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Fourier theory and C^*-algebras

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ABSTRACT

We discuss a number of results concerning the Fourier series of elements in reduced twisted group C^*-algebras of discrete groups, and, more generally, in reduced crossed products associated to twisted actions of discrete groups on unital C^*-algebras. A major part of the article gives a review of our previous work on this topic, but some new results are also included.

A Banach space X is said to have the metric approximation property (MAP) if the identity map on X is a point-norm limit of finite-rank linear contractions. As a result of work of M.D. Choi, E. Effros, E. Kirchberg and others in the 1970s, it was known that a C^*-algebra A is nuclear if and only if it has the completely positive approximation property (CPAP), i.e., the identity map on A is a point-norm limit of completely positive finite-rank linear contractions. In particular, A has the MAP whenever it is nuclear, and it was believed that the converse should also be true. It came therefore as a surprise when U. Haagerup was able to show in [1] that the reduced group C^*-algebra $C_r^*(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ is an example of a nonnuclear C^*-algebra having the MAP. In the course of the proof of this result, Haagerup actually showed the following facts, that for different reasons have exerted a lasting influence on the subsequent development of noncommutative harmonic analysis.

- Let $|\cdot|$ denote the word length function on $\mathbb{F}_2 = \langle a, b\rangle$ w.r.t. $S = \{a, b, a^{-1}, b^{-1}\}$. Then $|\cdot|$ is proper and negative definite. Equivalently, using Schoenberg's theorem, the functions on $\mathbb{F}_2$ given by $\psi_t(g) = e^{-t|g|}$ are vanishing at infinity and positive definite for every $t > 0$. Since ψ_t converges pointwise to 1 as $t \to 0$, this means, using more recent terminology [2], that $\mathbb{F}_2$ has the *Haagerup property*.

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- If $f \in C_c(\mathbb{F}_2)$, that is, if f is complex function on $\mathbb{F}_2$ having finite support, and $\lambda(f)$ denotes the associated left convolution operator acting on $\ell^2(\mathbb{F}_2)$, then the operator norm of $\lambda(f)$ satisfies

$$\|\lambda(f)\| \leq 2\left(\sum_{g\in\mathbb{F}_2} |f(g)|^2(1+|g|)^4\right)^{1/2} \quad \big(=2\,\|f(1+|\cdot|)^2\|_2\big).$$

Thus $\mathbb{F}_2$ has the *rapid decay property* (RD) in the sense of P. Jolissaint [3].
- If $\varphi : \mathbb{F}_2 \to \mathbb{C}$ is such that $K := \sup_{g\in\mathbb{F}_2} |\varphi(g)|(1+|g|)^2 < \infty$, then we have

$$\|\lambda(\varphi f)\| \leq 2K\,\|\lambda(f)\|$$

for every $f \in C_c(\mathbb{F}_2)$. This shows that φ gives rise to a *multiplier* of $C_r^*(\mathbb{F}_2)$.

Our work started from the desire to highlight the existence of a somewhat hidden track relating these results and related developments in noncommutative harmonic analysis to more traditional issues about the classical theory of Fourier series.

In Section 1 we collect some background material on classical Fourier series, group theory and the various operator algebras associated to (discrete) groups. The presentation covers more topics than strictly needed, with the purpose of providing a reference and a source of inspiration also for future works. In Sections 2 and 3, more biased towards our own contributions [4–8], we present some results illustrating various aspects of Fourier theory in (possibly twisted) discrete reduced group C^*-algebras and reduced C^*-crossed products.

Due to space and time limitations, we have omitted a discussion of a number of different themes, such as noncommutative L^p-spaces, groupoids and quantum groups, where a combination of Fourier theory and C^*-algebras also plays a nontrivial role. The interested reader may for instance consult [9–22] for a small sample.

1. Background

1.1. Classical Fourier series

The classical theory of Fourier series deals with periodic functions on the real line $\mathbb{R}$, i.e., with functions on the torus group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, usually identified with the interval $[-\pi, \pi)$.

Let $f : \mathbb{T} \to \mathbb{C}$ be a function in $L^1(\mathbb{T})$. For every $n \in \mathbb{Z}$, the *n-Fourier coefficient* of f is defined by

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t.$$

It is customary to use the notation $\widehat{f}(n)$ for c_n. The (formal) *Fourier series* of f at $t \in \mathbb{T}$ is given by

$$S[f](t) = \sum_{n\in\mathbb{Z}} c_n e^{int}$$

and, for $N \in \mathbb{N}$, its N-partial sum at $t \in \mathbb{T}$ is defined by

$$S_N[f](t) = \sum_{n=-N}^{N} c_n e^{int}.$$

The original idea of J.B. Fourier of replacing a function f with its Fourier series dates back to 1807 and was aimed at solving a problem of heat diffusion in a metal plate. Nowadays, some basic Fourier analysis, e.g. the Fourier transform $f \mapsto \widehat{f}$ provides a unitary operator from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$ and the L^2-convergence of $S_N[f]$ towards f for every f in $L^2(\mathbb{T})$, is a part of the tool box of many science students. Convergence of Fourier series in other ways gives rise to more delicate problems. Among the impressive body of results, we mention a few highlights.

In 1873, P. du Bois-Reymond showed the existence of a continuous function for which convergence fails at some point. In 1923, A. Kolmogorov produced an example of a function in $L^1(\mathbb{T})$ (but not in $L^2(\mathbb{T})$) having a Fourier series that was divergent almost everywhere (a.e.). He even showed in 1926 that the Fourier series can diverge everywhere. In particular, N. Lusin had asked in 1920 whether the Fourier series of any continuous function converged a.e. This problem was not answered before 1966, when L. Carleson indeed showed that the Fourier series of any function in $L^2(\mathbb{T})$ converges a.e. This result was soon extended by R. Hunt to $L^p(\mathbb{T})$, for any $p > 1$.

If $f \in C(\mathbb{T})$, then $S_N[f]$ converges uniformly (to f) as $N \to \infty$ whenever $S[f]$ is absolutely convergent, that is, whenever $\widehat{f} \in \ell^1(\mathbb{Z})$. This happens for example when f belongs to $C^1(\mathbb{T})$, and the speed of convergence is then known to increase with the smoothness of f, this being reflected in the decay rate of the Fourier coefficients. To the best of our knowledge, no precise characterization of those continuous functions having a uniformly convergent Fourier series is presently available.

Building upon the work of Abel, Cesáro, Poisson, Fejér, and others, there are some well-known procedures, often referred to as summation processes, to enforce uniform convergence by modifying the expression of the Fourier series: for a sequence (φ_N) of complex functions on $\mathbb{Z}$, one considers the (formal) series

$$M_N[f](t) := \sum_{n\in\mathbb{Z}} \varphi_N(n)\widehat{f}(n)e^{int}$$

for $f \in C(\mathbb{T})$ and $t \in \mathbb{T}$. For certain specific choices of (φ_N), one may argue that the series $M_N[f](t)$ makes sense for every t and that $M_N[f]$ converges uniformly to f as $N \to \infty$ for every $f \in C(\mathbb{T})$. (The same idea may also be used to ensure L^p-convergence when dealing with functions in $L^p(\mathbb{T})$ for $1 \le p < \infty$.)

For instance, the sequence of Cesáro means associated with the sequence of partial sums $(S_N[f])$ can be written as

$$\sigma_N[f](t) = \sum_{k\in\mathbb{Z}} \varphi_N(n)\widehat{f}(n)e^{int},$$

where $\varphi_N : \mathbb{Z} \to \mathbb{C}$ is given by $\varphi_N(n) = 1 - \frac{|n|}{N}$ if $|n| \le N - 1$ and 0 otherwise.[1] Fejér's theorem from 1904 then says that $\sigma_N[f]$ converges uniformly to f as $N \to +\infty$ for every $f \in C(\mathbb{T})$. (The L^p-version is also true.)

Similarly, replacing the sequence (φ_N) with the family $(\varphi_r)_{r\in(0,1)}$, where $\varphi_r(n) = r^{|n|}$ for $n \in \mathbb{Z}$, leads to the Abel–Poisson summability of the Fourier series of any $f \in C(\mathbb{T})$.

As we will discuss in this paper, Fourier series and a whole bunch of related constructions continue to make sense in the noncommutative framework of operator algebras associated with discrete groups and with (possibly twisted) actions of such groups. We will see that many of the results mentioned above dealing with uniform convergence admit a more or less straightforward generalization (the statement, not the proof!) to the C^*-algebraic setting.[2]

1.2. Groups

It may be argued that groups have their origin in the work of N.H. Abel and E. Galois on the (non-)solvability of polynomial equations of arbitrary degree by radicals. Group theory has been ever since one of the most fascinating areas of mathematics. Groups have played a role in basically every aspect of pure mathematics and in many applications to other branches of science, especially physics but also chemistry, geology, etc. The theory has been developing from the study of the simplest to the most convoluted examples.

Dealing with classical Fourier series, we have been considering functions on the torus group $\mathbb{T}$. One may replace $\mathbb{T}$ with any other locally compact Abelian group, as is done in abstract commutative harmonic analysis, and proceed along similar lines. However, when facing the problem of uniform convergence, there are some natural generalizations that go beyond the Abelian case.

In order to explain this point, we first represent $C(\mathbb{T})$ faithfully as multiplications operators on $L^2(\mathbb{T})$. Then we observe that the Fourier transform implements a $*$-isomorphism Φ from $B(L^2(\mathbb{T}))$ onto $B(\ell^2(\mathbb{Z}))$, that maps the space of operators associated with trigonometric polynomials onto the span S of all translation operators on $\ell^2(\mathbb{Z})$. Using the density of the trigonometric polynomials in $C(\mathbb{T})$, we deduce that Φ maps $C(\mathbb{T})$ onto the closure of S w.r.t. the operator norm, that is, onto the reduced group C^*-algebra $C_r^*(\mathbb{Z})$ (by definition). Now, it is clear, at least in principle, that we can replace $\mathbb{Z}$ with any discrete group G and consider the reduced group C^*-algebra $C_r^*(G)$ generated by (left) translations operators on $\ell^2(G)$. More generally, we will consider the twisted reduced group C^*-algebra $C_r^*(G,\sigma)$ associated with a 2-cocycle $\sigma : G\times G \to \mathbb{T}$. Since Fourier series make sense in this setting too, together with their summation processes (using multipliers), a natural goal is to upgrade our knowledge of the theory of Fourier series to this level of generality. However, the new scenario forces us to ask whether there are some kind of special requirements on the group G that play a role in the formulation of the results we are looking for. As we will see, depending on the various situations at hand, the group in question is often required to satisfy some additional properties, reflecting various geometrical and analytical features.

1.3. Operator algebras associated with groups

In his search of rigorous mathematical tools for a sound formulation of quantum theories, J. von Neumann started the field of operator algebras, i.e., the study of algebras of operators on Hilbert spaces. Pretty soon he realized that the richness of the world of groups allows very interesting constructions of operator algebras, an idea that has been exploited ever since by all the practitioners in this area of research. This trend has become even more fascinating in recent times, e.g. in connection with the massive work around the so-called Baum–Connes conjecture (with or without coefficients).

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1 Note that $\sigma_N[f] = F_N * f$, where $F_N(t) = \frac{1}{N+1}\left(\frac{\sin\left((N+1)\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right)^2$ is the Fejér kernel, satisfying $\widehat{F_N} = \varphi_N$.

2 Note that several classical results for Fourier series deal with pointwise convergence to (periodic) functions that are not necessarily continuous (e.g., consider the well-known Gibbs phenomenon). One might ask which operator algebraic framework is the best to discuss possible noncommutative generalizations involving non-continuous functions. As $L^\infty(\mathbb{T})$ translates as the group von Neumann algebra $\mathrm{vN}(\mathbb{Z})$ one option is to use $\mathrm{vN}(G)$ for some discrete group G. However, such a space might be too big. For instance, the smallest C^*-algebra containing the piecewise continuous functions on $\mathbb{T}$ is the space $\mathcal{B}(\mathbb{T})$ of complex-valued bounded Borel functions. But it is not clear to us what is the best noncommutative analog of such a space.

We now proceed to introduce some of the various operator algebras associated to groups. A (normalized) 2-*cocycle* on G with values in $\mathbb{T}$ is a map $\sigma : G \times G \to \mathbb{T}$ such that

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk) \quad \text{for all } g,h,k \in G,$$
$$\sigma(g,e) = \sigma(e,g) = 1 \quad \text{for all } g \in G.$$

It follows that $\sigma(g,g^{-1}) = \sigma(g^{-1},g)$ for all $g \in G$. The set of all such cocycles,[3] denoted by $Z^2(G,\mathbb{T})$, becomes an abelian group under pointwise product, the inverse operation corresponding to conjugation: $\sigma^{-1} = \overline{\sigma}$, where $\overline{\sigma}(g,h) = \overline{\sigma(g,h)}$, and the identity element being the trivial 2-cocycle on G denoted by 1.

An element $\beta \in Z^2(G,\mathbb{T})$ is called a *coboundary* whenever one has $\beta(g,h) = b(g)b(h)\overline{b(gh)}$ for all $g,h \in G$, for some function $b : G \to \mathbb{T}$ with $b(e) = 1$; in this case we write $\beta = \mathrm{d}b$ (such a b is uniquely determined up to multiplication by a character of G). The set of all coboundaries $B^2(G,\mathbb{T})$ is a subgroup of $Z^2(G,\mathbb{T})$. We denote elements in the quotient group $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ by $[\sigma]$ and write $\widetilde{\sigma} \sim \sigma$ when $[\widetilde{\sigma}] = [\sigma]$ for $\sigma, \widetilde{\sigma} \in Z^2(G,\mathbb{T})$.

An instructive and much studied class of examples is the following. Let $n \in \mathbb{N}$ and $\Theta \in M_n(\mathbb{R})$. Define $\sigma_\Theta \in Z^2(\mathbb{Z}^n,\mathbb{T})$ by

$$\sigma_\Theta(x,y) = e^{i x\cdot(\Theta y)}$$

for $x,y \in \mathbb{Z}^n$. Then $\sigma_\Theta \in B^2(\mathbb{Z}^n,\mathbb{T})$ whenever Θ is symmetric. Indeed, in this case, $\sigma_\Theta = \mathrm{d}b_\Theta$ where

$$b_\Theta(x) = e^{-i\frac{1}{2}x\cdot(\Theta x)}.$$

In general, we have $[\sigma_\Theta] = [\sigma_{\widetilde{\Theta}}]$, where $\widetilde{\Theta}$ denotes the skew-symmetric part of Θ. Every element in $H^2(\mathbb{Z}^n,\mathbb{T})$ may be written as $[\sigma_\Omega]$ for some skew-symmetric $\Omega \in M_n(\mathbb{R})$.

