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KOSTANT-SOURIAU-ODZIJEWICZ QUANTIZATION OF A MECHANICAL SYSTEM WHOSE CLASSICAL PHASE SPACE IS A SIEGEL DOMAIN

ELISABETTA BARLETTA, SORIN DRAGOMIR, AND FRANCESCO ESPOSITO

ABSTRACT. We adopt the Kostant-Souriau-Odzijewicz quantization scheme for quantizing both the quantizable observables and the classical states of a mechanical system whose classical phase space is the Siegel domain $\Omega_n = \{\zeta \in \mathbb{C}^n : \operatorname{Im}(\zeta_1) > |\zeta'|^2\}$. We compute the transition probability amplitude $a_{0\overline{0}}(\zeta, z)$ from the state $z \in \Omega_n$ to the state $\zeta \in \Omega_n$. When the system interacts with weak external fields ϵB, $B \in L^\infty(\Omega)$, $0 < \epsilon << 1$, we show that the corresponding transition probability amplitudes are $a_{0\overline{0}}(\zeta, z) + O(\epsilon)$. We refute A. Odzijewicz's assumption that the measure on phase space [associated to the reproducing kernel of $L^2 H(\Omega_n, \gamma)$] should coincide, up to a multiplicative constant, with the Liouville measure.

1. Introduction and statement of main results

By a result of A. Odzijewicz (cf. [13]) the classical states of a mechanical system whose phase space is a complex manifold M may be quantized, provided that M satisfies the topological requirements needed to support globally defined Kählerian metrics. Quantization is an anti-holomorphic embedding $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ built from the data (E, H) where $E \to M$ is a complex line bundle which is both holomorphic and equipped with a Hermitian bundle metric H. Here $\mathcal{M}$ is the complex Hilbert space of all top degree E-valued holomorphic $(n, 0)$-forms φ on M such that $\int_M H^*(\varphi, \varphi) < \infty$. The transition probability amplitude $a_{\beta\overline{\alpha}}(\zeta, z)$ from $z \in U_\alpha$ to $\zeta \in U_\beta$ is assigned a precise mathematical meaning once the classical states $z, \zeta \in M$ are identified with the corresponding coherent states $\mathcal{K}(z), \mathcal{K}(\zeta) \in \mathbb{CP}(\mathcal{M})$. Here $\{U_\alpha\}_{\alpha \in I}$ is an open cover of the complex manifold M, underlying a fixed trivialization atlas of E as a holomorphic vector bundle. The construction of $\mathcal{K}$ and its use in the calculation of the transition probability amplitudes relies on a number of structural assumptions such as

(i) $\mathcal{M}$ is sufficiently ample (cf. [13], p. 579),

(ii) the pullback by $\mathcal{K}$ of the Fubini-Study metric on $\mathbb{CP}(\mathcal{M})$ is a globally defined Kählerian metric on M,

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(iii) $(E,\, H)$ is a quantum bundle [the curvature form of the canonical Hermitian connection on $(E,\, H)$ is a symplectic structure on M],

(iv) the measure on the phase space got from the data $(E,\, H)$

$$K_{\alpha\overline{\alpha}}(\zeta,\,\zeta)\,\gamma_\alpha(\zeta)\,d\zeta^1_\alpha\wedge\cdots\wedge d\zeta^n_\alpha\wedge d\overline{\zeta}^1_\alpha\wedge\cdots\wedge d\overline{\zeta}^n_\alpha$$

(cf. e.g. [3], p. 21) coincides with the Liouville measure (up to a multiplicative constant). When applied to the Siegel domain $M = \Omega_n$, the trivial line bundle $E = \Omega_n \times \mathbb{C}$, and the family of Hermitian metrics

$$H_a\big(\sigma_0\,,\,\sigma_0\big) = \rho^a\,,\quad a > -1,$$

$$\rho(\zeta) = \mathrm{Im}(\zeta_1) - \big|\zeta'\big|^2\,,\quad \zeta\in\Omega_n\,,$$

A. Odzijewicz's quantization scheme turns out to be closely related to the theory of weighted Bergman kernels (cf. e.g. [14]-[15]) and the quantization of classical states is a map

$$\mathcal{K}_a : \Omega_n \to \mathbb{CP}\Big[L^2H\big(\Omega_n\,,\,\gamma_a\big)\Big]$$

determined by the weighted Bergman kernel $K_{\gamma_a}\big(\zeta,\,z\big)$ associated to the (admissible) weight $\gamma_a \in AW(\Omega_n)$, $\gamma_a = \rho^a$. As such, the problem of computing the transition probability amplitudes is reduced to that of the calculation of the weighted Bergman kernel, which is of course the novelty brought forth by A. Odzijewicz. Yet the explicit expression of the reproducing kernels is available only for a handful of particular domains (cf. [12], p. 47-51, [5], [9], for the unit ball and certain complex ovals in the unweighted case, and [14], [8], for the unit ball in the weighted case) while in general (e.g. for an arbitrary smoothly bounded strictly pseudoconvex domain $\Omega\subset\mathbb{C}^n$) only asymptotic information near the boundary is available (cf. [7] and [16] respectively for the unweighted and weighted cases). On the other hand, the weighted Bergman kernel $K_{\gamma_a}(\zeta,\,z)$ was explicitly computed in [2], by combining a very general method discovered by S. Saitoh (cf. [18]) with the use of an integral transform on the Siegel domain due to M.M. Djrbashian & A.H. Karapetyan (cf. [6]), thus allowing for the calculation of the transition probability amplitudes

$$a_{0\overline{0}}\big(\zeta,\,z\big) = \left[\frac{2\,\rho(z)^{\frac12}\,\rho(\zeta)^{\frac12}}{i\big(\overline{z}_1-\zeta_1\big) - 2\langle\zeta'\,,\,z'\rangle}\right]^{n+a+1}.$$

When the given mechanical system interacts with the external fields B [that is, by exploiting the classical approach by R. Penrose, [17], the Hermitian structure on E is deformed as $H\longmapsto e^B\,H$] the coherent states change

$$\mathcal{K} : \Omega_n \to \mathbb{CP}\Big[L^2H\big(\Omega_n\,,\,e^B\,\gamma_a\big)\Big]$$

and while the explicit weighted Bergman kernel $K_{e^B\gamma_a}\big(\zeta,\,z\big)$ isn't available, we may exploit a result by Z. Pasternak-Winiarski (cf. [15]) that the map

$$K : AW\big(\Omega_n\big) \to HA\big(\Omega_n\big),\quad \gamma\longmapsto K_\gamma\,,$$

[associating to every admissible weight γ the corresponding weighted Bergman kernel K_γ regarded as an element of the complex Fréchet space $HA(\Omega_n)$] is analytic, thus producing a series development

$$K_{(1+h)\gamma_a} = K_{\gamma_a} + \sum_{k=1}^{\infty} (-1)^k K^{(k)}_{1,\gamma_a}\big(h, \cdots, h\big),$$

$$h = e^B - 1 \in B_{\frac{1}{2}}(0) \subset L^\infty\big(\Omega_n\big).$$

As emphasized by A. Odzijewicz (cf. [13], p. 587) Z. Pasternak-Winiarski's methods are effective for the case of weak external fields ϵB, $0 < \epsilon << 1$, and we may indeed show that the transition probability amplitude from z to ζ (when identified with coherent states in $\mathbb{CP}\big[L^2 H(\Omega_n\, ,\, e^{\epsilon B}\gamma_a)\big]$) equals

$$\left[\frac{2\,\rho(z)^{\frac{1}{2}}\,\rho(\zeta)^{\frac{1}{2}}}{i\big(\overline{z}_1 - \zeta_1\big) - 2\langle\zeta'\, ,\, z'\rangle}\right]^{n+a+1} + \frac{8\,\epsilon}{(c_{n,a})^2}\,\rho(z)^{\frac{n+a+1}{2}}\,\rho(\zeta)^{\frac{n+a+1}{2}} \times$$

$$\times \int_{\Omega_n} \Big\{\Big[\rho(z)^{n+a+1}\big|K_{\gamma_a}(w,\, z)\big|^2 + \rho(\zeta)^{n+a+1}\big|K_{\gamma_a}(\zeta,\, w)\big|^2\Big]\, K_{\gamma_a}(\zeta,\, z) +$$

$$-\frac{c_{n,a}}{2}\, K_{\gamma_a}(w,\, z)\, K_{\gamma_a}(\zeta,\, w)\Big\}\; B(w)\,\rho(w)^a\, d\mu(w) + \mathrm{O}\big(\epsilon^2\big).$$

As another main purpose of the present paper we discuss A. Odzijewicz's structural assumptions (i)-(iv) above and find that, for $H \in \big\{H_a\, ,\, e^{\epsilon B} H_a \; : \; B \in L^\infty(\Omega_n),\ \epsilon > 0,\ a > -1\big\}$, the assumptions (i)-(iii) are satisfied, while the assumption (iv) is questionable and indeed only partially satisfied by the model investigated by us. We end the present introduction with a brief explanation of our finding. Let $\gamma \in AW\big(\Omega_n\big) \cap C^\infty\big(\Omega_n\big)$ be a smooth admissible weight. If $H\big(\sigma_0\, ,\, \sigma_0\big) = \gamma$ then $(E,\, H)$ is a quantum bundle and hence there is yet another Hermitian bundle metric $\hat{H}$ on E given by

$$\hat{H}\big(\sigma_0\, ,\, \sigma_0\big)_\zeta = \frac{(-i)^n\,\Omega_{0\overline{0}}(\gamma)_\zeta}{K_{0\overline{0}}(\zeta\, ,\, \zeta)}\, , \quad \zeta \in \Omega_n\, ,$$

(cf. e.g. (38) in [3], p. 21). Both H and $\hat{H}$ are sections in the complex line bundle $E^* \otimes E^*$ (with $H \neq 0$) so their quotient is well defined, providing the map

$$F : AW\big(\Omega_n\big) \cap C^\infty\big(\Omega_n\big) \to C^\infty\big(\Omega_n\big), \quad F\big(\gamma\big) = \frac{\hat{H}}{H}\, .$$

As it turns out, the assumption (iv) is equivalent to the requirement that $F(\gamma)_\zeta$ is a constant, both with respect to γ and ζ. Our result is that

$$F\big(\gamma_a\big) = \frac{1}{c_{n,a}}\,\Big(-\frac{a}{2\pi}\Big)^n\, , \quad a > -1,$$

so $F\big(\gamma_a\big)$ is a constant function, whilst F is not. This however suffices for recovering $a_{0\overline{0}}\big(\zeta,\, z\big)$ by averaging $a_{0\overline{0}}\big(w,\, z\big)\, a_{0\overline{0}}\big(\zeta,\, w\big)$ [the transition probability amplitude from z to ζ with simultaneous transition through w] over $w \in \Omega_n$.

