Electron. Commun. Probab. **21** (2016), no. 18, 1–15.
DOI: 10.1214/16-ECP4411
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Spectral densities related to some fractional stochastic differential equations

Mirko D'Ovidio* Enzo Orsingher† Ludmila Sakhno‡

Abstract

In this paper we consider fractional higher-order stochastic differential equations of the form

$$\left(\mu + c_\alpha \frac{d^\alpha}{dt^\alpha}\right)^\beta X(t) = \mathcal{E}(t), \quad \mu > 0,\ \beta > 0,\ \alpha \in (0,1) \cup \mathbb{N}$$

where $\mathcal{E}(t)$ is a Gaussian white noise. We obtain explicitly the covariance functions and the spectral densities of the stochastic processes satisfying the above equations.

Keywords: higher-order heat equations; Weyl fractional derivatives; airy functions; spectral densities.
AMS MSC 2010: 60K99; 60G60.
Submitted to ECP on July 7, 2015, final version accepted on February 17, 2016.

1 Introduction

In this paper we consider fractional stochastic ordinary differential equations of different form where the stochastic component is represented by a Gaussian white noise. Some of the fractional equations considered here are related to the higher-order heat equations and thus are connected with pseudo-processes.

The first part of the paper considers the following stochastic differential equation

$$\left(\mu + \frac{d^\alpha}{dt^\alpha}\right)^\beta X(t) \,=\, \mathcal{E}(t), \qquad \beta > 0,\, 0 < \alpha < 1,\ \mu > 0,\ t > 0 \tag{1.1}$$

where $\frac{d^\alpha}{dt^\alpha}$ represents the Weyl fractional derivative. We obtain a representation of the solution to (1.1) in the form

$$X(t) \,=\, \frac{1}{\Gamma(\beta)} \int_0^\infty \mathcal{E}(t-z) \int_0^\infty s^{\beta-1} e^{-s\mu}\, h_\alpha(z,s)\, ds dz \tag{1.2}$$

where $h_\alpha(z,s)$, $z,s \geq 0$, is the density function of a positively-skewed stable process $H_\alpha(s)$, $s \geq 0$ of order $\alpha \in (0,1)$, that is with Laplace transform

$$\int_0^\infty e^{-\xi z} h_\alpha(z,s) dz = e^{-s\xi^\alpha}, \quad \xi \geq 0.$$

*Sapienza University of Rome, Italy. E-mail: mirko.dovidio@uniroma1.it
†Sapienza University of Rome, Italy. E-mail: enzo.orsingher@uniroma1.it
‡Taras Shevchenko National University of Kyiv, Ukraine. E-mail: lms@univ.kiev.ua

For (1.2), we obtain the spectral density

$$f(\tau) = \frac{\sigma^2}{\left(\mu^2 + 2|\tau|^\alpha \mu \cos\frac{\pi\alpha}{2} + |\tau|^{2\alpha}\right)^\beta}, \quad \tau \in \mathbb{R} \tag{1.3}$$

and the related covariance function.

The second type of stochastic differential equations we consider has the form

$$\left(\mu + (-1)^n \frac{d^{2n}}{dt^{2n}}\right)^\beta X(t) \;=\; \mathcal{E}(t), \quad \beta > 0,\ \mu > 0,\ n \geq 1,\ t \in \mathbb{R} \tag{1.4}$$

where $\mathcal{E}(t)$ is a Gaussian white noise. The representation of the solution to (1.4) is

$$X(t) \;=\; \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} \mathcal{E}(t+x) \int_0^\infty w^{\beta-1} e^{-\mu w} u_{2n}(x,w) dw dx \tag{1.5}$$

where $u_{2n}(x,w)$, $x \in \mathbb{R}$, $w \geq 0$ is the fundamental solution to $2n$-th order heat equation

$$\frac{\partial u}{\partial w}(x,w) = (-1)^{n+1} \frac{\partial^{2n} u}{\partial x^{2n}}(x,w) \tag{1.6}$$

The covariance function of the process (1.5) can be written as

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw\, w^{2\beta-1} e^{-\mu w}\, u_{2n}(h,w) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n}(h, W_{2\beta}) \tag{1.7}$$

where $W_{2\beta}$ is a gamma r.v. with parameters μ and 2β. The spectral density $f(\tau)$ associated with (1.7) has the fine form

$$f(\tau) = \frac{\sigma^2}{(\mu + \tau^{2n})^{2\beta}}, \quad \tau \in \mathbb{R}. \tag{1.8}$$

