DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2013

Website: www.aimSciences.org
pp. **771–780**

ON THE UNIQUENESS OF BLOW-UP SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract. This paper contains new uniqueness results of the boundary blow-up viscosity solutions of second order elliptic equations, generalizing a well known result of Marcus-Veron for the Laplace operator.

1. **Introduction and main results.** This paper is concerned with the uniqueness of positive solutions of fully nonlinear second order elliptic equations

$$F(x, u, Du, D^2u) = f(x) \tag{1}$$

in a domain Ω satisfying the boundary blow-up condition

$$u(x) \to \infty \quad \text{as} \quad \operatorname{dist}(x, \partial\Omega) \to 0. \tag{2}$$

The solutions of this boundary *blow-up* value problem, or *large solutions*, are intended in *viscosity* sense; see Section 2 for definitions.

The fully nonlinear second order operator F will satisfy the uniform ellipticity structure condition

$$\mathcal{P}^-_{\lambda,\Lambda}(X-Y)-\gamma|\eta-\xi| \le F(x,t,\eta,X)-F(x,t,\xi,Y) \le \mathcal{P}^+_{\lambda,\Lambda}(X-Y)+\gamma|\eta-\xi| \tag{3}$$

for all x, t, ξ, η, X, Y and the superlinear monotonicity assumption

$$F(x,u,\xi,X) - F(x,v,\xi,X) \le -\delta(u-v)^s \quad \text{if } v<u, \text{ where } s>1 \text{ and } \delta>0 \tag{4}$$

for all x, ξ, X. Moreover

$$F(x, 0, 0, 0) = 0 \quad \text{for all } x \in \Omega \tag{5}$$

including a non-zero additive term of this kind in $f(x)$. In this case, setting $G(x, u, \xi, X) = F(x, u, \xi, X) - F(x, 0, 0, 0)$, (3) and (4) would be satisfied with G in the place of F. In the sequel we will say that F satisfies the structure conditions (SC) if (3) $\div$ (5) hold true.

In a previous paper [12] the first and the third author proved, together with existence and uniqueness of entire solutions, the existence of boundary blow-up solutions under various assumptions about the dependence of F on x. Our paper was a generalization of Esteban-Quaas-Felmer [11], based on interior estimates which provide the local uniform convergence of approximating solutions.

The issue of uniqueness was considered by Dong-Kim-Safonov in [9], to which we refer for a nice history of the problem. In that paper uniqueness of classical and

2010 *Mathematics Subject Classification.* Primary: 35J25, 35B50; Secondary: 35D40.
Key words and phrases. Elliptic equations, fully nonlinear equations, maximum principle, viscosity solutions, blow-up.
The first author is supported by FARB 2009-10 University of Salerno.

L^p-strong boundary blow-up solutions is proved for semilinear equations $Lu = u^s$, where L is a second order uniformly elliptic operator, in a domain Ω satisfying "the uniform exterior ball condition". The authors notice that a similar result can be obtained when L is replaced by a fully nonlinear operator F of Bellman type.

Here, we consider a different regularity condition on the boundary of domains Ω, called "local graph property", see Definition 2.5 below, and introduced by Marcus-Veron [15] to show the uniqueness of blow-up solutions of equation $\Delta u = u^s$. Moreover we investigate the problem in the larger class of viscosity solutions, although our method, which does not use informations about the boundary behavior of solutions, works for F independent of x. In this respect, solutions of Bellman type equations with constant coefficients are in fact classical solutions and therefore are covered by [9], under the uniform exterior ball condition. This is not, generally, the case of Isaacs type equations with constant coefficients, which are instead included in the present paper, see Remark 1 and the examples just below. Let $1 \le \varphi(k) \to 1$ as $k \to 1^+$. In the sequel we need the following additional assumption on F:

$$\liminf_{k\to 1^+} \Big(F(x,t,\xi,X) - \varphi(k) F(x, \tfrac{t}{k}, \tfrac{\xi}{k}, \tfrac{X}{k}) \Big) \le 0 \tag{6}$$

uniformly with respect to $(x,t,\xi,X) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^n$.