Let $f \in C_{FF}$ be a quantizable observable on Ω_n i.e. $f : \Omega_n \to \mathbb{R}$ is a C^∞ function such that $\big[X_f\, , \, \overline{Z}\big] \in T^{0,1}(\Omega_n)$ for any $Z \in T^{1,0}(\Omega_n)$, where X_f is the symplectic gradient of f [with respect to the symplectic structure $\omega_a = \operatorname{curv}\big(E, \, H_a\big)$]. Together with [10] we consider

$$\hat{f} = \frac{1}{i}\Big[\nabla_{X_f} + 4\pi i\, f\Big] \otimes \mathcal{L}_{X_f} \tag{1.1}$$

and show that (1.1) is a linear operator of $\mathcal{M}_a \approx L^2 H(\Omega_n\, , \, \gamma_a)$ into itself, satisfying

$$\frac{1}{i}\, d\Big[X_f \rfloor H^*(\varphi, \, \psi)\Big] = H^*\big(\hat{f}(\varphi), \, \psi\big) - H^*\big(\varphi, \, \hat{f}(\psi)\big) \tag{1.2}$$

for any $\varphi, \, \psi \in \mathcal{M}_a$. Following the approach by A. Odzijewicz (cf. [13]) one may of course integrate over Ω_n in (1.2) so that to obtain

$$i^{n^2-1} \int_{\Omega_n} d\Big[X_f \rfloor H^*(\varphi, \, \psi)\Big] = \langle \hat{f}(\varphi), \, \psi \rangle_{\mathcal{M}_a} - \langle \varphi, \, \hat{f}(\psi) \rangle_{\mathcal{M}_a}$$

yet no version of Stokes' theorem is available in the case at hand.

2. Weighted Bergman kernels, Banach manifolds of weights, Winiarski's theorem, Gawędzki's lemma

Let $\Omega_n = \{\zeta \in \mathbb{C}^n \, : \, \rho(\zeta) > 0\}$ be the Siegel domain i.e.

$$\rho(\zeta) = \operatorname{Im}\big(\zeta_1\big) - \big|\zeta'\big|^2, \quad \zeta = \big(\zeta_1\, , \, \zeta'\big), \quad \zeta_1 \in \mathbb{C}, \quad \zeta' \in \mathbb{C}^{n-1}\, .$$

A measurable function $\gamma \, : \, \Omega_n \to [0, +\infty)$ is a *weight* on Ω_n. The set of all weights on Ω_n is denoted by $W(\Omega_n)$. For every $\gamma \in W(\Omega_n)$ let $L^2(\Omega_n, \, \gamma)$ consist of all measurable functions $f \, : \, \Omega_n \to \mathbb{C}$ such that $\int_{\Omega_n} |f(\zeta)|^2 \, \gamma(\zeta) \, d\mu(\zeta) < \infty$. Here μ is the Lebesgue measure on $\mathbb{R}^{2n}$. Let $L^2 H(\Omega_n, \, \gamma)$ denote the space of all holomorphic functions $f : \Omega_n \to \mathbb{C}$ [i.e. $\overline{\partial} f = 0$ in Ω_n] such that $f \in L^2(\Omega_n, \, \gamma)$. A weight $\gamma \in W(\Omega_n)$ is *admissible* if 1) $L^2 H(\Omega_n, \, \gamma)$ is a closed subspace of $L^2(\Omega_n, \, \gamma)$, and 2) the evaluation functional

$$\delta_\zeta : L^2 H(\Omega_n, \, \gamma) \to \mathbb{C}, \quad \delta_\zeta(f) = f(\zeta),$$

is continuous for any $\zeta \in \Omega_n$. Let $AW(\Omega_n)$ be the set of all admissible weights on Ω. If $\gamma \in AW(\Omega_n)$ then $L^2 H(\Omega_n, \, \gamma)$ is a Hilbert space with the inner product

$$\big(f, \, g\big)_\gamma = \int_{\Omega_n} f(\zeta) \, \overline{g(\zeta)} \, \gamma(\zeta) \, d\mu(\zeta).$$

By the Riesz representation theorem, for every $\zeta \in \Omega_n$ there is a unique $k_{\zeta, \gamma} \in L^2 H(\Omega_n, \, \gamma)$ such that $\delta_\zeta(f) = \big(f, \, k_{\zeta, \gamma}\big)_\gamma$ for any $f \in L^2 H(\Omega_n, \, \gamma)$. The function

$$K_\gamma : \Omega_n \times \Omega_n \to \mathbb{C}, \quad K_\gamma(\zeta, \, z) = \overline{k_{\zeta, \gamma}(z)}\, , \quad \zeta, \, z \in \Omega_n,$$

is the γ-*Bergman kernel* of Ω_n. By a result in [2] if $\gamma_a(\zeta) = \rho(\zeta)^a$ with $a > -1$ then $\{\gamma_a \, : \, a > -1\} \subset AW(\Omega_n)$ and the corresponding γ_a-Bergman kernel is given by

$$K_{\gamma_a}\big(\zeta, \, z\big) = \frac{2^{n-1+a} \, c_{n,a}}{\big[i(\overline{z}_1 - \zeta_1) - 2\langle \zeta', \, z' \rangle\big]^{n+1+a}}\, , \tag{2.1}$$

$$c_{n,a} = \pi^{-n} \, (a+1) \cdots (a+n).$$

Note that $C^\infty\big(\Omega_n, \mathbb{R}_+\big) \subset AW(\Omega_n)$ with $\mathbb{R}_+ = (0, +\infty)$. Indeed $\gamma^{-1} \in L^1_{\rm loc}\big(\Omega_n\big)$ for any continuous function $\gamma : \Omega_n \to (0, +\infty)$ (and one applies Corollary 3.1 in [14], p. 6). By a result in [15] the set $W(\Omega_n)$ may be organized as an infinite dimensional Banach manifold, modeled on $L^\infty(\Omega_n)$, the Banach algebra of all real valued essentially bounded functions $g : \Omega_n \to \mathbb{R}$, such that $AW(\Omega_n)$ is an open subset of $W(\Omega_n)$. We briefly recall the construction in [15], as follows. Let us set $U(\Omega_n) = \{g \in L^\infty(\Omega_n) \,:\, \operatorname{ess\,inf}_{\zeta \in \Omega_n} g(\zeta) > 0\}$ [an open subset of $L^\infty(\Omega_n)$]. For every $\gamma \in W(\Omega_n)$ we consider the map

$$\Phi_\gamma : U(\Omega_n) \to W(\Omega), \quad \Phi_\gamma(g) = g\,\gamma,$$

and set $U(\Omega_n, \gamma) = \Phi_\gamma\big[U(\Omega_n)\big]$. One endows $W(\Omega_n)$ with a topology τ for which $\mathcal{B} = \{\Phi_\gamma(X) \,:\, \gamma \in W(\Omega_n),\ X \subset U(\Omega_n),\ X \text{ open}\}$ is a base of open sets, and then $\{\Phi_\gamma^{-1} : U(\Omega_n\,, \gamma) \to U(\Omega_n) \,\big|\, \gamma \in W(\Omega_n)\}$ is an analytic atlas on $\big(W(\Omega_n),\ \tau\big)$ organizing it as a Banach manifold. Here analyticity means that transition functions

$$F_{\gamma,\,\varphi} = \Phi_\gamma^{-1} \circ \Phi_\varphi : U(\Omega_n) \to U(\Omega_n), \quad \gamma,\, \varphi \in W(\Omega_n),$$

are analytic i.e. for any $g \in U(\Omega_n)$ there is a ball $B \subset L^\infty(\Omega_n)$ such that $g + B \subset U(\Omega_n)$ and there is a sequence $\{a_m\}_{m \in \mathbb{N}}$ of continuous multi-linear maps $a_m : L^\infty(\Omega_n)^m \to L^\infty(\Omega_n)$ such that

$$F_{\gamma,\,\varphi}\big(g + h\big) = F_{\gamma,\,\varphi}\big(g\big) + \sum_{m=1}^{\infty} a_m\big(h,\, \cdots,\, h\big), \quad h \in B,$$

and the series $\sum_{m=1}^\infty a_m\big(h,\, \cdots,\, h\big)$ converges uniformly on B. Let $HA(\Omega_n)$ be the space of all functions $f : \Omega_n \times \Omega_n \to \mathbb{C}$ which are holomorphic in the first n variables, and anti-holomorphic in the last n variables. $HA(\Omega_n)$ is organized as a Fréchet space, whose underlying locally convex topology is determined by the following separating family of semi-norms

$$\|f\|_A = \sup_{(\zeta,\, z) \in A \times A} \big|f(\zeta,\, z)\big|, \quad A \subset \Omega_n\,, \quad A \text{ compact}.$$

By a result in [15] the map $\gamma \in AW(\Omega_n) \mapsto K_\gamma \in HA(\Omega_n)$ is analytic i.e. its local expression [with respect to the local chart Φ_γ^{-1}]

$$g \in U(\Omega_n) \mapsto K_{g\,\gamma} \in HA(\Omega_n)$$

is analytic, for any $\gamma \in AW(\Omega_n)$.

