstract Using the wave-packet approach to neutrino oscillations, we analyze quantum-memory-assisted entropic uncertainty relations and show that uncertainty and the non-local advantage of quantum coherence are anti-correlated. Furthermore, we explore the hierarchy among three different definitions of NAQC, those based on l_1-norm, relative entropy and skew information coherence measures, and we find that the coherence content detected by the l_1-norm-based NAQC overcomes the other two. The connection between QMA-EUR and NAQC could provide a better understanding of the physical meaning of the results so far obtained and suggest their extension to quantum field theory.

1 Introduction

Coherence and quantum correlations play a central role in quantum information, quantum computation and cryptography [1], and recently the extension to particle physics of these topics has attracted much attention in view of potential applications in the above-cited fields. Different definitions and quantifications of coherence and of quantum correlations have been introduced, and it seems that they can find a natural and clear placing inside the complete complementarity relations (CCR) approach [2–4].

During the years, many studies have been carried on quantum correlation in neutrino systems [5–16], but only recently the attention has been placed on the study of uncertainty relation in neutrino oscillations [17, 18].

The uncertainty principle is one of the cardinal point of quantum mechanics. It provides a limit to our ability to accurately predict the measurement outcomes for a pair of incompatible observables of a quantum system. The principle was expressed for the first time by Heinsenberg, in 1927, for position and momentum [19] and was formulated by Kennard [20] in terms of variances. Afterward, it was generalized to any pair of incompatible observables and with the advent of Shannon information theory, the uncertainty relations have been expressed in terms of entropy. The first to wonder whether the uncertainty principle could be expressed in terms of entropy was Everett [21], who found Hirschman's support [22] and over the years many studies have addressed this topic [23–27].

Recently, advances have been made that allow a generalization of such studies to the case in which the parts of the considered system can be correlated in a non-classical way [28, 29]. In fact, the correlation between subsystems can be used to reduce the uncertainty below the usual limits, and the entropic uncertainty relations has been generalized to the case of quantum memory—quantum-memory-assisted entropic uncertainty relations (QMA-EUR).

In this paper, we consider the QMA-EUR in the physical context of neutrino oscillations. Moreover, to investigate what happens when correlations between systems are present, we consider the non-local advantage of quantum coherence (NAQC) [30], which has been proved to be the strongest one inside the whole hierarchy of quantum correlations, overtaking even the Bell non-locality. NAQC occurs in a bipartite system when the average coherence of the conditional state of a subsystem B, after a local measurements on A, exceeds the coherence limit of the single subsystem. Several definitions of NAQC have been formulated, which differ in the coherence measures adopted.

In this work, we analyze the relation between NAQC and uncertainty, using the relative entropy as coherence measure. Then, we note that the three criteria $N_\alpha(\rho_{AB}) > C_\alpha$ corresponding to $\alpha = l_1, re, sk$ (respectively, l_1-norm, relative entropy and skew information) may identify different regions of NAQC correlation, so NAQC is not only related to the form of the quantum state, but also to the choice of coherence measures. The question is whether there exists a hierarchy between these three different NAQC definitions in the case of neutrino oscillations, and assuming the value of the physical parameters of three different experiments, we provide a positive answer.

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At last, we want to remark that uncertainty is directly related to observational aspects and in particular to variances. Therefore, the connection with NAQC, highlighted in Ref. [17] and in this paper, could on the one hand clarify the physical meaning of the results obtained about in neutrino oscillations [4, 14, 16]; on the other hand, relying on the approach of [31, 32] helps to extend this result to quantum field theory in terms of variances of Hermitian operators (flavor changes), analogously to what done in Ref. [33]. The plan of the paper is as follows: in Sect. 2, we recall the notions of QMA-EUR and of NAQC and find their expression in terms of neutrino oscillation probabilities. We also investigate the relation between the NAQC and uncertainty using a wave packet approach, finding that the uncertainty is anti-correlated with the NAQC. In Sect. 3, we further analyze the NAQC in neutrino oscillations. It comes out that in this context a hierarchy between the three different NAQC is present. Section 4 is devoted to conclusions and outlook.

2 QMA-EUR and NAQC

Let us suppose Bob prepares a bipartite system ρ_{AB}, where the two parts A and B are correlated. Then, he sends part A to Alice and keeps part B as a quantum memory. After receiving A, Alice operates on it deciding to measure one of the observables P and R and tells to Bob her choice. Based on Alice's measurement choice, Bob is able to guess her outcomes with minimal deviation limited by the uncertainty's lower bound by means of the part B which is correlated with A.