A σ-*projective unitary representation* U of G on a (non-zero) Hilbert space $\mathcal{H}$ is a map from G into the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ such that

$$U(g)U(h) = \sigma(g,h)U(gh) \quad \text{for all } g,h \in G.$$

We then have $U(e) = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) and

$$U(g)^* = \overline{\sigma(g,g^{-1})}U(g^{-1}) \quad \text{for all } g \in G.$$

If $b : G \to \mathbb{T}$ satisfies $b(e) = 1$ and we set $\widetilde{U} = b\,U$, then $\widetilde{U}$ becomes a $\widetilde{\sigma}$-projective unitary representation of G on $\mathcal{H}$ with 2-cocycle $\widetilde{\sigma} \sim \sigma$ given by $\widetilde{\sigma} = (\mathrm{d}b)\sigma$. Such a representation $\widetilde{U}$ is called a *perturbation* of U (by b). If $\omega \in Z^2(G,\mathbb{T})$ and V is some ω-projective unitary representation of G on $\mathcal{K}$, one may form the $\sigma\omega$-projective tensor product representation $U \otimes V$ acting on $\mathcal{H} \otimes \mathcal{K}$ in the obvious way. One may also form the conjugate $\overline{\sigma}$-projective representation $\overline{U}$ of U, which acts as U on the conjugate Hilbert space $\overline{\mathcal{H}}$ of $\mathcal{H}$.

Let $\sigma \in Z^2(G,\mathbb{T})$. There are several (unitarily equivalent) ways to define the left (resp. right) *regular σ-projective unitary representation* of G on $\ell^2(G)$. In this paper, we will work with the ones defined by

$$(\Lambda_\sigma(g)\xi)(h) = \sigma(g,g^{-1}h)\,\xi(g^{-1}h),$$
$$(\rho_\sigma(g)\xi)(h) = \sigma(h,g)\,\xi(hg),$$

for $\xi \in \ell^2(G)$ and $g,h \in G$. Letting $\{\delta_h\}_{h\in G}$ denote the canonical basis of $\ell^2(G)$, we note that

$$\Lambda_\sigma(g)\delta_h = \sigma(g,h)\delta_{gh}, \quad g,h \in G.$$

In particular, we have $\Lambda_\sigma(g)\delta_e = \delta_g$. We also note that the following commutation relations

$$\Lambda_\sigma(g)\rho_{\overline{\sigma}}(h) = \rho_{\overline{\sigma}}(h)\Lambda_\sigma(g)$$

hold for all $g,h \in G$.

The *reduced twisted group C^*-algebra* $C_r^*(G,\sigma)$ (resp. the *twisted group von Neumann algebra* $\mathrm{vN}(G,\sigma)$) is defined as the C^*-subalgebra (resp. von Neumann subalgebra) of $B(\ell^2(G))$ generated by the set $\Lambda_\sigma(G)$. Hence, $C_r^*(G,\sigma)$ (resp. $\mathrm{vN}(G,\sigma)$) is the closure in the operator norm (resp. weak operator) topology of the $*$-algebra $\mathbb{C}(G,\sigma) :=\mathrm{Span}(\Lambda_\sigma(G))$. It is not difficult to see that instead of Λ_σ, up to (spatial) $*$-isomorphism we could have equally well used ρ_σ, or $\Lambda_{\sigma'}$ for any $\sigma' \sim \sigma$, in these definitions.

The twisted group C^*-algebras of the form $C_r^*(\mathbb{Z}^n,\sigma_\Theta)$, for $\Theta \in M_n(\mathbb{R})$, are often called *noncommutative tori* (because $C_r^*(\mathbb{Z}^n,\sigma_\Theta)$ is $*$-isomorphic to $C(\mathbb{T}^n)$ in the case where Θ is symmetric).

As usual, we set $\delta = \delta_e$, which is a cyclic (=generating) vector for all these algebras. The (normal) state τ on these algebras given by restricting the vector state ω_δ associated to δ is easily seen to be tracial. Further, τ is faithful as δ is separating for $\mathrm{vN}(G,\sigma)$: indeed, if $x \in \mathrm{vN}(G,\sigma)$ and $x\delta = 0$, then, using the commutation relations, we get $x\delta_h = x\rho_{\overline{\sigma}}(h)^*\delta = \rho_{\overline{\sigma}}(h)^*x\delta = 0$ for all $h \in G$, so $x = 0$. In particular, $\mathrm{vN}(G,\sigma)$ is finite as a von Neumann algebra.

We note (cf. [23, Corollary 1], [24, Proposition 1.3]) that $\mathrm{vN}(G,\sigma)$ is a factor if and only if *Kleppner's condition* is satisfied, that is, the conjugacy class of each non-trivial σ-regular element in G has infinite cardinality. Here, $g \in G$ is called σ-regular

[3] Sometimes also called multipliers in the literature. Notice, however, that we will use this term later with a different meaning.

whenever $\sigma(g,h)=\sigma(h,g)$ for all $h\in G$ that commute with g. It is also known [25] that Kleppner's condition holds if and only if $C_r^*(G,\sigma)$ has trivial center, if and only if $C_r^*(G,\sigma)$ is prime.

We also mention for completeness that we have $\mathrm{vN}(G,\sigma)'=\rho_{\overline{\sigma}}(G)''$. This follows by applying (pre-)Tomita–Takesaki theory to the pair $(\mathrm{vN}(G,\sigma),\delta)$: its J-operator is easily seen to be given by $(J_\sigma\xi)(g)=\overline{\sigma}(g,g^{-1})\overline{\xi(g^{-1})}$ for $g\in G$. As $J_\sigma\Lambda_\sigma(g)J_\sigma=\rho_{\overline{\sigma}}(g)$ for $g\in G$, we get

$$\mathrm{vN}(G,\sigma)'=J_\sigma\mathrm{vN}(G,\sigma)J_\sigma=(J_\sigma\Lambda_\sigma(G)J_\sigma)''=\rho_{\overline{\sigma}}(G)''.$$

Also in the vein of (pre)-Tomita–Takesaki theory, we may consider $\mathrm{vN}(G,\sigma)$ as a Hilbert algebra [26] with respect to the inner product $\langle x,y\rangle:=\tau(y^*x)=(x\delta,y\delta)$. Denoting by $\|\cdot\|_2$ the associated norm on $\mathrm{vN}(G,\sigma)$, the map $x\to\widehat{x}:=x\delta$ is an isometry, called the *Fourier transform*, from $(\mathrm{vN}(G,\sigma),\|\cdot\|_2)$ into $(\ell^2(G),\|\cdot\|_2)$, sending $\Lambda_\sigma(g)$ to δ_g for each $g\in G$.

The range of the Fourier transform is the subspace of $\ell^2(G)$ given by

$$\mathcal{U}(G,\sigma):=\{\widehat{x}\in\ell^2(G)\mid x\in\mathrm{vN}(G,\sigma)\}.$$

It may be turned into a Hilbert algebra by setting $\widehat{x}\cdot\widehat{y}:=\widehat{xy}$ and $\widehat{x}^*:=\widehat{x^*}$ for $x,y\in\mathrm{vN}(G,\sigma)$. A computation gives that $\widehat{x}^*(g)=\overline{\sigma}(g,g^{-1})\overline{\widehat{x}(g^{-1})}$ for each $g\in G$. Moreover, the product $\widehat{x}\cdot\widehat{y}$ may be expressed as a *twisted* convolution product. To explain this, consider first $\xi,\eta\in\ell^2(G)$. The σ*-convolution product* $\xi*_\sigma\eta$ is then defined as the complex function on G given by

$$(\xi*_\sigma\eta)(h)=\sum_{g\in G}\xi(g)\eta(g^{-1}h)\sigma(g,g^{-1}h)$$

for $h\in G$. As $|(\xi*_\sigma\eta)(h)|\le(|\xi|*|\eta|)(h)$ for each $h\in G$, it is straightforward to check that $\xi*_\sigma\eta$ is a well defined bounded function on G, satisfying

$$\|\xi*_\sigma\eta\|_\infty\le\||\xi|*|\eta|\|_\infty\le\|\xi\|_2\|\eta\|_2.$$

Now, if $x\in\mathrm{vN}(G,\sigma)$ and $\eta\in\ell^2(G)$, one checks by direct computation that $x\eta=\widehat{x}*_\sigma\eta$. Hence, we get

$$\widehat{x}\cdot\widehat{y}=\widehat{xy}=xy\delta=x\widehat{y}=\widehat{x}*_\sigma\widehat{y}$$

for all $x,y\in\mathrm{vN}(G,\sigma)$, thus justifying our comment above.

Let $x\in\mathrm{vN}(G,\sigma)$. The value $\widehat{x}(g)=(x\delta)(g)$ is called the *Fourier coefficient* of x at $g\in G$. Considering τ as the normalized "Haar functional" on $\mathrm{vN}(G,\sigma)$, we have indeed

$$\widehat{x}(g)=(x\delta,\delta_g)=(x\delta,\Lambda_\sigma(g)\delta)=\tau(x\Lambda_\sigma(g)^*).$$

Moreover, we have

$$\|\widehat{x}\|_\infty\le\|\widehat{x}\|_2=\|x\|_2\le\|x\|.$$

The (formal) *Fourier series* of x is defined as $\sum_{g\in G}\widehat{x}(g)\Lambda_\sigma(g)$. Notice that this series does not necessarily converge[4] in the weak operator topology. However, we have

$$x=\sum_{g\in G}\widehat{x}(g)\Lambda_\sigma(g)\quad(\text{convergence w.r.t. }\|\cdot\|_2).$$

The Fourier series representation of $x\in\mathrm{vN}(G,\sigma)$ is unique. More generally, if $\xi:G\to\mathbb{C}$ and $\sum_{g\in G}\xi(g)\Lambda_\sigma(g)$ converges to some $x\in\mathrm{vN}(G,\sigma)$ w.r.t. $\|\cdot\|_2$, then one deduces easily that $\xi\in\ell^2(G)$ and $\xi=\widehat{x}$.

We will be interested in the following self-adjoint subspace of $C_r^*(G,\sigma)$:

$$CF(G,\sigma)=\left\{x\in C_r^*(G,\sigma)\mid\sum_{g\in G}\widehat{x}(g)\Lambda_\sigma(g)\ \text{ is convergent in operator norm}\right\}.$$

Since $\|\cdot\|_2\le\|\cdot\|$, it follows that the Fourier series of an element x in $CF(G,\sigma)$ converge to x in operator norm.

Let $f\in\ell^1(G)$. The series $\sum_{g\in G}f(g)\Lambda_\sigma(g)$ is clearly absolutely convergent in operator norm and we shall denote its sum in $C_r^*(G,\sigma)$ by $\pi_\sigma(f)$. Then we have $\widehat{\pi_\sigma(f)}=f$, so $\pi_\sigma(f)\eta=f*_\sigma\eta$ for each $\eta\in\ell^2(G)$. Moreover,

$$\left\{x\in\mathrm{vN}(G,\sigma)\mid\widehat{x}\in\ell^1(G)\right\}=\pi_\sigma(\ell^1(G))\subset CF(G,\sigma).\text{ [5]}$$

It is also easy to check that $\ell^1(G)$ is a $*$-subalgebra of the Hilbert algebra $\mathcal{U}(G,\sigma)$. It becomes a unital Banach $*$-algebra with respect to the ℓ^1-norm $\|\cdot\|_1$, the unit being given by δ and its involution by $f^*(g)=\overline{\sigma}(g,g^{-1})\overline{f(g^{-1})}$ for $g\in G$. Now, as

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[4] Here, and in the sequel, since we consider possibly uncountable groups, convergence of any series indexed over G always means unconditional convergence, that is, w.r.t. to the net of finite subsets of G.

[5] In the special case where $G=\mathbb{Z}$, it can be shown that $CF(\mathbb{Z},1)=\pi_1(\ell^1(\mathbb{Z}))$, i.e., the Fourier series of a function f in $C(\mathbb{T})$ is unconditionally convergent (w.r.t. $\|\cdot\|_\infty$) if and only if it is absolutely convergent, see e.g. [27, Theorem 3.34]. However, the Fourier series of $f\in C(\mathbb{T})$ may be uniformly convergent in the usual sense (i.e., $\lim_{N\to\infty}S_N[f]$ exists w.r.t. $\|\cdot\|_\infty$) without being absolutely convergent.

the map π_σ gives a faithful $*$-representation of $\ell^1(G,\sigma)$ on $\ell^2(G)$, the enveloping C^*-algebra [28] of $\ell^1(G,\sigma)$ is simply the completion of $\ell^1(G,\sigma)$ w.r.t. the norm

$$\|f\|_{\max} := \sup_\pi\{\|\pi(f)\|\}$$

where the supremum is taken over all non-degenerate $*$-representations of $\ell^1(G,\sigma)$ on Hilbert spaces. This C^*-algebra is denoted by $C^*(G,\sigma)$ and called the *full twisted group C^*-algebra* associated with (G,σ). As usual, we will identify $\ell^1(G,\sigma)$ with its canonical image in $C^*(G,\sigma)$. We note that if $\widetilde{\sigma} \in Z^2(G,\mathbb{T})$ is such that $\widetilde{\sigma} \sim \sigma$, so $\widetilde{\sigma} = (db)\sigma$ for some $b : G \to \mathbb{T}$ with $b(e) = 1$, then $C^*(G,\widetilde{\sigma}) \simeq C^*(G,\sigma)$, the $*$-isomorphism being given at the ℓ^1-level by the map $f \to bf$.

