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Chaotic Solutions in non Linear Economic - Financial models

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Abstract. Following Mulligan and Sala-i-Martin (1993) we study a general class of endogenous growth models formalized as a non linear autonomous three-dimensional differential system. We consider the abstract model. By using the Shilnikov Theorem statements, we determine the parameters space in which the condition for the existence of a homoclinic Shilnikov orbit and Smale horseshoe chaos are true.

The Lucas model (1998) can be considered as an application of the general result. The series expression of the homoclinic orbit is derived by the undetermined coefficient method. We show the optimality for the solutions path based on the Shilnikov Theorem. Some economic implications of this analysis are discussed.
Keywords: homoclinic Shilnikov bifurcation, Smale horseshoe chaos..

1 Introduction

We consider a class of endogenous growth two sector models as formulated by Mulligan B. and X. Sala-i-Martin (1993). Some examples of this class are well known and deeply study in recent literature (see inter al. D. Fiaschi, S. Sordi, 2002). A lot of research has been done in indeterminacy results and in the conditions of existence and stability of cycles, in a special way Hopf bifurcations (see G. Benhabib and R. Perli, 1994, p.124; P. Mattana and B. Venturi, 1999; M. Boldrin, K. Nishimura, T, Shigoka, M. Yano, 2000; D. Fiaschi, S. Sordi, 2002; P. Mattana, 2004; S. Slobodyan, 2005; K. Nishimura , T.Shigoka, 2008; A. Antoci, M. Galeotti, and P. Russu, 2011; G.Bella, P. Mattana and B.Venturi, 2013).

In particular, our analysis focuses on the context in which the application system, the Lucas model, admits only one steady state which corresponding, after a change of variables, in standard way, to an equilibrium point of a non linear three-dimensional autonomous system.

As described by Guckenheimer J. and Holmes P. (1983), and Wiggins S. (1990) usually a chaotic attractor has two or more fixed points: one determines the location and the structure of the attractor, and another is used to build a suspended flow which forms the spine of the attractor. However, as reported in recently papers one equilibrium point is still possible to form a chaotic attractor.

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(In order to study the long-run properties of the equilibrium) we treat this class as a general dynamical system. We give the conditions under which the Shilnikov chaos occurs in a appropriated parameter set. Using Cardano formula and series solution of the differential equations, the eigenvalues problem and the rigorous proof of the existence of the homoclinic orbit are pursued and applied to the Lucas model.

The work develops as follows. The second Section introduces the considered class of generalized two sector models of endogenous growth, as a dynamical system. We refer to the original paper of B. Mulligan and X. Sala-i-Martin,1993 and R. Lucas 1988 for an appropriate economic description of the system and its application. The third Section is devoted to characterize the parameter set in which the Shilnikov Theorem statements hold. We give a rigorous proof of the emergence of a homoclinic Shilnikov orbit. In view of its evaluation , in the first we found the set in which the system has a saddle-focus (of index 2) and in the second, we determined the coefficients of the series expression of the stable and unstable manifolds of such equilibrium point (the saddle-focus). As application of these results we consider the Lucas model. At the end we show the optimality for the solutions path based on the Shilnikov Theorem. Numerical simulation demonstrate that there is a route to chaos. Some economic implications of this analysis are discussed.

2 The Generalized Class of Two Sector Models of Endogenous Growth

We review the generalized class of two sector models of endogenous growth, with externalities, as formulated by B. Mulligan and X. Sala-i-Martin (1993).The model deal with the maximization of a standard utility function:

$$\int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \tag{2.1}$$

where c is per-capita consumption, ρ is a positive discount factor and σ is the inverse of the intertemporal elasticity of substitution. The constraints to the growth process are represented by the following equations

$$\dot{k} = A((h(t)^{\alpha_h} u(t)^{\alpha_u})(\nu(t)^{\alpha_\nu} k(t)^{\alpha_k})\hat{h}(t)^{\alpha_{\hat{h}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t) (2.2)$$
$$\dot{h} = B((h(t)^{\beta_h}(1-u(t)^{\beta_u}))((1-\nu(t)^{\beta_\nu} k(t)^{\beta_k})\hat{h}(t)^{\beta_{\hat{h}}} k(t)^{\beta_{\hat{k}}} - \tau_h h(t)$$