The quantum memory can reduce the uncertainty, and the usual EUR, $H(P) + H(R) \geq -\log_2 c$, expressed in terms of Shannon entropy [29], is modified, in terms of von-Neumann entropy, as:

$$S(P|B) + S(R|B) \geq -\log_2 c(P|R) + S(A|B). \tag{1}$$

The quantities in Eq. (1) are the following:

- $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy of systemic state ρ_{AB} with $S(\rho_{AB}) = -tr(\rho_{AB}\log_2\rho_{AB})$, $\rho_B = tr_A(\rho_{AB})$
- $S(X|B) = S(\rho_{XB}) - S(\rho_B)$ is the conditional von Neumann entropy of $\rho_{XB} = \sum_i (|\psi_i^X\rangle_A\langle\psi_i^X| \mathcal{I}_B)\rho_{AB}(|\psi_i^X\rangle_A\langle\psi_i^X|\mathcal{I}_B)$ (that is the state of B after performing a measurement on A of the observable X with eigenstates $|\psi_i^X\rangle$),
- $c(P|R) = \max_{j,k}|\langle\psi_j^P|\phi_k^R\rangle|^2$ represents the maximal overlap between the eigenstates $|\psi_j^P\rangle$ and $|\phi_k^R\rangle$ of the observables P and R.

Since the correlations reduce uncertainty and the NAQC represents the strongest quantifier of quantum correlations, we verify its role within QMA-EUR. Several definitions of NAQC have been provided, based on different coherence measures. Given a state ρ in the reference basis $\{|i\rangle\}$, a measure of coherence takes the form:

$$C_D(\rho) = \min_{\delta\in I} D(\rho, \delta), \tag{2}$$

that is the minimum distance between ρ and the set of incoherent states I. $D(\rho, \delta)$ is a distance measure between two quantum states. For example, one can consider $D(\rho, \delta) = ||\rho - \delta||_{l_1}$, with $||.||_{l_1}$ is the l_1-norm or $D(\rho, \delta) = S(\rho||\delta)$, the quantum relative entropy. By minimizing over the set of incoherent states, one can obtain two bona fide measures of coherence [34] as:

$$C_{l_1}(\rho) = \sum_{i\neq j}|\langle i|\rho|j\rangle|, \quad C_{re}(\rho) = S(\rho_{diag}) - S(\rho), \tag{3}$$

where $S(\rho)$ is the von Neumann entropy of ρ and ρ_{diag} is the matrix of the diagonal elements of ρ.

Another possible coherence measure we can consider is the skew information [35]:

$$C_{sk}(\rho) = -\frac{1}{2}Tr\{[\sqrt{\rho}, K]^2\}, \tag{4}$$

where K is an Hermitian operator in a Hilbert space of dimension N.

Mondal et al. [30] defined the NAQC of a bipartite state ρ_{AB} considering the average coherence of the post-measurement state $\{p_{B|\Pi_i^a}, \rho_{B|\Pi_i^a}\}$ of B after a local measurement Π_i^a on A:

$$N_\alpha(\rho_{AB}) = \frac{1}{2}\sum_{i\neq j, a=\pm} p_{B|\Pi_i^a} C_\alpha^{\sigma_j}(\rho_{B|\Pi_i^a}), \tag{5}$$

where $\Pi_i^\pm = \frac{I\pm\sigma_i}{2}$, with I and σ_i, $(i = 1, 2, 3)$ being the identity and the three Pauli operators; $p_{B|\Pi_i^a} = \mathrm{Tr}(\Pi_i^a\rho_{AB})$, $\rho_{B|\Pi_i^a} = \mathrm{Tr}_A(\Pi_i^a\rho_{AB})/p_{B|\Pi_i^a}$. $C_\alpha^{\sigma_j}(\rho_{B|\Pi_i^a})$ is the coherence of the conditional state of B with respect to the eigenbasis of σ_j, with $\alpha = l_1, re, sk$. For a one-qubit state ρ, the sum of $C_\alpha(\rho)$ with respect to the three mutually unbiased bases is upper-bounded, respectively, by $C_{l_1} = \sqrt{6}$, $C_{re} = 2.23$ and $C_{sk} = 2$. If $N_\alpha(\rho_{AB}) > C_\alpha$, ρ_{AB} is said to have acquired NAQC.