Theorem 1.1. *Let Ω be a domain of class C_{gr} and $f \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$. If $F = F(u, Du, D^2u)$ is an operator (independent of x) satisfying the structure conditions (SC) and (6), then problem (1), (2) has at most one non-negative solution.*

Remark 1. Condition (6) on F is satisfied with $\varphi(k) = k^\alpha$ in the case of operators

$$F = F_1(\xi, X) - |t|^{s-1} t$$

such that F_1 is positively homogeneous of degree $\alpha \in (0, s]$, $s > 0$, i.e. $F_1(k\xi, kX) = k^\alpha F_1(\xi, X)$ for all $k > 0$ and all $(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n$. Observe indeed that

$$F = F_1(\xi, X) - t^s = k^\alpha F_1(\tfrac{\xi}{k}, \tfrac{X}{k}) - t^s \le \ k^\alpha F_1(\tfrac{\xi}{k}, \tfrac{X}{k}) - k^\alpha \tfrac{t^s}{k^s} = k^\alpha F(\tfrac{t}{k}, \tfrac{\xi}{k}, \tfrac{X}{k}).$$

when $t \ge 0$ and $k \ge 1$. As one can see in Remark 2 below, $f \le 0$ is a sufficient condition to have non-negative solutions.

By Theorem 1.1 and Remark 1 we have uniqueness of non-negative blow-up solutions for the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) + \gamma |Du| - |u|^{s-1} u = f(x)$$

and more generally for Bellman and Isaacs type equations like

$$\sup_j \inf_i \{ Tr(A^{ij} D^2 u) + \langle b_{ij}, Du \rangle \} - |u|^{s-1} u = f(x).$$

with $A^{ij} \in \mathcal{S}^n$ such that $\lambda I \le A^{ij} \le \Lambda I$ and $|b_{ij}| \le \gamma \in \mathbb{R}_+$.

Following Marcus-Veron [16], we also consider more general operators, acting on the convex cone of non-negative continuous functions, which are obtained adding a "positive semilinearity", namely

$$F = F_1(\xi, X) + ct^\alpha - t^s \tag{7}$$

where c is a positive constant, $0 < \alpha < s$ and F_1 is positively homogeneous of degree $\beta \in [\alpha, s]$, see Remark 1.

Theorem 1.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ of class C_{gr}. Let F be an uniformly elliptic operator satisfying (3) and (5) of type (7), with $c \in \mathbb{R}$ and F_1 positively homogeneous of degree $\beta \in [\alpha, s]$. Let also $f \in C(\Omega) \cap L^p(\Omega)$. If $f \leq 0$, then problem*

$$F_1(Du, D^2u)+c|u|^{\alpha-1}u-|u|^{s-1}u = f(x) \text{ in } \Omega, \quad u(x) \to \infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0. \quad (8)$$

has at most one positive solution.

Note that the case $c \leq 0$ is already provided by Theorem 1.1.

2. **Preliminaries.** Let Ω be a domain (open connected set) of $\mathbb{R}^n$. By $\mathcal{S}^n$ we denote the set of $n \times n$ real symmetric matrices equipped with the usual partial order: $X \geq Y$ means $\langle X\xi, \xi\rangle \geq \langle Y\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^n$, where $\langle \cdot, \cdot\rangle$ is the Euclidean inner product.

An operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \mapsto \mathbb{R}$ is *degenerate elliptic* if it is nondecreasing in its matrix argument, namely $F(x, u, \xi, X) \geq F(x, u, \xi, Y)$ for $X \geq Y$, and *uniformly elliptic* if there exist two constants $\Lambda \geq \lambda > 0$, called ellipticity constants, such that

$$\mathcal{P}^-_{\lambda,\Lambda}(X - Y) \leq F(x, u, \xi, X) - F(x, u, \xi, Y) \leq \mathcal{P}^+_{\lambda,\Lambda}(X - Y) \quad (9)$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$.

Here $\mathcal{P}^{\pm}_{\lambda,\Lambda}$ are the *Pucci's extremal operators*, defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = \sup_{\lambda I \leq A \leq \Lambda I} Tr(AX) \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(X) = \inf_{\lambda I \leq A \leq \Lambda I} Tr(AX), \quad (10)$$

where $Tr(\cdot)$ is the trace of a matrix. From (10) it follows the subadditivity, resp. superadditivity, of $\mathcal{P}^+_{\lambda,\Lambda}$, resp. $\mathcal{P}^-_{\lambda,\Lambda}$, and the equality $\mathcal{P}^+_{\lambda,\Lambda}(X) = -\mathcal{P}^-_{\lambda,\Lambda}(-X)$.

A second order partial differential equation

$$F(x, u, Du, D^2u) = f(x) \quad (11)$$

is said to be *fully nonlinear uniformly elliptic* when the condition (9) holds. We will assume the continuity of the real valued mappings F and f.