For $n = 1$, (1.6) is the classical heat equation, $u_2(x,w) = \frac{e^{-\frac{x^2}{4w}}}{\sqrt{4\pi w}}$ and, from (1.7) we obtain an explicit form of the covariance function in terms of the modified Bessel functions. In connection with the equations of the form (1.6) the so-called pseudo-processes, first introduced at the beginning of the Sixties ([7]), have been constructed. The solutions to (1.6) are sign-varying and their structure has been explored by means of the steepest descent method ([11, 1]) and their representation has been recently given in [14].

For the fractional odd-order stochastic differential equation

$$\left(\mu + \kappa \frac{d^{2n+1}}{dt^{2n+1}}\right)^\beta X(t) = \mathcal{E}(t), \quad n = 1, 2, \ldots, \quad \kappa = \pm 1,\ t \in \mathbb{R} \tag{1.9}$$

the solution has the structure

$$X(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} \mathcal{E}(t+x) \int_0^\infty dw\, w^{\beta-1} e^{-\mu w} u_{2n+1}(x,w) dw dx \tag{1.10}$$

where $u_{2n+1}(x,w)$, $x \in \mathbb{R}$, $w \geq 0$ is the fundamental solution to

$$\frac{\partial u}{\partial w}(x,w) = \kappa \frac{\partial^{2n+1} u}{\partial x^{2n+1}}(x,w), \quad \kappa = \pm 1. \tag{1.11}$$

The solutions u_{2n+1} and u_{2n} are substantially different in their behaviour and structure as shown in [14] and [8].

A special attention has been devoted to the case $n = 1$ (and $\kappa = -1$) for which (1.10) takes the interesting form

$$X_3(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} \mathcal{E}(t+x) \int_0^{\infty} w^{\beta-1} e^{-\mu w} \frac{1}{\sqrt[3]{3w}} Ai\left(\frac{x}{\sqrt[3]{3w}}\right) dw dx \tag{1.12}$$

where $Ai(\cdot)$ is the first-type Airy function. The process X_3 can also be represented as

$$X_3(t) = \frac{1}{\mu^{\beta}} \mathbf{E}\mathcal{E}(t + Y_3(W_\beta)) \tag{1.13}$$

where the mean $\mathbf{E}$ is defined in formula (1.19) below, Y_3 is the pseudo-process related to equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} \tag{1.14}$$

and W_β is a Gamma-distributed r.v. independent from Y_3 and possessing parameters β, μ. Therefore, the covariance function of X_3 has the following form

$$\mathbb{E}X_3(t)X_3(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}\left[\frac{1}{\sqrt[3]{3W_{2\beta}}} Ai\left(\frac{h}{\sqrt[3]{3W_{2\beta}}}\right)\right] \tag{1.15}$$

where $W_{2\beta}$ is the sum of two independent r.v.'s W_β.

For the solution to the general odd-order stochastic equation we obtain the covariance function

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}\left[u_{2n+1}(h, W_{2\beta})\right] \tag{1.16}$$

Of course, the Fourier transform of (1.16) becomes, for $\kappa = \pm 1$,

$$f(\tau) = \frac{\sigma^2}{\mu^{2\beta}} \int_{\mathbb{R}} e^{i\tau h}\, \mathbb{E}\left[u_{2n+1}(h, W_{2\beta})\right] dh = \frac{\sigma^2}{(\mu + i\kappa\tau^{2n+1})^{2\beta}}. \tag{1.17}$$

Stochastic fractional differential equations similar to those treated here have been analysed in [2], [4] and [6]. In our paper we consider equations where different operators are involved. Such operators are defined as fractional powers ($\beta > 0$) of operators of order α, for $\alpha \in (0, 1) \cup \mathbb{N}$. The equations we deal with and involving the white noise $\mathcal{E}(t)$ can be interpreted as integral equations. We define as usual (see [18, pag. 110])

$$X(f) = \int \mathcal{E}(s) f(s) ds$$

so that, for each $f, g \in L^2(dx)$, we have that

$$\mathbb{E}X(f)X(g) = \sigma^2 \int f(x) g(x) dx. \tag{1.18}$$

Thus, by considering integral equations, we do not care about assumptions such as sample continuity and differentiability. Moreover, for the sake of clarity we introduce the following conditional expectation

$$\mathbf{E}[\mathcal{E}(t + Y(W))] = \int \mathcal{E}(t+y) \mathbb{P}(Y(W) \in dy) \tag{1.19}$$

where the expectation is performed w.r.t. the probability measure of $Y(W)$. Throughout the paper we consider Y given by:

- the stable subordinator of order $\alpha \in (0, 1]$, denoted by H_α;

- the pseudo-processes of order $2n$ and $2n+1$ with $n \in \mathbb{N}$, denoted by Y_{2n} and Y_{2n+1}.

