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Orders of Criticality in Voting Games

Abstract

In this paper we focus on the problem of investigating the blackmail power of players in simple games, which is the possibility of players of threatening coalitions to cause them loss using arguments that are (apparently) unjustified. To this purpose, we extend the classical notion of criticality of players in order to characterize situations where players may gain more power over the members of a coalition thanks to the collusion with other players.

Keywords Voting Games, Blackmailing Power, Semivalues

1 Introduction

We consider a Parliament that produced a majority coalition. If the majority corresponds to a *minimal winning coalition* all the parties result *critical*, i.e. each of them is able to destroy the majority when leaving, but we may face also a different situation in which not all the parties are critical, i.e. the majority corresponds to a *quasi-minimal winning coalition*. A similar situation was typical in the Eighties when the Italian governments included five parties, namely Christian Democracy (*Democrazia Cristiana* - DC), Italian Socialist Party (*Partito Socialista Italiano* - PSI), Italian Social-Demoratic Party (*Partito Socialista Democratico Italiano* - PSDI), Italian Republican Party (*Partito Repubblicano Italiano* - PRI), Italian Liberal Party (*Partito Liberale Italiano* - PLI); for some years only DC was critical (in the last years also PSI became critical), but all the parties received ministries and/or departments and in 1981 the premiership of the government was given to Giovanni Spadolini, the leader of PRI.

At a first glance, the situation may seem unusual because all the parties received a quota of the power even if the non-critical parties should receive nothing. On the other hand, it is possible to notice that the number of seats of the parties allowed for a minimal winning coalition after the leaving of a non-critical party, generating a situation that reduced the importance of critical parties. For instance, in case of a minimal winning coalition of four parties each of them is critical and should receive a quarter of the power; but if the majority includes five parties and only one is critical it may leave a small amount of power to the four non-critical parties getting much more than one quarter of the power. This could be a way to justify the power assigned to the parties, i.e. compensate them in order that they do not leave the majority. This situation was considered, under a different viewpoint in Chessa and Fragnelli (2014); they accounted for the possibility of

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the parties of forming a different majority coalition that excludes another party, which in its turn may propose another majority coalition that does not include the party that started the process. In this way, the hypotheses on which the bargaining set (see Aumann and Maschler, 1964) relies are satisfied, so the elements in the bargaining set are suitable for measuring the power of the parties.

In the present Italian political situation, we may consider the group *Alleanza Liberal-popolare-Autonomie* (ALA) that is represented only in the Senate, where the majority supporting the Italian government is very unstable, differently from the Lower Chamber, *Camera dei Deputati*, where the majority is stable. The ALA group had in mind to support the approval of some reforms proposed by the government, acting as a critical party, but the presence of this group had the consequence that other members of the majority, the minority of the Democratic Party, decided to support the reforms in order to avoid the criticality of the ALA group.

We can say that the critical parties have a *first order of criticality*, while the non-critical ones have a higher order of criticality.

The aim of this paper is to provide a formal definition of the second order critical players, which may be extended to further orders, and analyze some properties, proposing an allocation of the power.

The paper is organized as follows. We start recalling in Section 2 some notations and general definitions; Section 3 deals with higher orders of criticality, where a player i becomes critical for a coalition M only if other players (not critical for the same coalition) leave M before i; in Section 4 we strengthen the notion of criticality of a player i to a coalition M adding the further constraint that the threaten of i to leave M is made "credible" only if i has another opportunity to form a winning coalition with players out of M; Section 5 concludes.

2 Preliminaries

A *cooperative game with transferable utility* (TU-game) is a pair (N, v), where $N = \{1, 2, ..., n\}$ denotes the *finite set of players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\varnothing) = 0$. $v(S)$ is the worth of coalition $S \subseteq N$, i.e. what players in S may obtain standing alone.