Definition 2.1. Given a function $u : \Omega \mapsto \mathbb{R}$, the second order *superjet* $J^{2,+}u(x)$, respectively *subjet* $J^{2,-}u(x)$, of u at $x \in \Omega$ is the convex set (possibly empty) of all pairs $(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n$ such that

$$u(y) \leq u(x) + \langle \xi, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \quad \text{as } y \to x,$$

$$\textit{resp. } u(y) \geq u(x) + \langle \xi, y - x\rangle + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \quad \text{as } y \to x.$$

Note that $J^{2,+}u(x) = -J^{2,-}(-u)(x)$. If u is twice differentiable at $x \in \Omega$ then

$$J^{2,+}u(x) = \left\{(Du(x), X) : \ X \geq D^2u(x)\right\},$$
$$J^{2,-}u(x) = \left\{(Du(x), X) : \ X \leq D^2u(x)\right\}.$$

Definition 2.2. An upper semicontinuous function $u : \Omega \mapsto \mathbb{R}$ is a *viscosity subsolution* of (11), for short $F[u] \geq f$, if

$$F(x, u(x), \xi, X) \geq f(x) \quad \forall x \in \Omega \text{ and } (\xi, X) \in J^{2,+}u(x).$$

Similarly a lower semicontinuous function $u : \Omega \mapsto \mathbb{R}$ is a *viscosity supersolution* of (11), for short $F[v] \leq f$, if

$$F(x, u(x), \xi, X) \leq f(x) \quad \forall x \in \Omega \text{ and } (\xi, X) \in J^{2,-}u(x).$$

Finally $u \in C(\Omega)$ is a *viscosity solution* of (11), for short $F[u] = f$ or also $F = f$, if it is simultaneously a viscosity sub and supersolution.

It is evident that a classical solution of (11), i.e. a $C^2(\Omega)$ function satisfying pointwise the equation, is also a viscosity solution. Conversely a twice differentiable viscosity solution is a classical one. We refer to [2], [5], [13], [14] for major details on viscosity solutions of nonlinear equations.

Lemma 2.3. *Let $\tau \in [0,1]$ and F be an operator satisfying (3) and (4). If u is a supersolution of (11) and $w \geq 0$ is a solution of $\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) + \gamma|Dw| - \tau\delta w^s = 0$, then the function $u + w$ is in turn a supersolution of (11).*

Proof. By regularity results for convex operators (see [1] and Sections 6.2, 8.1 in [2]) we have $w \in C^{2,a}$, with $0 < a < 1$, so w is a classical solution.

Let $(\xi, X) \in J^{2,-}(u + w)(x)$, then $(\xi - Dw, X - D^2 w) \in J^{2,-}u(x)$ and

$$\begin{aligned} F(x, u(x) + w(x), \xi, X) &\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2 w(x)) + \gamma|Dw(x)| \\ &\quad + F(x, u(x) + w(x), \xi - Dw(x), X - D^2 w(x)) \\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(D^2 w(x)) + \gamma|Dw(x)| - \delta w^s \\ &\quad + F(x, u(x), \xi - Dw(x), X - D^2 w(x)) \leq f(x). \qquad \square \end{aligned}$$

In the sequel $p_0 \in (\frac{n}{2}, n)$ will be the exponent of Escauriaza [10] (see also Crandall-Swiech [7]) in order that the Alexandroff-Bakelman-Pucci Maximum Principle holds true with $p > p_0$ in the form (GMP)

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C d^{2-\frac{n}{p}} \|f^-\|_{L^p(\Omega)} \tag{12}$$

for solutions $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ of the maximal equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 u) + \gamma|Du| \geq f,$$

where $d = \mathrm{diam}(\Omega) < +\infty$ and C a positive constant depending on $n, \lambda, \Lambda, p, \gamma d$. This result can be generalized to viscosity solutions, see Swiech [20], Lemma 1.4.

We will use the following Generalized Comparison Principle (GCP), which is deduced by the Maximum Principle of [12], Lemma 3.2.

Lemma 2.4. *Let Ω be a domain of $\mathbb{R}^n$ and F be an uniformly elliptic operator satifying (SC) and independent of x. Suppose that u and v are continuous solutions, resp., of $F[u] \geq f$ and $F[v] \leq g$ in viscosity sense, where $f, g \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$ for some $p > p_0$. Then for any $y \in \Omega$ and any ball B_R centered at y we have*

$$(u-v)^+(y) \leq \limsup_{x \to \partial\Omega \cap B_R} (u-v)^+ + C_0 \left(\frac{1+\gamma R}{R^2}\right)^{\frac{1}{s-1}} + C_1 \|(f-g)^-\|_{L^p(\Omega \cap B_R)}, \tag{13}$$

where $C_0 = C_0(n, \Lambda, s, \delta)$ and $C_1 = C_1(n, p, \lambda, \Lambda, \gamma, R)$ are positive constants.Here, if $\partial\Omega \cap B_R = \emptyset$, one reads $\limsup_{x \to \partial\Omega \cap B_R} (u - v)^+ = 0$.