where k is physical capital, h is human capital, $\alpha_k$and $\alpha_h$being the private share of physical and the human capital in the output sector, β_k and β_h being the corresponding shares share in the education sector, u and v are the fraction of aggregate human and physical capital used in the final output sector at instant t (and conversely, $(1-u)$ and $(1-v)$ are the fractions used in the education sector), A and B are the level of the technology in each sector, τ is a discount factor, $\alpha_{\hat{k}}$ is a positive externality parameter in the production of physical capital, $\alpha_{\hat{h}}$is a positive externality parameter in the production of human capital. The equalities $\alpha_k + \alpha_h = 1$ and $\beta_k + \beta_h = 1$ ensure that there are constant returns to scale at the private level. At the social level, however, there may be increasing, constant or decreasing returns depending on the signs of the externality parameters. All other parameters $\omega = (\alpha_k,\ \alpha_{\hat{k}}, \alpha_h,$ $\alpha_{\hat{h}}, \beta_k, \beta_{\hat{k}}, \beta_h\ , \beta_{\hat{h}},\ \sigma, \gamma, \delta, \rho)$ live inside the following set $\Omega \subset (0,\ 1)\times(0,\ 1)\times(0,$ $1)\times(0,\ 1)\times(0,\ 1)\times(0,\ 1)\times(0,\ 1)\ \times \mathrm{R}^4_+$

The representative agent's problem (1.1)-(1.2) is solved by defining the current value Hamiltonian.

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_1(A((h(t)^{\alpha_h} u(t)^{\alpha_u})(\nu(t)^{\alpha_\nu} k(t)^{\alpha_k})\hat{h}(t)^{\alpha_{\hat{h}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t)) +$$
$$+\lambda_2(B((h(t)^{\beta_h}(1-u(t)^{\beta_u}))((1-\nu(t)^{\beta_\nu} k(t)^{\beta_k})\hat{h}(t)^{\beta_{\hat{h}}} k(t)^{\beta_{\hat{k}}} - \tau_h h(t)) \quad (2.3)$$

where λ_1 and λ_2 are co-state variables which can be interpreted as shadow prices of the accumulation. The solution candidate comes from the first-order necessary conditions (for an interior solution) obtained from the Maximum Principle, with the usual transversality condition

$$\lim_{t\to\infty} \left[e^{-\rho t}\left(\lambda_1 k + \lambda_2 h\right)\right] = 0 \quad (2.4)$$

consider only the competitive equilibrium solution. After eliminating $v(t)$ the rest of the first order conditions and accumulation constraints entail four first order non linear differential equations in four variables: two controls (c and u) and two states (k and h). By using new variables, since h, k and c grow at a constant rate and u is a constant, Mulligan B.-Sala-i-Martin X.(1993) have transformed a system of ordinary differential equations for c, u, k and h, , into a system of three first order ordinary differential equations.

Setting $A = B = 1$ and

$$x_1 = h^{\frac{a_{\hat{h}}}{(a_{\hat{h}}-1)}} k\ ; \quad x_2 = u; \quad x_3 = \frac{c}{k} \quad (2.5)$$

we get:

$$\begin{aligned}
\dot{x}_1 &= \phi_1(x_1, x_2, x_3, \alpha_k, \alpha_{\hat{k}}, \alpha_h, a_{\hat{h}}, \beta_k, \beta_{\hat{k}}, \beta_h, \beta_{\hat{h}}, \sigma, \gamma, \delta, \rho) \\
\dot{x}_2 &= \phi_2(x_1, x_2, x_3, \alpha_k, \alpha_{\hat{k}}, \alpha_h, \alpha_{\hat{h}}, \beta_k, \beta_{\hat{k}}, \beta_h, \beta_{\hat{h}}, \sigma, \gamma, \delta, \rho) \\
\dot{x}_3 &= \phi_3(x_1, x_2, x_3, \alpha_k, \alpha_{\hat{k}}, \alpha_h, \alpha_{\hat{h}}, \beta_k, \beta_{\hat{k}}, \beta_h, \beta_{\hat{h}}, \sigma, \gamma, \delta, \rho)
\end{aligned} \quad (2.6)$$

where the ϕ_i with $i = 1, 2, 3$ are complicated nonlinear functions ; which depend of the parameters $(x_1, x_2, x_3, \alpha_k, \alpha_{\hat{k}}, \alpha_h, \alpha_{\hat{h}}, \beta_k, \beta_{\hat{k}}, \beta_h, \beta_{\hat{h}}, \sigma, \gamma, \delta, \rho)$ of the model.