A TU-game (N, v) is *simple* when $v : 2^N \to \{0, 1\}$, with $S \subseteq T \Rightarrow v(S) \leq v(T)$[5] and $v(N) = 1$. If $v(S) = 0$ then S is a *losing* coalition, while if $v(S) = 1$ then S is a *winning* coalition. Given a winning coalition S, if $S \setminus \{i\}$ is losing then $i \in N$ is a *critical player* for S. When a coalition S contains at least one critical player for it, S is a *quasi-minimal winning* coalition; when all the players of S are critical, it is a *minimal winning* coalition. A simple game may be defined also assigning the set of winning coalitions or the set of minimal winning coalitions.

A particular class of simple games is represented by the *weighted majority games*. A vector of weights $(w_1, w_2, ..., w_n)$ is associated to the players that leads to the following definition of the characteristic function of the corresponding weighted majority game (N, w):

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}, \quad S \subseteq N,$$

[5] This property is called *monotonicity*.

where q is the *majority quota*. A *weighted majority situation* is often denoted as $[q; w_1, w_2, ..., w_n]$. Usually, we ask that the game is *proper* or *N-proper*, i.e. if S is winning then $N \setminus S$ is losing; for this aim it is sufficient to choose $q > \frac{1}{2}\sum_{i\in N} w_i$. Note that a simple game may not correspond to any weighted majority situation.

Given a TU-game (N, v), an *allocation* is a n-dimensional vector $(x_i)_{i\in N} \in \mathbb{R}^N$ assigning to player $i \in N$ the amount x_i; an allocation $(x_i)_{i\in N}$ is *efficient* if $x(N) = \sum_{i\in N} x_i = v(N)$. A *solution* is a function ψ that assigns an allocation $\psi(v)$ to every TU-game (N, v) belonging to a given class of games $\mathcal{G}$ with player set N.

For simple games, and in particular for weighted majority games, a solution is often called a *power index*, as each component x_i may be interpreted as the percentage of power assigned to player $i \in N$. In the literature, several power indices were introduced; among others, we recall the following definitions.

The *Shapley-Shubik index* (Shapley and Shubik, 1954), ϕ, is the natural version for simple games of the Shapley value (Shapley, 1953). It is defined as the average of the marginal contributions of player i w.r.t. all the possible orderings and it can be written as:

$$\phi_i(v) = \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}\, m_i(S),\ i \in N,$$

where n and s denote the cardinalities of the set of players N and of the coalition S, respectively and $m_i(S) = v(S) - v(S \setminus \{i\})$ denotes the marginal contribution of player $i \in N$ to coalition $S \subseteq N, S \ni i$.

The normalized *Banzhaf index* (Banzhaf, 1965), β, is similar to the Shapley-Shubik index, but it considers the marginal contributions of a player to all possible coalitions, independently from the order of the players; first, we define:

$$\beta_i^*(v) = \frac{1}{2^{n-1}} \sum_{S\subseteq N, S\ni i} m_i(S),\ i \in N;$$

then, by normalization we get:

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j\in N} \beta_j^*(v)},\ i \in N.$$

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a vector of n non-negative numbers such that $\sum_{k=1}^n p_k \binom{n}{k} = 1$ with the interpretation that p_s is the probability that a coalition of size s forms. We denote by $\pi^{\mathbf{p}}$ the *semivalue* (Dubey, Neyman and Weber, 1981) engendered by the vector $\mathbf{p}$. Hence,

$$\pi_i^{\mathbf{p}}(v) = \sum_{S\subseteq N, S\ni i} p_s m_i(S). \tag{1}$$

Notice that both the Shapley-Shubik index ϕ and the Banzhaf index β^* can be defined as particular semivalues $\pi^{\mathbf{p}^\phi}$ and $\pi^{\mathbf{p}^\beta}$, respectively, where the probability vector $\mathbf{p}^\phi$ is such that $p_s^\phi = \frac{(s-1)!(n-s)!}{n!}$ and $\mathbf{p}^\beta$ is such that $p_s^\beta = \frac{1}{2^{n-1}}$. Notice that for a (monotonic) simple game v relation (1) can be simply written as