Proof. Since F is independent of x, by means of the Jensen's approximations, we may use the structure conditions (SC) just as for smooth functions, see e.g. [4]-[8], to deduce that $w = (u - v)^+$ is a viscosity subsolution of

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2 w) + \gamma|Dw| - \delta w^s = -(f(x) - g(x))^- \quad in\ \Omega.$$

From this, reasoning as in Lemma 3.2 of [12] and using GMP (12), for any ball B_r centered at y of radius $r < R$ we get

$$\sup_{\Omega\cap B_r} w \le \limsup_{x\to\partial\Omega\cap B_R} w + C_0\left(\frac{R(1+\gamma R)^{\frac{1}{2}}}{R^2-r^2}\right)^{\frac{2}{s-1}} + C_1\|(f-g)^-\|_{L^p(\Omega\cap B_R)}, \quad (14)$$

from which (13) follows, letting $r \to 0^+$. □

Remark 2. If $f \ge g$, letting $R \to \infty$, from Lemma 2.4 we obtain the Comparison Principle (CP):

$$(u-v)^+(y) \le \limsup_{x\to\partial\Omega} (u-v)^+. \quad (15)$$

Note also that Ω is possibly unbounded in Lemma 2.4. Nonetheless no assumption is made on the growth of u and v at infinity.

Definition 2.5. (Marcus-Veron) A domain Ω satisfies the *local graph property* at $P \in \partial\Omega$ if there exist a neighborhood Q_P and a function $\psi \in C(\mathbb{R}^{n-1})$ such that

$$Q_P \cap \Omega = \{x \in Q_P : y_n < \psi(y')\}$$

in a coordinate system $y \equiv (y', y_n)$ obtained by rotation from $x \equiv (x', x_n)$.

Remark 3. We may assume that Q_P is a spherical cylinder

$$Q_P = \{x \in \mathbb{R}^n\ ,\ |y'| < \rho,\ |y_n| < \sigma\} \quad (16)$$

centered at P, of radius $\rho > 0$ and finite height $2\sigma > 0$, as well as $|\psi(y')| < \sigma$ in $\overline{Q}_P$ so that

$$\overline{Q}_P \cap \Omega = \{x \in \mathbb{R}^n\ ,\ |y'| \le \rho,\ -\sigma \le y_n < \psi(y')\}. \quad (17)$$

Here $x = Ry + x(P)$ for an orthogonal matrix R (i.e. $R^{-1} = R^T$). As in [15], the class of domains satisfying the local graph property at every $P \in \partial\Omega$ will be denoted by C_{gr}.

3. **Uniqueness of blow-up solutions.** Let Q_P be a spherical cylinder centered at P as in (16). We start recalling that a non-negative viscosity solution $w_P \equiv w \in C(Q_P)$ of the boundary blow-up problem

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + \gamma|Dw| - \delta w^s = 0 \text{ in } Q_P, \quad w(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial Q_P) \to 0 \quad (18)$$

is provided by Theorem 1.6 of [12] and by [1], Cor. 1.3, $w \in C^{2,a}(Q_P)$ for $a \in (0,1)$.

The main tool to show the uniqueness will be the comparison principle (13).

Proposition 1. *Let Ω be a domain of $\mathbb{R}^n$ satisfying the local graph property at $x_P \in \partial\Omega$, and Q_P the cylinder of Remark 3. Assume that $F : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mapsto \mathbb{R}$ satisfies the structure conditions (SC) and the comparison principle (15) holds true with $Q_P \cap \Omega$ in place of Ω. If there exists a viscosity subsolution $u \in C(\overline{Q}_P \cap \Omega)$ of (1) such that*

$$u(x) \to +\infty \ \ \textit{locally uniformly} \ \ \textit{as}\ x \to \Gamma_1 \equiv Q_P \cap \partial\Omega, \quad (19)$$

then the problem

$$F(x, v, Dv, D^2v) = f(x) \quad \textit{in } Q_P \cap \Omega, \quad (20)$$

$$v(x) \to \infty \textit{ locally uniformly as } x \to Q_P \cap \partial\Omega, \quad (21)$$

$$v = 0 \quad \textit{on } \Gamma_2 \equiv \partial Q_P \cap \Omega, \quad (22)$$

has a viscosity solution $v \in C(\overline{Q}_P \cap \Omega)$ for every $f \in C(\Omega) \cap L^p_{loc}(\mathbb{R}^n)$.