3 Shilnikov Theorem and The Emergence of a Homoclinic Orbit.

In order to verify that our system satisfies the Shilnikov Theorem statements, we follow strictly D. Shang M.Han, 2005. In the first we determine the parameter space in which our system has a homoclinic orbit. We remember that a homoclinic orbit is a transversal intersection between the stable manifold with the unstable manifold of a hyperbolic equilibrium point (connects a saddle to itself). Under regularity conditions (continuity since the second order) the model (2.6), has at least one stationary point $P^*(x_1^*,\, x_2^*,\, x_3^*)$.

Lemma 1. *In Ω exists a parameters subset $\overset{\wedge}{\Omega}$ such that the equilibrium point $P^*(0,0,0)$ is a saddle focus of index 2.*

Proof. By using Cardano's formula, we determine a parameters space in which the solutions (roots) $r_i,\ i = 1,2,3$ of the polynomial characteristic of the Jacobian matrix J, evaluated in the stationary point $J^* = J(P^*)$ satisfies the following conditions

$$r_1 = -\frac{\hat{a}}{3} + u + v \tag{3.1}$$

$$r_{2,3} = -\frac{\hat{a}}{3} - \frac{u+v}{2} \pm \sqrt{3}\frac{u-v}{2}i$$

where $i = \sqrt{-1}$ is the imaginary root, $u = \sqrt[3]{-\frac{m}{2} + \sqrt{\Delta}}$ and $v = \sqrt[3]{-\frac{m}{2} - \sqrt{\Delta}}$, with, $l = \frac{3\hat{b} - \hat{a}^2}{3}$ and $m = \hat{c} + \frac{2\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{3}$, $\hat{a} = -\mathrm{Tr}(\mathbf{J}^*)$, $\hat{b}$ =B($\mathbf{J}^*$), and $\hat{c} = -\mathrm{Det}(\mathbf{J}^*)$, whereas $\Delta = \left(\frac{l}{3}\right)^3 + \left(\frac{m}{2}\right)^2$ is the discriminant. For the scope of our paper, a saddle-focus (of index 2) emerges when

$$\Delta > 0 \tag{3.2}$$

$$\sqrt[3]{-\frac{m}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{m}{2} - \sqrt{\Delta}} < -\frac{2\hat{a}}{3}$$

that is explicitly

$$\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6}\right)^2 > \left(\frac{\hat{a}^2 - 3\hat{b}}{9}\right)^3 \tag{3.3}$$

Thus (3.2) holds the characteristic equation has one real root and a conjugate pair of complex , and the real root is positive (negative) since $\mathrm{Det}(\mathbf{J}^*) > 0 (< 0)$.

To ensure that the real part of the complex conjugate roots is positive (negative) and that the equilibrium point is a saddle focus (of index 2) it is further required that :

$$\sqrt[3]{-\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6}\right) + \sqrt{\Delta}} + \sqrt[3]{-\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6}\right) - \sqrt{\Delta}} < -\frac{2\hat{a}}{3} \tag{3.4}$$

In other word when (3.1) (3.2) (3.3) are satisfied the characteristic equations of the Jacobian J^*in $\overset{\wedge}{\Omega}$ has one positive real and two complex conjugate eigenvalues whose real parts is negative: then the equilibrium point P^*in $\overset{\wedge}{\Omega}$ is a saddle focus and the real eigenvalue is bigger than the absolute value of the real part of the complex conjugate eigenvalues.

Lemma 2. *In $\overset{\wedge}{\Omega} \subset \Omega$ the system (2.6) has an homoclinic Shilnikov orbit.*

Proof. We compute the stable and unstable manifolds of the saddle focus equilibrium point to construct the Shilnikov type homoclinic orbit in an analytic style.

Theorem 1. *The system (2.6) exhibits a Smale horseshoe type of chaos. In other words (2.6) has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map defined near the homoclinic orbit.*

Proof. Theorem 1 is a direct application of the Shilnikov Theorem (see Guckenheim-Holmes1983, pp.151-152). We only have to verify that the assumptions of Shilnikov theorem are satisfied.