$$\pi_i^{\mathbf{p}}(v) = \sum_{S\in W_i(v)} p_s m_i(S), \tag{2}$$

where $W_i(v)$ is the set of winning coalitions containing player $i \in N$. Moreover, the value $m_i(S) = 1$ only if i is critical for S in v, for each $S \in W_i(v)$. Consequently, the value $\pi_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i to play a critical role in v. Every semivalue $\pi^{\mathbf{p}}$ satisfies the *symmetry property* (i.e., $\pi_i^{\mathbf{p}}(v) = \pi_j^{\mathbf{p}}(v)$ for each game v and every pair of *symmetric* players $i, j \in N$, i.e. such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$)) and the *null player property* (i.e., $\pi_i^{\mathbf{p}}(v) = 0$ for each game v and every *null* player $i \in N$, i.e. such that $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$).

In the following, we mainly refer to the Banzhaf index because it is more probability-oriented, as it does not account the ordering in which the players enter a coalition, which is usual in majority coalitions.

3 Second and Higher Orders of Criticality

In this section, we introduce the formal definition of *order of criticality* for a player. Given a winning coalition $M \subseteq N$, a critical player $i \in M$ may be called *critical of the first order* for coalition M. Now we deal with the other players in M.

Definition 1 *Let $M \subseteq N$, with $|M| \geq 3$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is* Critical Of the Second Order (COSO) for coalition M, via player $j \in M \setminus \{i\}$ *iff $v(M \setminus \{i, j\}) = 0$ with $v(M \setminus \{j\}) = 1$.*

The interpretation is the following: player i is not critical for M, but there exists in M another player j, different from i, s.t. M becomes a losing coalition when both the players leave. From the definition, we immediately get the following proposition.

Proposition 1 *If player $i \in M$ is critical of the second order for coalition M, via player $j \in M$, then player j is critical of the second order for coalition M, via player i.*

The result is a straightforward consequence of the definition, and does not require a proof.

Remark 1 *Note that when there are critical players of the second order, they are at least two, but they can be also more as in the following example.*

Example 1 *Consider the weighted majority situation [51; 40, 8, 5, 5, 5]; the first party is the unique critical one, while the other four parties are critical of the second order, even if the last three parties are critical only via the second party.*

Definition 1 can be extended to further orders as follows.

Definition 2 *Let $k \geq 2$ be an integer, let $M \subseteq N$, with $|M| \geq k + 2$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is* critical of the order $k + 1$ for coalition M, via coalition $K \subseteq M \setminus \{i\}$, *with $|K| = k$ iff*

$$v(M \setminus K) - v(M \setminus (K \cup \{i\})) = 1, \tag{3}$$

and K is the set of minimal cardinality satisfying (3), *i.e.*

$$v(M \setminus (T \cup \{i\})) = 1, \tag{4}$$

for any $T \subset K$ with $|T| < k$.

The interpretation is similar to the previous one: Player i is not critical for M, but there exists in M a coalition K, not including i, s.t. M becomes losing when all the players in $K \cup \{i\}$ leave.

It should also be noticed that the notion of minimal cardinality is crucial to unambiguously assign the order of criticality of a player in a coalition.

Example 2 *Consider the weighted majority situation [31; 21,5,3,3,3,3,2]; player 7 becomes critical whenever either a coalition involving player 2 and any of the players 3-6 are involved (they are the coalition K in the definition), or three of the players 3-6 are involved. According to Definition 2, player 7 is critical of the third order.*

Notice also that the above definition encompasses the definitions for lower orders. In particular, we obtain first order criticality when (3) is satisfied by $K = \varnothing$ of 0-cardinality, leading to $v(M) - v(M \setminus \{i\}) = 1$. For second order criticality consider $K = \{j\}$, then, by (3), $v(M \setminus \{j\}) = 1$ and $v(M \setminus \{i, j\}) = 0$. Moreover, since $\{j\}$ is the set of minimal cardinality which makes i critical, then $v(M \setminus \{i\}) = 1$.

We can derive a more general result than Proposition 1.