Remark 4. Following [15], by condition (21) we mean $v(x) \to +\infty$ as $\mathrm{dist}(x, A) \to 0$ for every $A \subset\subset \Gamma_1$ in the relative topology.

Proof. Following [15], with the notations of (16) consider an approximation from below of

$$\Theta \equiv Q_P \cap \Omega = \{x \in \mathbb{R}^n : |y'| < \rho, -\sigma < y_n < \psi(y')\},$$

where $x = Ry + x(P)$ and $R^{-1} = R^T$, assuming $\psi > 0$, as we may, using a monotone increasing sequence of smooth positive functions $\psi_j \to \psi$ as $j \to \infty$.
Correspondingly, let

$$\begin{aligned}
\Theta_j &= \{x \in \mathbb{R}^n : |y'| < \rho,\ -\sigma < y_n < \psi_j(y')\}\\
\Gamma_{1j} &= \{x \in \mathbb{R}^n : |y'| < \rho,\ y_n = \psi_j(y')\}\\
\Gamma_{2j} &= \{x \in \mathbb{R}^n : |y'| = \rho,\ -\sigma \le y_n < \psi_j(y')\}\\
&\cup \{x \in \mathbb{R}^n : |y'| < \rho,\ y_n = -\sigma\}
\end{aligned}$$

Let also $\Gamma_{2j} = \Gamma'_{2j} \cup \Gamma''_{2j}$ where

$$\begin{aligned}
\Gamma'_{2j} &= \{x \in \Gamma_{2j} :\ |y'| = \rho,\ \psi_j(y') - \tfrac{1}{j} \le y_n < \psi_j(y')\}\\
\Gamma''_{2j} &= \{x \in \Gamma_{2j} :\ |y'| = \rho,\ -\sigma \le y_n \le \psi_j(y') - \tfrac{1}{j}\}\\
&\cup \{x \in \Gamma_{2j} : |y'| < \rho,\ y_n = -\sigma\}
\end{aligned}$$

By Theorem 4.1 of [5] we can find a continuous viscosity solution of the problem

$$\begin{aligned}
F(x, v_{j,k}(x), Dv_{j,k}(x), D^2 v_{j,k}(x)) &= f(x) \quad \textit{in } \Theta_j\\
v_{j,k}(x) &= k \quad \textit{on } \Gamma_{1j}\\
v_{j,k}(x) &= j\left(y_n - \psi_j(y') + \tfrac{1}{j}\right)k \quad \textit{on } \Gamma'_{2j}\\
v_{j,k}(x) &= 0 \quad \textit{on } \Gamma''_{2j}.
\end{aligned}$$

Here we are using the same boundary conditions of [15], Theorem 2.2. Then by construction for any fixed $j \in \mathbb{N}$ the sequence $(v_{j,k})_{k\in\mathbb{N}}$ is increasing, with respect to $k \in \mathbb{N}$, on $\partial\Theta_j$ and so, by the comparison principle, is also increasing in Θ_j. On the other side, from Proposition 3.3 in [12] we have uniform boundedness in compact sets K of Θ_j, say

$$\sup_K |v_{j,k}| \le C$$

for a positive constant $C = C(n, \lambda, \Lambda, p, \delta, K)$. Moreover by (SC)

$$\begin{aligned}
F(x, v_{j,k}, \xi, X) &\le \mathcal{P}^+_{\lambda,\Lambda}(X) + \gamma|\xi| + F(x, v_{j,k}, 0, 0)\\
&\le \mathcal{P}^+_{\lambda,\Lambda}(X) + \gamma|\xi| + \max_{\substack{x\in K\\ |t|\le C}} |F(x, t, 0, 0)|,\\
F(x, v_{j,k}, \xi, X) &\ge \mathcal{P}^-_{\lambda,\Lambda}(X) - \gamma|\xi| + F(x, v_{j,k}, 0, 0)\\
&\ge \mathcal{P}^-_{\lambda,\Lambda}(X) - \gamma|\xi| - \max_{\substack{x\in K\\ |t|\le C}} |F(x, t, 0, 0)|.
\end{aligned}$$

By using Hölder estimates (see Caffarelli-Cabré [2] and Sirakov [19]), Ascoli-Arzelá theorem and stability results for viscosity solutions (see Proposition 4.11 in [2], Theorem 3.8 in [3]), we deduce that

$$v_{j,\infty} = \lim_{k\to\infty} v_{j,k}$$

is a solution of (20) in Θ_j.