4 Application: The Lucas Model

The general model just presented collapses to Lucas's model (1988) that is analyzed by Benhabib and Perli (1994), Mattana and Venturi (1999) and Mattana (2004) when depreciation is neglected and the following restrictions are imposed

$$\alpha_\nu = \alpha_{\hat{k}} = 0; \beta_{\hat{k}} = \beta_{\hat{h}} = \beta_\nu = \beta_k = 0; \alpha_\nu = \alpha_h = 1 - \alpha_k; \beta_u = \beta_h \tag{4.1}$$

The equations of the Lucas's model can be formalized in R^3 in the following form

$$\begin{array}{c} \dot{x}_1 = x_1^\beta x_2^{\beta-1} - x_1 x_3 + \psi \frac{(\beta-1)}{\beta}(1 - x_2) \\ \dot{x}_2 = \eta x_2^2 + \psi \frac{(\beta-1)}{\beta} x_2 + x_1 x_3 \\ \dot{x}_3 = \phi x_2^{1-\beta} x_1^{\beta-1} x_3 - \frac{\rho}{\sigma} x_3 + x_3^2 \end{array} \tag{4.2}$$

as a system of three first order differential equations where

$$\phi = \frac{\beta - \sigma}{\sigma} \quad \eta = \frac{\delta(\beta-1)}{\beta} \quad \psi = \frac{\delta(1-\beta+\gamma)}{\beta-1} \qquad \xi = \frac{\rho}{\sigma} \tag{4.3}$$

A stationary (equilibrium) point $P^*$of the system is any solution of

$$\begin{array}{c} x_1^{*1-\beta} x_2^{*\ \beta-1} - x_1^* x_3^* + \psi(1 - x_2) x_1^* = 0 \\ \eta x_2^2 + \psi \frac{(\beta-1)}{\beta} x_2 - x_2^*\ x_3^* = 0 \\ \phi x_2^{*1-\beta} x_1^{*\beta-1} x_3^* - \ \xi x_3^* + x_3^{*2} = 0 \end{array} \tag{4.4}$$

Then, we solved the system in (4.4) and we get the following steady state values

$$x_1^* = x_2^* \left[\frac{\beta\rho - \delta\sigma\,(1-u^*) + \delta(\beta-\gamma)}{\beta\,(\beta-\sigma)} \right]^{1/(\beta-1)} \tag{4.5a}$$

$$x_2^* = \frac{(1-\beta)\ \ (\rho-\delta)}{\delta - [\gamma - \sigma(1-\beta+\gamma)]} \tag{4.5b}$$

$$x_3^* = \eta x_2^* + \delta \frac{(1-\beta+\gamma)}{\beta} \tag{4.5c}$$

where $\phi = \frac{\beta-\sigma}{\sigma}$ simplifies the notation.

The system (4.2) possesses an interior steady-state characterized by the stationary values in (4.5.a), (4.5.b) *and* (4.5.c) for x_1^*, x_2^* and x_3^*. It is well-known that many theoretical results relating to the system depend upon the eigenvalues of the Jacobian matrix evaluated at the stationary point.

Let J be the Jacobian matrix and $P^*(x_1^*,\, x_2^*,\, x_3^*)$ the stationary point ($J(P^*) = J^*$, see appendix A). The "feasible" restrictions in the parameters are satisfied if and only if the parameters lie in one of the following subsets

Remark 1. i) if $\omega \in \Omega^1$, J^* has one negative eigenvalue and two eigenvalues with positive real parts. (This means that the competitive equilibrium path is locally unique). ii) $\omega \in \Omega^2$, J^* has one positive eigenvalue and two eigenvalues with negative real parts. iii) $\omega \in \Omega^3$ there exist two subsets Ω_3^A *and* Ω_3^B, and such that:
if $\omega \in \Omega_3^A$ J^**has one eigenvalue with a positive real part and two eigenvalues with negative real parts.* $\Omega_3^A = \left\{ \rho \in (\delta, -\psi), \sigma \in (0.1, \rho/\psi), \gamma \in (\frac{(1-\beta)(\rho-\delta)}{\delta}, \overset{\leadsto}{\gamma}) \right\}$
where $\overset{\leadsto}{\gamma}$ is the Hopf bifurcation value found in Mattana P. and Venturi B.(1999);
if $\omega \in \Omega_3^B$ J^**has three eigenvalues with positive real parts:*
$\Omega_3^B = \left\{ \rho \in (\delta, -\psi), \sigma \in (0.1, \rho/\psi), \gamma \in (\overset{\leadsto}{\gamma}, \beta) \right\}$. *So, there is either a continuum of equilibria converging towards the steady-state or no stable transitional paths at all.*

We focuses our attention in the set Ω_3^A and we rigorously prove that our system in this subset satisfies all conditions stated in the Shilnikov Theorem.