Proposition 2 *If player $i \in M$ is critical of the order $k+1$ for coalition M, via coalition $K \subset M$, then each player $j \in K$ is critical of the order $k+1$ for coalition M, via coalition $K \cup \{i\} \setminus \{j\}$.*

Proof Define $K' = K \setminus \{j\}$. Since i is critical for M via coalition $K = K' \cup \{j\}$, then

$$v(M \setminus (K' \cup \{j\})) - v(M \setminus (K' \cup \{i, j\})) = 1 \ . \tag{5}$$

We want to show that j is critical for M via $K' \cup \{i\}$. Now $v(M \setminus (K' \cup \{i, j\})) = 0$ by (5), and, since $|K'| < k$, $v(M \setminus (K' \cup \{i\})) = 1$ by (4). Therefore

$$v(M \setminus (K' \cup \{i\})) - v(M \setminus (K' \cup \{i, j\})) = 1 \ .$$

We need to verify the minimality of $K' \cup \{i\}$. Consider $T \subset K' \cup \{i\}$. There are two cases: (a) $i \in T$, then

$$v(M \setminus (T \cup \{j\})) = v(M \setminus [T \cup \{j\} \setminus \{i\}] \cup \{i\}) = 1 \ ,$$

since $|T \cup \{j\} \setminus \{i\}| < k$. (b) $i \notin T$, then $T \cup \{j\} \subset K$ and $v(M \setminus (T \cup \{j\})) \geq v(M \setminus K) = 1$ and equation (4) is always satisfied. ■

Remark 2 *A null player is never critical.*

Remark 3 *Note that when there are critical players of order $k+1$, they are at least $k+1$, but they can be also more as in the following example.*

Example 3 *Consider the weighted majority situation [51; 44, 3, 3, 3, 3, 3, 3]; in this case the first one is the unique critical party, while the other six parties are critical of the fourth order; note that there are no parties critical of order 2 or 3.*

In view of Propositions 1 and 2 and Remark 2 we obtain the following corollary.

Corollary 1 *Let $M \subseteq N$ be a winning coalition, then the players in M may be partitioned into those who are critical of some order and those who are never critical.*

Proposition 3 *Let $i \in M$ be a player critical of the order $k+1$ for coalition M, via coalition $K \subset M$; if a player $j \in K$ leaves the coalition, then i is a player critical of the order k for coalition $M \setminus \{j\}$, via coalition $K \setminus \{j\}$.*

Proof It is sufficient to note that $M \setminus (K \cup \{i\}) = (M \setminus \{j\}) \setminus ((K \setminus \{j\}) \cup \{i\})$ so both of them are losing and $|K \setminus \{j\}| = |K| - 1$. ■

After defining the various orders of criticality, we want to provide an index to measure how much a player may profit from being critical. The first step is to measure the power of a player w.r.t. a given coalition, accounting also his order; then we may aggregate the power of a player w.r.t. all coalitions he may belong to.

We want now to compute the probability for a player $i \in N$ to be COSO in v for some coalitions via another player $j \in N$. First, consider a coalition $S \in 2^{N \setminus \{i,j\}}$ with $v(S \cup \{i,j\}) = 1$ and define $C_{ij}(S)$ as follows:

$$C_{ij}(S) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\} - v(S).$$

By monotonicity of v we have four possible cases as shown in Table 1.

Table 1: Possible cases for player i (j) to be COSO for coalition $S \cup \{i,j\}$ via player j (i).

	$v(S \cup \{i\})$	$v(S \cup \{j\})$	$v(S)$	$C_{ij}(S)$
1)	0	1	0	0
2)	1	0	0	0
3)	1	1	1	0
4)	1	1	0	1

The only case in which i is COSO for $S \cup \{i,j\}$ via j is the last one (4) and $C_{ij}(S) = 1$. Note also that, in general, $C_{ij}(S) = C_{ji}(S)$.