Any non-degenerate $*$-representation of $\ell^1(G,\sigma)$ extends uniquely to a non-degenerate $*$-representation of $C^*(G,\sigma)$, and we will always use the same symbol to denote the extension. There is a bijective correspondence $U \to \pi_U$ between σ-projective unitary representations of G and non-degenerate $*$-representations of $C^*(G,\sigma)$ determined by

$$\pi_U(f) = \sum_{g\in G} f(g)U(g)$$

for $f \in \ell^1(G)$, (the series above being obviously absolutely convergent in operator norm), the inverse correspondence being given by $U_\pi(g) = \pi(\delta_g)$ for $g \in G$. As $\pi_{\Lambda_\sigma} = \pi_\sigma$, we get

$$C_r^*(G,\sigma) = \overline{\pi_\sigma(\ell^1(G,\sigma))}^{\|\cdot\|} = \pi_\sigma(C^*(G,\sigma)).$$

When G is amenable, then π_σ is faithful on $C^*(G,\sigma)$ [29], that is, $C^*(G,\sigma)$ is canonically $*$-isomorphic to $C_r^*(G,\sigma)$. An interesting question is whether the converse holds true. Of course, this is certainly the case if σ is a coboundary, but puzzingly this seems open when $\sigma \not\sim 1$.

The dual space of $C^*(G,\sigma)$ may be identified as a subspace $B(G,\sigma)$ of $\ell^\infty(G)$ through the linear injection $\Phi : \omega \to \varphi_\omega$, where $\varphi_\omega(g) := \omega(\delta_g)$ for $\omega \in C^*(G,\sigma)^*$ and $g \in G$. We equip $B(G,\sigma)$ with the transported norm $\|\Phi(\omega)\| := \|\omega\|$. Now, if ω is a positive linear functional on $C^*(G,\sigma)$, then φ_ω is σ-positive definite according to the following definition: a complex function φ on G is called *σ-positive definite* (σ-p.d.) whenever we have

$$\sum_{i,j=1}^n \overline{c_i}c_j\varphi(g_i^{-1}g_j)\overline{\sigma}(g_i, g_i^{-1}g_j) \geq 0$$

for all $n \in \mathbb{N}$, $c_1,\ldots,c_n \in \mathbb{C}$ and $g_1,\ldots,g_n \in G$.

Mimicking the untwisted case, one checks readily that φ is σ-p.d. if and only if there exists some σ-projective unitary representation U of G on some Hilbert space $\mathcal{H}$ and some $\xi \in \mathcal{H}$ (that may be chosen to be cyclic for U) such that $\varphi(g) = (U(g)\xi,\xi)$ for $g \in G$; it follows that φ is bounded with $\|\varphi\|_\infty = \|\xi\|^2 = \varphi(e)$. Further, as $(\pi_U(f)\xi,\xi) = \sum_{g\in G} f(g)\varphi(g)$ for all $f \in \ell^1(G)$, we also get an unambiguously defined positive linear functional L_φ on $C^*(G,\sigma)$ via $L_\varphi(x) := (\pi_U(x)\xi,\xi)$, which satisfies that $\Phi(L_\varphi) = \varphi$. Denoting by $P(G,\sigma)$ the cone of all σ-p.d. functions on G, we now see that $B(G,\sigma) = \mathrm{Span}(P(G,\sigma))$. By considering the universal $*$-representation of $C^*(G,\sigma)$, one deduces that $B(G,\sigma)$ consists precisely of all coefficient functions associated to σ-projective unitary representations of G.

We remark that if φ is σ-p.d. and ψ is ω-p.d. for some $\omega \in Z^2(G,\mathbb{T})$ then $\varphi\psi$ is $\sigma\omega$-p.d. Hence we have $B(G,\sigma)B(G,\omega) \subset B(G,\sigma\omega)$. In particular, $B(G,\sigma)$ is not a priori an algebra w.r.t. to pointwise multiplication (except when $\sigma = 1$, in which case it is usually called the Fourier–Stieltjes algebra [30] of G and denoted by $B(G)$). It is not a priori closed under complex conjugation either: if $\varphi \in P(G,\sigma)$, then $\overline{\varphi} \in P(G,\overline{\sigma})$. Similarly, if $\widetilde{\varphi}(g) := \sigma(g,g^{-1})\varphi(g^{-1})$ for $g \in G$, then $\widetilde{\varphi} \in P(G,\overline{\sigma})$. Hence $\varphi^* \in P(G,\sigma)$, where $\varphi^*(g) := \overline{\sigma}(g,g^{-1})\overline{\varphi}(g^{-1})$ for $g \in G$. This corresponds to the fact that $L_{\varphi^*} = (L_\varphi)^*$ is then also a positive linear functional on $C^*(G,\sigma)$.

As $C_r^*(G,\sigma)$ is a quotient of $C^*(G,\sigma)$, we may identify its dual space as a closed subspace $B_r(G,\sigma)$ of $B(G,\sigma)$. It consists of the span of all σ-p.d. functions on G associated to unitary representations of G which are weakly contained in Λ_σ (that is, such that the associated representation of $C^*(G,\sigma)$ is weakly contained in π_σ [28]). Further, the predual $\mathrm{vN}(G,\sigma)_*$ of $\mathrm{vN}(G,\sigma)$ can be regarded as a closed subspace of the dual of $C_r^*(G,\sigma)$, hence as a closed subspace $A(G,\sigma)$ of $B_r(G,\sigma)$, that may be described as the set of all coefficient functions of Λ_σ. For instance, as easily seen, we have $\ell^2(G) \subset A(G,\sigma)$. When $\sigma = 1$, one recovers the so-called Fourier algebra [30] $A(G)$ of G.

The general problem of deciding when two (full or reduced) twisted group C^*-algebras associated to the same group are $*$-isomorphic is undoubtedly hard. For some results in this direction based on K-theoretical considerations, see e.g. [31–36] and references therein. The von Neumann algebraic version of this problem is essentially open, with one notable exception. As follows from Connes' work on injective factors [37,38], if G is countably infinite and amenable, and $\mathrm{vN}(G,\sigma)$ is a factor, then it is the hyperfinite II_1-factor; in particular, $\mathrm{vN}(G,\sigma)$ and $\mathrm{vN}(G,\omega)$ are then $*$-isomorphic whenever both σ and ω satisfy Kleppner's condition.

As thoroughly discussed in the book of N. Brown and N. Ozawa [39], many approximation properties for G are reflected in some analogous properties for $C_r^*(G)$ and/or $C^*(G)$, and/or $\mathrm{vN}(G)$. To keep our exposition at a reasonable size, we will not repeat here the definitions of all involved concepts. We advice the reader to consult [39] for information whenever necessary. It is not difficult to see that the proofs of these results can be extended to cover the twisted case. For example, we have:

Theorem 1.1. *The following conditions are equivalent.*

(1) *G is amenable.*
(2) *$C^*(G, \sigma)$ is nuclear.*
(3) *$C_r^*(G, \sigma)$ is nuclear.*
(4) *$\mathrm{vN}(G, \sigma)$ is semidiscrete.*
(5) *$\mathrm{vN}(G, \sigma)$ is injective.*

Theorem 1.2. *The following conditions are equivalent.*

(1) *G has the Haagerup Property.*
(2) *$\mathrm{vN}(G, \sigma)$ has the Haagerup Property.*

Theorem 1.3. *The following conditions are equivalent.*

(1) *G is weakly amenable.*
(2) *$C_r^*(G, \sigma)$ has the CBAP.*
(3) *$\mathrm{vN}(G, \sigma)$ has the $W*$ CBAP.*

Theorem 1.4. *The following conditions are equivalent.*

(1) *G has the AP of Haagerup and Kraus.*
(2) *$C_r^*(G, \sigma)$ has the OAP.*
(3) *$C_r^*(G, \sigma)$ has the SOAP.*
(4) *$\mathrm{vN}(G, \sigma)$ has the $W*$ OAP.*

Theorem 1.5. *The following conditions are equivalent.*

(1) *G is exact.*
(2) *$C_r^*(G, \sigma)$ is exact.*
(3) *$C_u^*(G, \sigma)$ is nuclear.*
(4) *$\mathrm{vN}(G, \sigma)$ is weakly exact.*

In item (3) of Theorem 1.5, $C_u^*(G, \sigma)$ denotes the *twisted uniform Roe algebra* associated with (G, σ), which, by definition, is the C^*-subalgebra of $B(\ell^2(G))$ generated by $\Lambda_\sigma(G)$ and $\ell^\infty(G)$. The equivalences between (1), (2) and (3) in Theorem 1.5 have been checked in [40].

We recall that if G is amenable, then it is weakly amenable, and if it weakly amenable, then it has property AP of Haagerup and Kraus. Moreover, property AP for G implies its exactness. All the opposite implications are false in general. Counterexamples for the first two cases are mentioned in [39], while V. Lafforgue and M. de la Salle have shown in [41] that the (linear, thus) exact group $\mathrm{SL}(3, \mathbb{Z})$ fails to have the AP.

As in [39], let $\Lambda_{\mathrm{cb}}(G) \in [1, \infty]$ denote the Cowling–Haagerup content of G, so G is weakly amenable if and only if $\Lambda_{\mathrm{cb}}(G) < \infty$. Say that G has the *complete metric approximation property* (CMAP) if it is weakly amenable and $\Lambda_{\mathrm{cb}}(G) = 1$. It was open for a while whether every countable discrete group with the Haagerup property had the CMAP, but a counterexample was given in [42, Cor. 2], namely the (standard, restricted) wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ (alternatively, see [39, Theorem 12.2.11 and Corollary 12.3.7]). The other half of this conjecture of M. Cowling is still open, that is, it is unknown whether the CMAP implies the Haagerup property. In 2014, due to the work of D. Osajda [43], it became clear that the first half of Cowling's conjecture can fail dramatically: there exist groups with the Haagerup property that are non-exact, so, in particular, that are not weakly amenable.

It should also be mentioned that S. Knudby [44,45] (see also [46]) has recently introduced the *weak Haagerup property* for groups, a property possessed by every group having the Haagerup property or being weakly amenable. He has also defined a weak Haagerup property for finite von Neumann algebras and showed that a (discrete) group has the weak Haagerup property if and only if its group von Neumann algebra has the weak Haagerup property. We note that this result may also be extended without trouble to the twisted case.

Finally, we recall that a Banach space X is said to have the (Grothendieck) *approximation property* (AP) if on any compact subset of X the identity operator id_X on X can be uniformly approximated by bounded finite rank operators. A stronger requirement is that X has the *bounded approximation property* (BAP), that is, id_X is the point-norm limit of a net (T_α) of bounded finite rank operators satisfying $\sup_\alpha \|T_\alpha\| < \infty$. We will see in Section 2.3 how the BAP (and the MAP) for $C_r^*(G, \sigma)$ may be characterized in terms of G. A similar characterization of the AP for $C_r^*(G, \sigma)$ seems unknown.

1.4. Twisted crossed products

We consider a *unital, discrete, twisted C^*-dynamical system* $\Sigma = (A, G, \alpha, \sigma)$. Thus, A is a C^*-algebra with unit 1, G is a discrete group with identity e and (α, σ) is a *twisted action* of G on A, that is, α is a map from G into the group $\mathrm{Aut}(A)$ of $*$-automorphisms of A and σ is a map from $G \times G$ into $\mathcal{U}(A)$, the unitary group of A, satisfying

$$\begin{aligned} \alpha_g \circ \alpha_h &= \mathrm{Ad}(\sigma(g,h)) \circ \alpha_{gh} \\ \sigma(g,h)\sigma(gh,k) &= \alpha_g(\sigma(h,k))\sigma(g,hk) \\ \sigma(g,e) &= \sigma(e,g) = 1 \end{aligned}$$

for all $g, h, k \in G$.

To any such twisted C^*-dynamical system $\Sigma = (A, G, \alpha, \sigma)$ one may associate its *full twisted crossed product* $C^*(\Sigma)$ and its *reduced twisted crossed product* $C_r^*(\Sigma)$. General references are [47] (when σ is trivial), [29] (when σ takes its values in $Z(A)$, the center of A) and [33,48]. Our exposition follows [5] and makes essential use of Hilbert (right) C^*-modules. The reader may for instance consult [49] for unexplained terminology about such modules.

A *covariant homomorphism* of Σ is a pair (π, u), where π is a $*$-homomorphism of A into a C^*-algebra C and u is a map of G into $\mathcal{U}(C)$, satisfying

$$u(g)\, u(h) = \pi(\sigma(g,h))\, u(gh)$$

and the covariance relation

$$\pi(\alpha_g(a)) = u(g)\, \pi(a)\, u(g)^* \tag{1}$$

for all $g, h \in G$, $a \in A$. If $C = \mathcal{L}(X)$ is the C^*-algebra of adjointable operators on some (right) Hilbert C^*-module X, then (π, u) is called a *covariant representation* of Σ on X.

The vector space $C_c(\Sigma)$ of functions from G into A with finite support becomes a unital $*$-algebra when equipped with the operations

$$\begin{aligned} (f_1 * f_2)(h) &= \sum_{g\in G} f_1(g)\, \alpha_g\big(f_2(g^{-1}h)\big)\, \sigma(g, g^{-1}h), \\ f^*(h) &= \sigma(h, h^{-1})^*\, \alpha_h\big(f(h^{-1})\big)^*. \end{aligned}$$

The *full C^*-algebra* $C^*(\Sigma)$ is generated by (a copy of) $C_c(\Sigma)$ and has the universal property that whenever $(\phi, u) : A \to C$ is a covariant homomorphism of Σ, then there exists a unique $*$-homomorphism $\phi \times u : C^*(\Sigma) \to C$ such that

$$(\phi \times u)(f) = \sum_{g\in G} \phi(f(g))\, u(g) \quad \text{for all } f \in C_c(\Sigma).$$

Any representation π of A on some Hilbert B-module Y induces a (left) *regular covariant representation* $(\widetilde{\pi}, \widetilde{\lambda}_\pi)$ of Σ on the B-module $B^G := \ell^2(G) \otimes B$. Considering A itself as a (right) Hilbert A-module in the obvious way and letting $\ell : A \to \mathcal{L}(A)$ denote left multiplication, we may then form the regular covariant representation of Σ

$$\Lambda = \widetilde{\ell} \times \widetilde{\lambda}_\ell\ : C^*(\Sigma) \to \mathcal{L}(A^G).$$

The *reduced C^*-algebra of* Σ is then defined as the C^*-subalgebra of $\mathcal{L}(A^G)$ given by

$$C_r^*(\Sigma) = \Lambda\big(C^*(\Sigma)\big).$$

It is often more convenient to consider the Hilbert A-module

$$A^\Sigma = \Big\{\xi : G \to A \mid \sum_{g\in G} \alpha_g^{-1}\big(\xi(g)^*\, \xi(g)\big) \text{ is norm-convergent in } A\Big\}$$

endowed with the A-valued inner product

$$\langle \xi, \eta\rangle_\alpha = \sum_{g\in G} \alpha_g^{-1}\big(\xi(g)^* \eta(g)\big),$$

the right action of A being given by $(\xi \cdot a)(g) = \xi(g)\, \alpha_g(a)$.