In the first we translate the unique equilibrium point P^* in the origin W^*,we get

$$\frac{dw_i}{dt} = f_i(w_1, w_2, w_3) \; with \quad i = 1, 2, 3 \tag{4.6}$$

and we make use of the normal form (see Appendix B and Mattana and Venturi, 1999).

Lemma 3. *If* $\omega \in \Omega_3^A$ *the equilibrium point* $W^*(0,0,0)$ *is a saddle focus.*

The Jacobian J^*in Ω_3^A has one positive real and two complex conjugate eigenvalues whose real parts is negative: then the equilibrium point W^*in Ω_3^A is a saddle focus and the real eigenvalue is bigger than the absolute value of the real part of the complex conjugate eigenvalues. By using Cardano Formula we have verified analytically, and numerically the statement.

Lemma 4. *In* Ω_3^A *the system (4.6) has an homoclinic Shilnikov orbit* Γ*.*

Proof. We show that the **equilibrium point** $W^*(0,0,0)$ of system (4.6) is **doubly asymptotic** with respect to time t along the solution manifold. See Appendix B for details.

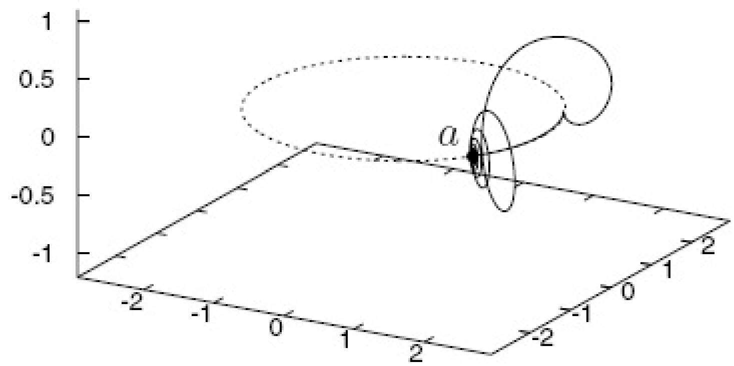

Figure 1: The homoclinic orbit

In Figure 1. we have the graph of the Homoclinic Shilnikov Orbit Γ .

Remark 2. In the set Ω_3^B the Jacobian J^* has one positive real and two complex conjugate eigenvalues whose real parts is positive. In this situation the model is expanding and thus it cannot have homoclinic orbits .

Theorem 2. *The system (4.6) exhibits a Smale horseshoe type of chaos. In other words (4.6) has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map defined near the homoclinic orbit.*

Proof. By lemma1 the equilibrium point W^**is a saddle focus in* Ω_3^A *and the real eigenvalue* $r_1 \in \mathbb{R}$ *is bigger than the real part of the complex conjugate eigenvalues* $r_{2/3} = -p \pm iq$ *and* $r_1 p > 0$. *with a further constraint* $|r_1| > |p|$. *By lemma 2 the system has a homoclinic Shilnikov orbit* Γ *in* Ω_3^A. *It follows directly from the Shilnikov Theorem that if the third-order autonomous system (4.6) has a saddle-focus (of index 2) in the unique equilibrium points,* W^* *with eigenvalues associated to* J^* *given by* $r_1 \in \mathbb{R}$ *and* $r_{2/3} = -p + iq \in \mathbb{C}$, *such that* $r_1 p > 0$. *with a further constraint* $|r_1| > |p|$, *and there exists a homoclinic orbit* Γ *connecting* W^*, *then the Shilnikov map, defined in a neighborhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its discrete dynamics. and for any sufficiently small* C^1*-perturbation* $\mathbf{g}$ *of* $\mathbf{f}$ *the perturbed system* $\frac{dw_i}{dt} = g_i(w_1, w_2, w_3)$ *with* $i = 1, 2, 3$ *exhibits a Smale horseshoe type of chaos has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map defined near the homoclinic orbit.*

5 Transversality Conditions

Proposition 1. *The transversality conditions are satisfied on the homoclinic orbit Γ.*

As shown in BP the transversality conditions are satisfied on the balanced growth paths. Let W^* $(\beta^*, \delta^*, \rho^*, \sigma^*, \gamma^*)$ be the only steady state in Ω_3^A. Let U in R^3be a small open neighborhood of W^*.So for each $(\beta, \delta, \rho, \sigma, \gamma) \in \Omega_3^A$, if we choose U sufficiently small, each path inside, starting from a point in the homoclinic orbit Γ, satisfies the transversality conditions. It follows directly from continuity arguments (the theorem of the permanence of the sign for continuous functions).