We also denote by W the Gamma r.v. W_β with parameters μ and β such that $W_1 + W_2 \stackrel{d}{=} W_{2\beta}$.

Pseudo-processes have been developed in a series of papers dating back to the Sixties ([3, 9], [7] for the even-order case, [12] for pseudo-processes related to equations with two space derivatives) and recently by Orsingher [13] for the third-order case, Lachal [8] for the general case and also Smorodina and Faddeev [17].

2 Fractional powers of fractional operators

In this section we consider the following generalization of the Gay and Heyde equation (see [4])

$$\left(\mu + \frac{d^\alpha}{dt^\alpha}\right)^\beta X(t) \; = \; \mathcal{E}(t), \qquad \beta > 0, \, 0 < \alpha < 1, \; \mu > 0, \; t > 0 \tag{2.1}$$

where $\mathcal{E}(t)$, $t \in \mathbb{R}$, is a Gaussian white noise for which (1.18) holds true. Then, we have that $\mathbb{E}\mathcal{E}(t)\mathcal{E}(s) \; = \sigma^2 \delta(t-s)$ where δ is the Dirac function. The fractional derivative appearing in (2.1) must be meant, for $0 < \alpha < 1$, as

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^t \frac{f(s)}{(t-s)^\alpha} ds = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(t) - f(t-w)}{w^{\alpha+1}} dw.$$

For $\alpha = 1$ we have that

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{d}{dt} f(t)$$

as usual. Consult, for example, [16, pag. 111] for information on fractional derivatives of this form, called also Marchaud derivatives. For $\lambda \geq 0$, we introduce the Laplace transform

$$\mathcal{L}\left[\frac{d^\alpha f}{dt^\alpha}\right](\lambda) = \int_0^\infty e^{-\lambda t} \frac{d^\alpha}{dt^\alpha} f(t) dt = \lambda^\alpha \mathcal{L}[f](\lambda) \tag{2.2}$$

which can be immediately obtained by considering that

$$\mathcal{L}\left[\frac{d^\alpha f}{dt^\alpha}\right](\lambda) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(\mathcal{L}[f](\lambda) - e^{-w\lambda} \mathcal{L}[f](\lambda)\right) \frac{dw}{w^{\alpha+1}} \tag{2.3}$$

where we used the fact that

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-wx}\right) \frac{dw}{w^{\alpha+1}}, \quad \alpha \in (0,1), \; x \geq 0.$$

Lemma 2.1. *The following relationship holds in a generalized m.s. sense*

$$e^{z\frac{d}{dt}} \mathcal{E}(t) = \mathcal{E}(t+z). \tag{2.4}$$

Proof. In view of the Taylor expansion

$$f(x) \; = \sum_{k=0}^\infty f^{(k)}(x_0) \frac{(x-x_0)^k}{k!} \tag{2.5}$$

with $x_0 = t$ and $x = t + z$ we can write

$$e^{z\frac{d}{dt}} f(t) \; = \sum_{k=0}^\infty \frac{z^k}{k!} \frac{d^k}{dt^k} f(t) = \; f(t+z) \tag{2.6}$$

which holds for a bounded and continuous function $f : [0,\infty) \mapsto [0,\infty)$. Since we can find an orthonormal set, say $\{\phi_j\}_{j\in\mathbb{N}}$, for which (2.6) holds true $\forall\, j$ and a sequence of r.v.'s $\{a_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \mathbb{E} \left\| \mathcal{E} - \sum_{j=1}^{N} a_j \phi_j \right\|_2 = 0, \tag{2.7}$$

we can write (2.4). Since $\mathcal{E}$ is a generalized white noise with second moment as in (1.18) we get the claim. □