A nice covariant representation $(\ell_\Sigma, \lambda_\Sigma)$ of Σ on A^Σ is then given by

$$\begin{aligned} \big[\ell_\Sigma(a)\, \xi\big](h) &= a\, \xi(h), \\ \big[\lambda_\Sigma(g)\, \xi\,\big](h) &= \alpha_g\big(\xi(g^{-1}h)\big)\, \sigma(g, g^{-1}h). \end{aligned}$$

Setting $\Lambda_\Sigma = \ell_\Sigma \times \Lambda_\Sigma$ and identifying A with $\ell_\Sigma(A)$ gives

$$\Lambda_\Sigma(f) = \sum_{g\in G} f(g)\, \lambda_\Sigma(g), \quad \text{for } f \in C_c(\Sigma).$$

As Λ_Σ is easily seen to be unitarily equivalent to Λ, we have

$$C_r^*(\Sigma) \simeq \Lambda_\Sigma(C^*(\Sigma)) \subset \mathcal{L}(A^\Sigma).$$

Let $\xi_0 \in A^\Sigma$ be defined by $\xi_0(e) = 1$ and $\xi_0(g) = 0$ for $g \neq e$. Then

$$\Lambda_\Sigma(f)\,\xi_0 = f \quad \text{for } f \in C_c(\Sigma).$$

Hence, setting $\widehat{x} = x\,\xi_0 \in A^\Sigma$ for $x \in C_r^*(\Sigma)$, we get $\widehat{\Lambda_\Sigma(f)} = f$ for each $f \in C_c(\Sigma)$. The (injective) linear map $x \to \widehat{x}$ from $C_r^*(\Sigma)$ into A^Σ is called the *Fourier transform*. Moreover, we say that $\widehat{x}(g) \in A$ is the *Fourier coefficient* of $x \in C_r^*(\Sigma)$ at $g \in G$. For any $x \in C_r^*(\Sigma)$, we have

$$\|\widehat{x}\|_\infty \leq \|\widehat{x}\|_\alpha \leq \|x\|$$

where $\|\widehat{x}\|_\infty := \sup_g \|\widehat{x}(g)\|$ and $\|\widehat{x}\|_\alpha = \left\| \sum_g \alpha_g^{-1}\big(\widehat{x}(g)^*\,\widehat{x}(g)\big) \right\|^{1/2}$.

The canonical conditional expectation E from $C_r^*(\Sigma)$ onto A is simply given by

$$E(x) = \widehat{x}(e)$$

for $x \in C_r^*(\Sigma)$. It is faithful, and some of its useful properties are:

$$E(\lambda_\Sigma(g)) = 0 \quad \text{whenever } g \neq e,$$
$$\widehat{x}(g) = E\big(x\,\lambda_\Sigma(g)^*\big),$$
$$E(x^*x) = \langle \widehat{x}, \widehat{x} \rangle_\alpha,$$
$$E\big(\lambda_\Sigma(g) x \lambda_\Sigma(g)^*\big) = \alpha_g(E(x))$$

where $x \in C_r^*(\Sigma)$ and $g \in G$.

For $x \in C_r^*(\Sigma)$, its (formal) *Fourier series* is defined as

$$\sum_{g \in G} \widehat{x}(g)\,\Lambda_\Sigma(g).$$

Notice that we have placed the Fourier coefficients on the left side in this formula. One can also work with a right version, but this is basically only a matter of convention.

1.5. Equivariant representations

The equivariant representations of a twisted system Σ interplay in a non-trivial way with its covariant representations. In particular, it allows to give useful generalizations of the classical Fell absorption property. We refer to [5,6] for more details.

An *equivariant representation* of $\Sigma = (A, G, \alpha, \sigma)$ on a Hilbert A-module X is a pair (ρ, v) where $\rho : A \to \mathcal{L}(X)$ is a representation of A on X and $v : G \to \mathcal{I}(X)$ (the group of all $\mathbb{C}$-linear, invertible, bounded maps from X into itself) satisfying

(i) $\rho(\alpha_g(a)) = v(g)\,\rho(a)\,v(g)^{-1}\,,\ g \in G\,,\ a \in A$
(ii) $v(g)\,v(h) = \mathrm{ad}_\rho(\sigma(g,h))\,v(gh)\,,\ g, h \in G$
(iii) $\alpha_g\big(\langle x\,,\ x'\rangle\big) = \langle v(g)x\,,\ v(g)x'\rangle\,,\ g \in G\,, x, x' \in X$
(iv) $v(g)(x \cdot a) = (v(g)x) \cdot \alpha_g(a)\,,\ g \in G,\ x \in X,\ a \in A.$

In (ii) above, $\mathrm{ad}_\rho(\sigma(g,h)) \in \mathcal{I}(X)$ is defined by

$$\mathrm{ad}_\rho(\sigma(g,h))\,x = \big(\rho(\sigma(g,h))\,x\big) \cdot \sigma(g,h)^*, \quad g, h \in G,\ x \in X.$$

Turning X into an A-A bimodule by setting $a \cdot x = \rho(a)x$, $a \in A,\ x \in X$, we have

$$\mathrm{ad}_\rho(\sigma(g,h))\,x = \sigma(g,h) \cdot x \cdot \sigma(g,h)^*, \quad g, h \in G,\ x \in X.$$

To stress some relationship with other notions, the pair $(v, \mathrm{ad}_\rho(\sigma))$ is a kind of (α, σ)-compatible action of G on X that, when $\sigma = 1$, gives back a so-called α–α compatible action. There is also a close connection between equivariant representations of Σ and C^*-correspondences over $C_r^*(\Sigma)$ that can be further exploited [8].

Some examples are as follows:

(i) $\ell : A \to \mathcal{L}(A)$ and $\alpha : G \to \mathrm{Aut}(A) \subset \mathcal{I}(A)$ give the *trivial equivariant representation* (ℓ, α) of Σ.

(ii) Let (ρ, v) be an equivariant representation of Σ on X. The *induced* equivariant representation $(\check{\rho}, \check{v})$ on X^G is given by

$$\big(\check{\rho}(a)\,\xi\big)(h) = \rho(a)\,\xi(h), \qquad \big(\check{v}(g)\xi\big)(h) = v(g)\,\xi(g^{-1}h) \quad \text{for } a \in A,\ \xi \in X^G, \text{ and } g, h \in G.$$

(iii) More generally, if w is a unitary representation of G on some Hilbert space $\mathcal{H}$, then $(\rho \otimes \iota, v \otimes w)$ is an equivariant representation of Σ on $X \otimes \mathcal{H}$. Identifying X^G with $X \otimes \ell^2(G)$, we have

$$(\check{\rho}, \check{v}) = (\rho \otimes \iota, v \otimes \lambda).$$

(iv) $(\check{\ell}, \check{\alpha})$ is the *regular equivariant representation* of Σ. It acts on A^G via

$$\big[\check{\ell}(a)\,\xi\big](h) = a\,\xi(h), \qquad \big[\check{\alpha}(g)\,\xi\big](h) = \alpha_g\big(\xi(g^{-1}h)\big).$$

Tensoring an equivariant representation with a covariant representation is possible. If (ρ, v) is an equivariant representation of Σ on a Hilbert A-module X and (π, u) is a covariant representation of Σ on a Hilbert B-module Y, one may then form the covariant representation $(\rho\dot{\otimes}\pi\,,\ v\dot{\otimes}u)$ of Σ on the internal tensor product Hilbert B-module $X\ \otimes_\pi Y$. It acts on simple tensors in $X\ \otimes_\pi Y$ as follows:

$$\big[(\rho\dot{\otimes}\,\pi)(a)\big]\,(x\dot{\otimes}\,y) = \rho(a)x\dot{\otimes}\ y, \qquad \big[(v\dot{\otimes}\,u)(g)\big]\,(x\dot{\otimes}\,y) = v(g)x\dot{\otimes}\ u(g)y.$$

The following properties hold:

- $(\ell\dot{\otimes}\,\pi) \times (\alpha\dot{\otimes}\,u)\ \simeq\ \pi \times u$;
- Fell's absorption principle (I): $(\rho\dot{\otimes}\,\ell_\Sigma)\ \times\ (v\dot{\otimes}\,\lambda_\Sigma)\ \simeq\ \widetilde{\rho}\ \times\ \widetilde{\lambda}_\rho$.
- Fell's absorption principle (II): Let $\pi' : \mathcal{L}(X^G) \to \mathcal{L}(X^G\ \otimes\ _\pi Y)$ denote the amplification map, so that $\check{\rho}\dot{\otimes}\,\pi = \pi' \circ \check{\rho} : A \to \mathcal{L}(X^G\ \otimes\ _\pi Y)$. Then

$$(\check{\rho}\dot{\otimes}\,\pi) \times (\check{v}\dot{\otimes}\,u)\ \simeq\ \pi' \circ (\widetilde{\rho} \times \widetilde{\lambda}_\rho).$$

2. Fourier series in twisted group algebras

Throughout this section, G denotes a discrete group and we let $\sigma\ \in\ Z^2(G, \mathbb{T})$. When F is subset of G, we denote its characteristic function by χ_F. If f is complex function on G, we set $\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$.

2.1. Norm-convergence and decay properties

Let $\mathcal{L}$ be a subspace of $\ell^2(G)$ containing the space $C_c(G)$ of complex functions on G having finite support, and let $\|\cdot\|'$ be a norm on $\mathcal{L}$.

The pair (G, σ) is said to have the *$\mathcal{L}$-decay property* (w.r.t. $\|\cdot\|'$) if, for every $\xi \in \mathcal{L}$ and every $\varepsilon > 0$, there exists a finite subset F_0 of G such that $\|\xi\,\chi_F\|' < \varepsilon$ for all finite subsets F of G that are disjoint from F_0, and, moreover, the linear map $f \mapsto \pi_\sigma(f)$ from $(C_c(G), \|\cdot\|')$ to $(C^*_r(G, \sigma), \|\cdot\|)$ is bounded. Our interest in this notion lies in the following result.

Theorem 2.1. *Assume (G, σ) has the $\mathcal{L}$-decay property. Then we have*

$$\mathcal{L}^\vee := \big\{x \in \mathrm{vN}(G, \sigma) \mid \widehat{x} \in \mathcal{L}\big\}\ \subset\ \mathrm{CF}(G, \sigma).$$

For example, consider $\kappa : G \to [1, +\infty)$ and equip the space

$$\mathcal{L}^2_\kappa := \{\xi : G \to \mathbb{C} \mid \xi\kappa \in \ell^2(G)\}$$

with the norm $\|\xi\|_{2,\kappa} = \|\xi\kappa\|_2$. We then say that (G, σ) is *κ-decaying* when (G, σ) has the $\mathcal{L}^2_\kappa$-decay property. This amounts to requiring that there exists some $C > 0$ such that

$$\|\pi_\sigma(f)\| \leq C_\sigma\,\|f\kappa\|_2 \quad \text{for all } f \in C_c(G).$$

Any countable group is κ-decaying for some suitably chosen κ. However, one would like to do this in a way such that the subspace $(\mathcal{L}^2_\kappa)^\vee$ of $CF(G, \sigma)$ becomes large. For example, assume G is finitely generated and L is a word-length function on G. Then one can show that there exists some $t_0 \geq 0$ such that (G, σ) is e^{tL}-decaying whenever $t > t_0$. But this is not very useful unless t_0 can be chosen to be zero. This can be easily achieved if G has subexponential growth. As we will see, following Haagerup, this can also be achieved under much weaker hypotheses.

For a non-empty finite subset $E \subset G$, define its *twisted Haagerup content* (w.r.t. σ) as

$$c_\sigma(E) = \sup\big\{\|\pi_\sigma(f)\| \mid \mathrm{supp}(f) \subset E,\ \|f\|_2 \leq 1\big\}.$$

Also set $c(E) = c_1(E)$. Then we have $1 \leq c_\sigma(E) \leq c(E) \leq |E|^{1/2}$, and $c(E) = |E|^{1/2}$ whenever G is amenable.

Definition 2.2. Assume G is countable and $L : G \to [0, +\infty)$ is proper. Then G is said to have *polynomial* H_σ*-growth* (w.r.t. L) whenever there exist positive constants K, p such that

$$c_\sigma\big(\{g \in G \mid L(g) \leq r\}\big)\ \leq\ K\,(1 + r)^p$$

for all $r \geq 0$. Similarly, G is said to have *subexponential* H_σ*-growth* if, for any $b > 1$, there exists $r_1 \geq 0$ such that

$$c_\sigma\big(\{g \in G \mid L(g) \leq r\}\big)\ <\ b^r$$

for all $r \geq r_1$. When $\sigma = 1$ we just talk about polynomial and subexponential H-growth, respectively.

Assume that G is countably infinite and let $L : G \to [0, \infty)$ be proper. Then it can be shown that G has polynomial (resp. subexponential) H_σ-growth whenever it has polynomial (resp. subexponential) H-growth, and that it has subexponential H_σ-growth whenever it has polynomial H_σ-growth. Moreover, if $L = \ell$ is a proper length function on G and G has (usual) polynomial growth (w.r.t. ℓ) then G has polynomial H_σ-growth (w.r.t. ℓ). Finally, if G is amenable, then polynomial (resp. subexponential) H-growth (w.r.t. ℓ) reduces to polynomial (resp. subexponential) growth.

Proposition 2.3. *Assume that G is countably infinite and $L : G \to [0, \infty)$ is proper.*
If G has polynomial H_σ-growth (w.r.t. L), then (G, σ) is $(1+L)^s$-decaying for some $s > 0$, and the Fourier series of $x \in C_r^*(G, \sigma)$ is then norm-convergent (to x) whenever

$$\sum_{g\in G} |\widehat{x}(g)|^2 (\,1 + L(g))^{2s} < \infty.$$

If G has subexponential H_σ-growth (w.r.t. L), then (G, σ) is e^{tL}-decaying for every $t > 0$, and the Fourier series of $x \in C_r^*(G, \sigma)$ is then norm-convergent (to x) whenever

$$\sum_{g\in G} |\widehat{x}(g)|^2\, e^{tL(g)} < \infty,$$

for some $t > 0$.

Remark 2.4. If G is a countably infinite group with a proper length function, then analogous statements concerning norm-approximation of elements in the (untwisted) uniform Roe algebra $C_u^*(G)$ by band truncations have been obtained in [50].