Proposition 2. *The transversality conditions hold near the homoclinic orbit where the Shilnikov Theorem is true.*

Proof. Let g_i , $i = 1, 2, 3,$ be a C^1perturbation of f_i, $i = 1, 2, 3,$ where the Shilnikov Theorem is true near the homoclinic orbit. Then for each $(\bar{\beta}, \bar{\delta}, \bar{\rho}, \bar{\sigma}, \bar{\gamma}) \in \Gamma$ in Ω_3^A there exists a constant $\mathbf{L}$ such that

$$|\mathbf{f}(\bar{w}(t)) - \mathbf{g}(\mathbf{w}(t))| < \mathbf{L}\, |\bar{w}(t) - \mathbf{w}(t)| \tag{5.1}$$

i.e., in vectorial form the distance between a path starting in the homoclinic Shilnikov orbit Γ and a Smale horseshoe chaotic path of $\mathbf{g}$ can be arbitrary small. From proposition *1* the transversality conditions are satisfied on the homoclinic orbit Γ then their are satisfied also in the chaotic solutions. We can choose an arbitrary small open set U of f a path starting in the homoclinic orbits in which there is a path that exhibits a Smale horseshoe chaos. But the Shilnikov Theorem stated that for any sufficiently small C^1-perturbation g of f, the perturbed system exhibits a Smale horseshoe chaos. Then the transversality condition is satisfied.

6 Conclusions

This paper aims to give a contribution of research to conditions which determine a chaotic behavior in the long-run properties of an economic model . Investigations of this kind are important in economic theory since help mapping the regions of the parameters space in correspondence of which the capacity of the models to produce indications on future economic outcomes starting from given fundamentals is drastically impaired. The aim of the present paper is to point out some basic ideas that may be useful to prove the transition to bounded and complex behavior, and to explain how the presence of an Homoclinic Shilnikov orbit and chaos in a model of a general class of economic-financial models can be interesting from an economic and dynamic point of view.

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7 Appendix A

As shown in the text, Luca's model gives rise to the following system of first-order differential equations

$$\begin{aligned} \dot{x}_1 &= x_1^{\beta} x_2^{1-\beta} - x_1 x_3 + \psi(1-x_2)x_1; \\ \dot{x}_2 &= \eta x_2^2 + \psi \tfrac{(\beta-1)}{\beta} x_2 - x_2 x_3; \\ \dot{x}_3 &= \phi x_2^{1-\beta} x_1^{\beta-1} x_3 - \ \xi x_3 + x_3^2; \end{aligned} \tag{A.1}$$

where $\phi = \frac{\beta-\sigma}{\sigma}$, $\eta = \frac{\delta(\beta-\gamma)}{\beta}$, $\psi = \frac{\delta(1-\beta+\gamma)}{\beta-1}$, $\xi = \frac{\rho}{\sigma}$.

The system has the single equilibrium point: $P^*(x_1^*, x_2^*, x_3^*)$

$$\begin{aligned} x_1^* &= x_2^* \left[\tfrac{\beta\xi - \delta(1-\beta+\gamma) + \delta(\beta-\gamma)x_2^*}{\beta\phi} \right]^{1/(\beta-1)} \\ x_2^* &= \tfrac{(1-\beta)(\rho-\delta)}{\delta - [\gamma - \sigma(1-\beta+\gamma)]} \\ x_3^* &= \eta x_2^* + \delta \tfrac{(1-\beta+\gamma)}{\beta} \end{aligned} \tag{A.2}$$

The Jacobian matrix J associated with the system (A.1) evaluated at the unique equilibrium point P^* is given by $J(P^*)$:

$$J(P^*) = \begin{bmatrix} J_{11}^* & \frac{x_1^*}{x_2^*}(J_{11} + \psi x_2^*) & -x_1^* \\ 0 & -\eta x_2^* & x_2^* \\ \frac{J_{11}\phi x_3^*}{x_1^*} & \frac{J_{11}\phi x_3^*}{x_2^*} & x_3^* \end{bmatrix} \tag{A.3}$$

where

$$J_{11}^* = \frac{(\beta-1)[\gamma\rho - \delta\sigma(1-\beta+\gamma)]}{\beta[\gamma - \sigma(1-\beta+\gamma)]} \tag{A.4}$$

and

$$Tr(J^*) = \frac{\delta(2\beta-\gamma)}{\beta} x_2^* \tag{A.5}$$

$$Det(J^*) = J_{11}^* x_2^* x_3^* \frac{\delta(\gamma - \sigma(1-\beta+\gamma))}{\sigma(\beta-1)} \tag{A.6}$$

$$B(J^*) = J_{11}^* x_3^* + \frac{\delta^2(\beta-\gamma))}{\beta} x_2^{*2} \tag{A.7}$$

8 Appendix B.

The Shilnikov type homoclinic orbit in an analytic style.