Next consider the sequence $(v_{j,\infty})_{j\in\mathbb{N}}$. Since $v_{j+1,k} \leq v_{j,k}$ on $\partial\Theta_j$, we have $v_{j+1,\infty} \leq v_{j,+\infty}$ on $\partial\Theta_j$ so that, again by the comparison principle, the sequence $(v_{j,\infty})_{j\in\mathbb{N}}$ is monotone decreasing in Θ_j and, by reasoning as before to show that $v_{j,\infty}$ are solutions, in turn converges locally uniformly to a solution v of (20) in Θ.

It is easy to check that $v = 0$ on Γ_2, which is regular enough in order that the boundary condition is satisfied with continuity, see [6]. In order to prove (21), let us observe that for all $k \in \mathbb{N}$

$$v_{j,\infty} \geq v_{j,k} = k \;\; on \;\; \Gamma_{1j}\,.$$

Since u is bounded on $\partial\Theta_j$, then $u \leq v_{j,\infty}$ on Γ_{1j}, as well as $u \leq w$ on Γ_{2j}, by (18). Moreover from Lemma (2.3) the function $v_{j,\infty} + w$ is a supersolution of (20) in Θ_j and hence by the comparison principle

$$u \leq v_{j,\infty} + w \;\; in \; \Theta_j.$$

Passing to the limit as $j \to \infty$ we obtain

$$u \leq v + w \tag{23}$$

in Θ, from which condition (21) follows. □

Corollary 1. *Suppose that the assumptions of Proposition 1 are satisfied for positive functions $u = u_i \in C(\overline{Q}_P \cap \Omega)$, $i = 1, 2$. Let $Q_P^* \subset\subset Q_P$ be a spherical cylinder centered at P. If F is independent of x, then there exists a positive constant C such that*

$$|u_2 - u_1| \leq C \;\; in \; Q_P^* \cap \Omega. \tag{24}$$

Proof. Since F is independent of x, the comparison principle holds true by Remark 2. Therefore, from (23) we have $u_2 \leq v + w$ in $Q_P \cap \Omega$, where, up to a rotation, we may suppose the axis of the cylinder Q_P parallel to x_n, see (17). Since w is bounded in Q_P^*, we get then

$$u_2 \leq v + C \;\; in \; Q_P^* \cap \Omega \tag{25}$$

with $C = \sup_{Q_P^*} w$. On the other side, consider $v_h(x', x_n) = v(x', x_n - h)$ for sufficiently small $h > 0$: v_h is continuous in $\overline{Q_P^h \cap \Omega}$ and $v_h = 0$ on Γ_2^h. Here Q_P^h and Γ_i^h, $i = 1, 2$, result from the corresponding sets Q_P and Γ_i moved up by h along the axis of Q_P. Then

$$v_h \leq u_i \;\; on \; \partial(Q_P^h \cap \Omega) \tag{26}$$

Since F is independent of x, the function v_h satisfies the equation

$$F[v_h] = f_h \;\; in \; Q_P^h \cap \Omega. \tag{27}$$

where $f_h(x', x_n) = f(x', x_n - h)$.

Therefore, fixing $y \in Q_P \cap \Omega$, choosing $h > 0$ small enough in order that $y \in Q_P^h \cap \Omega$ and applying Lemma 2.4 in $Q_P^h \cap \Omega$, for any $R > 0$ we get

$$\begin{aligned}(v_h - u_i)^+(y) \leq & \limsup_{x\to\partial(Q_P^h\cap\Omega)\cap B_R} (v_h - u_i)^+ \\ & + C_0 \left(\frac{1+\gamma R}{R^2}\right)^{\frac{1}{s-1}} + C_1 \|(f - f_h)^-\|_{L^p(Q_P^h\cap\Omega\cap B_R)} \\ \leq & \; C_0 \left(\frac{1+\gamma R}{R^2}\right)^{\frac{1}{s-1}} + C_1 \|(f_h - f)^-\|_{L^p(\Omega\cap B_R)}.\end{aligned}$$

Letting $h \to 0^+$, we have

$$(v - u_i)^+(y) \le C_0 \left(\frac{1 + \gamma R}{R^2} \right)^{\frac{1}{s-1}}$$

and as $R \to \infty$, since $y \in Q_P \cap \Omega$ is arbitrary,

$$v \le u_i \quad in\ Q_P \cap \Omega. \tag{28}$$

From (25) and (28) the result follows. □

Proof of Theorem 1.1 Let u_1, u_2 be non-negative blow-up solutions of $F[u] = f(x)$ in Ω. Let $\varepsilon > 0$ small enough. Setting $k_\varepsilon = 1 + \varepsilon$, $u_{1\varepsilon} = (1 + \varepsilon)u_1$ and using (6), we have