2.2. Twisted multipliers

For $\omega \in Z^2(G, \mathbb{T})$, let V be an ω-projective unitary representation of G on a Hilbert space $\mathcal{H}$. Letting $\iota_{\mathcal{H}}$ denote the trivial representation of G on $\mathcal{H}$, the following twisted version of Fell's classical absorption principle is easily seen to hold:

$$\Lambda_\sigma \otimes V \cong \Lambda_{\sigma\omega} \otimes \iota_{\mathcal{H}}. \tag{2}$$

Using this property, one can deduce the following generalization of Haagerup's fundamental lemma [1, Lemma 1.1].

Lemma 2.5. *Let $\varphi \in B(G, \omega)$ and write*

$$\varphi(g) = (V(g)\eta_1, \eta_2) \quad \textit{for all } g \in G,$$

for some ω-projective unitary representation V on a Hilbert space $\mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{H}$.
Then there exists a completely bounded normal map $\widetilde{M}_\varphi : \mathrm{vN}(G, \sigma\omega) \to \mathrm{vN}(G, \sigma)$ such that

$$\widetilde{M}_\varphi(\Lambda_{\sigma\omega}(g)) = \varphi(g)\, \Lambda_\sigma(g) \quad \textit{for all } g \in G,$$

and we have $\|\widetilde{M}_\varphi\| \le \|\widetilde{M}_\varphi\|_{cb} \le \|\varphi\| \le \|\eta_1\|\,\|\eta_2\|$.
By restriction, one obtains a completely bounded map $M_\varphi : C_r^*(G, \sigma\omega) \to C_r^*(G, \sigma)$ satisfying $\|\widetilde{M}_\varphi\| = \|M_\varphi\|$ and $\|\widetilde{M}_\varphi\|_{cb} = \|M_\varphi\|_{cb}$.
If $\varphi \in P(G, \omega)$, then $\widetilde{M}_\varphi$ and M_φ are completely positive, and we have

$$\|\widetilde{M}_\varphi\| = \|M_\varphi\| = \|\varphi\| = \varphi(e).$$

Remark 2.6. A similar result holds if one replaces V with an ω-projective uniformly bounded representation of G in $\mathrm{GL}(\mathcal{H})$, the group of invertible bounded operators on $\mathcal{H}$. Note that if G is amenable, so $C^*(G, \omega)$ is nuclear, then any uniformly bounded ω-projective representation of G is similar to an ω-projective unitary representation.

Definition 2.7 *(cf. [1, Def. 1.6] when $\sigma = \omega = 1$)***.** Let $\varphi : G \to \mathbb{C}$ and consider the linear map $m_\varphi : \mathbb{C}(G, \omega) \to \mathbb{C}(G, \sigma)$ given by

$$m_\varphi(\pi_\omega(f)) = \pi_\sigma(\varphi f), \quad \text{for all } f \in C_c(G).$$

Equivalently, m_φ is the linear map determined by

$$m_\varphi(\Lambda_\omega(g)) = \varphi(g)\, \Lambda_\sigma(g), \quad \text{for all } g \in G.$$

We say that φ is a (σ, ω)-*multiplier* if m_φ is bounded w.r.t. the operator norms on $\mathbb{C}(G, \omega)$ and $\mathbb{C}(G, \sigma)$, in which case we denote by M_φ the (unique) extension of m_φ to an element in $B(C_r^*(G, \omega), C_r^*(G, \sigma))$. Note that M_φ is then the unique element in this space satisfying

$$M_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), \quad \text{for all } g \in G.$$

We denote the subspace of $\ell^\infty(G)$ consisting of all (σ,ω)-multipliers on G by $MA(G,\sigma,\omega)$. We set $MA(G,\sigma) := MA(G,\sigma,\sigma)$ and $MA(G) := MA(G,1)$.

Adapting the arguments of Haagerup–de Cannière given in the proof of [51, Proposition 1.2], one can show the following result.

Proposition 2.8. *Let $\varphi : G \to \mathbb{C}$. Then the following four conditions are equivalent:*

(1) *$\varphi\psi \in A(G,\omega)$ for all $\psi \in A(G,\sigma)$.*
(2) *There exists a (unique) normal operator $\widetilde{M}_\varphi$ from $\mathrm{vN}(G,\omega)$ to $\mathrm{vN}(G,\sigma)$ such that*

$$\widetilde{M}_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), \quad g \in G.$$

(3) *$\varphi \in MA(G,\sigma,\omega)$.*
(4) *$\varphi\psi \in B_r(G,\omega)$ for all $\psi \in B_r(G,\sigma)$.*

Denote by a_φ (resp. b_φ) the bounded linear map from $A(G,\sigma)$ (resp. $B_r(G,\sigma)$) to $A(G,\omega)$ (resp. $B_r(G,\omega)$) obtained from multiplication with $\varphi \in MA(G,\sigma,\omega)$. Then $\widetilde{M}_\varphi$ is the transpose of a_φ, we have

$$\|M_\varphi\| = \|\widetilde{M}_\varphi\| = \|a_\varphi\| = \|b_\varphi\|$$

and $MA(G,\sigma,\omega)$ becomes a Banach space under the norm $|||\varphi||| := \|M_\varphi\|$.

We note that the space $MA(G,\sigma)$ is independent of σ. As this was left open in [4], we sketch the proof.

Proposition 2.9. *$MA(G,\sigma) = MA(G)$ (and the norm is independent of σ).*

Proof. Let $W : \ell^2(G) \to \ell^2(G) \otimes \ell^2(G)$ denote the isometry satisfying $W\delta_g = \delta_g \otimes \delta_g$ for all $g \in G$ and let λ denote the left regular representation of G. Let $\Phi_\sigma : C_r^*(G,\sigma) \to C_r^*(G,\sigma) \otimes C_r^*(G)$ denote the $*$-homomorphism satisfying

$$\Phi_\sigma(\Lambda_\sigma(g)) = \Lambda_\sigma(g) \otimes \lambda(g)$$

for each $g \in G$ and let $\Psi_\sigma : C_r^*(G) \to C_r^*(G,\sigma) \otimes C_r^*(G,\overline{\sigma})$ denote the $*$-homomorphism satisfying

$$\Psi_\sigma(\lambda(g)) = \Lambda_\sigma(g) \otimes \Lambda_{\overline{\sigma}}(g)$$

for each $g \in G$. The existence of both these homomorphisms follows readily from the twisted version of Fell's absorption principle given in (2).

Assume first $\varphi \in MA(G,\sigma)$ and let $M_\varphi^\sigma \in B(C_r^*(G,\sigma))$ denote the associated map satisfying $M_\varphi^\sigma(\Lambda_\sigma(g)) = \varphi(g)\,\Lambda_\sigma(g)$ for each $g \in G$. Let also $m_\varphi : \mathbb{C}(G) \to \mathbb{C}(G)$ be the linear map determined by

$$m_\varphi(\lambda(g)) = \varphi(g)\,\lambda(g) \quad \text{for all } g \in G.$$

Then one checks that

$$W^*(M_\varphi^\sigma \otimes \mathrm{id}_{C_r^*(G,\overline{\sigma})})\Psi_\sigma(x)W = m_\varphi(x)$$

for all $x \in \mathbb{C}(G)$. It follows that $\varphi \in MA(G)$ and $\|M_\varphi\| \le \|M_\varphi^\sigma\|$.

Next, assume that $\varphi \in MA(G)$ and let $M_\varphi \in B(C_r^*(G))$ denote the associated map satisfying $M_\varphi(\lambda(g)) = \varphi(g)\,\lambda(g)$ for each $g \in G$. Let also $m_\varphi^\sigma : \mathbb{C}(G,\sigma) \to \mathbb{C}(G,\sigma)$ be the linear map determined by

$$m_\varphi^\sigma(\Lambda_\sigma(g)) = \varphi(g)\,\Lambda_\sigma(g) \quad \text{for all } g \in G.$$

Then one checks that

$$W^*(\mathrm{id}_{C_r^*(G,\sigma)} \otimes M_\varphi)\Phi_\sigma(x)W = m_\varphi^\sigma(x)$$

for all $x \in \mathbb{C}(G,\sigma)$. It follows that $\varphi \in MA(G,\sigma)$ and $\|M_\varphi^\sigma\| \le \|M_\varphi\|$. $\square$

Still following Haagerup–de Cannière [51], we also introduce

$$M_0A(G,\sigma,\omega) := \{\varphi \in MA(G,\sigma,\omega) \mid M_\varphi \text{ is a completely bounded map}\}$$

and equip this space with the norm $\|\varphi\|_{cb} = \|M_\varphi\|_{cb}$. We set $M_0A(G,\sigma) := M_0A(G,\sigma,\sigma)$ and $M_0A(G) := M_0A(G,1)$. It can be shown [4, Prop. 4.3] that $M_0A(G,\sigma) = M_0A(G)$, and the cb-norm of $\varphi \in M_0A(G,\sigma)$ is actually independent of σ.

Remark 2.10. Completely bounded multipliers are closely related to (Herz–)Schur multipliers [52,53]. Recall that a kernel $K : G \times G \to \mathbb{C}$ is a *Schur multiplier* on $B(\ell^2(G))$ if, for every $A \in B(\ell^2(G))$ with associated matrix $[A(s,t)]$ w.r.t. the canonical ONB of $\ell^2(G)$, the matrix $[K(s,t)A(s,t)]$ also represents an element in $B(\ell^2(G))$. In that case, the associated linear operator S_K, mapping $B(\ell^2(G))$ into itself, is necessarily completely bounded, with $\|S_K\|_{cb} = \|S_K\|$. If $\varphi : G \to \mathbb{C}$ and $K_\varphi(s,t) = \varphi(st^{-1})$ is the associated kernel, then $\varphi \in M_0A(G)$ if and only if K_φ is a Schur multiplier, in which case we have $\|\varphi\|_{cb} = \|S_{K_\varphi}\|$.

The existence of (completely bounded) twisted multipliers is guaranteed by the following proposition, that follows immediately from Lemma 2.5.

Proposition 2.11. *We have*

$$B(G,\omega) \subset M_0A(G,\sigma,\sigma\omega) \subset MA(G,\sigma,\sigma\omega)$$

and $|||\varphi||| \le \|\varphi\|_{cb} \le \|\varphi\|$ *for all* $\varphi \in B(G,\omega)$. *If* $\varphi \in P(G,\omega)$, *then* $|||\varphi||| = \|\varphi\| = \varphi(e)$.

Thus, we get

$$B(G,\overline{\sigma}\omega) \subset M_0A(G,\sigma,\omega) \subseteq MA(G,\sigma,\omega).$$

Further, using Proposition 2.8, we obtain

$$A(G,\sigma)B(G,\omega) \subset A(G,\sigma\omega) \quad \text{and} \quad B_r(G,\sigma)B(G,\omega) \subset B_r(G,\sigma\omega)$$

which are the twisted analogues of Eymard's result [30] saying that $A(G)$ and $B_r(G)$ are ideals in the Fourier–Stieltjes algebra $B(G)$.

Choosing $\omega = 1$ in Proposition 2.11, we get

Corollary 2.12. *We have*

$$B(G) \subset M_0A(G,\sigma) = MA_0(G) \subset MA(G,\sigma) = MA(G)$$

and $|||\varphi||| \le \|\varphi\|_{cb} \le \|\varphi\|$ *for all* $\varphi \in B(G)$. *If* $\varphi \in P(G)$, *then* $|||\varphi||| = \|\varphi\| = \varphi(e)$.

Remark 2.13. A result of C. Nebbia [54] says that G is amenable if and only if $B(G) = MA(G)$, in which case $\|\varphi\| = |||\varphi|||$ for all $\varphi \in B(G)$. On the other hand, M. Bożejko has shown [55] that G is amenable if and only if $B(G) = M_0A(G)$. Hence, $MA_0(G) = MA(G)$ whenever G is amenable. It seems unknown whether the converse is true.

Remark 2.14. If $\varphi \in \ell^2(G)$, then $\varphi \in A(G,\sigma) \subset B_r(G,\omega) \subset B(G,\omega)$, and we have $\|\varphi\| \le \|\varphi\|_2$. Proposition 2.11 gives that

$$\ell^2(G) \subset M_0A(G,\sigma,\sigma\omega) \subset MA(G,\sigma,\sigma\omega)$$

with $|||\varphi||| \le \|\varphi\|_{cb} \le \|\varphi\|_2$ for all $\varphi \in \ell^2(G)$. Since this holds for arbitrary σ and ω, we deduce that

$$\ell^2(G) \subset M_0A(G,\sigma,\omega) \subset MA(G,\sigma,\omega).$$

Remark 2.15. In an analogous way, one may consider multipliers on full twisted group C^*-algebras. In the untwisted case, G. Pisier [56, Corollary 8.7] shows that all multipliers from $C^*(G)$ into itself are completely bounded, and that the set of such multipliers coincides with $B(G)$. In the twisted case, it is not immediate that the same description holds. One can show without difficulty that any element of $B(G,\omega)$ induces a multiplier from $C^*(G,\sigma\omega)$ into $C^*(G,\sigma)$, which is completely bounded. However, it is not clear to us how to proceed to deduce that any multiplier, or at least that any completely bounded multiplier, is given in this way. The problem is that there is no substitute for the trivial representation of a group in the twisted setting.

2.3. Summation processes

Consider $\varphi \in MA(G,\sigma)$ and $x \in C_r^*(G,\sigma)$. The Fourier series of $M_\varphi(x)$, that is,

$$\sum_{g\in G} \varphi(g)\widehat{x}(g)\Lambda_\sigma(g),$$

converges to $M_\varphi(x)$ in $\|\cdot\|_2$-norm, but it does not necessarily converge in operator norm. We therefore introduce the space $MCF(G,\sigma)$ consisting of all functions $\varphi : G \to \mathbb{C}$ having the property that the series $\sum_{g\in G}\varphi(g)\widehat{x}(g)\Lambda_\sigma(g)$ converges in operator norm for all $x \in C_r^*(G,\sigma)$. For example, we have

$$\ell^2(G) \subset MCF(G,\sigma).$$

Indeed, if $\varphi \in \ell^2(G)$ and $x \in C_r^*(G,\sigma)$, then $\varphi\widehat{x} \in \ell^1(G)$, so the Fourier series of $M_\varphi(x)$ is absolutely convergent.

The first assertion of the following proposition is a simple consequence of the closed graph theorem.