Let $E = \Omega_n \times \mathbb{C}^n$ be the trivial complex line bundle over Ω_n and let $T_0 = 1_E$ [so that $\big\{T_0\big\}$ is a fixed trivialization atlas]. Let $\sigma_0 \in \mathcal{O}(\Omega_n\,, E)$ be the holomorphic section defined by $\sigma_0(\zeta) = T_0^{-1}(\zeta,\, 1) = (\zeta,\, 1)$. If H is a Hermitian bundle metric on E we set $\gamma = H\big(\sigma_0\,, \sigma_0\big)$. To every pair of E-valued holomorphic $(n, 0)$-forms φ, ψ on Ω_n we associate a scalar (n, n)-form $H^*(\varphi\,, \psi)$ given by

$$H^*(\varphi\,, \psi) = \Psi_\alpha\, \overline{\Phi}_\alpha\, \gamma_\alpha\, d\zeta^1 \wedge \cdots \wedge d\zeta^n \wedge d\overline{\zeta}^1 \wedge \cdots \wedge d\overline{\zeta}^n\,,$$

$$\varphi = \Psi\, \sigma_0 \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n\,, \quad \psi = \Phi\, \sigma_0 \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n\,,$$

$$\Psi\,,\ \Phi \in \mathcal{O}(\Omega_n).$$

Let $\mathcal{M}$ consist of all $\varphi \in \mathcal{O}\big(\Lambda^{n,0}(\Omega_n) \otimes E\big)$ such that $\int_{\Omega_n} H^*(\varphi\,,\,\varphi) < \infty$. By a result of K. Gawędzki (cf. [10]) 1) $\mathcal{M}$ is a complex Hilbert space with the inner product

$$(\varphi\,,\,\psi)_{\mathcal{M}} = i^{n^2} \int_{\Omega_n} H^*(\varphi\,,\,\varphi),$$

and 2) $\big|\delta^0_\zeta(\varphi)\big| \leq C\,\|\varphi\|_{\mathcal{M}}$ for some constant $C > 0$, where

$$\delta^0_\zeta : \mathcal{M} \to \mathbb{C}, \quad \delta^0_z(\varphi) = \Psi_\alpha(\zeta), \quad \|\varphi\|_{\mathcal{M}} = \sqrt{(\varphi\,,\,\varphi)_{\mathcal{M}}}\,, \quad \varphi \in \mathcal{M}.$$

Consequently there is a unique $k_{\zeta,\,\overline{0}} \in \mathcal{M}$ such that

$$\delta^0_\zeta(\varphi) = \big(\varphi\,,\,k_{\zeta,\,\overline{0}}\big)_{\mathcal{M}}$$

for every $\varphi \in \mathcal{M}$. For the representation of $k_{\zeta,\,\overline{0}}$ we adopt the notation

$$k_{\zeta,\,\overline{0}} = \overline{K_{0\overline{0}}(\zeta\,,\,\cdot\,)}\,\sigma_0 \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n$$

thus producing the kernel $K_{0\overline{0}}(\zeta,\,z)$.

Lemma 2.1. i) $\mathcal{M} \approx L^2 H(\Omega_n\,,\,\gamma)$ (*an isomorphism of Hilbert spaces*).

ii) $K_{0\overline{0}}(\zeta, z) = 2^{-n} K_\gamma(\zeta,\,z)$.

Proof. (i)-(ii) follow from Example 5 in [3], p. 35. □

Lemma 2.2. *If* $\gamma \in \{\gamma_a\,:\,a > -1\}$ *then* $k_{\zeta,\,\overline{0}} \neq 0$ *for any* $\zeta \in \Omega_n$. *Consequently the map*

$$\mathcal{K}_a : \Omega_n \to \mathbb{C}P(\mathcal{M}_a), \quad \mathcal{K}_a(\zeta) = \big[k_{\zeta,\,\overline{0}}\big], \quad \zeta \in \Omega_n\,,$$

is well defined.

Here $[\varphi] = \{\lambda\varphi\,:\,\lambda \in \mathbb{C}^*\}$ is the projective ray represented by $\varphi \in \mathcal{M} \setminus \{0\}$. Let us set

$$g = \frac{\partial^2 \log K_{0\overline{0}}(\zeta,\,\zeta)}{\partial\zeta^j\,\partial\overline{\zeta}^k}\,d\zeta^j \odot d\overline{\zeta}_k\,.$$

Then g is a symmetric $(0,2)$-tensor field on Ω_n. When $\gamma \in \{\gamma_a\,:\,a > -1\}$ the notation is specialized to $g = g_a$. According to the terminology in [13], $\mathcal{M}$ is *sufficiently ample* if

$$\forall\, z^1\,,\,z^2 \in \Omega_n\,, \quad \exists\,\varphi_1\,,\,\varphi_2 \in \mathcal{M}\,:\,\det \begin{bmatrix} \Psi_1\big(z^1\big) & \Psi_1\big(z^2\big) \\ \Psi_2\big(z^1\big) & \Psi_2\big(z^2\big) \end{bmatrix} \neq 0.$$

Lemma 2.3. $\mathcal{M}_a$ *is sufficiently ample.*

Proof. Let $z,\,\zeta \in \Omega_n$ such that $\mathcal{K}_a(z) = \mathcal{K}_a(\zeta)$ i.e. $k_{\zeta,\,\overline{0}} = \lambda\,k_{z,\,\overline{z}}$ for some $\lambda \in \mathbb{C}^*$. Then [by (ii) in Lemma 2.1]

$$K_{\gamma_a}(\zeta,\,w) = \overline{\lambda}\,K_{\gamma_a}(z,\,w), \quad w \in \Omega_n\,. \tag{2.2}$$

In particular for $w' = 0$

$$\left(\frac{\overline{\zeta}_1 - w_1}{\overline{z}_1 - w_1}\right)^{n+1+a} = \frac{1}{\lambda}$$

on the halfplane $\mathrm{Im}(w_1) > 0$. Yet $w_1 \mapsto \big(\overline{\zeta}_1 - w_1\big)\big(\overline{z}_1 - w_1\big)^{-1}$ is a constant function if and only if $\zeta_1 = z_1$. Consequently (2.2) reads

$$\Big[i\big(\overline{w}_1 - \zeta_1\big) - 2\langle \zeta' \,,\, w'\rangle\Big]^{n+1+a} = \frac{1}{\lambda}\Big[i\big(\overline{w}_1 - \zeta_1\big) - 2\langle z' \,,\, w'\rangle\Big]^{n+1+a}\,, \quad w \in \Omega_n\,.$$

In particular for $w' = \zeta'$

$$\left[\frac{\overline{w}_1 - \zeta_1 + 2i|\zeta'|^2}{\overline{w}_1 - \zeta_1 + 2i\langle z' \,,\, \zeta'\rangle}\right]^{n+1+a} = \frac{1}{\lambda}$$

yielding $|\zeta'|^2 = \langle z' \,,\, \zeta'\rangle$. By interchanging z and ζ in the previous argument one also obtains

$$|z'|^2 = \langle \zeta' \,,\, z'\rangle. \tag{2.3}$$

Multiplication of the last two identities gives $|\langle z' \,,\, \zeta'\rangle|^2 = |z'|^2|\zeta'|^2$ which holds if and only if z' and ζ' are collinear i.e. $\zeta' = \mu\, z'$ for some $\mu \in \mathbb{C}^*$. Finally substitution into (2.3) yields $\mu = 1$ i.e. $\zeta' = z'$. Summing up, $\mathcal{K}_a$ is injective. By Proposition 2 in [13], p. 582, injectivity of $\mathcal{K}_a$ and sufficient ampleness of $\mathcal{M}_a$ are equivalent. Q.e.d. □

Lemma 2.4. i) $\big(E,\, H_a\big)$ *is a quantum bundle.*

ii) g_a *is a Kählerian metric on* Ω_n.

Proof. i) Let ∇ be the canonical Hermitian connection of $\big(E,\, H_a\big)$ i.e. 1) $\nabla H_a = 0$ and 2) $\nabla^{0,1} = \overline{\partial}_E$. Let $\omega = \mathrm{curv}(\nabla)$ be the curvature 2-form of $\big(E,\, \nabla\big)$. Then (cf. e.g. [11])

$$\omega = -\frac{1}{2\pi i}\,\partial\overline{\partial}\,\log\gamma_a$$

[hence $d\omega = 0$] or

$$\big[\omega_{j\overline{k}}\big]_{1\le j,k\le n} = -\frac{1}{2\pi i}\,\frac{a}{\rho(\zeta)^2}\,\times \tag{2.4}$$

$$\times\begin{bmatrix} -\dfrac{1}{4} & \dfrac{1}{2i}\,\zeta_\beta \\ & \\ -\dfrac{1}{2i}\,\overline{\zeta}_\alpha & -\big[\rho(\zeta)\,\delta_{\alpha\beta} + \overline{\zeta}_\alpha\,\zeta_\beta\big]\end{bmatrix}_{2\le\alpha,\beta\le n}\,,$$

$$\det\left[\frac{\partial^2\gamma_a}{\partial\zeta_j\,\partial\overline{\zeta}_k}\right] = \frac{(-a)^n}{4\,\rho(\zeta)^{n+1}}\,,$$

so that ω is nondegenerate. Q.e.d.

ii) Let $K(\zeta,\, z)$ and

$$g_0 = \frac{\partial^2\log K(\zeta,\,\zeta)}{\partial\zeta_j\,\partial\overline{\zeta}_k}\,d\zeta_j\odot d\overline{\zeta}_k$$

be respectively the ordinary (unweighted) Bergman kernel and metric of Ω_n. Then

$$K\big(\zeta,\, z\big) = \frac{2^{n-1}\,c_{n,0}}{\big[i(\overline{z}_1 - \zeta_1) - 2\langle\zeta',\, z'\rangle\big]^{n+1}}\,, \quad c_{n,0} = \pi^{-n}\,n!\,,$$

$$K\big(\zeta,\,\zeta\big) = \frac{\pi^{-n}\,n!}{4\,\rho(\zeta)^{n+1}}\,, \quad g_0 = -(n+1)\frac{\partial^2\rho}{\partial\zeta_j\,\partial\overline{\zeta}_k}\,d\zeta_j \odot d\overline{\zeta}_k\,,$$

and (ii) follows by observing that

$$K_{0\overline{0}}\big(\zeta,\,\zeta\big) = \frac{2^{-(n+2)}\,c_{n,a}}{\rho(\zeta)^{n+1+a}}\,, \quad g_a = \frac{n+1+a}{n+1}\,g_0\,,$$

i.e. g_a is the ordinary Bergman metric, in another guise. Q.e.d. But then ω is (up to a multiplicative constant) the Kähler 2-form of g_a [providing yet another proof to statement (i)]. $\square$