2.1 QMA-EUR and NAQC in neutrino oscillations

At time t, the state for a two-flavor neutrino of initial flavor α is:

$$\rho^{\alpha}_{AB}(t) = |a_{\alpha\alpha}(t)|^2|10\rangle\langle 10| + a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t)|01\rangle\langle 10| + |a_{\alpha\beta}(t)|^2|01\rangle\langle 01| + a_{\alpha\alpha}(t)a^*_{\alpha\beta}(t)|10\rangle\langle 01| \tag{6}$$

where $\alpha, \beta = e, \mu, \tau$. We choose as incompatible observables $(P, R) = (\sigma_x, \sigma_y)$. In this case, the maximal overlap is $c(P, R) = \frac{1}{2}$. Starting by Eq. (6) (see Appendix 1), we find that entropic uncertainty U^{α} and the uncertainty's lower bound U^{α}_b are the LHS and the RHS of Eq. (1), respectively. They can be expressed in terms of oscillation probabilities[1] as:

$$U^{\alpha}(t) = S(\rho^{\alpha}_{PB}(t)) + S(\rho^{\alpha}_{RB}(t)) - 2S(\rho^{\alpha}_{B}(t)) = 2(P_{\alpha\alpha}(t)\log_2 P_{\alpha\alpha}(t) + P_{\alpha\beta}(t)\log_2 P_{\alpha\beta}(t) + 1) \tag{7}$$

$$U^{\alpha}_b(t) = S(\rho^{\alpha}_{AB}(t)) - S(\rho^{\alpha}_{B}(t)) - \log_2 c(P, R) = P_{\alpha\alpha}(t)\log_2 P_{\alpha\alpha}(t) + P_{\alpha\alpha}(t)\log_2 P_{\alpha\beta}(t) + 1 \tag{8}$$

Since the QMA-EUR is expressed in terms of von Neumann entropy, we consider the entropy-based NAQC to investigate the relation between these quantities. The expression of the entropy-based NAQC in terms of oscillation probabilities is:

$$N(\rho^{\alpha}_{AB})(t) = 2 - P_{\alpha\alpha}(t)\log_2 P_{\alpha\alpha}(t) - P_{\alpha\beta}(t)\log_2 P_{\alpha\beta}(t) \tag{9}$$

By these equations, it is simple to obtain the result:

$$U^{\alpha}(t) = 2U^{\alpha}_b(t) = 2[3 - N(\rho^{\alpha}_{AB}(t))]. \tag{10}$$

Equation (10) shows how a stronger quantum correlation will lead to a reduced uncertainty.

In Fig. 1, the entropic uncertainty, the uncertainty's lower bound and the entropy-based NAQC are plotted using the experimental parameters from the Daya Bay [36, 37], KamLAND [38, 39] and MINOS [40, 41] experiments. These experiments studied neutrino oscillations with the aim of producing precision measurements of the neutrino mixing angle and of the neutrino squared mass difference. Daya-Bay and KamLAND are electron-antineutrino disappearance experiments, and their oscillation parameters are $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$ for Daya-Bay and $\tan^2 2\theta_{12} = 0.47$ and $\Delta m^2_{12} = 7.49 \times 10^{-5} eV^2$ for KamLAND. MINOS is a muonic neutrino disappearance experiment, and its oscillation parameters are $\sin^2 2\theta_{23} = 0.95$ and $\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2$. We used these parameters by replacing them in the oscillation probability formula, Eq. (30), in terms of which the NAQCs and the QMA-EUR are expressed.

The wave packet approach confirms the results by Wang et al. [17], obtained by means a plane-wave approximation, according to which the uncertainty is completely anti-correlated with the quantum correlations: the stronger the quantum correlation, the smaller the uncertainty. In particular, we have shown in [14] that in the wave packet approach the asymptotic trend of the NAQC depends on the mixing angle, with NAQC attaining its maximum value at great distances if the value of the mixing angle overcomes a threshold[2]. This happens in KamLAND and MINOS experiments, and consequently we see from Fig. 1 that the entropy uncertainty and its lower bound go to zero.

3 Hierarchy among α-based NAQCs in neutrino oscillations

In this section, we further analyze the NAQC in neutrino oscillations showing some interesting results, which arise from the comparison among the NAQCs based on the three different coherence measures mentioned in Sect. 2.

The expression of the entropy-based NAQC in terms of oscillation probabilities is given by Eq. (9), and analogously, following Ref. [30], we determine the expressions of the l_1-norm-based NAQC and of the skew information-based NAQC:

$$N^{l_1}(\rho^{\alpha}_{AB}(t)) = 2 + 2\sqrt{P_{\alpha\alpha}(t)P_{\alpha\beta}(t)}, \tag{11}$$

and

$$N^{sk}(\rho^{\alpha}_{AB}(t)) = 2 + 4P_{\alpha\alpha}(t)P_{\alpha\beta}(t). \tag{12}$$

In Fig. 2, we compare the plots of the NAQC obtained using the three different coherence measures and referring to MINOS, Daya Bay and KamLAND experiments.