Proposition 2.16. *We have* $MCF(G,\sigma) \subset MA(G,\sigma)$. *Moreover,*

$$MCF(G,\sigma) = \left\{\varphi \in MA(G,\sigma) \mid M_\varphi \text{ maps } C_r^*(G,\sigma) \text{ into } CF(G,\sigma)\right\}$$

and, for all $\varphi \in MCF(G,\sigma)$ *and all* $x \in C_r^*(G,\sigma)$, *we have*

$$M_\varphi(x) = \sum_{g\in G}\varphi(g)\widehat{x}(g)\Lambda_\sigma(g) \quad (\textit{convergence in operator norm}).$$

In analogy with [1, Lemma 1.7], we have:

Proposition 2.17. *Assume (G, σ) is κ-decaying and let $\psi \in \mathcal{L}^\infty_\kappa$, that is, $\psi : G \to \mathbb{C}$ satisfies $\|\psi\kappa\|_\infty < \infty$. Then $\psi \in MCF(G, \sigma)$.*

Indeed, for each $x \in C^*_r(G, \sigma)$, we have $\widehat{M_\psi(x)} = \psi\widehat{x} \in \mathcal{L}^2_\kappa$, so Theorem 2.1 gives that the series $\sum_{g\in G} \psi(g)\widehat{x}(g)\Lambda_\sigma(g)$ converges in operator norm.

Definition 2.18. A net $\{\varphi_\alpha\}$ in $MCF(G, \sigma)$ is called a *Fourier summing net* for (G, σ) when the identity map on $C^*_r(G, \sigma)$ is the point-norm limit of $\{M_{\varphi_\alpha}\}$.

Note that a Fourier summing net gives *a summation process for Fourier series*. Indeed, if $x \in C^*_r(G, \alpha)$, then the series $\sum_{g\in G} \varphi_\alpha(g)\widehat{x}(g)\Lambda_\sigma(g)$ is convergent in operator norm for every α; moreover, we have

$$\lim_\alpha \left\| \sum_{g\in G} \varphi_\alpha(g)\widehat{x}(g)\Lambda_\sigma(g) - x \right\| = 0.$$

Note also if a net $\{\varphi_\alpha\}$ in $MCF(G, \sigma)$ is *bounded*, that is, if $\sup_\alpha |\!|\!|\varphi_\alpha|\!|\!| < +\infty$, then it gives a Fourier summing net for (G, σ) if (and only if) $\{\varphi_\alpha\}$ converges pointwise to 1.

Remark 2.19. In general, it is not obvious how to construct Fourier summing nets. When G is finitely generated, letting L denote some word length function on G, it seems reasonable to consider the "Gaussian" net given by $\varphi_t = e^{-tL^2}$ for each $t > 0$. Then $\{\varphi_t\} \subset \ell^2(G)$ and $\varphi_t \to 1$ pointwise on G. However, it is not clear under which conditions the net $\{M_{\varphi_t}\}$ converges to the identity map on $C^*_r(G, \sigma)$ in the point-norm topology. It will if the net $\{\varphi_t\}$ is bounded, and this happens for instance when L can be chosen so that L^2 is negative definite, but this in turn will force G to be amenable.

On the other hand, it can be argued that if G is amenable and $\sigma = 1$, the existence of a Fourier summing net $\{\varphi_\alpha\}$ consisting of normalized functions in $\ell^2(G)$ satisfying $\|M_{\varphi_\alpha}\| \le 1$ can only be achieved when each φ_α is assumed to be positive definite.

The following result, which may be deduced from Corollary 2.12, goes back to G. Zeller-Meier [29].

Theorem 2.20. *Let G be amenable, and let $\{\varphi_\alpha\}$ be any net of normalized positive-definite functions in $\ell^2(G)$ converging pointwise to 1. Then $\{\varphi_\alpha\}$ is a bounded Fourier summing net for (G, σ), satisfying $|\!|\!|\varphi_\alpha|\!|\!| = 1$ for all α.*

Example 2.21. Assume that G is amenable and pick a Følner net $\{F_\alpha\}$ for G. Set

$$\varphi_\alpha(g) = \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|} \quad \text{for each } g \in G.$$

Then one easily checks that $\{\varphi_\alpha\} \subset C_c(G)$ satisfies all conditions in Theorem 2.20. Thus, for every $x \in C^*_r(G, \sigma)$, we have

$$\sum_{g\in F_\alpha\cdot F_\alpha^{-1}} \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|}\, \widehat{x}(g)\Lambda_\sigma(g) \underset{\alpha}{\longrightarrow} x \quad \text{(in operator norm).} \tag{3}$$

In particular, this shows that $C^*_r(G, \sigma)$ has the CPAP (cf. Theorem 1.1). When $G = \mathbb{Z}$ and $F_n = \{0, 1, \ldots, n-1\}$ for each $n \in \mathbb{N}$, Eq. (3) just gives the classical Fejér summation theorem. In the case where $G = \mathbb{Z}^2$, a similar result was proven by N. Weaver [57], using different methods.

Next, we take a look at Abel–Poisson (and Gauss) summation on noncommutative tori.

Theorem 2.22. *Let $G = \mathbb{Z}^n$ and $\sigma \in Z^2(\mathbb{Z}^n, \mathbb{T})$. For $p = 1, 2$, let $|\cdot|_p$ denote the usual p-norm on $\mathbb{Z}^n$. Let L denote either $|\cdot|_1$, $|\cdot|_2$ or $|\cdot|_2^2$. For $r \in (0, 1)$, set $\varphi_r = r^L$. Then $\{\varphi_r\}_{r\to 1^-}$ is a bounded Fourier summing net for $(\mathbb{Z}^n, \sigma)$. Hence,*

$$\lim_{r\to 1^-} \left\| \sum_{g\in\mathbb{Z}^n} r^{L(g)}\widehat{x}(g)\Lambda_\sigma(g) - x \right\| = 0$$

for all $x \in C^*_r(\mathbb{Z}^n, \sigma)$.

Remark 2.23. The existence of Abel–Poisson summation processes can be shown to exist for many classes of nonamenable groups. This is for example true for many countable groups with the Haagerup property: note first that if G is countable, then it has the Haagerup property if (and only if) it has a *Haagerup function* L, that is, a proper negative definite real-valued function L (that may in fact be chosen to be a length function); now, if G has subexponential H-growth w.r.t. to such Haagerup function L, then $\{r^L\}_{r\to 1^-}$ is a bounded Fourier summing net for (G, σ). The same conclusion also holds if G is a Gromov hyperbolic group and L is a word length function (see [4, Sect. 5]).

Having Example 2.21 in mind, the following definition is natural.

Definition 2.24. We say that (G, σ) has the *Fejér property* if there exists a Fourier summing net $\{\varphi_\alpha\}$ for (G, σ) that lies in $C_c(G)$. Moreover, we say that (G, σ) has the *bounded Fejér property* if, in addition, $\{\varphi_\alpha\}$ can be chosen to bounded; if $\sup_\alpha |||\varphi_\alpha||| = 1$, then we say that (G, σ) has the *metric Fejér property*. When $\sigma = 1$, we just suppress it from our notation.

Remark 2.25. It makes no difference if we replace $C_c(G)$ with $\ell^2(G)$ in Definition 2.24.

Theorem 2.26. *The following conditions are equivalent:*

(1) *G has the metric Fejér property (resp. the bounded Fejér property).*
(2) *(G, σ) has the metric Fejér property (resp. the bounded Fejér property).*
(3) *$C_r^*(G, \sigma)$ has the MAP (resp. the BAP).*

Proof. The equivalence between (1) and (2) is an immediate consequence of Proposition 2.9. It is clear that (2) $\Rightarrow$ (3) under both alternatives. The converse implication may be shown by adapting Brown–Ozawa's proof of [39, Theorem 12.3.10], where they show that G is weakly amenable whenever $C_r^*(G)$ has the CBAP. We sketch the argument, which is related to the one given in the second part of the proof of Proposition 2.9. Letting λ denote the left regular representation of G on $\ell^2(G)$, we recall that Φ_σ denotes the $*$-homomorphism from $C_r^*(G, \sigma)$ into $C_r^*(G, \sigma) \otimes C_r^*(G)$ that satisfies $\Phi_\sigma(\Lambda_\sigma(g)) = \Lambda_\sigma(g) \otimes \lambda(g)$ for each $g \in G$. Moreover, we let W be the isometry from $\ell^2(G)$ into $\ell^2(G) \otimes \ell^2(G)$ satisfying $W\delta_g = \delta_g \otimes \delta_g$ for each $g \in G$.

Assume that $\{T_\alpha\}$ is a net of finite rank operators in $B(C_r^*(G, \sigma))$ such that the identity map on $C_r^*(G, \sigma)$ is the point-norm limit of this net, and assume also that, for some $C > 0$, we have $\|T_\alpha\| \le C$ for every α. Then one checks that for each α, the function φ_α on G given by

$$\varphi_\alpha(g) = \tau\big(T_\alpha(\Lambda_\sigma(g))\Lambda_\sigma(g)^*\big) \quad \text{for each } g \in G$$

(where τ is the canonical tracial state of $C_r^*(G, \sigma)$) lies in $\ell^2(G)$. Hence, $\varphi_\alpha \in MCF(G, \sigma)$. Further, as $T_\alpha(\Lambda_\sigma(g)) \to \Lambda_\sigma(g)$ for each $g \in G$, it follows that φ_α converges pointwise to 1. Finally, one verifies that

$$M_{\varphi_\alpha}(x) = W^*\big(T_\alpha \otimes \mathrm{id}_{C_r^*(G)}\big)\Phi_\sigma(x)\, W$$

for each α and each $x \in C_r^*(G, \sigma)$. Hence, we get $|||\varphi_\alpha||| \le \|T_\alpha\| \le C$ for each α. So $\{\varphi_\alpha\}$ is a bounded Fourier summing net for (G, σ). $\square$

Example 2.21 shows that G has the metric Fejér property whenever G is amenable. On the other hand, by Haagerup's result in [1], $\mathbb{F}_n$ has the metric Fejér property for each $n \ge 2$. More generally, if G is countable with the Haagerup property and L is a Haagerup function on G such that G has subexponential H-growth (w.r.t. L), then G has the metric Fejér property. In particular, if G has a Haagerup length function such that G has the RD-property (w.r.t. L), then G has the metric Fejér property (see also [58] for the untwisted case).

For a group G, it is clear that the CMAP is stronger than the metric Fejér property, weak amenability is stronger than the bounded Fejér property, and the AP of Haagerup and Kraus is stronger than the Fejér property.

An interesting example is $SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$. It is known that it does not have the Haagerup property [2,39], but it has the AP of Haagerup and Kraus [59]. Hence, it has the Fejér property. On the other hand, it follows from Haagerup's work in [60] that it does not have the bounded Fejér property (hence that it is not weakly amenable, see [61] for another proof of this fact).

Here is a list of problems.

Problem 1. Find examples showing that the bounded Fejér property is strictly weaker than weak amenability. Find also examples showing that the metric Fejér property is strictly weaker than the CMAP.

Comment: To approach Problem 1, one may try to answer Problem 2 and/or Problem 3.

Problem 2. Let H be a (nontrivial and) finite group. The wreath product $H \wr \mathbb{F}_2$ is known to have the Haagerup property without having the CMAP [42]. Does $H \wr \mathbb{F}_2$ have the metric Fejér property?

Problem 3. The iterated wreath product $(H \wr \mathbb{F}_2) \wr \mathbb{Z}$ has the Haagerup property, but is not weakly amenable [42]. Does $(H \wr \mathbb{F}_2) \wr \mathbb{Z}$ have the bounded Fejér property?

Problem 4. For a countably infinite G with Kazhdan property (T), is it possible to have the metric Fejér property? More generally, does there always exist a Fourier summing net for such a group G (or for (G, σ))?

Problem 5. Does any exact group have the Fejér property? What about $SL(3, \mathbb{Z})$? If $SL(3, \mathbb{Z})$ does not have the Fejér property, does it have a Fourier summing net?

3. Fourier series in twisted crossed products

Throughout this section, $\Sigma = (A, G, \alpha, \sigma)$ denotes a unital, discrete, twisted C^*-dynamical system, as considered in Section 2.

3.1. Decay subspaces

Our exposition follows [6]. As in the scalar case, we start by defining

$$CF(\Sigma) = \left\{ x \in C_r^*(\Sigma) \,\middle|\, \sum_{g\in G} \widehat{x}(g)\Lambda_\Sigma(g) \text{ is convergent w.r.t. } \|\cdot\| \right\}.$$

It is easy to see that $x \in CF(\Sigma)$ when $\widehat{x} \in \ell^1(G, A)$, i.e., when $\|\widehat{x}\|_1 = \sum_{g\in G} \|\widehat{x}(g)\| < \infty$.

Next, we look at other suitable decay subspaces. Let $\kappa : G \to [1, +\infty)$. Set

$$\ell^2_\kappa(G, A) = \{\xi : G \to A \mid \|\xi\kappa\|_2 < \infty\}$$

where $\|\xi\|_2 = \big(\sum_{g\in G} \|\xi(g)\|^2\big)^{1/2}$, and equip this space with the norm $\|\xi\|_{2,\kappa} = \|\xi\kappa\|_2$. If G is κ-decaying, then one can show that $\ell^2_\kappa(G, A)$ is a decay subspace of A^Σ (w.r.t. $\|\cdot\|_{2,\kappa}$). Thus we obtain:

Proposition 3.1. *Suppose that G is κ-decaying. If $x \in C_r^*(\Sigma)$ and $\widehat{x} \in \ell^2_\kappa(G, A)$, then $x \in CF(\Sigma)$.*

Corollary 3.2. *Assume $L : G \to [0, +\infty)$ is a proper function.*

If G has polynomial H-growth (w.r.t. L), then there exists some $s > 0$ such that the Fourier series of $x \in C_r^*(\Sigma)$ is norm-convergent (to x) whenever

$$\sum_{g\in G} \|\widehat{x}(g)\|^2 \, (1 + L(g))^{2s} < \infty\,.$$

If G has subexponential H-growth (w.r.t. L), then the Fourier series of $x \in C_r^*(\Sigma)$ is norm-convergent (to x) whenever

$$\sum_{g\in G} \|\widehat{x}(g)\|^2 \, e^{tL(g)} < \infty$$

for some $t > 0$.