Theorem 2.5. *$\mathcal{K}_a : \Omega_n \to \mathbb{CP}\big[L^2H\big(\Omega_n\,,\,\gamma_a\big)\big]$ is an anti-holomorphic embedding and g_a is the pullback by $\mathcal{K}_a$ of the Fubini-Study metric on $\mathbb{CP}\big[L^2H\big(\Omega_n\,,\,\gamma_a\big)\big]$, for any $a > -1$.*

Proof. By Lemma 2.3, the complex Hilbert space $\mathcal{M}_a$ is sufficiently ample, while by Lemma 2.4 the $(0,2)$-tensor field g_a is a Kählerian metric. Then, by a result of A. Odzijewicz (cf. Proposition 3 in [13], p. 583) $K_a : \Omega_n \to \mathbb{CP}\big(\mathcal{M}_a\big)$ is an anti-holomorphic embedding. Q.e.d. $\square$

3. Transition probability amplitudes

The transition probability amplitude $a_{0\overline{0}}(\zeta,\,z)$ [from the state $z \in \Omega_n$ to the state $\zeta \in \Omega_n$] is

$$a_{0\overline{0}}\big(\zeta,\,z\big) = \left\langle \frac{k_{z,\,\overline{0}}}{\big\|k_{z,\,\overline{0}}\big\|}\,,\,\frac{k_{\zeta,\,\overline{0}}}{\big\|k_{\zeta,\,\overline{0}}\big\|}\right\rangle = \frac{K_{0\overline{0}}(\zeta,\,z)}{K_{0\overline{0}}(z,\,z)^{1/2}K_{0\overline{0}}(\zeta,\,\zeta)^{1/2}}$$

cf. (2.20) in [13], p. 584, or [3], p. 9. Substitution from (2.1) yields

$$a_{0\overline{0}}(\zeta, z) = \left[\frac{2\,\rho(z)^{1/2}\rho(\zeta)^{172}}{i(\overline{z}_1 - \zeta_1) - 2\langle\zeta'\,,\,z'\rangle}\right]^{n+1+a}\,.$$

If we set

$$d = \frac{4\,\rho(z)\,\rho(\zeta)}{\big|i\big(\overline{z}_1 - \zeta_1\big) - 2\,\langle\zeta'\,,\,z'\rangle\big|^2}$$

then $0 < d \le 1$ and d^{n+1+a} is the transition probability density. A comparison of the quantizations $\big\{\mathcal{K}_a\ :\ a > -1\big\}$ shows that, for two fixed classical states $z,\,\zeta \in \Omega_n$, the transition probability density decreases indefinitely as a function of the parameter $a > -1$ [i.e. $\lim_{a\to+\infty}\big|a_{0\overline{0}}(\zeta,\,z)\big|^2 = 0$].

The transition probability amplitude from z to ζ with *simultaneous transition* though w is $a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta,\,w)$ (cf. [13], p. 584). We wish to show that averaging over $w \in \Omega$ with respect to the Liouville measure

$$d\,\mu_L = (-i)^n\,\Omega_{0\overline{0}}\big(\gamma_a\big)\,d\zeta^1 \wedge \cdots \wedge d\zeta^n \wedge d\overline{\zeta}^1 \wedge \cdots \wedge d\overline{\zeta}^n$$

gives $a_{0\overline{0}}(\zeta, z)$. Here $\Omega_{0\overline{0}}\big(\gamma_a\big) = \det\big[\omega_{j\overline{k}}\big]$ and $\omega = \operatorname{curv}\big(E,\,H_a\big)$. The precise statement is Theorem 3.1 below.

We need some preparation on the Liouville form associated to the Hermitian bundle $\big(E,\,H\big)$ with $H\big(\sigma_0\,,\,\sigma_0\big) = \gamma$, for an arbitrary C^∞ weight $\gamma \in W(\Omega_n)$. Let

$\omega = \operatorname{curv}(E,\, H)$ and let us set $\Omega_{0\overline{0}}(\gamma) = \det\big[\omega_{j\overline{k}}\big]$. Let us consider the field $\hat{H}$ of sesquilinear forms on E given by

$$\hat{H}\big(\sigma_0\,,\,\sigma_0\big)_\zeta = (-i)^n\,\frac{\Omega_{0\overline{0}}(\gamma)_\zeta}{K_{0\overline{0}}(\zeta\,,\,\zeta)}\,, \quad \zeta \in \Omega_n\,.$$

Note that

$$(-i)^n\,\Omega_{0\overline{0}}(\gamma) = \frac{1}{(4\pi)^n}\det\left[\frac{\partial^2 \log\gamma}{\partial\zeta^j\,\partial\overline{\zeta}^k}\right].$$

If $\gamma \in \{\gamma_a \,:\, a > -1\}$ then [by the proof of statement (ii) in Lemma 2.4]

$$(-2i)^n\frac{\Omega_{0\overline{0}}(\gamma_a)_\zeta}{K_{\gamma_a}(\zeta\,,\,\zeta)} = \frac{1}{c_{n,a}}\left(-\frac{a}{2\pi}\right)^n\,\rho(\zeta)^a$$

hence $\hat{H}_a$ is definite and then [by eventually replacing $\hat{H}_a$ by $-\hat{H}_a$] a Hermitian bundle metric on E. For any $\varphi = \Psi\,\sigma_0 \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \in \mathcal{M}_a$

$$\langle\varphi\,,\,k_{\zeta,\,\overline{0}}\rangle_{\mathcal{M}_a} = \Psi(\zeta).$$

In particular for $\varphi = k_{z,\,\overline{0}}$

$$\big\langle k_{z,\,\overline{0}}\,,\,k_{\zeta,\,\overline{0}}\big\rangle_{\mathcal{M}_a} = \overline{K_{0\overline{0}}\big(z,\,\zeta\big)}$$

or

$$i^{n^2}\int_{\Omega_n} H^*\big(k_{z,\,\overline{0}}\,,\,k_{\zeta,\,\overline{0}}\big) = K_{0\overline{0}}\big(\zeta,\,z\big)$$

yielding

$$2^{-n}i^{n^2}\int_{\Omega_n} K_{\gamma_a}(w,z)\,K_{\gamma_a}(\zeta,w)\,\gamma_a(w)\,dw^{1\cdots n\,\overline{1}\cdots\overline{n}} = K_{\gamma_a}(\zeta,\,z)$$

or [dividing by $K_{\gamma_a}(z,z)^{1/2}K_{\gamma_a}(\zeta,\zeta)^{1/2}$]

$$a_{0\overline{0}}(\zeta,z) = 2^{-n}i^{n^2}\,\times \tag{3.1}$$

$$\times\int_{\Omega_n} a_{0\overline{0}}(w,z)\,a_{0\overline{0}}(\zeta,w)\,K_{\gamma_a}(w,w)\,\gamma_a(w)\,dw^{1\cdots n\,\overline{1}\cdots\overline{n}}\,.$$

Here $dw^{1\cdots n\,\overline{1}\cdots\overline{n}}$ is short for $dw^1 \wedge \cdots \wedge dw^n \wedge d\overline{w}^1 \wedge \cdots \wedge d\overline{w}^n$.

Theorem 3.1. *Let $E = \Omega_n \times \mathbb{C}$ be the trivial complex line bundle, endowed with the Hermitian bundle metric H_a [i.e. $H_a(\sigma_0\,,\,\sigma_0) = \rho^a$] with $a \in (-1,+\infty)\setminus\{0\}$, and let $\mathcal{K}_a : \Omega_n \to \mathbb{CP}\big[L^2H(\Omega_n\,,\,\gamma_a)\big]$ be the corresponding quantization of classical states. Then*

$$a_{0\overline{0}}(\zeta,\,z) = i^{n^2}\int_{\Omega_n} a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta,\,w)\,d\tilde{\mu}_L(w)$$

i.e. the transition probability amplitude $a_{0\overline{0}}(\zeta,\,z)$ [relying upon the identification $z \approx \mathcal{K}_a(z)$ and $\zeta \approx \mathcal{K}_a(\zeta)$] is the average over $w \in \Omega_n$ of the transition probability amplitude from z to ζ with simultaneous transition through w, with respect to the measure

$$d\tilde{\mu}_L = c_{n,a}\left(-\frac{2\pi}{a}\right)^n d\mu_L\,.$$

Proof. Follows from

$$(-i)^n \, \Omega_{0\overline{0}}\big(\gamma_a\big)_\zeta = \frac{1}{c_{n,a}} \Big(-\frac{a}{4\pi}\Big)^n \, K_{\gamma_a}(\zeta, \zeta) \, \gamma_a(\zeta).$$

□

4. External fields

By following the ideas of R. Penrose (cf. [17]) the external fields interacting (at the quantum level) with the given mechanical system are mathematically described by a deformation of the holomorphic and metric structures of the given complex line bundle E. To keep matters simple, we only consider external fields B producing a deformation of the metric tensor H_a i.e. we endow E with the Hermitian metric $e^B H_a$ while keeping the holomorphic structure of E fixed. We seek to compute the transition probability amplitudes after each classical state $\zeta \in \Omega_n$ is identified with the coherent state $\mathcal{K}(\zeta) \in \mathbb{CP}\big[L^2 H\big(\Omega_n \, , \, e^B \, \gamma_a\big)\big]$. This amounts to the calculation of the weighted Bergman kernel $K_{e^B \gamma_a}\big(\zeta, \, z\big)$.