From Fig. 2, we can observe how in the case of neutrino oscillations the l_1-norm-based NAQC is able to capture more quantum resource with respect to the others. Furthermore, it represents an upper limit for the other two measures. These observations confirm our previous results about the large distances behavior of NAQC [14] for all three cases. In Fig. 2, we also observe a different trend of the plots relating to the KamLAND experiment compared to the other two. In fact, it can be seen from Fig. 2b that there is a non-monotonous growth of the minima of the NAQC. In particular, notice how the second minimum is lower than the first. Indeed,

[1] The expression of neutrino oscillation probability in the wave packet approach is obtained in Appendix 5

[2] Note that threshold values are different if we consider NAQC based on different coherence measures.

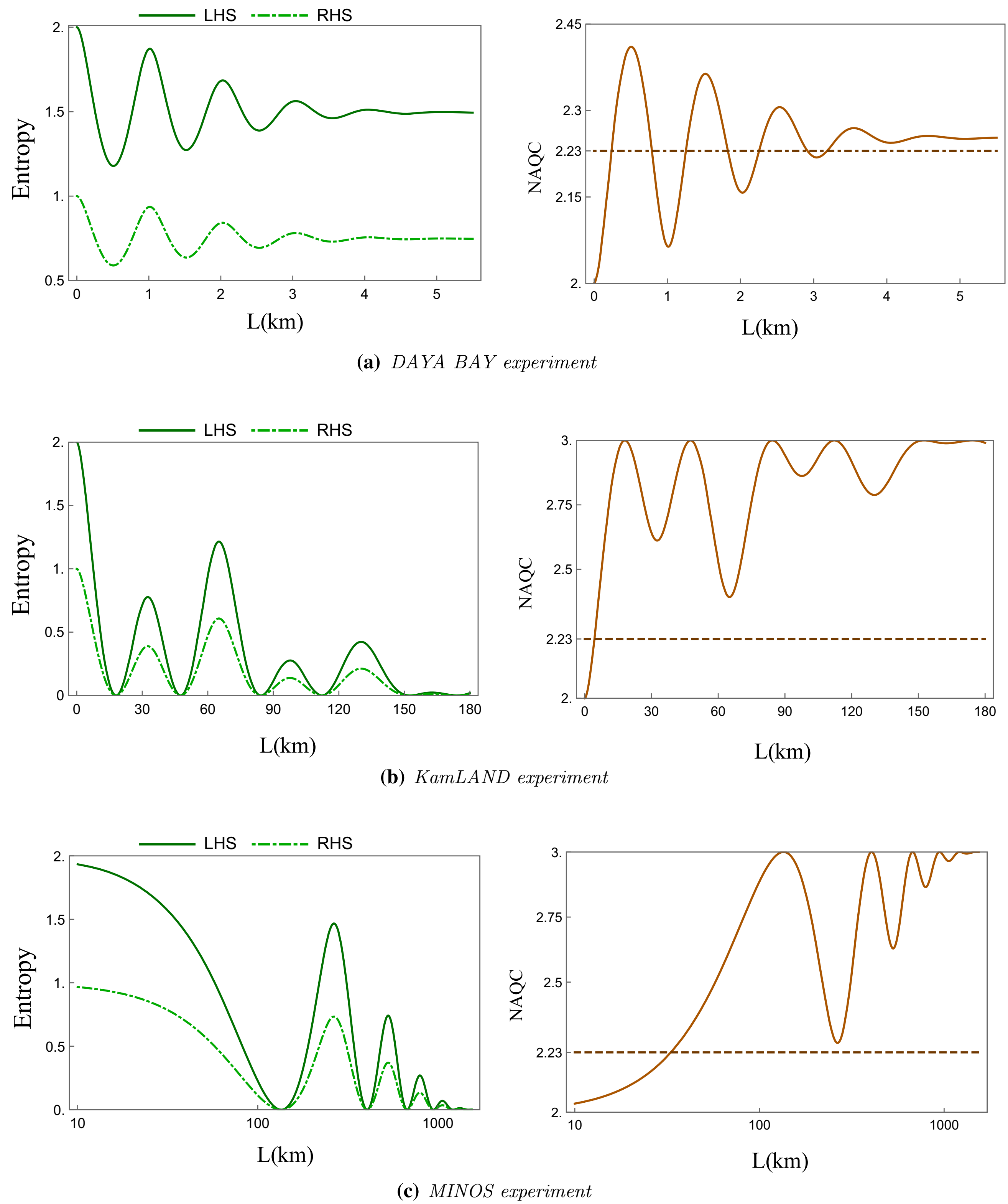


Fig. 1 QMA-EUR (on the left panel) and NAQC (on the right panel) as a function of the distance for: **a** Daya Bay, **b** KamLAND, **c** MINOS. The experimental parameters used are $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$ for Daya Bay, $\tan^2 2\theta_{12} = 0.47$ and $\Delta m^2_{12} = 7.49 \times 10^{-5} eV^2$ for KamLAND and $\sin^2 2\theta_{23} = 0.95$ and $\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2$ for MINOS