$$F[u_{1\varepsilon}] \le \varphi(k_\varepsilon)F[u_1] + o(\varepsilon) = \varphi(k_\varepsilon)f(x) + o(\varepsilon), \tag{29}$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Take, for every $P \in \partial\Omega$, a spherical cylinder Q^*_P centered at P of radius ρ^* and height $2\sigma^*$, as in Corollary 1. By (24) we have constructed an open covering $\{Q^*_P\}_{P\in\partial\Omega}$ of $\partial\Omega$ such that

$$1 - \frac{C_P}{u_1} \le \frac{u_2}{u_1} \le 1 + \frac{C_P}{u_1} \quad in\ Q^*_P \cap \Omega. \tag{30}$$

Since $u_1 \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, then

$$u_2(x) \le (1 + \varepsilon)u_1(x) = u_{1\varepsilon}(x), \tag{31}$$

in N^*_P, an open neighborhood of $Q^*_P \cap \partial\Omega$.

Collecting all N^*_P we obtain a neighborhood N_ε of $\partial\Omega$ where (31) holds true.

Let $\Omega_\varepsilon = \{u_2 > (1 + \varepsilon)u_1 = u_{1\varepsilon}\}$. We claim that there is a sequence $\varepsilon \to 0^+$ such that $\Omega_\varepsilon = \emptyset$.

By contradiction, suppose $\Omega_\varepsilon \ne \emptyset$ for infinitely many $\varepsilon \to 0^+$. Since $\overline{\Omega}_\varepsilon \subset \Omega$ and therefore $u_2 = u_{1\varepsilon}$ on $\partial\Omega_\varepsilon$, using (29) and recalling that $F[u_2] \ge f$, we have by (13)

$$(u_2 - u_{1\varepsilon})^+(y) \le C_0 \left(\frac{1 + \gamma R}{R^2} \right)^{\frac{1}{s-1}} + C_1 \left((\varphi(k_\varepsilon) - 1)\|f^+\|_{L^p(\Omega\cap B_R(y))} + o(\varepsilon) \right) \tag{32}$$

for all $y \in \Omega_\varepsilon$ and $R > 0$. Thus, letting $\varepsilon \to 0^+$ and then $R \to \infty$, we get $u_2 \le u_1$ in Ω, which contradicts $\Omega_\varepsilon \ne \emptyset$ and proves the claim.

Hence $u_2 \le (1 + \varepsilon)u_1$ in Ω for a sequence $\varepsilon \to 0^+$ and taking this limit we have $u_2 \le u_1$ in Ω. Interchanging u_1 and u_2 we finish the proof. □

4. **A generalization.** In this Section, we consider an uniformly elliptic operator F satisfying (3) and (5) of form

$$F = \tilde{F}(x, t, \xi, X) + c|t|^{\alpha-1}t - |t|^{s-1}t \tag{33}$$

where $c \ge 0$, $\alpha \in (0, s)$ and $\tilde{F}$ is positively homogeneous of degree $\beta \in [\alpha, s]$:

$$F_1(x, kt, k\xi, kX) = k^\beta F_1(x, t, \xi, X) \quad \forall\, k > 0,\ \forall\, (x, t, \xi, X) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^n.$$

In the spite of Marcus-Veron [16] we have in mind for instance fully non linear second order operators like

$$\tilde{F} = \sup_i \inf_j \{Tr(A^{ij}X) + \langle b_{ij}, \xi\rangle + c_{ij}t\}$$

which is positively homogeneous of degree $\beta = 1 = \alpha < s$.

Note that, if $\tilde{F}$ is non-increasing in t and $c \le 0$, then (4) is also satisfied, but this fails to hold, in general. However, we can state a comparison principle in all cases.

For references about maximum principles and related methods see [17] and [18].

Lemma 4.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ and F be an uniformly elliptic operator satisfying (3) and (5) of form (33) with $\tilde{F}$ positively homogeneous of degree $\beta \in [\alpha, s]$ for $t > 0$ and $c \geq 0$. Suppose that u and v are continuous subsolutions and supersolutions, respectively, of $F = f$ in viscosity sense, where $f \in C(\Omega)$ and $f \leq 0$. In addition we assume $u, v \in C^1(\Omega)$ and F independent of x. Suppose $v > 0$ in Ω, then*

$$\limsup_{x\to\partial\Omega} (u - v) \leq 0 \quad \Rightarrow \quad u \leq v \quad \text{in } \Omega. \tag{34}$$

Remark 5. If we assume at least one of u and v to be $C^2(\Omega)$, then we do not need to assume F independent of x.