We need a preparation on the bilinear forms H and $\hat{H}$, associated to the (admissible) weight $e^B \, \gamma$ with $\gamma \in C^\infty\big(\Omega_n \, , \, \mathbb{R}_+\big)$ and $B \in C^\infty(\Omega_n \, , \, \mathbb{R})$, touching upon the field equations proposed by A. Odzijewicz (cf. equation (3.3) in [13], p. 587) which we implicitly refute [as Odzijewicz's structural assumption (iv) as stated in Section § 1 of the present paper, isn't realized in general]. Starting from

$$\det \left[\frac{\partial^2 \log \big(e^B \gamma\big)}{\partial \zeta_j \, \partial \overline{\zeta}_k} \right]_\zeta = (2\pi)^n \, F\big(e^B \, \gamma\big)_\zeta \, e^{B(\zeta)} \, \gamma(\zeta) \, K_{e^B \gamma}(\zeta, \, \zeta),$$

$$\det \left[\frac{\partial^2 \log \gamma}{\partial \zeta_j \, \partial \overline{\zeta}_k} \right]_\zeta = (2\pi)^n \, F(\gamma)_\zeta \, \gamma(\zeta) \, K_\gamma(\zeta, \, \zeta),$$

one obtains

$$\det \left[\frac{\partial^2 \log \gamma}{\partial \zeta_j \, \partial \overline{\zeta}_j} + \frac{\partial^2 B}{\partial \zeta_j \, \partial \overline{\zeta}_k} \right] = \frac{F\big(e^B \gamma\big)}{F(\gamma)} \, \det \left[\frac{\partial^2 \log \gamma}{\partial \zeta_j \, \partial \overline{\zeta}_j} \right] e^{B+A} \tag{4.1}$$

where we have set

$$e^{A(\zeta)} = \frac{K_{e^B \gamma}(\zeta, \, \zeta)}{K_\gamma(\zeta, \, \zeta)} \, , \quad \zeta \in \Omega_n \, , \tag{4.2}$$

i.e. the function $A : \Omega_n \to \mathbb{R}$ describes the deformation of the weighted Bergman kernel produced by the deformation $B : \Omega_n \to \mathbb{R}$ of the metric structure. The dependence of A on B appears to play a fundamental role in the application of the theory to scalar massive conformal particles (cf. [13], p. 590-596). A. Odzijewicz's suggestion is that (4.1) be looked as a field equation for B and that, once a solution B is known, the potential A should be computed from (4.1) as a function of B.

When $\gamma \in \{\gamma_a : a > -1\}$ equations (4.1)-(4.2) read

$$\det \begin{bmatrix} \dfrac{\partial^2 B}{\partial \zeta_1 \, \partial \overline{\zeta}_1} - \dfrac{a}{4\rho^2} & \dfrac{\partial^2 B}{\partial \zeta_1 \, \partial \overline{\zeta}_\beta} + \dfrac{a}{2i\rho^2} \, \zeta_\beta \\ \\ \dfrac{\partial^2 B}{\partial \zeta_\alpha \, \partial \overline{\zeta}_\beta} - \dfrac{a}{2i\rho^2} \, \overline{\zeta}_\alpha & \dfrac{\partial^2 B}{\partial \zeta_\alpha \, \partial \overline{\zeta}_\beta} - \dfrac{a}{\rho^2} \big[\rho \, \delta_{\alpha\beta} + \overline{\zeta}_\alpha \, \zeta_\beta \big] \end{bmatrix} =$$

$$= \frac{c_{n,a}}{4} \, (-2\pi)^n \, F\big(e^B \gamma_a\big) \, \rho^{-(n+1)} \, e^{B+A},$$

$$e^{A(\zeta)} = \frac{4}{c_{n,a}} \, \rho(\zeta)^{-(n+1+a)} \, K_{e^B \gamma_a}\big(\zeta, \, \zeta\big).$$

As to the calculation of the kernel $K_{e^B \gamma_a}$ we build on the results in [15] i.e. for any $g \in U(\Omega_n)$ and any $h \in B_{i(g)/2}(0)$

$$K_{(g+h)\gamma_a} = K_{g\gamma_a} + \sum_{m=1}^{\infty} (-1)^m \, K^{(m)}_{g, \, \gamma_a}\big(h, \, \cdots, \, h\big), \tag{4.3}$$

$$K^{(m)}_{g, \, \gamma_a}\big(h_1 \, , \, \cdots \, , \, h_m\big)(\zeta, \, z) =$$

$$= \int_{\Omega_n} K_{g\gamma_a}\big(w_1 \, , \, z\big) \, h_1\big(w_1\big) \, \gamma_a\big(w_1\big) \, d\mu(w_1) \cdot$$

$$\cdot \int_{\Omega_n} K_{g\gamma_a}\big(w_2 \, , \, w_1\big) \, h_2\big(w_2\big) \, \gamma_a\big(w_2\big) \, d\mu(w_2) \cdot$$

$$\vdots$$

$$\cdot \int_{\Omega_n} K_{g\gamma_a}\big(w_{m-1} \, , \, w_{m-2}\big) h_{m-1}\big(w_{m-1}\big) \, \gamma_a\big(w_{m-1}\big) \, d\mu(w_{m-1}) \cdot$$

$$\cdot \int_{\Omega_n} K_{g\gamma_a}\big(w_m \, , \, w_{m-1}\big) \, h_m\big(w_m\big) \, K_{g\gamma_a}\big(\zeta, \, w_m\big) \, \gamma\big(w_m\big) \, d\mu(w_m),$$

for any $h_1 \, , \cdots , \, h_m \in L^\infty(\Omega_n)$. Here $i(g) = \operatorname{ess\,inf}_{z \in \Omega_n} g(z)$ and $B_r(0)$ is the ball of radius $r > 0$ and center 0 in $L^\infty(\Omega_n)$. In particular

$$K^{(1)}_{g, \, \gamma_a}(h)(\zeta, \, z) = \int_{\Omega_n} K_{g\gamma_a}\big(w, \, z\big) \, h\big(w\big) \, K_{g\gamma_a}\big(\zeta, \, w\big) \, \gamma(w) \, d\mu(w).$$

We shall only study the case of weak external fields ϵB with $0 < \epsilon << 1$. The corresponding weight function is $e^{\epsilon B} \gamma_a$ and we ought to apply (4.3) for $g \equiv 1$ [so that $i(g) = 1$] and $1 + h = e^{\epsilon B}$ i.e. [by $h = \epsilon \, B + \mathrm{O}(\epsilon^2)$]

$$K^{(m)}_{1, \, \gamma_a}\big(h, \, \cdots \, , \, h\big) = \mathrm{O}\big(\epsilon^m\big),$$

$$K_{e^{\epsilon B} \, \gamma_a} = K_{\gamma_a} - \epsilon \, K^{(1)}_{1, \, \gamma_a}\big(B\big) + \mathrm{O}\big(\epsilon^2\big),$$

or

$$K_{e^{\epsilon B} \gamma_a}\big(\zeta, \, \zeta\big) = K_{\gamma_a}\big(\zeta, \, \zeta\big) + \tag{4.4}$$

$$- \epsilon \int_{\Omega_n} \big| K_{\gamma_a}\big(\zeta, \, w\big) \big|^2 \, B(w) \, \rho(w)^a \, d\mu(w) + \mathrm{O}(\epsilon^2).$$

This is of course legitimate provided that $\| e^{\epsilon B} - 1 \|_\infty < 1/2$ for ϵ sufficiently small.

Proposition 4.1. *Let $B : \Omega_n \to \mathbb{R}$. The following statements are equivalent*

i) $B \in L^\infty(\Omega_n)$.

ii) *There is $\epsilon_0 > 0$ such that $\|e^{\epsilon B} - 1\|_\infty < 1/2$ for any $0 < \epsilon \leq \epsilon_0$.*

To prove Proposition 4.1 we first characterize such $\epsilon_0 > 0$ i.e.

Lemma 4.2. *Let $B : \Omega_n \to \mathbb{R}$ and $\epsilon_0 > 0$. The following statements are equivalent*

a) $\forall\, 0 < \epsilon \leq \epsilon_0 \ : \ \|e^{\epsilon B} - 1\|_\infty < \dfrac{1}{2}$.

b) $\exists\, M \in I(\epsilon_0) := \left(0, \ \dfrac{1}{\epsilon_0} \log \dfrac{3}{2}\right) \subset \mathbb{R} \ :$

$$\frac{1}{\epsilon_0} \log\left(2 - e^{\epsilon_0 M}\right) \leq B(\zeta) \leq M \quad a.e. \ \zeta \in \Omega_n \, . \tag{4.5}$$

Proof. (a) $\Longrightarrow$ (b). Let us consider the set

$$\mathcal{E}_\epsilon = \left\{ K > 0 \ : \ \left| e^{\epsilon B(\zeta)} - 1 \right| \leq K \ \text{ a.e. } \ \zeta \in \Omega_n \right\}.$$

If $1/2 > \inf \mathcal{E}_\epsilon$ then $1/2$ is not a lower bound of $\mathcal{E}_\epsilon$ i.e. there is $0 < K_0 < 1/2$ such that

$$\log\left(1 - K_0\right) \leq \epsilon\, B(\zeta) \leq \log\left(1 + K_0\right) \ \text{ a.e. } \ \zeta \in \Omega_n \, . \tag{4.6}$$

Let us set $M = \dfrac{1}{\epsilon_0} \log\left(1 + K_0\right)$ so that $K_0 = e^{\epsilon_0 M} - 1$ and (4.6) may be written

$$\log\left(2 - e^{\epsilon_0 M}\right) \leq \epsilon\, B(\zeta) \leq \epsilon_0 M. \tag{4.7}$$

Note that $0 < K < 1/2$ is equivalent to $M \in I(\epsilon_0)$. Finally (4.7) for $\epsilon = \epsilon_0$ yields (4.5). Q.e.d.

(b) $\Longrightarrow$ (a). Let $0 < \epsilon \leq \epsilon_0$ and let $M \in I(\epsilon_0)$ such that (4.5) is fulfilled. Next, let us set $K_0 := e^{\epsilon_0 M} - 1$ so that [by the very choice of M] $0 < K_0 < 1/2$ and (4.5) may be written

$$\frac{1}{\epsilon_0} \log(1 - K_0) \leq B(\zeta) \leq \frac{1}{\epsilon_0} \log(1 + K_0) \ \text{ a.e. } \ \zeta \in \Omega_n \, .$$

Consequently, for any such ζ

$$\log(1 - K_0) \leq \frac{\epsilon}{\epsilon_0} \log(1 - K_0) \leq \epsilon\, B(\zeta) \leq \frac{\epsilon}{\epsilon_0} \log(1 + K_0) \leq \log(1 + K_0)$$

so that $\left| e^{\epsilon B(\zeta)} - 1 \right| \leq K_0$ i.e. $K_0 \in \mathcal{E}_\epsilon$. Finally

$$\left\| e^{\epsilon B} - 1 \right\|_\infty = \inf \mathcal{E}_\epsilon \leq K_0 < \frac{1}{2} \, .$$

Q.e.d. □

Next we prove Proposition 4.1.