Theorem 2.2. *Let us consider the equation* (2.1)*, then a generalized m.s. solution is*

$$\begin{aligned} X(t) =& \frac{1}{\mu^\beta} \mathbf{E}[\mathcal{E}(t - H_\alpha(W_\beta))], \quad \beta > 0,\ 0 < \alpha < 1,\ \mu > 0 \\ =& \frac{1}{\Gamma(\beta)} \int_0^\infty dz \int_0^\infty ds\, s^{\beta-1} e^{-s\mu}\, h_\alpha(z,s)\, \mathcal{E}(t-z) \end{aligned} \tag{2.8}$$

Proof. The solution to the equation (2.1) can be obtained as follows

$$\begin{aligned} X(t) &= \left(\frac{d^\alpha}{dt^\alpha} + \mu \right)^{-\beta} \mathcal{E}(t) \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\mu - s\frac{d^\alpha}{dt^\alpha}} \mathcal{E}(t)\, ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\mu} \left\{ e^{-s\frac{d^\alpha}{dt^\alpha}} \mathcal{E}(t) \right\} ds. \end{aligned} \tag{2.9}$$

The first step in (2.9) can be justified on the basis of the arguments in Renardy and Rogers [15, pag. 417)] where the representation of fractional power operators is dealt with.

Now, for the stable subordinator $H_\alpha(t)$, $t > 0$, we have that

$$\begin{aligned} e^{-s\frac{d^\alpha}{dt^\alpha}} \mathcal{E}(t) &= \mathbf{E} e^{-H_\alpha(s)\frac{d}{dt}} \mathcal{E}(t) \\ &= \int_0^\infty dz\, h_\alpha(z,s)\, e^{-z\frac{d}{dt}} \mathcal{E}(t) \\ &= \int_0^\infty dz\, h_\alpha(z,s)\, \mathcal{E}(t-z) \end{aligned} \tag{2.10}$$

where $h_\alpha(z,s)$ is the probability law of $H_\alpha(s)$, $s > 0$. In the last step of (2.10) we used the translation property (2.4). Therefore,

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty \mathcal{E}(t-z) \int_0^\infty s^{\beta-1} e^{-s\mu} h_\alpha(z,s) ds dz \tag{2.11}$$

is the representation of the solution to the fractional equation (2.1). □

Remark 2.3. With (2.7) and (1.18) in mind, notice that a representation of (2.8) is given by

$$X(t) = \frac{1}{\mu^\beta} \sum_{j\in\mathbb{N}} a_j\, \mathbb{E}[\phi_j(t - H_\alpha(W_\beta))], \quad t > 0. \tag{2.12}$$

Remark 2.4. For the case $\alpha \uparrow 1$, $h_\alpha(z,s) \to \delta(z-s)$ where δ is the Dirac delta function and from (2.4) we infer that

$$X(t) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\mu s} s^{\beta-1}\, \mathcal{E}(t-s)\, ds \tag{2.13}$$

is a generalized solution to

$$\left(\mu+\frac{d}{dt}\right)^{\beta} X(t) \;=\; \mathcal{E}(t). \tag{2.14}$$

Consult on this point [6]. A direct proof is also possible because from (2.9) we have that

$$\begin{aligned} X(t) \;&=\; \frac{1}{\Gamma(\beta)}\int_0^\infty s^{\beta-1}\, e^{-\mu s} e^{-s\frac{d}{dt}}\mathcal{E}(t)\, ds \\ &=\; \frac{1}{\Gamma(\beta)}\int_0^\infty s^{\beta-1}\, e^{-\mu s}\mathcal{E}(t-s)\, ds. \end{aligned} \tag{2.15}$$

In the last step we applied (2.4).

Remark 2.5. For $\alpha=1$ and $\beta=1$, we observe that (2.1) coincides with the Langevin equation and (2.15) can be reduced to the following form of the Ornstein-Uhlenbeck process

$$X(t)=\int_{-\infty}^{t} e^{-\mu(t-s)}\mathcal{E}(s)ds$$

with covariance function

$$\mathbb{E}[X(t+h)X(t)]=\frac{\sigma^2}{2\mu}e^{-\mu|h|}.$$

Our next step is the evaluation of the Fourier transform of the covariance function of the solution to the differential equation (2.1). Let

$$f(\tau)=\int_{-\infty}^{+\infty} e^{i\tau h} Cov_X(h)dh$$

where

$$Cov_X(h)=\mathbb{E}[X(t+h)X(t)]$$

with $\mathbb{E}X(t)=0$.