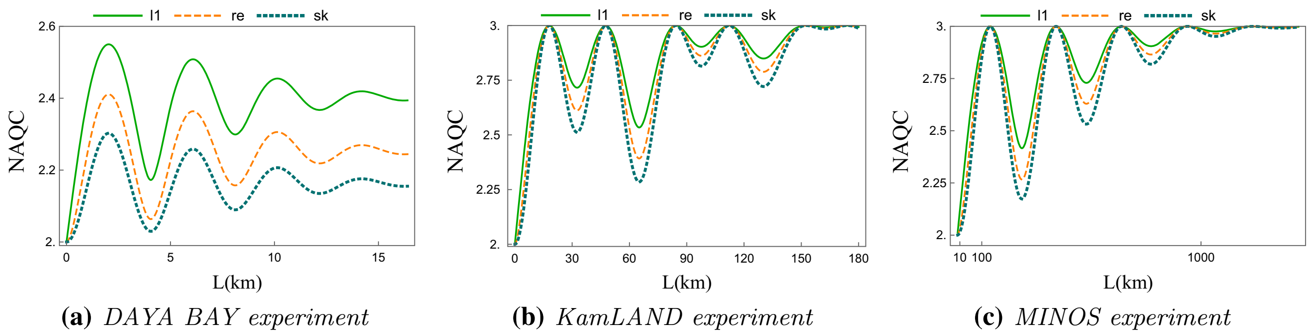
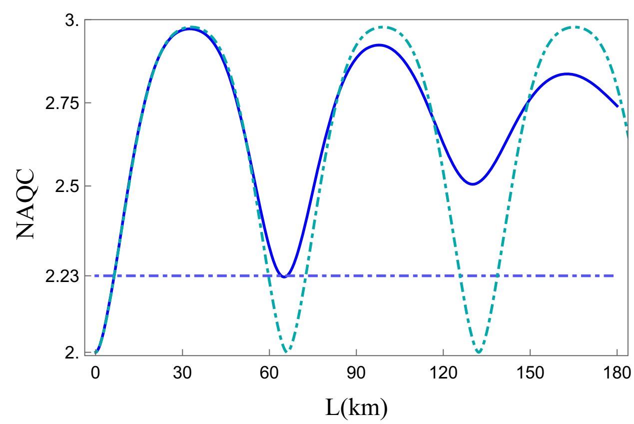


Fig. 2 Comparison among the α-based NAQCs, ($\alpha = l_1, re, sk$) for Daya Bay, KamLAND and MINOS experiments

Fig. 3 Entropy-based NAQC as a function of the distance for a mixing angle of 20°. The solid and the dashed lines represent the plot for the wave packet and plane wave approaches, respectively



in [16] we found, in addition to the thresholds determining different asymptotic trends of the NAQCs, a further effect of varying the value of this angle. In an intermediate range of values of this angle, in fact, the monotonicity of NAQC is lost, and this is just the case of KamLAND experiment. Furthermore, we evaluated the mixing angle for which the minimum of the entropy-based NAQC coincides with the bound value 2.23, see Fig. 3. This minimum is positioned at L equal to π times the oscillation length, and the value of the mixing angle corresponding to it turns out to be 20°.

Figure 3 also shows a difference between the plots obtained with the plane wave and the wave packet approaches. In fact, by using plane waves we can reach a NAQC for certain regular range of distances. This does not happen in the case of wave packets, where above a certain distance we can always reach a NAQC.

4 Conclusions

The convergence between topics of quantum information and the physics of elementary particles, in particular of neutrino oscillations has led in this last field to the development of a wide and systematic study of different definitions and quantifiers of coherence and quantum correlations. In this paper, using a wave-packet approach,[3] we considered the quantum-memory-assisted entropic uncertainty relation in the physical context of neutrino oscillations, investigating the relation between the non-local advantage of quantum coherence and uncertainty. We found that the uncertainty is anti-correlated with the NAQC, a result in accordance with what obtained by Wang et al. [17], in the plane-wave approximation. However, the richer structure generated by the wave-packet approach suggested that entropic uncertainty can go to zero asymptotically in the distance for sufficiently high values of the mixing angle.

Given the several definitions of NAQC based on three different coherence measures - l_1-norm, relative entropy and skew information—we then proceeded to further analyze the hierarchy among them. In particular, the l_1-norm-based NAQC resulted to be more able to capture quantum resources with respect the other two quantities. It is obviously interesting to investigate whether this hierarchy is confined to the instance of neutrino oscillations, or if it is true in any context. This question will be one subject of future investigations.