Let $\mathbf{p} = (p_0, \ldots, p_{n-1})$ be a probability vector as defined in Section 2. If we want to compute the probability that i is COSO for some coalitions via j, we should compute the following expression:

$$\Gamma^{\mathbf{p}}_{ij}(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} p_{s+1} C_{ij}(S).$$

By Proposition 1, it immediately follows that $\Gamma^{\mathbf{p}}_{ij}(v) = \Gamma^{\mathbf{p}}_{ji}(v)$ for each $i, j \in N$. Following the same approach used to define semivalues (see relation (1)), we can also compute the total probability that a player i is COSO for some coalition via some other player as the following summation:

$$C^{\mathbf{p}}_i(v) = \sum_{j \in N \setminus \{i\}} \Gamma^{\mathbf{p}}_{ij}(v).$$

Now, consider the game (N, v^{ij}) such that for each $S \in 2^{N \setminus \{i,j\}}$:

$$v^{ij}(S \cup \{i,j\}) = v^{ij}(S \cup \{i\}) = v^{ij}(S \cup \{j\}) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\}$$

and

$$v^{ij}(S) = v(S).$$

The game (N, v^{ij}) represents a coalitional situation where the role of i (resp. j) is negatively influenced by j (resp. i), that is the value of each coalition M containing either i or j is lowered to the worst value between $v(M \cup \{j\} \setminus \{i\})$ and $v(M \cup \{i\} \setminus \{j\})$.

It is easy to check that $\Gamma^{\mathbf{p}}_{ij}(v) = \pi^{\mathbf{p}}_i(v^{ij})$.

Example 4 *Consider the simple game (N, v) with $N = \{1, 2, 3, 4\}$ whose minimal winning coalitions are $\{1, 2, 3\}$ and $\{1, 2, 4\}$. Note that 3 (resp. 4) is COSO for $\{1, 2, 3, 4\}$ via 4 (resp. 3), and no other player is COSO via another player for some coalition. Taking the vector $\mathbf{p}^\beta$ as the probability vector yielding the Banzhaf index $\pi^{\mathbf{p}^\beta}$ (see Section 2), we have that $\Gamma^{\mathbf{p}^\beta}_{34}(v) = \Gamma^{\mathbf{p}^\beta}_{43}(v) = \frac{1}{8}$ and $\Gamma^{\mathbf{p}^\beta}_{ij}(v) = 0$ for all the other i and j in N with $\{i, j\} \neq \{3, 4\}$. We have that $v^{34}(S) = v^{43}(S) = v(S)$ for each $S \in 2^N$. So, $\Gamma^{\mathbf{p}^\beta}_{34}(v) = \Gamma^{\mathbf{p}^\beta}_{43}(v) = \pi^{\mathbf{p}^\beta}_3(v^{34}) = \pi^{\mathbf{p}^\beta}_4(v^{43}) = \frac{1}{8}$. In addition, we also have that $v^{ij}(S) = 0$ for each $S \in 2^N$ and for all i and j with $\{i, j\} \neq \{3, 4\}$; so, $\Gamma^{\mathbf{p}^\beta}_{ij}(v) = \pi^{\mathbf{p}^\beta}_i(v^{ij}) = 0$ for all i and j with $\{i, j\} \neq \{3, 4\}$.*

In this example, the total probability to be COSO is $C^{\mathbf{p}^\beta}_1(v) = C^{\mathbf{p}^\beta}_2(v) = 0$ and $C^{\mathbf{p}^\beta}_3(v) = C^{\mathbf{p}^\beta}_4(v) = \frac{1}{8}$.

Now, consider a game (N, v) and take a coalition M such that $i \in M$ and $K \subseteq M \setminus \{i\}$ with $|K| = k$, $k \geq 2$. Similarly as above, we can compute the value

$$C_{iK}(M) = \min\{v(M \setminus T) : T \subset K \cup \{i\}\} - v(M \setminus (K \cup \{i\})).$$

that is equal to 1 iff i is critical of order $k + 1$ for coalition M via coalition $K \subset M$. Consequently, if we want to compute the probability that i is critical of order $k + 1$ for some coalitions via coalition K, we should compute the following expression:

$$\Gamma^{\mathbf{p}}_{iK}(v) = \sum_{S \in 2^{N \setminus \{i\}} \text{ with } k \subseteq M} p_s C_{ij}(S).$$

Once again, note that $\Gamma^{\mathbf{p}}_{ij,k}(v) = \Gamma^{\mathbf{p}}_{ji,k}(v)$.