Proof. (i) $\Longrightarrow$ (ii). For $B = 0$ the statement is obvious. Let $B \in L^\infty(\Omega_n)$, $B \neq 0$, and let us consider the planar domain

$$\mathcal{D} = \left\{ (x, y) \in \left(0,\, \frac{\log \frac{3}{2}}{\|B\|_\infty} \right) \times \mathbb{R} \, : \, \|B\|_\infty < y < \frac{1}{x} \log \frac{3}{2} \right\}.$$

Clearly $\mathcal{D} \neq \emptyset$. Let then $\big(\epsilon_0\, ,\, M\big) \in \mathcal{D}$ so that

$$\frac{1}{\epsilon_0} \log \frac{3}{2} > M > \|B\|_\infty = \operatorname{ess\,sup}_{z \in \Omega_n} \big|B(z)\big| = \inf \mathcal{E}_B\, ,$$

$$\mathcal{E}_B := \big\{ K > 0 \, : \, \big|B(\zeta)\big| \leq K \;\; \text{a.e. } \; \zeta \in \Omega_n \big\},$$

i.e. M is not a lower bound of $\mathcal{E}_B$. Hence there is $0 < K < M$ such that $\big|B(\zeta)\big| \leq K$ a.e. $\zeta \in \Omega_n$. It follows that $\big|B(\zeta)\big| < M$ a.e. $\zeta \in \Omega_n$. Note that

$$\frac{1}{\epsilon_0} \log \big(2 - e^{\epsilon_0 M}\big) < -M. \tag{4.8}$$

Indeed (4.8) is equivalent to $t^2 - 2t + 1 > 0$ with $t = e^{\epsilon_0 M} > 1$. Then

$$\frac{1}{\epsilon_0} \log \big(2 - e^{\epsilon_0 M}\big) < B(\zeta) < M \;\; \text{a.e. } \; \zeta \in \Omega_n$$

i.e. ϵ_0 obeys to (b) in Lemma 4.1. Q.e.d.

(ii) $\Longrightarrow$ (i). Let $\epsilon_0 > 0$ as in (ii) of Proposition 4.1. Then [by Lemma 4.2] ϵ_0 satisfies (b), as well, and $-\dfrac{1}{\epsilon_0} \log \big|2 - e^{\epsilon_0 M}\big| \geq M$ implies

$$\big|B(\zeta)\big| \leq -\frac{1}{\epsilon_0} \log \big(2 - e^{\epsilon_0 M}\big) \;\; \text{a.e. } \; \zeta \in \Omega_n$$

so that $B \in L^\infty(\Omega_n)$. Q.e.d. □

Theorem 4.3. *Let $B \in C^\infty(\Omega_n) \cap L^\infty(\Omega_n)$ and let $\epsilon_0 > 0$ as in Proposition 4.1. Let $E \to \Omega_n$ be the trivial complex line bundle equipped with the Hermitian inner product $H_{B,\,\epsilon}$ given by $H_{B,\,\epsilon}\big(\sigma_0\, ,\, \sigma_0\big) = e^{\epsilon\, B} \gamma_a$ with $0 < \epsilon \leq \epsilon_0$ and $a > -1$. Let $\mathcal{K}_{B,\,\epsilon} : \Omega_n \to \mathbb{CP}\big[L^2 H(\Omega_n\, ,\, e^{\epsilon B} \gamma_a)\big]$ be the quantization of classical states, springing from the data $(E,\, H_{B,\,\epsilon})$. Then the transition probability amplitude $a_{0\overline{0}}(\zeta,\, z)$ from $z \approx \mathcal{K}_{B,\,\epsilon}(z)$ to $\zeta \approx \mathcal{K}_{B,\,\epsilon}(\zeta)$ is*

$$a_{0\overline{0}}(\zeta,\, z) = \Big[\frac{2\, \rho(z)^{1/2} \rho(\zeta)^{1/2}}{i(\overline{z}_1 - \zeta_1) - 2\langle \zeta'\, ,\, z' \rangle} \Big]^{n+1+a} + \epsilon\, G(\zeta, z) + \mathrm{O}(\epsilon^2) \tag{4.9}$$

where the $\mathrm{O}(\epsilon)$ *term is given by*

$$G(\zeta,\, z) = \frac{8}{(c_{n,a})^2}\, \rho(\zeta)^{(n+1+a)/2}\, \rho(z)^{(n+1+a)/2} \, \times \tag{4.10}$$

$$\times \int_{\Omega_n} \Big\{ \Big[\rho(z)^{n+1+a} \Big| K_{\gamma_a}\big(w,\, z\big) \Big|^2 + \rho(\zeta)^{n+1+a} \Big| K_{\gamma_a}\big(w,\, \zeta\big) \Big|^2 \Big] K_{\gamma_a}\big(\zeta,\, z\big) +$$

$$- \frac{c_{n,a}}{2}\, K_{\gamma_a}\big(w,\, z\big)\, K_{\gamma_a}\big(\zeta,\, w\big) \Big\}\, B(w)\, \rho(w)^a\, d\mu(w).$$

Proof. One starts from

$$a_{0\overline{0}}(\zeta,\, z) = \frac{K_{e^{\epsilon B}\gamma_a}\big(\zeta,\, z\big)}{K_{e^{\epsilon B}\gamma_a}\big(z,\, z\big)^{1/2}\, K_{e^{\epsilon B}\gamma_a}\big(\zeta,\, \zeta\big)} \tag{4.11}$$

together with (4.4). The truncated Taylor development [i.e. $\sqrt{a+\epsilon\, b} = \sqrt{a} + \epsilon b/[2\sqrt{a}] + \mathrm{O}\big(\epsilon^2\big)$] gives

$$K_{e^{\epsilon B}\gamma_a}\big(\zeta,\, \zeta\big)^{1/2} = \frac{\sqrt{c_{n,a}}}{2}\, \rho(\zeta)^{-(n+1+a)/2} +$$

$$-\frac{\epsilon}{\sqrt{c_{n,a}}}\, \rho(\zeta)^{(n+1+a)/2} \int_{\Omega_n} \big|K_{\gamma_a}\big(\zeta,\, w\big)\big|^2\, B(w)\, \rho(w)^a\, d\mu(w) + \mathrm{O}(\epsilon^2).$$

Thus [by dropping the terms of order $\mathrm{O}(\epsilon^2)$ and higher]

$$K_{e^{\epsilon B}\gamma_a}\big(\zeta,\, \zeta\big)^{1/2} K_{e^{\epsilon B}\gamma_a}\big(z,\, z\big)^{1/2} =$$

$$= \frac{1}{2}\, \rho(\zeta)^{-(n+1+a)/2}\, \rho(z)^{-(n+1+a)/2}\, \times$$

$$\times \Bigg\{ \frac{c_{n,a}}{2} - \epsilon \int_{\Omega_n} \Big[\rho(z)^{n+1+a} \Big| K_{\gamma_a}\big(z,\, w\big) \Big|^2 +$$

$$+ \rho(\zeta)^{n+1+a} \Big| K_{\gamma_a}\big(\zeta,\, w\big) \Big|^2 \Big]\, B(w)\, \rho(w)^a\, d\mu(w) \Bigg\}.$$

The truncated Taylor development [i.e. $(a+\epsilon b)/(c+\epsilon d) = a/c - c^{-2} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \epsilon + \mathrm{O}(\epsilon^2)$] of (4.11) leads to (4.9)-(4.10). Q.e.d. □

We end the section with a comment on *Andreotti-Vesentini external fields*. Let $\mathcal{C}$ be the set of all nondecreasing convex functions $\lambda(t)$, $0 \leq t < +\infty$. The results in [1] rely on a Carleman type inequality for the operator $\overline{\partial}$. With respect to the ordinary Carleman inequality (cf. e.g. [4]) the parameter is $\lambda \in \mathcal{C}$ and the weight of integration is $e^{\lambda(\Phi)}$ for some C^∞ function $\Phi : \Omega_n \to [0, +\infty)$ chosen such that E be W-elliptic with respect to the data $(e^{\lambda(\Phi)} H,\, g)$ (for some Hermitian bundle metric H on E, some complete Hermitian metric g on Ω_n, ad any $\lambda \in \mathcal{C}$, cf. [1], p. 95). Central to the discussion in [1] is thus the deformation $e^{\lambda(\Phi)}\, H$ of the metric structure, so that $B = \lambda(\Phi)$ may be physically interpreted as an external field. Our calculation of the transition probability amplitudes of the given mechanical system, interacting at the quantum level with the weak external fields $\epsilon\, B$, required that $B \in L^\infty(\Omega_n)$. As to Andreotti-Vesentini external fields $\lambda(\Phi)$ we may prove

Proposition 4.4. *Let $\Phi : \Omega_n \to [0, +\infty)$ be a C^∞ function. The following statements are equivalent*

i) $\Phi \in L^\infty(\Omega_n)$.

ii) $\lambda(\Phi) \in L^\infty(\Omega_n)$ *for any* $\lambda \in \mathcal{C}$.

iii) *There is a nonconstant* $\lambda \in \mathcal{C}$ *such that* $\lambda(\Phi) \in L^\infty(\Omega_n)$.

Proof. (i) $\Longrightarrow$ (ii). Let $\Phi \in L^\infty(\Omega_n)$ and let us consider the closed interval $J = \big[0,\ \|\Phi\|_\infty\big] \subset \mathbb{R}$. Let $\lambda \in \mathcal{C}$ and let us set $\Lambda = \sup_{t\in J}|\lambda(t)|$. We claim that $\|\lambda(\Phi)\|_\infty \le \Lambda$. The proof is by contradiction i.e. we assume that $\Lambda < \|\lambda(\Phi)\|_\infty = \inf \mathcal{E}_{\Lambda(\Phi)}$. In particular

$$\Lambda < K \ \text{ for any } \ K \in \mathcal{E}_{\lambda(\Phi)}. \tag{4.12}$$

Yet $\Lambda \ge |\lambda(t)|$ for any $t \in J$ and $\Phi(z) \in J$ for a.e. $z \in \Omega$, so that $\Lambda \in \mathcal{E}_{\lambda(\Phi)}$, contradicting (4.12).