Finally, we reiterate what was observed in the introduction, i.e., that the anti-correlation between QMA-EUR and NAQC could represent a useful basis for a better understanding of the physical meaning of the various results recently obtained on NAQC in

3 The relevance of the wave packet approach has been discussed in connection with recent experiments in Refs. [42, 43].

neutrino oscillations [4, 14, 16]. Indeed, uncertainty relations could be generally formulated in terms of variances and thus are potentially more connected to quantities with an operational meaning, also in view of possible implementation of quantum protocols with neutrinos. We also envisage that the extension of the present results to quantum field theory, which we plan to perform relying on the approach of [31, 32], could be carried out in terms of variances of Hermitian operators (flavor charges), analogously to what done in Ref. [33].

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Data Availability This manuscript has no associated data or the data will not be deposited.

Appendix A: QMA-EUR and NAQC in terms of neutrino oscillation probability

We consider the state for a neutrino of flavor α:

$$|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)|\nu_\alpha\rangle + a_{\alpha\beta}(t)|\nu_\beta\rangle. \tag{13}$$

The corresponding density matrix in the orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is given by Eq. (6):

$$\rho^{\alpha}_{AB}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{\alpha\beta}(t)|^2 & a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t) & 0 \\ 0 & a_{\alpha\alpha}(t)a^*_{\alpha\beta}(t) & |a_{\alpha\alpha}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

A.1 QMA-EUR

We rewrite Eq. (1) for this density matrix, by considering as incompatible observables P and R the Pauli matrix σ_x and σ_y. So we have to evaluate:

$$S(\sigma_x|B) + S(\sigma_y|B) \geq -\log_2 c(\sigma_x|\sigma_y) + S(A|B) \tag{15}$$

where

$$\begin{aligned} S(\sigma_x|B) &= S(\rho_{\sigma_x B}) - S(\rho_B) \\ S(\sigma_y|B) &= S(\rho_{\sigma_y B}) - S(\rho_B) \end{aligned} \tag{16}$$

with:

$$\rho_{\sigma_x B} = \sum_{i=1,2} (|x_i\rangle\rangle x_i|)\rho_{AB}(|x_i\rangle\rangle x_i|) = \frac{1}{2}\begin{pmatrix} |a_{\alpha\alpha}(t)|^2 & 0 & 0 & a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t) \\ 0 & |a_{\alpha\beta}(t)|^2 & a_{\alpha\alpha}(t)a^*_{\alpha\beta} & 0 \\ 0 & a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t) & |a_{\alpha\alpha}(t)|^2 & 0 \\ a_{\alpha\alpha}(t)a^*_{\alpha\beta} & 0 & 0 & |a_{\alpha\beta}(t)|^2 \end{pmatrix} \tag{17}$$

$$\rho_{\sigma_y B} = \sum_{i=1,2} (|y_i\rangle\rangle y_i|)\rho_{AB}(|y_i\rangle\rangle y_i|) = \frac{1}{2}\begin{pmatrix} -|a_{\alpha\alpha}(t)|^2 & 0 & 0 & -a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t) \\ 0 & |a_{\alpha\beta}(t)|^2 & -a_{\alpha\alpha}(t)a^*_{\alpha\beta} & 0 \\ 0 & -a_{\alpha\beta}(t)a^*_{\alpha\alpha}(t) & |a_{\alpha\alpha}(t)|^2 & 0 \\ -a_{\alpha\alpha}(t)a^*_{\alpha\beta} & 0 & 0 & -|a_{\alpha\beta}(t)|^2 \end{pmatrix} \tag{18}$$

$|x_i\rangle$ and $|y_i\rangle$ are the eigenstates of σ_x and σ_y, respectively. ρ_B is given by the partial trace with respect to A of ρ_{AB}:

$$\rho_B = \begin{pmatrix} |a_{\alpha\alpha}(t)|^2 & 0 \\ 0 & |a_{\alpha\beta}(t)|^2 \end{pmatrix} \tag{19}$$

By considering the spectral representation $\rho = \sum_i \lambda_i |i\rangle\langle i|$, we can evaluate the von Neumann entropy as $S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$, where λ_i are the eigenvalues of ρ. Hence, we find $S(\rho_{\sigma_x B}) = -2 \cdot \frac{1}{2}(|a_{\alpha\alpha}(t)|^2 + |a_{\alpha\beta}(t)|^2)\log_2[\frac{1}{2}(|a_{\alpha\alpha}(t)|^2 + |a_{\alpha\beta}(t)|^2)]$. However, $|a_{\alpha\alpha}(t)|^2 = P_{\alpha\alpha}$ and $|a_{\alpha\beta}(t)|^2 = P_{\alpha\beta}$ are nothing more than survival and transition probability, respectively, such that $P_{\alpha\alpha} + P_{\alpha\beta} = 1$. Thus, $S(\rho_{\sigma_x B}) = 1$. We also find $S(\rho_B) = -P_{\alpha\alpha}\log_2 P_{\alpha\alpha} - P_{\alpha\beta}\log_2 P_{\alpha\beta}$. From Eq. (16), it is clear that $S(\sigma_x|B) = 1 + P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\beta}\log_2 P_{\alpha\beta}$. In the same way, we can evaluate $S(\sigma_y|B) = S(\sigma_x|B)$, $S(A|B) = P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\beta}\log_2 P_{\alpha\beta}$ and $c(\sigma_x, \sigma_y) = \frac{1}{2}$.