To apply the Shilnikov theorem to the system $(A.1)$, we have to prove that the system has a homoclinic Shilnikov orbit at the equilibrium point P^*. If the parameters lie in the following subsets:
$\Omega_3^A = \left\{ \rho \in (\delta, -\psi), \sigma \in (0.1, \rho/\psi), \gamma \in (\frac{(1-\beta)(\rho-\delta)}{\delta}, \tilde{\tilde{\gamma}}) \right\}$,
where $\tilde{\tilde{\gamma}}$ is the Hopf bifurcation value found in Mattana and Venturi (1999). By using Cardano Formula, and numerical evaluation we shown that the singular equilibrium point $P^* \in \Omega_3^A$ is a hyperbolic saddle focus of index 2 .

In other words, the eigenvalues of the Jacobian matrix of the system (A.1) evaluated in $P^*$are of the form $r = r_1$ and $r_{2/3} = -p \pm iq$,a saddle focus, with $r_1 > 0, p > 0,\ q \neq 0$ and $r_1 > p > 0$. We remember that a homoclinic orbit joining the **equilibrium point** $P^*$of system $(A.1)$ is **doubly asymptotic** with respect to time t along the solution manifold.

We translate the equilibrium point P^* in the origin $W^*(0, 0, 0)$ and we put the system (A.1) in normal form

Body Math

$$\begin{aligned} \dot{w_1} &= r\ w_1 + F_{1a}w_1w_2 + F_{1b}w_1w_3 + F_{1c}w_2w_3 + F_{1d}w_1^2 + F_1ew_2^2 + F_{1f}w_3^2; \\ \dot{w_2} &= pw_2 \ -qw_3 + F_{2a}w_1w_2 + F_{2b}w_1w_3 + F_{2c}w_2w_3 + F_{2d}w_1^2 + F_{2e}w_2^2 + F_{2f}\ w_3^2 \\ \dot{w_3} &= qw_2 + pw_3 + F_{3a}w_1w_2 + F_{3b}w_1w_3 + F_{3c}w_2w_3 + F_{3d}w_1^2 + F_{3e}w_2^2 + F_{3f}\ w_3^2 \end{aligned} \tag{B.1}$$

We compute the stable and unstable manifolds of the saddle focus equilibrium point W^* to construct the Shilnikov type homoclinic orbit in an analytic style. Parameter values are set as $\beta = 0.76$, $\rho = 0.055$, $\delta = 0.05499$, $\sigma = 0.1$ and $\gamma = 0.\ 042$. So let's begin with the **analytic expression** of the **one-dimensional unstable manifold** associated with **the real eigenvalue** r_1 where $a_m.b_m.c_\mathrm{m}$.are undetermined coefficients.

So for t¡0 the trajectory will tend to zero (to steady state) along the unstable manifolds.

$w_1(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{krt};$

$w_2(t) = b_0 + \sum_{k=1}^{\infty} b_k e^{krt};$

$w_3(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{krt}$

When $k = 4$ we get:

$w_1(t) = \xi e^{0,05307121t} - \xi^2 0,10856 e^{0,10614t} - \xi^4 0,00064 e^{0,159214t} - \xi^8 1,5E-08\ e^{0,212285t}$

$w_2(t) = -0,275059245\xi^2\ e^{0,106142t} + 0,5911103\ \xi^4 e^{0,159214t}\ - \xi^8 83,1E-07\ e^{0,212285t}$

$w_3(t) = -0,3075101\xi^2\ e^{0,106142t} - 0,008405\xi^4\ e^{0,159214t} + \xi^8 80,0003196 e^{0,212285t}$

We choose $\xi \leq 1$.