(iii) $\Longrightarrow$ (i). Let $\lambda \in \mathcal{C}$ be a nonconstant function such that $\lambda(\Phi) \in L^\infty(\Omega_n)$. We set as customary $\mathcal{E}_\Phi = \{K > 0 \,:\, |\Phi(z)| \le K \ \text{a.e. } z \in \Omega_n\}$ so that $\|\Phi\|_\infty = \inf \mathcal{E}_\Phi$. Let $K \in \mathcal{E}_\Phi$ and let us set $S_K = \{z \in \Omega_n \,:\, |\Phi(z)| > K\}$, so that $\mu(S_K) = 0$. It is an elementary matter that $S_K = \emptyset$. Indeed, if $S_K \ne \emptyset$ then let $z_0 \in S_K$. As Φ is continuous, the property $|\Phi(z_0)| > K$ will persist over a whole neighborhood U of z_0 in Ω_n. Thus $U \subset S_K$ and consequently $\mu(S_K) > 0$, a contradiction. Therefore $\|\Phi\|_\infty$ is the least upper bound of $\{|\Phi(z)| \,:\, z \in \Omega_n\}$ and $\|\Phi\|_\infty < \infty$. Indeed, it were $\|\Phi\|_\infty = +\infty$ then $\lim_{\nu\to\infty}|\Phi(z_\nu)| = +\infty$ for some sequence $\{z_\nu\}_{\nu\ge1} \subset \Omega_n$. As λ is convex and nonconstant one would have $\lim_{\nu\to\infty}\lambda(\Phi(z_\nu)) = +\infty$, hence $\lambda(\Phi) \notin L^\infty(\Omega_n)$, a contradiction. Q.e.d. $\square$

5. Quantizable observables

We recall a few notions of symplectic geometry, needed through this section. For every $f \in C^\infty\big(\Omega_n\,,\mathbb{R}\big)$ let $X_f \in \mathfrak{X}(\Omega_n)$ be the symplectic gradient of f i.e. $\omega_a(X_f\,,\cdot) = df$ where $\omega_a = \operatorname{curv}(E,\, H_a)$. Let $\mathcal{H}(\Omega_n)$ [respectively $\mathcal{H}_{\rm loc}(\Omega_n)$] be the space of all Hamiltonian (respectively locally Hamiltonian) vector fields on Ω_n. If $\eta(X) = \big[\omega_a(X,\cdot)\big]_{\rm dR}$ then $0 \to \mathcal{H}(\Omega_n) \hookrightarrow \mathcal{H}_{\rm loc}(\Omega_n) \xrightarrow{\eta} H^1\big(\Omega_n\,,\mathbb{R}\big) \to 0$ is a short exact sequence of vector spaces and linear maps. Given $s > 0$ the parabolic dilation $\delta_s : \overline{\Omega}_n \to \overline{\Omega}_n$ is $\delta_s(\zeta) = (s^2\,\zeta_1\,,\, s\,\zeta')$. The map $H : \Omega_n \times [0,1] \to \Omega_n$, $H(\zeta,t) = \delta_{1-t}(\zeta)$, is a homotopy $H : 1_{\Omega_n} \simeq 0$ i.e. Ω_n is contractible. Hence $H^1(\Omega_n\,,\mathbb{R}) = 0$ and $\mathcal{H}(\Omega_n) \approx \mathcal{H}_{\rm loc}(\Omega_n)$ (an isomorphism of vector spaces). The Poisson bracket of $f,\, g \in C^\infty\big(\Omega_n\,,\mathbb{R}\big)$ is $\{f,\, g\} = \omega_a\big(X_f\,,\, X_g\big)$. By $X_{\{f,g\}} = -\big[X_f\,,\, X_g\big]$ the Poisson algebra $\big(C^\infty(\Omega_n\,,\mathbb{R}),\ \{\,\cdot\,,\,\cdot\,\}\big)$ and $\mathfrak{X}(\Omega_n)$ are isomorphic Lie algebras. Let $F := T^{0,1}(\Omega_n)$ be the canonical Kähler polarization of Ω_n. The space of F-stable sections in E is then $\mathcal{O}(E)$. Let C_{FF} be the Lie algebra of all quantizable observables i.e.

$$C_{FF} = \Big\{ f \in C^\infty\big(\Omega_n\,,\mathbb{R}\big) \,:\, \big[X_f\,,\,\overline{Z}\big] \in T^{0,1}(\Omega_n),\ \ \forall\, Z \in T^{1,0}(\Omega_n)\Big\}.$$

For every E-valued $(n,0)$-form $\varphi = \Psi\,\sigma_0 \otimes dz^{1\cdots n}$ on Ω_n we set

$$\hat{f}(\varphi) = \frac{1}{i}\Big\{\Big[\nabla_{X_f}(\Psi\sigma_0) + 4\pi i\, f\,\Psi\,\sigma_0\Big] \otimes dz^{1\cdots n} + \Psi\,\sigma_0 \otimes \mathcal{L}_{X_f} dz^{1\cdots n} \tag{5.1}$$

where ∇ is the canonical Hermitian connection of $(E,\, H_a)$ and $\mathcal{L}$ is the Lie derivative.

Theorem 5.1. *Let $E \to \Omega_n$ be the trivial complex line bundle with the Hermitian metric H_a. Then for any quantizable observables $f,\, g \in C_{FF}$*

i) $\hat{f}\big(\mathcal{M}_a\big) \subset \mathcal{M}_a, \quad i\,\{f,\, g\} = \big[\hat{f},\, \hat{g}\big]$,

ii) $\hat{f}(\varphi) = \dfrac{1}{i}\Big\{\Big[Z^j\,\dfrac{\partial}{\partial z^j} + \dfrac{\partial Z^j}{\partial z^j} + g\Big]\,\Psi\Big\}\,\sigma_0 \otimes dz^{1\cdots n}$ *where*

$$Z = \frac{1}{2}\big(X_f - i\,J\,X_f\big) = Z^j\,\frac{\partial}{\partial z^j} \in \mathcal{O}\big(T^{1,0}(\Omega_n)\big),$$

$$Z^1 = \frac{8\pi\rho}{a}\Big\{2i\,\mathrm{Im}(z^1)\,f_{\overline{z}^1} - \sum_{j=2}^{n} \overline{z}^j\,f_{\overline{z}^j}\Big\}, \tag{5.2}$$

$$Z^j = \frac{4\pi\rho}{a}\Big\{2\,z^j\,f_{\overline{z}^1} + i\,f_{\overline{z}^j}\Big\}, \quad 2 \leq j \leq n, \tag{5.3}$$

$$g := Z\big(\log\gamma_a\big) + 4\pi i\,f \in \mathcal{O}(\Omega_n).$$

iii) *For any* $\varphi,\, \psi \in \mathcal{M}_a$

$$\frac{1}{i}\,d\Big[X_f \rfloor H^*(\varphi,\, \psi)\Big] = H^*\big(\hat{f}(\varphi),\, \psi\big) - H^*\big(\varphi,\, \hat{f}(\psi)\big).$$

Given an arbitrary holomorphic line bundle $\pi : E \to \Omega_n$ together with a fixed trivialization atlas $\{T_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}\}_{\alpha\in I}$, let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ be the corresponding transition functions, so that E is identified (up to an isomorphism) by the cohomology class $\big[g_{\alpha\beta}\big] \in H^1\big(\Omega,\, \mathcal{O}^*\big)$. Let us consider

$$\tau : \mathcal{O}\big(T^{1,0}(\Omega_n)\big) \to H^1\big(\Omega_n\,,\, \mathcal{O}\big), \quad \tau(Z) = \big[Z\big(\log\, g_{\alpha\beta}\big)\big].$$

Let H_{FF} consist of all $X \in \mathcal{H}(\Omega_n)$ such that $X = Z + \overline{Z}$ for some $Z \in \mathrm{Ker}(\tau)$. Then $0 \to \mathbb{R} \hookrightarrow C_{FF} \xrightarrow{\Gamma} H_{FF} \to 0$ is exact [here $\Gamma(f) = X_f$]. If E is the trivial complex line bundle over Ω_n and $I = \{0\}$, $T_0 = 1_E$, then τ is constant [equals the identity in $H^1(\Omega_n\,,\, \mathcal{O})$] so that H_{FF} consists of all Hamiltonian vector fields X on Ω_n such that $X = Z + \overline{Z}$ for some $Z \in \mathcal{O}\big(T^{1,0}(\Omega_n)\big)$. If $f \in C_{FF}$ then $X_f \in H_{FF}$. Also $\Gamma\big(C_{FF}\big) = H_{FF}$. Starting from

$$\omega_a\big(X_f\,,\,\cdot\,\big) = df, \quad \omega_a = -\frac{1}{2\pi i}\,\partial\,\overline{\partial}\,\log\gamma_a\,,$$

one has

$$i\big(\partial\,\overline{\partial}\,\log\gamma_a\big)\big(Z,\, \overline{W}\big) = 2\pi\big(\overline{\partial}f\big)\overline{W}$$

for any $W \in T^{1,0}(\Omega_n)$. In particular for $W = \partial/\partial z^k$

$$\frac{i}{2}\,\frac{\partial^2\gamma_a}{\partial z^j\,\partial\overline{z}^k}\,Z^j = 2\pi\,\frac{\partial f}{\partial\overline{z}_j} \tag{5.4}$$

yielding $\partial g/\partial\overline{z}_k = 0$ where we have set

$$g = Z\big(\log\gamma_a\big) + 4\pi i\,f. \tag{5.5}$$

Note that (5.4) reads

$$\frac{1}{2}\,\omega_{j\overline{k}}\,Z^j = \frac{\partial f}{\partial\overline{z}_k} \tag{5.6}$$

where $\big[\omega_{j\overline{k}}\big]$ is given by (2.4). Equations (5.6) may be solved for Z [the $(1,0)$-component of the symplectic gradient X_f] without explicitly inverting the $n \times n$

matrix $\left[\omega_{j\overline{k}}\right]$ (by a trick in tensor calculus). Indeed [by taking into account (2.4)] (5.6) may be written

$$Z^1 + \frac{2}{i}\sum_{\alpha=2}^{n} \overline{z}_\alpha Z^\alpha = \frac{16\pi i}{a}\,\rho^2\, f_{\overline{z}_1}\,, \tag{5.7}$$

$$z_\beta\, Z^1 - 2i\big(\rho\,\delta_{\alpha\beta} + \overline{z}_\alpha\, z_\beta\big)\, Z^\alpha = \frac{8\pi}{a}\,\rho^2\, f_{\overline{z}_\beta}\,, \quad 2 \leq \beta \leq n. \tag{5.8}$$