At this point, we are able to write the left-hand side and the right-hand side of Eq. (15), corresponding, respectively, to the entropic uncertainty and to the uncertainty's lower bound as:

$$U^{\alpha} = 2(P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\beta}\log_2 P_{\alpha\beta} + 1) \tag{20}$$

$$U_b^{\alpha} = P_{\alpha\alpha}\log_2 P_{\alpha\alpha} + P_{\alpha\alpha}\log_2 P_{\alpha\beta} + 1 \tag{21}$$

A.2 NAQC

Here we show the basic steps to write the NAQC in terms of neutrino probability.

Following [30], we decompose our density matrix, Eq. (6), as:

$$\rho_{AB} = \frac{1}{4}(I_4 + \mathbf{r}\cdot\boldsymbol{\sigma}\otimes I_2 + I_2\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \sum_{i,j} t_{ij}\sigma_i\otimes\sigma_j), \tag{22}$$

where $\mathbf{r}\equiv(r_x, r_y, r_z)$, $\mathbf{s}\equiv(s_x, s_y, s_z)$ and t_{ij} are the correlation matrix elements. The decomposition coefficients can be found as: $r_i = \mathrm{Tr}[\rho_{AB}(\sigma_i\otimes I_2)]$, $s_i = \mathrm{Tr}[\rho_{AB}(I_2\otimes\sigma_i)]$ and $t_{ij} = \mathrm{Tr}[\rho_{AB}(\sigma_i\otimes\sigma_j)]$, $(i, j = x, y, z)$, where σ_i are the Pauli matrices. We need these coefficients to evaluate Eq. (5), i.e.,

$$N_{\alpha}(\rho_{AB}) = \frac{1}{2}\sum_{i\neq j, a=\pm} p_{B|\Pi_i^a} C_{\alpha}^{\sigma_j}(\rho_{B|\Pi_i^a}), \tag{23}$$

Here, $p_{B|\Pi_i^a} = \frac{1}{2}\gamma_{ja}$ and $C_{\alpha}^{\sigma_j}(\rho_{B|\Pi_i^a})$ has a different definition depending on the coherence measure used:

- *l_1-norm:* $C_{l_1}^{\sigma_j} = \sqrt{\frac{\sum_{i\neq j}\alpha_{ik_a}^2}{\gamma_{k_a}^2}}$
- *relative entropy:* $C_{re}^{\sigma_j} = \sum_{p=+,-}\lambda_{k_a}^p\log_2\lambda_{k_a}^p - \beta_{jk_a}^p\log_2\beta_{jk_a}^p$
- *skew information:* $C_{sk}^{\sigma_j} = \frac{(\sum_{i\neq j}\alpha_{ik}^2)(1-\sqrt{1-(2\lambda_{k_a}^{\pm}-1)^2})}{\gamma_{k_a}^2(2\lambda_{k_a}^{\pm}-1)^2}$

where $\alpha_{ij_a} = s_i + (-1)^a t_{ij}$, $\gamma_{k_a} = 1 + (-1)^a r_k$, $\lambda_{i_a}^{\pm} = \frac{1}{2}\pm\frac{\sqrt{\sum_j\alpha_{ji_a}^2}}{2\gamma_{i_a}}$ and $\beta_{ij_a}^{\pm} = \frac{1}{2}\pm\frac{\alpha_{ija}}{2\gamma_{ja}}$.

Following these instructions, it is simple to obtain the expressions given by Eqs. (9,11,12) for the NAQCs.