However, there are some indications that one had better consider the subspace of A^Σ given by

$$A^\Sigma_\kappa = \{\xi : G \to A \mid \xi\kappa \in A^\Sigma\}$$

equipped with the norm $\|\xi\|_{\alpha,\kappa} = \|\xi\kappa\|_\alpha$. The problem is then to find conditions ensuring that Σ is A^Σ_κ-decaying, meaning that for some $C > 0$, we have

$$\|\Lambda_\Sigma(f)\| \le C\, \|f\|_{\alpha,\kappa} \quad \text{for all } f \in C_c(G, A).$$

We have checked that such an inequality holds in the case where A is commutative, G is κ-decaying and α is trivial. It would be interesting to know if/when these conditions may be relaxed.

3.2. Multipliers and summation properties

This subsection is based on [5,6]. Let $T : G \times A \to A$ be a map that is linear in the second variable. For each $g \in G$, let $T_g : A \to A$ be the linear map given by

$$T_g(a) = T(g, a), \quad a \in A.$$

For each $f \in C_c(\Sigma)$, define $T \cdot f \in C_c(\Sigma)$ by

$$\big(T \cdot f\big)(g) = T_g(f(g)), \quad g \in G.$$

We say that T is a (reduced) *multiplier* of Σ whenever there exists a bounded linear map $M_T : C_r^*(\Sigma) \to C_r^*(\Sigma)$ such that $M_T\big(\Lambda_\Sigma(f)\big) = \Lambda_\Sigma(T \cdot f)$, that is

$$M_T\Big(\sum_{g\in G} f(g)\,\lambda_\Sigma(g)\Big) = \sum_{g\in G} T_g(f(g))\,\lambda_\Sigma(g)$$

for all $f \in C_c(\Sigma)$. We then set $|||T||| = \|M_T\|$.

We note that for any $x \in C_r^*(\Sigma)$, we have $\widehat{M_T(x)}(g) = T_g(\widehat{x}(g))$ for all $g \in G$.

We let $MA(\Sigma)$ denote the normed linear space (w.r.t. $|\!|\!| \cdot |\!|\!|$) consisting of all (reduced) multipliers of Σ and let $M_0A(\Sigma)$ denote the subspace of $MA(\Sigma)$ consisting of completely bounded multipliers, that is, of multipliers satisfying $\|M_T\|_{cb} < \infty$.

As an example, consider $\varphi : G \to \mathbb{C}$ and set $T^\varphi(g, a) = \varphi(g)a$. If $T^\varphi \in MA(\Sigma)$, then $\varphi \in MA(G)$. We do not know whether the converse holds. Anyhow, it can be shown that $T^\varphi \in M_0A(\Sigma)$ if and only if $\varphi \in M_0A(G)$, in which case we have

$$|\!|\!|T^\varphi|\!|\!| \le \|M_{T^\varphi}\|_{cb} \le \|M_\varphi\|_{cb}.$$

Also, if $\varphi \in P(G)$, then T_φ is completely positive and $|\!|\!|T_\varphi|\!|\!| = \varphi(e)$.

Our next result shows that multipliers arise as coefficients of equivariant representations.

Theorem 3.3. *Let (ρ, v) be an equivariant representation of Σ on a Hilbert A-module X and let $x, y \in X$. Define $T : G \times A \to A$ by*

$$T(g, a) = \big\langle x\,,\ \rho(a)\, v(g)\, y\big\rangle \quad \text{for } a \in A,\ g \in G.$$

Then $T \in M_0A(\Sigma)$, with $|\!|\!|T|\!|\!| \le \|M_T\|_{\rm cb} \le \|x\|\,\|y\|$. Moreover, if $x = y$, then M_T is completely positive and

$$|\!|\!|T|\!|\!| = \|M_T\|_{\rm cb} = \|x\|^2.$$

The proof relies on Fell's absorption principle (I). With the help of this result one may construct Fejér-like summation processes for Fourier series of elements in $C_r^*(\Sigma)$ in many cases.

Remark 3.4. Let T be as in the previous theorem.

(a) Let $Z = \{z \in X \mid \rho(a)\,z = z \cdot a \text{ for all } a \in A\}$ denote the *central part of* X. Then we have $T(g, a) = \big\langle x, v(g)y\big\rangle a$ if $y \in Z$, while $T(g, a) = a\big\langle x, v(g)y\big\rangle$ if $x \in Z$.

(b) Let w be a unitary representation of G on a Hilbert space $\mathcal{H}$ and pick $\xi, \eta \in \mathcal{H}$. Considering $(\rho, v) = (\ell \otimes \mathrm{id}, \alpha \otimes w)$ on $X = A \otimes \mathcal{H}$ and $x = 1 \otimes \xi$, $y = 1 \otimes \eta$ gives

$$T(g, a) = \big\langle 1\,,\ a\,\alpha_g(1)\big\rangle\,\langle \xi, w(g)\eta\rangle = \langle \xi, w(g)\eta\rangle\, a.$$

Using Fell's absorption principle (II), we can prove:

Theorem 3.5. *Let (ρ, v) be an equivariant representation of Σ on a Hilbert A-module X and let $\xi, \eta \in X^G$. Define $\check{T} : G \times A \to A$ by*

$$\check{T}(g, a) = \langle\, \xi\,,\ \check{\rho}(a)\, \check{v}(g)\, \eta\,\rangle\,.$$

Then $\check{T}$ is a completely bounded rf-multiplier *of Σ, that is, there exists a completely bounded map $\Phi_T : C_r^*(\Sigma) \to C^*(\Sigma)$ such that*

$$\Phi_T\big(\Lambda_\Sigma(f)\big) = T \cdot f$$

for all $f \in C_c(\Sigma)$, satisfying $\|\Phi_T\|_{cb} \le \|\xi\|\,\|\eta\|$.

We therefore say that Σ has the *weak approximation property* if there exists an equivariant representation (ρ, v) of Σ on some Hilbert A-module X and nets $\{\xi_i\}$, $\{\eta_i\}$ in X^G, both having finite support, satisfying

- there exists some $M > 0$ such that $\|\xi_i\| \cdot \|\eta_i\| \le M$ for all i;
- for all $g \in G$ and $a \in A$ we have

$$\lim_i \|\big\langle \xi_i\,,\ \check{\rho}(a)\check{v}(g)\eta_i\big\rangle - a\| = 0.$$

If one can choose $\eta_i = \xi_i$ for each i, Σ is said to have the *positive weak approximation property*. We add the qualifying word *central* if the η_i's and the ξ_i's can be chosen to lie in the central part of X^G. We also note that if (ρ, v) is chosen to be (ℓ, α), then one recovers *Exel's approximation property* for Σ.

From our previous theorem, one can deduce the following result, previously known when Σ has Exel's approximation property [62,63].

Theorem 3.6. *Assume that Σ has the weak approximation property. Then Σ is regular that is, the canonical $*$-homomorphism from $C^*(\Sigma)$ onto $C_r^*(\Sigma)$ is faithful. Moreover, $C^*(\Sigma) \simeq C_r^*(\Sigma)$ is nuclear if and only if A is nuclear.*

When A is abelian and σ is scalar-valued, the weak approximation property coincides with Exel's approximation property. In fact, we have:

Theorem 3.7. *Assume that A is abelian. Then the following conditions are equivalent:*

(a) *Σ has the approximation property.*

(b) *The action α is amenable in the sense of* [64] *(see also* [65]*).*

(c) *Σ has the central weak approximation property.*

If $\sigma \in Z^2(G, \mathbb{T})$, then these conditions are also equivalent to

(d) *Σ has the weak approximation property.*

Note that the equivalence of (a) and (b) is due to Exel and Ng [63] in the untwisted case.
The following permanence result illustrates that the weak approximation property is a useful concept.

Proposition 3.8. *Let B be a C^*-subalgebra containing the unit of A. Assume that σ takes values in $\mathcal{U}(B)$ and B is invariant under α_g for each $g \in G$; thus, the system $\Sigma_B := (B, G, \alpha_{|B}, \sigma)$ makes sense. If Σ has the weak approximation property and there exists an equivariant conditional expectation $E : A \to B$, then Σ_B has also the weak approximation property.*

Applying Proposition 3.8, we get:

Example 3.9. Assume G is exact and $\sigma \in Z^2(G, \mathbb{T})$. Let H be an amenable subgroup of G and let α be the action of G on $A = \ell^\infty(G)$ by left translations. Then it is well-known that α is amenable [65,39], so Σ has the approximation property. Let β denote the natural (left) action of G on $A_H = \ell^\infty(G/H)$. Then $\Sigma_H = (A_H, \beta, G, \sigma)$ has the (weak) approximation property and $C^*(\Sigma_H) \simeq C_r^*(\Sigma_H)$ is nuclear.

We now turn our attention to summation processes on $C_r^*(\Sigma)$. We set

$$MCF(\Sigma) = \big\{T \in MA(\Sigma) \mid M_T(x) \in CF(\Sigma) \text{ for all } x \in C_r^*(\Sigma)\big\}.$$

Thus, when $T \in MCF(\Sigma)$, the series

$$\sum_{g\in G} T_g(\widehat{x}(g))\lambda_\Sigma(g)$$

is norm-convergent (to $M_T(x)$) for all $x \in C_r^*(\Sigma)$.

Definition 3.10. A *Fourier summing net* for Σ is a net $\{T_i\} \subset MCF(\Sigma)$ such that

$$\lim_i \|M_{T_i}(x) - x\| = 0, \quad \text{for all } x \in C_r^*(\Sigma).$$

Such a net is called a *bounded Fourier summing net* if it also satisfies $\sup_i \|\!|T_i|\!\| < \infty$.

One should not expect that a Fourier summing net for Σ always exists. For example, it will follow from Proposition 3.13 that a Fourier summing net for $(\mathbb{C}, G, \mathrm{id}, \sigma)$ cannot exist if G is not exact.
In aim to study the ideal structure of $C_r^*(\Sigma)$, along lines initiated by Zeller-Meier [29] (when G is amenable) and by Exel [62,66] (when Σ has the approximation property or G is exact), the following notion turns out to be useful:

Definition 3.11. A Fourier summing net $\{T_i\}$ for Σ is said to *preserve the invariant ideals* of A if, for each α-invariant ideal $J \subset A$ and each $a \in J$, we have

$$T_i(g, a) \in J \quad \text{for all } i \text{ and all } g \in G.$$

We also need to introduce another concept, exactness of Σ, as introduced by A. Sierakowski [67] when σ is trivial. We first recall some notation.
For an α-invariant ideal $J \subset A$, set $\Sigma/J = (A/J, G, \dot{\alpha}, \dot{\sigma})$, where $(\dot{\alpha}, \dot{\sigma})$ denotes the twisted action of G on A/J naturally associated with (α, σ). We will let $\widetilde{J}$ denote the kernel of the canonical $*$-homomorphism from $C_r^*(\Sigma)$ onto $C_r^*(\Sigma/J)$. It is not difficult to show that

$$\widetilde{J} = \big\{x \in C_r^*(\Sigma) \mid \widehat{x}(g) \in J \text{ for all } g \in G\big\}.$$

Moreover, if $\langle J\rangle$ denotes the ideal generated by J in $C_r^*(\Sigma)$, then we have

$$E(\langle J\rangle) = J \quad \text{and} \quad \langle J\rangle \subset \widetilde{J}.$$

Definition 3.12. Σ is called *exact* when $\langle J\rangle = \widetilde{J}$ for every α-invariant ideal J of A.

As shown by Exel [68], Σ is exact whenever G is exact. We also have:

Proposition 3.13. *Assume there exists a Fourier summing net $\{T_i\}$ for Σ that preserves the invariant ideals of A. Then Σ is exact, and $C_r^*(\Sigma)$ is exact if and only if A is exact.*

An ideal $\mathcal{J}$ of $C_r^*(\Sigma)$ is called *induced* whenever it is generated by an α-invariant ideal of A. It is called *E-invariant* whenever $E(\mathcal{J}) \subset \mathcal{J}$ or, equivalently, whenever $E(\mathcal{J}) = \mathcal{J} \cap A$.

Proposition 3.14. *Assume that G is exact or that there exists a Fourier summing net for Σ that preserves the invariant ideals of A. Then the map $J \mapsto \langle J \rangle$ is a bijection between the set of all invariant ideals of A and the set of all E-invariant ideals of $C_r^*(\Sigma)$.*

If $T \in MA(\Sigma)$, we say that T has *finite G-support* whenever $T_g \neq 0$ for all but finitely many g's in G.

Definition 3.15. Σ is said to have the *Fejér property* if there exists a Fourier summing net $\{T_i\}$ for Σ such that each T_i has finite G-support. If, in addition, such a net can be chosen to be bounded, then Σ is said to have the *bounded Fejér property*.

It follows from the work of Zeller-Meier [29] that Σ has the bounded Fejér property whenever G is amenable and σ is central. (See also [69] for a short proof when $G = \mathbb{Z}$ and $\sigma = 1$.) Using a Følner net for G, as in Example 2.21, one easily deduces that this also holds for any σ. More generally, we have:

Theorem 3.16. *Assume that G is weakly amenable, or that Σ has the weak approximation property. Then Σ has the bounded Fejér property.*

Remark 3.17. We expect that summation processes also make sense in many C^*-algebras that have a crossed product like structure. For instance, let $\mathcal{O}_n$ denote the Cuntz algebra generated by isometries $S_1, \ldots, S_n$, with mutually orthogonal ranges summing up to 1. Then the following hold (see [70]):

(1) Each element x in the dense $*$-subalgebra generated by $S_1, \ldots, S_n$ has a representation

$$x = \sum_{i=1}^{N} (S_1^*)^i x_{-i} + x_0 + \sum_{i=1}^{N} x_i S_1^i$$

where, for each i, $x_i \in \mathcal{F}^n$, the canonical UHF subalgebra of type n^∞. This representation is unique if one requires, for each $i \geq 1$, that $x_i = x_i P_i$ and $x_{-i} = P_i x_{-i}$, where $P_i = S_1^i (S_1^i)^*$ is the final projection of S_1^i.

(2) The linear maps E_i defined by $E_i(x) = x_i$ extend to continuous, contractive linear maps from $\mathcal{O}_n$ to $\mathcal{F}^n$.

(3) The generalized Cesáro sums

$$\sigma_N(x) = \sum_{k=1}^{N} \Big(1 - \frac{|k|}{N}\Big) (S_1^*)^k E_{-k}(x) + \sum_{k=0}^{N} \Big(1 - \frac{|k|}{N}\Big) E_k(x) S_1^k$$

converge to x in norm when $N \to \infty$.