Let us contract (5.8) by $\overline{z}^\beta$ so that to obtain

$$|z'|^2\, Z^1 - 2i\big(\rho + |z'|^2\big)\,\overline{z}_\alpha\, Z^\alpha = \frac{8\pi}{a}\,\rho^2\,\overline{z}^\beta\, f_{\overline{z}_\beta}\,. \tag{5.9}$$

At this point we may solve (5.7) and (5.9) for the unknowns Z^1 and $V = \overline{z}_\alpha Z^\alpha$. We obtain

$$Z^1 = \frac{8\pi\rho}{a}\Big[2i\,\mathrm{Im}(z_1)\, f_{\overline{z}_1} - \overline{z}^\beta\, f_{\overline{z}_\beta}\Big]$$

[thus proving (5.2)] and

$$V = \frac{4\pi\rho}{a}\Big[2|z'|^2\, f_{\overline{z}_1} + i\,\overline{z}^\beta\, f_{\overline{z}_\beta}\Big]\,. \tag{5.10}$$

Next [by (5.2) and (5.10)]

$$Z^1 - 2i\, V = \frac{16\pi i}{a}\,\rho^2\, f_{\overline{z}_1}\,. \tag{5.11}$$

Finally equation (5.8) may be written as

$$\big(Z^1 - 2i\, V\big)\, z_\beta - \frac{8\pi}{a}\,\rho^2\, f_{\overline{z}_\beta} = 2i\rho\, Z^\beta$$

and substitution from (5.11) yields (5.3). Q.e.d. Let

$$\hat{f} : C^\infty\Big(\Lambda^{n,0}(\Omega_n)\otimes E\Big) \to C^\infty\Big(\Lambda^{n,0}(\Omega_n)\otimes E\Big)$$

be defined by (5.1). By a result of A. Odzijewicz (cf. [13]) $\hat{f}$ maps $\mathcal{M}_a \approx LH(\Omega_n\,,\,\gamma_a)$ into itself. Note that

$$\nabla_{X_f}\big(\Psi\,\sigma_0\big) = \big[Z(\Psi) + \Psi\, Z(\log\gamma_a)\big]\,\sigma_0\,,$$

$$\mathcal{L}_{X_f}\, dz^{1\cdots n} = \sum_{j=1}^{n} \frac{\partial Z^j}{\partial z^j}\, dz^{1\cdots n}\,,$$

yielding (ii) in Theorem 5.1. Starting from the identity

$$X \rfloor \big(\alpha_1\wedge\cdots\wedge\alpha_p\big) =$$

$$= \sum_{j=1}^{p}(-1)^{j+1}\alpha_j(X)\,\alpha^1\wedge\cdots\wedge\widehat{\alpha^j}\wedge\cdots\wedge\alpha^p$$

[for any $X \in \mathfrak{X}(\Omega_n)$ and any $\alpha_j \in \Omega^1(\Omega_n)$] a calculation shows that

$$X_f \rfloor H^*(\varphi,\,\psi) = \Psi\,\overline{\Phi}\,\gamma_a \sum_{j=1}^{n}(-1)^j\Big[Z^j\, dz^{1\cdots\hat{j}\cdots n}\wedge d\overline{z}^{1\cdots n} +$$

$$+ (-1)^n\,\overline{Z^j}\, dz^{1\cdots n}\wedge d\overline{z}^{1\cdots\hat{j}\cdots n}\Big]$$

hence [by taking the exterior derivative]

$$d\big[X_f \rfloor H^*(\varphi,\, \psi)\big] =$$

$$= \Big\{ X_f\big(\Psi\, \overline{\Phi}\, \gamma_a\big) + 2\, \Psi\, \overline{\Phi}\, \gamma_a\, \mathrm{Re}\Big(\frac{\partial Z^j}{\partial z^j}\Big)\Big\}\, dz^{1\cdots n\, \overline{1}\cdots \overline{n}}\, .$$

Yet

$$X_f(\Psi\, \overline{\Phi}\, \gamma_a) = \Big[Z(\Psi)\, \overline{\Phi} + \Psi\, \overline{Z(\Phi)} + (g + \overline{g})\, \Psi\, \overline{\Phi}\Big]\, \gamma_a$$

hence

$$d\big[X_f \rfloor H^*(\varphi,\, \psi)\big] = \Big\{ Z(\Psi)\overline{\Phi} + \Psi\overline{Z(\Phi)} +$$

$$+ \Psi\overline{\Phi}\Big[\frac{\partial Z^j}{\partial z^j} + g + \Big(\frac{\partial Z^j}{\partial z^j} + g\Big)^{-}\,\Big]\Big\}\, \gamma_a\, dz^{1\cdots n\, \overline{1}\cdots\overline{n}} =$$

$$= i\Big\{ H^*\big(\hat{f}(\varphi),\, \psi\big) - H^*\big(\varphi,\, \hat{f}(\psi)\big)\Big\}.$$

Q.e.d. By Cartan's formula $d \circ i_{X_f} + i_{X_f} \circ d = \mathcal{L}_{X_f}$ [as $H^*(\varphi,\, \psi)$ is a top degree form on Ω_n]

$$d\Big[X_f \rfloor H^*(\varphi,\, \psi)\Big] = \mathcal{L}_{X_f}\, H^*(\varphi,\, \psi) = \mathrm{div}_a\big(Y\big)\, \gamma_a\, dz^{1\cdots n\, \overline{1}\cdots\overline{n}}$$

where $\mathrm{div}_a : \mathfrak{X}(\Omega_n) \to C^\infty(\Omega_n)$ is (up to a constant) the divergence operator with respect to the $\gamma_a\, d\mu(\zeta)$ and $Y = \Psi\, \overline{\Phi}\, X_f$. We expect that $\int_{\Omega_n} \mathrm{div}_a(Y)\, \gamma_a\, d\mu = 0$ provided that $\mathrm{div}_a(Y) \in L^1\big(\Omega_n\, ,\, \gamma_a\big)$ and $Y = \mathrm{O}\big(|\zeta|^{1-2n}\big)$ as $|\zeta| \to +\infty$ [these conditions would provide the domain of $\hat{f}$]. If the conjecture is true then [by (iii) in Theorem 5.1] $\hat{f}$ is a symmetric operator.

References

1. Andreotti, A., Vesentini, E.: Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Publications Mathématiques de l'I.H.É.S.* **25** (1965) 81–130.
2. Barletta, E., Dragomir, S.: On the Djrbashian kernel of a Siegel domain, *Studia Mathematica* **127** (1998) 47–63.
3. Barletta, E., Dragomir, S., Esposito, F.: Weighted Bergman kernels and mathematical physics, *Axioms* **48** (2020) 42 pp.
4. Carleman, T.: Sur un problème d'unicité pour les systèmes d'equations aux dérivées partielles à deux variables indépendantes, *Ark. Mat. Astr. och Fys.* **26B** (1939) 1–9.
5. D'Angelo, J.P.: An explicit computation of the Bergman kernel function, *J. Geom. Anal.* **4** (1994) 23–34.
6. Djrbashian, M.M., Karapetyan, A.H.: Integral representations for some classes of functions holomorphic in a Siegel domain, *J. Math. Anal. Appl.* **179** (1993) 91–109.
7. Fefferman, C.: The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* **26** (1974) 1–65.
8. Forelli, F., Rudin, W.: Projections on spaces of holomorphic functions in balls, *Indiana Univ. Math. J.* **24** (1974) 593–602.
9. Francsics, G., Hanges, N.: The Bergman kernel of complex ovals and multivariable hypergeometric functions, *J. Funct. Anal.* **142** (1996) 494–510.
10. Gawędzki, K.: *Fourier-like kernels in geometric quantization*, Instytut Matematyczny Polskiej Akademi Nauk, Warsaw, Poland, 1976, 5–78.
11. Kostant, B.: *Quantization and unitary representations. Part I: Prequantization*, Lectures in Modern Analysis and Applications III, Springer, Berlin-Heidelberg, 2009, 87–206.
12. Krantz, S.G.: *Function theory of several complex variables*, John Wiley & Sons, New York, 1982.

13. Odzijewicz, A.: On reproducing kernels and quantization of states, *Commun. Math. Phys.* **114** (1988) 577–597.
14. Pasternak-Winiarski, Z.: On weights which admit the reproducing kernel of Bergman type, *Internat. J. Math. & Math. Sci.* **15** (1992) 1–14.
15. Pasternak-Winiarski, Z.: On the dependence of the reproducing kernel on the weight of integration, *Journal of Functional Analysis* **94** (1990) 110–134.
16. Peloso, M.M.: Hankel operators on weighted Bergman spaces on strongly pseudoconvex domains, *Illinois J. Math.* **38** (1994) 223–249.
17. Penrose, R., MacCallum, M.A.H.: Twistor theory: an approach to the quantization of fields and space-time, *Phys. Rep.* **6** (1972) 241–316.
18. Saitoh, S.: Hilbert spaces induced by Hilbert space valued functions, *Proc. Amer. Math. Soc.* **89** (1983), 74–78.

Elisabetta Barletta: Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, 85100 Potenza, Italy
Email address: elisabetta.barletta@unibas.it

Sorin Dragomir: Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, 85100 Potenza, Italy
Email address: sorin.dragomir@unibas.it

Francesco Esposito: Dipartimento di Matematica e Fisica *Ennio de Giorgi*, Università del Salento, Lecce, Italy
Email address: francesco.esposito@unisa.it