Appendix B: wave packet description of neutrino oscillations

We briefly review the wave packet approach to neutrino oscillations [44, 45]. Let us consider a neutrino with flavor α, $(\alpha = e, \mu, \tau)$, propagating along x axis:

$$|\nu_{\alpha}\rangle(x, t) = \sum_j U_{\alpha j}^* \psi_j(x, t)|\nu_j\rangle, \tag{24}$$

where $U_{\alpha j}$ are the PMNS mixing matrix elements and $\psi_j(x, t)$ is the wave function of the mass eigenstate $|\nu_j\rangle$ with mass m_j. By assuming a Gaussian distribution for the momentum of the massive neutrino ν_j:

$$\psi_j(p) = (2\pi\sigma_p^{P\,2})^{-\frac{1}{4}}\exp\left\{-\frac{(p-p_j)^2}{4\sigma_p^{P\,2}}\right\} \tag{25}$$

where p_j is the average momentum and σ_p^P is the momentum uncertainty determined by the production process, we can write the wave function as:

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}}\int dp\,\psi_j(p)e^{ipx - iE_j(p)t}, \tag{26}$$

where $E_j(p) = \sqrt{p^2 + m_j^2}$ is the energy. We suppose that the Gaussian momentum distribution, Eq. (25), is strongly peaked around p_j, that is, we assume the condition $\sigma_p^P \ll E_j^2(p_j)/m_j$. This allows us to approximate the energy with:

$$E_j(p) \simeq E_j + v_j(p - p_j), \tag{27}$$

where $E_j = \sqrt{p_j^2 + m_j^2}$ is the average energy and $v_j = \frac{\partial E_j(p)}{\partial p}\Big|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet of the massive neutrino ν_j.

Using these approximations, we can perform an integration on p of Eq. (26), obtaining:

$$\psi_j(x,t) = (2\pi\sigma_x^{P\,2})^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4\sigma_x^{P\,2}}\right] \tag{28}$$

where $\sigma_x^P = \frac{1}{2\sigma_p^P}$ is the spatial width of the wave packet.

At this point, by substituting Eq. (28) in Eq. (24) it is possible to obtain the density matrix operator by $\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|$ which describes the neutrino oscillations in space and time. Although in laboratory experiments it is possible to measure neutrino oscillations in time through the measurement of both the production and detection processes, due to the long-time exposure in time of the detectors it is convenient to consider an average in time of the density matrix operator. In this way $\rho_\alpha(x)$ is the relevant density matrix operator and it can be obtained by a Gaussian time integration.

In the case of ultra-relativistic neutrinos, it is useful to consider the following approximations: $E_j \simeq E + \xi_P \frac{m_j^2}{2E}$, where E is the neutrino energy in the limit of zero mass and ξ_P is a dimensionless quantity that depends on the characteristics of the production process, $p_j \simeq E - (1 - \xi_P)\frac{m_j^2}{2E}$ and $v_j \simeq 1 - \frac{m_j^2}{2E_j^2}$. Considering these approximations, $\rho_\alpha(x)$ becomes:

$$\rho_\alpha(x) = \sum_{j,k} U^*_{\alpha j} U_{\alpha k} \exp\left[-i\frac{\Delta m^2_{jk} x}{2E} - \left(\frac{\Delta m^2_{jk} x}{4\sqrt{2}E^2\sigma_x^P}\right)^2 - \left(\xi_P \frac{\Delta m^2_{jk}}{4\sqrt{2}E\sigma_p^P}\right)^2\right] |\nu_j\rangle\rangle\nu_k|, \tag{29}$$

where $\Delta m^2_{jk} = m_j^2 - m_k^2$.

Taking into account that the detection process, described by the operator $\mathcal{O}_\beta(x - L)$, takes place at a distance L from the origin of the coordinates, the transition probability is given by:

$$P_{\nu_\alpha\to\nu_\beta}(L) = \mathrm{Tr}(\rho_\alpha(x)\mathcal{O}_\beta(x - L)) = \sum_{j,k} U^*_{\alpha j} U_{\alpha k} U^*_{\beta j} U_{\beta k} \exp\left[-2\pi i \frac{L}{L^{osc}_{jk}} - \left(\frac{L}{L^{coh}_{jk}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L^{osc}_{jk}}\right)^2\right], \tag{30}$$

where L^{osc}_{jk} is the oscillation length and L^{coh}_{jk} the coherence length, defined by:

$$L^{osc}_{jk} = \frac{4\pi E}{\Delta m^2_{jk}},\; L^{coh}_{jk} = \frac{4\sqrt{2}E^2}{|\Delta m^2_{jk}|}\sigma_x, \tag{31}$$

with $\sigma_x^2 = \sigma_x^{P\,2} + \sigma_x^{D\,2}$ and $\xi^2\sigma_x^2 = \xi_P^2\sigma_x^{P\,2} + \xi_D^2\sigma_x^{D\,2}$, where σ^D is the uncertainty of the detection process and ξ_D depends from the characteristics of the detection process.

We note that the wave packet description confirms the standard value of the oscillation length. The coherence length is the distance beyond which the interference of the massive neutrinos ν_j and ν_k is suppressed. The last term in the exponential of Eq. (31) implies that the interference of the neutrinos is observable only if the localization of the production and detection processes is smaller than the oscillation length.

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