We conclude this section with an example of possible application of the notions of criticality of first and second order to the analysis of the power of political parties in a realistic scenario.

Example 5 *Consider the political situation described in Section 1 and concerning the Italian Senate during the Eighties. More precisely, the distribution of seats among the political parties of the largest alliance in the Italian Senate during the IX Legislature (1979-1983) was the one shown in Table 2.*

At that time, the quota needed to have a majority within the Senate was 162. This leads to the weighted majority situation $[162; 145, 32, 9, 6, 2]$ on the player set $\{DC, PSI, PSDI, PRI, PLI\}$. Notice that a coalition is a winning one if it contains one of the following minimal winning coalitions $\{\{DC, PSI\}, \{DC, PSDI, PRI, PLI\}\}$. The symmetric relation of COSO exists between several pairs of players and for coalition $\{DC, PSI, PSDI, PRI, PLI\}$, specifically: PSI vs. PSDI, PSI vs. PRI and PSI vs. PLI.

Table 2: The distribution of seats in the largest alliance in the Italian Senate during the IX Legislature (1979-1983).

Party	seats
Democrazia Cristiana (DC)	145
Partito Socialista Italiano (PSI)	32
Partito Socialdemocratico Italiano (PSDI)	9
Partito Repubblicano Italiano (PRI)	6
Partito Liberale Italiano (PLI)	2

Using the probability vector $\mathbf{p}^{\beta}$*, we can compute the probability to be critical of the first order (i.e., the Banzhaf index) and to be COSO (using the index* $C^{\mathbf{p}^{\beta}}$*), as shown in Table 3.*

Table 3: The Banzhaf index and the total probability to be COSO in the Italian Senate.

Party	$\pi^{\mathbf{p}^{\beta}}$	$C^{\mathbf{p}^{\beta}}$
DC	$\frac{9}{16}$	0
PSI	$\frac{7}{16}$	$\frac{3}{16}$
PSDI	$\frac{1}{16}$	$\frac{1}{16}$
PRI	$\frac{1}{16}$	$\frac{1}{16}$
PLI	$\frac{1}{16}$	$\frac{1}{16}$

For the sake of completeness, we recall that in 1983 the PSI threatened to leave the five-parties alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise in order to avoid a new election. Maybe, the DC party had evaluated the threaten of the PSI as likely in view of the high index $C^{\mathbf{p}^{\beta}}$ *for the PSI party.*

4 Credible criticality

We now consider an alternative notion of criticality of the first order where the fact that i can threaten coalition M in a *credible* way is made possible by the fact that there exists another opportunity for i to be winning without the help of players in M. We want to remark that this hypothesis is different from that in Chessa and Fragnelli (2013) where i could ask for the help of some players in M, but not all.

Definition 3 *Let* $M \subseteq N$ *with* $v(M) = 1$*. A player* $i \in M$ *is said credibly critical or* credibly critical of the first order *for coalition* M *iff it satisfies the following two conditions: (c.1)* i *is critical for* M *(i.e.* $v(M \setminus \{i\}) = 0$*) and (c.2) there exists another coalition* $S \subseteq N \setminus M$ *that is winning together with* i*, i.e.* $v(S \cup \{i\}) = 1$*.*

Example 6 *Consider the simple game* (N, v) *with* $N = \{1, 2, 3\}$ *whose minimal winning coalitions are* $\{1, 2\}$ *and* $\{2, 3\}$*. Player* 2 *is credibly critical for coalition* $\{1, 2\}$ *(in fact* $v(1, 2) = 1$*,* $v(1) = 0$ *and* $v(2, 3) = 1$*), but players* 1 *and* 3 *are never credibly critical.*

The property of *credible criticality* of a player $i \in N$ for a coalition $M \subseteq N \setminus \{i\}$ can affect the ability of player i to gain power over a winning coalition $M \cup \{i\}$ by defeating the resistance of the other members in M to assign the marginal contribution $v(M \cup \{i\}) - v(M)$ to i.