As t$\rightarrow \infty$, the trajectory will tend to zero along the stable manifold. We choose $r_2 = -p + iq$ the complex eigenvalue

$w_1(t) = a_0 + \sum_{k=1}^{\infty} a_k(\varsigma,\eta)e^{k(-p+iq)t} = \sum_{k=2}^{\infty} [a_k^1(\varsigma,\eta) + ia_k^2(\varsigma,\eta)]e^{k(-p+iq)t}$

$w_2(t) = b_0 + \sum_{k=1}^{\infty} b_k e^{k(-p+iq)t} = \sum_{k=1}^{\infty} [b_k^1(\varsigma,\eta) + ib_k^2(\varsigma,\eta)]e^{k(-p+iq)t}$

$w_3(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{k(-p+iq)t} = \sum_{k=1}^{\infty} [c_k^1(\varsigma,\eta) + ic_k^2(\varsigma,\eta)]e^{k(-p+iq)t}$

$w_1(t) = e^{-2pt}\ [a_2^1(\varsigma,\eta)\cos(2q) + ia_2^1(\varsigma,\eta)sin(2q)] + i[a_2^2(\varsigma,\eta)\cos(2q) + \imath a_2^2(\varsigma,\eta)\sin(2q)] +$ ...

$w_2(t) = e^{-pt}[-\varsigma\cos(q) - i\varsigma\sin(q)] + i[\eta\cos(q) + \grave{\imath}\eta\sin(q)]+$
$+e^{-2pt}\,[b_2^1(\varsigma,\eta)\cos(2q)+ib_2^1(\varsigma,\eta)\sin((2q)]+i[b_2^2(\varsigma,\eta)\cos(2q)+\grave{\imath}b_2^2(\varsigma,\eta)\sin(2q)]....$
$w_3(t) = e^{-pt}[-\eta\cos(q) - i\eta\sin(q)] + i[\varsigma\cos(q) + \grave{\imath}\varsigma\sin(q)]$
$+e^{-2pt}\,[c_2^1(\varsigma,\eta)\cos(2q)+ib_2^1(\varsigma,\eta)\sin((2q)]+i[c_2^2(\varsigma,\eta)\cos(2q)+\grave{\imath}c_2^2(\varsigma,\eta)\sin(2q)]....$

a_{12}=$-3,7E-06$;b_{12}=$-0,004825$;c_{12}=$-0,00046$
a_{13}=$5,49E-10$;b_{13}=$4,03E-06$;c$_{13}$=$4,66E-07$
a_{22}=-1,1E-09;b_{22}=7,148E-07;c_{22}=$-1E-07$
a_{23}=-1,1E-23;b_{23}=-1,2E-20;c_{23}=-4,1E-20
Body Math When $k=3$ we get:

$w_1(t) = e^{-0,00302t}\,\varsigma^2([(3,7E{-}06)\cos(0,11053t){-}(-1,1E{-}09)\sin(0,11053t)]+$
$+\grave{\imath}[(-1,1E-09)\cos(0,11053t) + (3,7E-06)sin(0,11053t))+$
$+e^{-0,0045t}\,\varsigma^4([(5,49E-10)\cos(0,016579t) - (1,1E-23)\sin(0,016579t) +$
$i[(1,1E-23)\cos(0,016579t) + (5,49E-10)\sin(0,016579t)])...$

$w_2(t) = e^{-0,00151t}\varsigma([-\cos(0,005523t) - \sin(0,005523t)] + i[\cos(0,005523\ t) -$
$\sin(0,005523\ )])+$
$+e^{-0,00302t}\,\varsigma^2((([(-0,004825)\cos(0,11053t) - (7,148E-07)\sin(0,11053t)] +$
$i[(7,148E-07)\cos((0,11053t)) - 0,004825\sin(0,11053t]+$
$+.e^{-0,0045t}\,\varsigma^4([(4,03E-06)\cos(0,016579t) - (1,2E-20)\sin(0,016579t) +$
$i((-1,2E-20))\cos(0,016579t) + (4,03E-06)\sin(0,016579t))]$

$w_3(t) = e^{-0,00151t}\varsigma([-\cos(0,005523t) - \sin(0,005523t)] + i[\cos(0,005523\ t) -$
$\sin(0,005523\ )])+$
$+e^{-0,00302t}\,\varsigma^2((([(-0,00046)\cos(0,11053t) - (7,148E-07)\sin(0,11053t)] +$
$i[(7,148E-07)\cos((0,11053t)) - (0,00046)\sin(0,11053t]+$
$+.e^{-0,0045t}\,\varsigma^4([(4,66E-07)\cos(0,016579t) + (4,1E-20)\sin(0,016579t) +$
$i((-4,1E-20)\cos(0,016579t) + (4,66E-07)\sin(0,016579t))]$

We assume $\varsigma = \eta \le 1$.