3.3. The Fourier–Stieltjes algebra of a C^*-dynamical system

In analogy with the Fourier–Stieltjes algebra $B(G)$ of a group G, that may be described as the algebra of matrix coefficients of unitary representations of G, one may associate to a unital, discrete, twisted C^*-dynamical system $\Sigma = (A, G, \alpha, \sigma)$ its Fourier–Stieltjes algebra $B(\Sigma)$. We give here a short introduction to this subject, based on [8].

Following M. Walter [71,72], we first recall that if B is a C^*-algebra, one can consider its dual algebra $\mathcal{D}(B)$ consisting of all completely bounded maps from B into itself, the product being given by composition. Equipped with the completely bounded norm, $\mathcal{D}(B)$ becomes a Banach algebra, having an isometric conjugation $\Phi \to \overline{\Phi}$ given by

$$\overline{\Phi}(b) = (\Phi(b^*))^*$$

for each $b \in B$ (cf. [71, Proposition 1]).

Now, we may organize $M_0A(\Sigma)$ as a unital algebra with a conjugation by setting

$$\begin{aligned}
(T + T')(g, a) &:= T(g, a) + T'(g, a) \\
(\lambda T)(g, a) &:= \lambda T(g, a) \\
(T \times T')(g, a) &= T(g, T'(g, a)) \\
\overline{T}(g, a) &:= \sigma(g, g^{-1})^* \, \alpha_g\Big(T\big(g^{-1}, \alpha_g^{-1}(a^* \sigma(g, g^{-1})^*)\big)\Big)^* \\
I(g, a) &:= a
\end{aligned}$$

for all $g \in G$, $a \in A$ and $\lambda \in \mathbb{C}$. One then checks that $T \mapsto M_T$ is an injective unital algebra homomorphism from $M_0A(\Sigma)$ into $\mathcal{D}(C_r^*(\Sigma))$ that respects conjugation.

Setting $\|T\|_{cb} := \|M_T\|_{cb}$, we get a norm on $M_0A(\Sigma)$. As $\big\{M_T \mid T \in M_0A(\Sigma)\big\}$ is closed in $\mathcal{D}(C_r^*(\Sigma))$, it follows that $M_0A(\Sigma)$ becomes a Banach algebra with an isometric conjugation.

We recall that if (ρ, v) is an equivariant representation of Σ on a Hilbert A-module X and $x, y \in X$, then we have $T_{\rho,v,x,y} \in M_0A(\Sigma)$, where

$$T_{\rho,v,x,y}(g, a) = \big\langle x \,,\ \rho(a)\, v(g)\, y\big\rangle$$

for all $g \in G$ and $a \in A$. We define $B(\Sigma) \subset M_0A(\Sigma)$ to be the set of all multipliers of Σ of the form given above, thinking of it as the A-valued coefficient functions associated with equivariant representations of Σ. Exploiting the natural construction of direct sums and tensor products of equivariant representations, one verifies that $B(\Sigma)$ is a subalgebra of $M_0A(\Sigma)$, which is closed under conjugation (as $\overline{T}_{\rho,v,x,y} = T_{\rho,v,y,x}$). Using (b) in Remark 3.4, one sees that $B(G)$ embeds in $B(\Sigma)$. Elements of $B(\Sigma)$ may also be shown to give rise to completely bounded linear maps on $C^*(\Sigma)$.

An alternative description of $B(\Sigma)$ may be given using C^*-correspondences over $C_r^*(\Sigma)$.[6] Let Y be a C^*-correspondence over $C_r^*(\Sigma)$ and let $y, z \in Y$. Define $T^Y_{y,z} : G \times A \to A$ by $T^Y_{y,z}(g, a) = E\big(\langle y\,,\ [a\lambda(g)] \cdot z\rangle\, \lambda(g)^*\big)$ for all $g \in G$ and $a \in A$. Then we have

$$B(\Sigma) = \left\{ T^Y_{y,z} \mid Y \text{ is a C}^*\text{-correspondence over } C_r^*(\Sigma) \text{ and } y, z \in Y \right\}.$$

A similar result holds if one replaces $C_r^*(\Sigma)$ with $C^*(\Sigma)$.

Now, consider a function $\varphi : G \to A$. Following [64], φ is called AD-*positive definite* (w.r.t. Σ) if, for any $s_1, \ldots, s_n \in G$, the matrix $[\alpha_{s_i}\varphi(s_i^{-1}s_j)]$ is positive in $M_n(A)$. Recall that $T^\varphi : G \times A \to A$ is defined by $T^\varphi(g, a) = \varphi(g)\, a$ for $g \in G$ and $a \in A$. It can be shown that $T^\varphi \in M_0A(\Sigma)$ with M_{T^φ} being completely positive if and only if φ takes its values in $Z(A)$ and is AD-positive definite (w.r.t. Σ); moreover, M_{T^φ} is then an A-bimodule map of $C_r^*(\Sigma)$. (See [73] for a related result when $\sigma = 1$.)

More generally, one may introduce a notion of positive definiteness (w.r.t. Σ) for any map $T : G \times A \to$ that is linear in the second variable. This property is satisfied when $T = T_{\rho,v,x,x}$ for some equivariant representation (ρ, v) of Σ on X and some $x \in X$. Conversely, a Gelfand–Raikov type result holds: if T is positive definite (w.r.t. Σ), then it may be written as $T_{\rho,v,x,x}$ for some (ρ, v) and x as above. This allows us to describe $B(\Sigma)$ as the span of positive definite maps (w.r.t. Σ). We can also show that T is positive definite (w.r.t. Σ) if and only if $T \in M_0A(\Sigma)$ with M_T being completely positive. Finally, we propose a definition of amenability for Σ, by requiring the existence of a uniformly bounded net $\{T^i\}$ of finitely supported Σ-positive definite maps satisfying $\lim_i \|T^i(g, a) - a\| = 0$ for every $g \in G$ and $a \in A$, and show that this implies the regularity of Σ.

Many interesting questions arise. For example, when is $B(\Sigma) = M_0A(\Sigma)$? Given another system Σ', when is $B(\Sigma')$ isomorphic to $B(\Sigma)$? Does regularity of Σ imply its amenability (in the above sense)?

3.4. On C^*-simple groups and maximal ideals in certain crossed products

The topic of C^*-simple groups started with the seminal work of R. Powers [74], who showed that $C_r^*(\mathbb{F}_2)$ is simple with a unique tracial state. We recall that a discrete group G is called *C^*-simple* when $C_r^*(G)$ is simple, and it is said to have *the unique trace property* if $C_r^*(G)$ has a unique tracial state (the canonical one). For basic examples and properties, we refer to [75]. The problem whether C^*-simplicity is equivalent to the unique trace property remained elusive for quite long time. However, as shown recently by E. Breuillard, M. Kalantar, M. Kennedy and N. Ozawa in [76], G has the unique trace property if and only if its amenable radical is trivial, and it follows easily from this that every C^*-simple group has the unique trace property. A quite elementary proof of this fact has later been obtained by M. Kennedy [77] and, independently, by U. Haagerup [78], based on characterizations of C^*-simplicity and of the unique trace property in terms of Powers type averaging properties of $C_r^*(G)$. On the other hand, A. Le Boudec [79] has produced examples of non C^*-simple groups with the unique trace property.

The search for new C^*-simple groups continues to be an active area. A.Yu. Olshanskii and D.V. Osin have produced examples of C^*-simple groups without free subgroups [80]; these include the free Burnside groups $B(m, n)$ with $m \geq 2$ and n odd and large enough. Using their characterization of C^*-simplicity as being equivalent to the existence of a topologically free boundary action, Kalantar and Kennedy have shown [81] that the torsion-free Tarski monsters are C^*-simple. This is also true for other Tarski monster groups [76]. An intriguing question is whether the Thompson group T is C^*-simple or not. As shown by Haagerup and Olesen [82] (see also [76]), the C^*-simplicity of T would imply the non-amenability of the Thompson group F, thus solving a major open problem. On the other hand, T is known to have the unique trace property. So if T is not C^*-simple, it would provide another example of a group similar to those found by Le Boudec. An interesting class of groups with the unique trace property is the class of groups with the property BP introduced in [83]. To our knowledge, it is unknown whether such groups are necessarily C^*-simple. One may of course also study simplicity and uniqueness of the trace for twisted reduced group C^*-algebras, and quite a lot is known. We refer to [84,85] for an overview on this theme, as well as some new results.

In another direction, generalizing a result of P. de la Harpe and G. Skandalis in [86] (see also [87]), it is shown in [76] that if a C^*-simple group acts on a unital C^*-algebra in a minimal way (the only invariant ideals are the trivial ones), then the associated reduced C^*-crossed product is simple. In the case of a non-minimal system, the lattice of ideals of the reduced crossed product may be quite complicated. In a first step, one may try to describe the maximal ideals. In [7], we consider the case where the group lies in the class $\mathcal{P}$ consisting of all PH-groups introduced by Promislow [88] and of groups satisfying the "combinatorial" property (P_{com}) of Bekka–Cowling–de la Harpe [89]. See the diagram below for an illustration of the logical relationships between the various conditions appearing in the literature on this topic. (The unexplained terminology may be found in the references cited above, also noticing that OO and PT denote the classes of C^*-simple groups investigated in [80] and [90], respectively.)

[6] If Y is a (right) Hilbert B-module and ϕ is a representation of B on Y, the triple (Y, B, ϕ) is often called a C^*-correspondence over B, or sometimes a right Hilbert B-bimodule; one usually sets $b \cdot y = \phi(b)\, y$ for $b \in B$ and $y \in Y$.

Let $\Sigma = (A, G, \alpha, \sigma)$ denote a unital, discrete, twisted C^*-dynamical system, and let $\mathcal{U}_\Sigma$ denote the unitaries in $C_r^*(\Sigma)$. Our approach in [7] relies on the following notion.

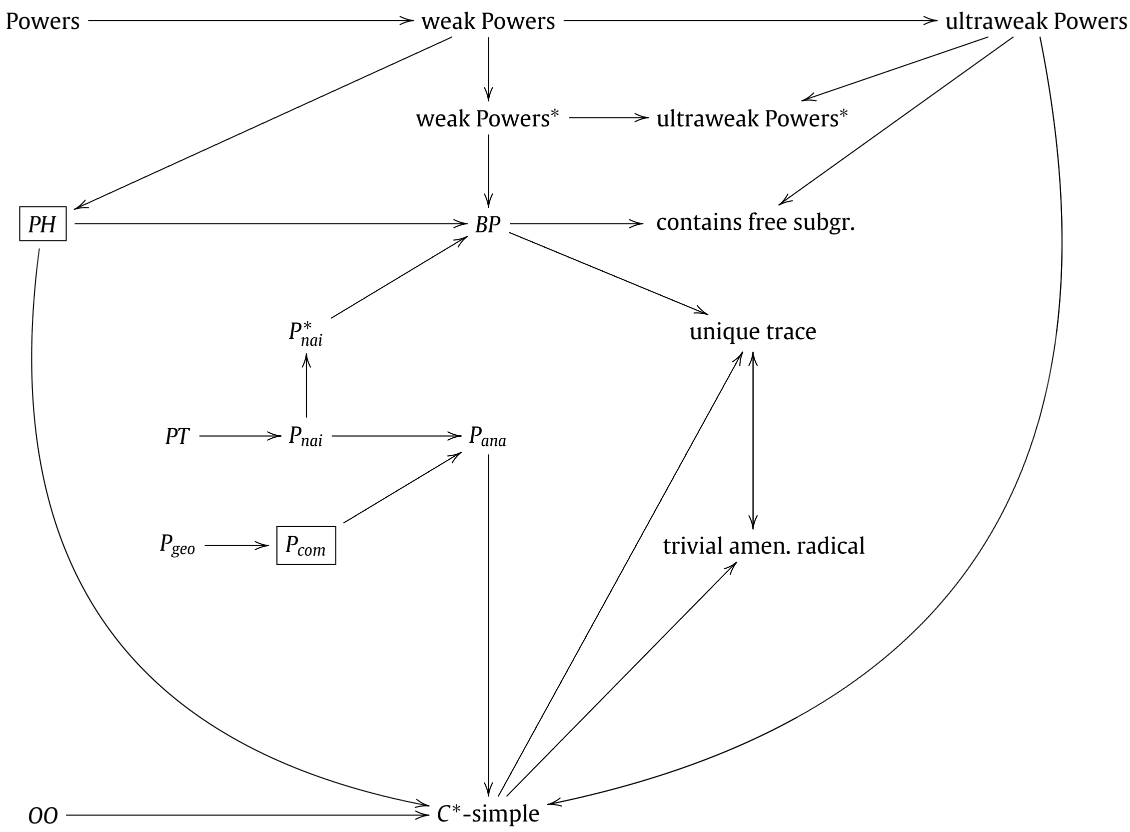
Definition 3.18. We say that Σ has *property (DP)* if $0 \in \overline{\mathrm{co}}\{vyv^* \mid v \in \mathcal{U}_\Sigma\}$ for every $y \in C_r^*(\Sigma)$ satisfying $y = y^*$ and $E(y) = 0$.

It is not difficult to check that Σ has property (DP) whenever $C_r^*(\Sigma)$ has the Dixmier property [91]. However, the converse implication is not true in general.

Theorem 3.19. *The following assertions hold:*

(1) *If G belongs to $\mathcal{P}$, then Σ has property (DP).*
(2) *If Σ has property (DP), then the restriction map gives a bijection between the set of all tracial states of $C_r^*(\Sigma)$ and the set of invariant tracial states of A.*
(3) *If the action is minimal and Σ has property (DP), then $C_r^*(\Sigma)$ is simple.*
(4) *If Σ is exact and has property (DP), then the map $\mathcal{J} \to \mathcal{J} \cap A$ gives a bijection between the set of maximal ideals of $C_r^*(\Sigma)$ and the set of maximal invariant ideals of A.*

After the first draft of this paper had been submitted for publication, R.S. Bryder and M. Kennedy have shown in [92] that if G is C^*-simple, then $C_r^*(\Sigma)$ satisfies a certain Powers type averaging property, related to property (DP). Using this, they are able to show that the bijection in item (4) above actually holds for any C^*-simple group. They also show that the bijection in item (2) above holds for any group having the unique trace property (thus in particular for any C^*-simple group). Finally, a twisted crossed product by an action of a C^*-simple group is simple if and only if the action is minimal.



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