More in general, we can think of a situation where the marginal contribution $v(M \cup \{i\}) - v(M)$ is assigned to player i only if i could potentially take part in another coalition $S \subseteq N \setminus M$ at least as powerful as $M \cup \{i\}$ and with $M \cap S = \varnothing$. For a simple game, this consideration leads us to the following definition of *credible* marginal contribution of a player i to a coalition $M \subseteq N \setminus \{i\}$:

$$\hat{m}_i^v(M) = \begin{cases} v(M \cup \{i\}) - v(M) & \text{if } v(N \setminus M) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Remark 4 *A credible marginal contribution exists and can be computed for each player i and each coalition $M \subseteq N \setminus \{i\}$ on each possible game (N, v). However, in the remaining of the paper we will focus on the computation of the quantity $\hat{m}_i^v(M)$ for simple games.*

Remark 5 *A more compact way to define the quantity $\hat{m}_i^v(S)$ for each $i \in N$ and $M \in 2^{N\setminus\{i\}}$ is as follows:*

$$\hat{m}_i^v(M) = \min\{v(M \cup \{i\}) - v(M), v(N \setminus M)\}. \tag{7}$$

Now, consider again a probability vector $\mathbf{p} = (p_0, \ldots, p_{n-1})$ as in the previous section. As a measure of the power that players may credibly claim in a simple game, we define the following *credible semivalue* engendered by the vector $\mathbf{p}$. Hence,

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N\setminus\{i\}}} p_s \hat{m}_i^v(S) = \sum_{S \in W_i(v)} p_s \hat{m}_i^v(S). \tag{8}$$

For simple games, $\hat{\pi}_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i to be credibly critical (under the probability vector $\mathbf{p}$). The next examples show that the vector of indices provided by a semivalue $\pi^{\mathbf{p}}$ engendered by a probability vector $\mathbf{p}$ can be drastically different from the one provided by the credible semivalue $\hat{\pi}^{\mathbf{p}}$ engendered by the same probability vector.

Example 7 *Consider the simple game (N, v) with $N = \{1, 2, 3\}$ whose unique minimal winning coalitions is $\{1, 2\}$. By the null player property, whatever semivalue $\pi^{\mathbf{p}}$ yields $\pi_3^{\mathbf{p}}(v) = 0$ and, by symmetry, $\pi_1^{\mathbf{p}}(v) = \pi_2^{\mathbf{p}}(v)$. Notice also that no player $i \in \{1, 2, 3\}$ is credibly critical. So, each credible semivalue yields $\hat{\pi}_i^{\mathbf{p}}(v) = 0$ for each $i \in \{1, 2, 3\}$.*

The next example shows that not only the values $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ can be very different, but even the ranking of players according to $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ need not be preserved.

Example 8 *Consider the simple game (N, v) with $N = \{1, 2, 3, 4, 5\}$ whose minimal winning coalitions are $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}$ and $\{3, 4, 5\}$. Consider the semivalue $\pi^{\mathbf{p}^\beta}$ corresponding to the Banzhaf value. Then, the Banzhaf value gives $\pi_1^{\mathbf{p}^\beta} = \pi_2^{\mathbf{p}^\beta} = \frac{6}{16}$ and $\pi_3^{\mathbf{p}^\beta} = \pi_4^{\mathbf{p}^\beta} = \pi_5^{\mathbf{p}^\beta} = \frac{4}{16}$. Now consider the notion of credibly criticality. Note that 1 and 2 are never credibly critical, whereas 3, 4 and 5 are credibly critical whenever they are critical. Consequently, $\hat{\pi}_1^{\mathbf{p}^\beta} = \hat{\pi}_2^{\mathbf{p}^\beta} = 0$ and $\hat{\pi}_3^{\mathbf{p}^\beta} = \hat{\pi}_4^{\mathbf{p}^\beta} = \hat{\pi}_5^{\mathbf{p}^\beta} = \frac{4}{16}$.*