Proof of Lemma 4.1. By contradiction, suppose $\Omega^+ \equiv \{x \in \Omega \,|\, u(x) > v(x)\} \neq \emptyset$.

Setting $u = e^U$ and $v = e^V$, by straightforward computation, we obtain, in viscosity sense,

$$\tilde{F}(x, 1, DU, D^2U + DU \otimes DU) + ce^{(\alpha-\beta)U} - e^{(s-\beta)U} \geq e^{-\beta U} f(x), \tag{35}$$

$$\tilde{F}(x, 1, DV, D^2V + DV \otimes DV) + ce^{(\alpha-\beta)V} - e^{(s-\beta)V} \leq e^{-\beta V} f(x), \tag{36}$$

where we have used the positive homogeneity of $\tilde{F}$.

Let $w = U - V$. Subtracting (36) from (35), as we may in viscosity setting when $\tilde{F}$ is independent of x, using (3) we have

$$\begin{aligned}\mathcal{P}^+_{\lambda,\Lambda}&(D^2w + (DU \otimes Dw + Dw \otimes DV)) + \gamma|Dw| \\ &\geq -c(e^{(\alpha-\beta)U} - e^{(\alpha-\beta)V}) + (e^{(s-\beta)U} - e^{(s-\beta)V}) + (e^{-\beta U} - e^{-\beta V})f(x),\end{aligned}$$

from which, since $c \geq 0$, $\alpha \leq \beta$, $f \leq 0$, letting $b(x) = \Lambda(|DU(x)| + |DV(x)|) + \gamma$,

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2w) + b(x)|Dw| \geq 0 \quad \text{in } \Omega^+.$$

But w is positive in Ω^+ and $\limsup_{x\to\partial\Omega^+} w \leq 0$, and this contradicts the maximum principle. Therefore $U \leq V$ and consequently $u \leq v$ in Ω. $\square$

We are ready to develop the program of the previous Section to establish an uniqueness result for the blow-up problem (1) & (2) with fully nonlinear uniformly elliptic operators of type (7), i.e. $\tilde{F} = F_1(\xi, X)$ in (33).

Proof of Theorem 1.2. Consider two positive solutions u_1, u_2 of problem (8). By standard viscosity results, see [2] and [20], u_1, u_2 have Hölder first derivatives.

Let $\varepsilon \in (0, 1)$. By the local graph property and the boundary blow-up condition, for every $P \in \partial\Omega$ we can find a spherical cylinder Q_P as (16) such that (17) holds true and $u_i \geq (\frac{c}{\varepsilon})^{\frac{1}{s-\alpha}}$ in $Q_P \cap \Omega$, $i = 1, 2$, so that by Lemma 4.1 the comparison principle (15) holds true with $Q_P \cap \Omega$ in place of Ω. Then

$$F_1(Du_i, D^2u_i) - (1 - \varepsilon)u_i^s \geq f(x).$$

Taking $\delta = \frac{1}{2}$, say, in the definition (18) of w, we conclude as in Corollary 1 that for $Q_P^* \subset\subset Q_P$ there exists C such that (24) holds true.

As in the proof of Theorem 1.1 we find a neighborhood $N_{\varepsilon'}$ of $\partial\Omega$ where (31) holds true and set $\Omega_{\varepsilon'} = \{x \in \Omega\,,\, u_2 > (1 + \varepsilon')u_1 = u_{1\varepsilon'}\}$. We infer that $\Omega_{\varepsilon'} = \emptyset$.

By contradiction, suppose $\Omega_{\varepsilon'} \neq \emptyset$. Setting $k_{\varepsilon'} = 1 + \varepsilon'$, using the positive homogeneity of $\tilde{F}$ and the fact that $f \leq 0$, from $F_1[u_1] + cu_1^\alpha - u_1^s \leq f$ we have

$$F_1[u_{1\varepsilon'}] + cu_{1\varepsilon'}^\alpha - u_{1\varepsilon'}^s \leq k_{\varepsilon'}^\beta f(x) \leq f(x)$$

so that, being $F_1[u_2] + cu_2^\alpha - u_2^s \geq f$, Lemma 4.1 yields $u_2 \leq u_{1\varepsilon'}$ in $\Omega_{\varepsilon'}$, against $\Omega_{\varepsilon'} \neq \emptyset$. From this $u_2 \leq (1+\varepsilon')u_1$ and, letting $\varepsilon' \to 0^+$, $u_2 \leq u_1$ in Ω.

Interchanging u_1 and u_2, we also have $u_1 \leq u_2$, as claimed. □

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Received September 2012; revised December 2012.

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