Proposition 4 *A credible semivalue $\hat{\pi}^{\mathbf{p}}$ satisfies the null player property and the symmetry property.*

Proof Both properties follow from the definition of credible marginal contribution as for the case of classical semivalues. ■

Proposition 5 *Consider a weighted majority game (N, v) with quota and weights $[q; w_1, \ldots, w_n]$. Let $\hat{\pi}^{\mathbf{p}}$ defined according to relation (8) on any probability vector $\mathbf{p}$. Then,*

$$w_i \geq w_j \Rightarrow \hat{\pi}_i^{\mathbf{p}} \geq \hat{\pi}_j^{\mathbf{p}} \ ,$$

for each $i, j \in N$

Proof Let $i, j \in N$ with $w_i \geq w_j$ and $S \in 2^{N \setminus \{i,j\}}$. Then, by definition of weighted majority game, $v(S \cup \{i\}) \geq v(S \cup \{j\})$. So, by relation (7), it immediately follows that

$$\begin{array}{l} \hat{m}_i^v(S) = \min\{v(S \cup \{i\}) - v(S), v(N \setminus S)\} \geq \\ \min\{v(S \cup \{j\}) - v(S), v(N \setminus S)\} = \hat{m}_j^v(S). \end{array} \tag{9}$$

Now take $S \in 2^{N \setminus \{i,j\}}$ and consider the credible marginal contribution $\hat{m}_i^v(S \cup \{j\})$ and $\hat{m}_j^v(S \cup \{i\})$. Then, $v(S \cup \{i\}) \geq v(S \cup \{j\})$. Again by relation (7) it follows that

$$\begin{array}{l} \hat{m}_i^v(S \cup \{j\}) = \min\{v(S \cup \{i, j\}) - v(S \cup \{j\}), v(N \setminus (S \cup \{j\}))\} \geq \\ \min\{v(S \cup \{i, j\}) - v(S \cup \{i\}), v(N \setminus (S \cup \{i\}))\} = \hat{m}_j^v(S \cup \{i\}), \end{array} \tag{10}$$

where the inequality follows from the fact that by definition of the weighted majority game, $v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i, j\}) - v(S \cup \{i\})$, and $v(N \setminus (S \cup \{j\})) \geq v(N \setminus (S \cup \{i\}))$, since $N \setminus (S \cup \{j\})$ is obtained by substituting j with i in $N \setminus (S \cup \{i\})$.

The proof follows by relations (9) and (10) and the fact that by relation (8)

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} \left(\hat{m}_i^v(S) + \hat{m}_i^v(S \cup \{j\}) \right) ,$$

for each $i \in N$. ■

The converse of Proposition 5 is not true, as shown by Example 7, where the game (N, v) can be generated by the weighted majority situation $[3; 2, 2, 0]$ and $\hat{\pi}^{\mathbf{p}}(v) = (0, 0, 0)$ for any credible semivalue $\hat{\pi}^{\mathbf{p}}$.

5 Concluding Remarks

In this paper, we introduced the concept of order of criticality for a party in a winning coalition, based on the minimal number of other parties that are necessary for making the coalition losing. Then, we defined a measure of the criticality that allows fixing the relevance of a party in a majority coalition; this measure accounts for the probability that a party is critical of a given order. Finally, we studied the credibility of a threat of a party considering the possibility of forming an alternative majority joining to some of the parties in the opposition. Possible developments of the credibility may account for the effectiveness of the possible alternative majorities. For instance, it is possible to take into account the ideological contiguity of the parties on a left-right axis (see Fragnelli et al, 2009) or the previous majorities that were formed in the past.

Another possible direction of investigation may be the analysis of credible criticality of higher orders; in this case, when player $i \in N$ is critical of second (or higher) order for a winning coalition $M \ni i$, we cannot use the same approach as for criticality of the first order because this implies that the game is not proper as the two disjoint coalitions $M \setminus \{i\}$ and $(N \setminus M) \cup \{i\}$ are winning. Consequently, the definition should allow to include in the alternative majority some players in $M \setminus \{i\}$.

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