Theorem 2.6. *The spectral density of* (2.8) *is*

$$f(\tau)=\frac{\sigma^2}{\left(\mu^2+2|\tau|^\alpha\mu\cos\frac{\pi\alpha}{2}+|\tau|^{2\alpha}\right)^{\beta}}, \quad \tau\in\mathbb{R},\ 0<\alpha<1,\ \beta>0. \tag{2.16}$$

Proof. The Fourier transform of the covariance function of (2.8) is given by

$$\begin{aligned} &\int_0^\infty e^{i\tau h}\mathbb{E}X(t)X(t+h)\, dh \\ &= \frac{1}{\Gamma^2(\beta)}\int_0^\infty e^{i\tau h}dh\int_0^\infty dz_1\int_0^\infty ds_1\int_0^\infty ds_2\int_0^\infty dz_2\, s_1^{\beta-1}\, s_2^{\beta-1} \\ &\qquad \times e^{-(s_1+s_2)\mu}h_\alpha(z_1,s_1)\; h_\alpha(z_2,s_2)\; \mathbb{E}\mathcal{E}(t-z_1)\mathcal{E}(t+h-z_2) \end{aligned}$$

where

$$\mathbb{E}\mathcal{E}(t-z_1)\mathcal{E}(t+h-z_2)=\sigma^2\delta((z_1-z_2)-h). \tag{2.17}$$

Thus,

$$\begin{aligned} \int_0^\infty e^{i\tau h}\mathbb{E}X(t)X(t+h)\, dh =& \frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty dz_1\int_0^\infty ds_1\int_0^\infty ds_2\int_0^\infty dz_2\, s_1^{\beta-1}\, s_2^{\beta-1} \\ &\times e^{-(s_1+s_2)\mu}h_\alpha(z_1,s_1)\; h_\alpha(z_2,s_2)\; e^{i\tau(z_1-z_2)}. \end{aligned}$$

By considering the characteristic function of a positively-skewed stable process with law h_α, we have that

$$\int_0^\infty e^{i\tau\, z_1} h_\alpha\left(z_1, s_1\right)\, dz_1 \;=\; e^{-(-i\tau)^\alpha s_1} = e^{-s_1|\tau|^\alpha e^{-i\frac{\pi}{2}\, sgn\, \tau}}, \tag{2.18}$$

and

$$\int_0^\infty e^{-i\tau z_2} h_\alpha\left(z_2, s_2\right) dz_2 \;=\; e^{-(i\tau)^\alpha s_2} = e^{-s_2|\tau|^\alpha e^{i\frac{\pi}{2}\, sgn\, \tau}}. \tag{2.19}$$

Thus, we obtain that

$$\begin{aligned}
&\int_0^\infty e^{i\tau h}\mathbb{E}X(t)X(t+h)\, dh \\
&= \frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty ds_1 \int_0^\infty ds_2\, s_1^{\beta-1}\, s_2^{\beta-1} e^{-(s_1+s_2)\mu}\, e^{-(i\tau)^\alpha s_2 - (-i\tau)^\alpha s_1} \\
&= \frac{\sigma^2}{\left(\mu + |\tau|^\alpha\, e^{-\frac{i\pi\alpha}{2}\, sgn\, \tau}\right)^\beta \left(\mu + |\tau|^\alpha\, e^{\frac{i\pi\alpha}{2}\, sgn\, \tau}\right)^\beta} \\
&= \frac{\sigma^2}{\left(\mu^2 + 2|\tau|^\alpha \mu \cos\frac{\pi\alpha}{2} + |\tau|^{2\alpha}\right)^\beta}.
\end{aligned}$$

□

Remark 2.7. In the special case $\alpha = 1$ the result above simplifies and yields

$$f(\tau) = \frac{\sigma^2}{(\mu^2+\tau^2)^\beta}. \tag{2.20}$$

We note that for $\beta = 1$, (2.20) becomes the spectral density of the Ornstein-Uhlenbeck process. Processes with the spectral density f are dealt with, for example, in [2] where also space-time random fields governed by stochastic equations are considered. The covariance function is given by

$$\begin{aligned}
Cov_X(h) =& \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau h} f(\tau) d\tau \\
=& \frac{\sigma^2}{2\pi}\int_{\mathbb{R}} e^{-i\tau h}\left(\frac{1}{\Gamma(\beta)}\int_0^\infty z^{\beta-1}e^{-z\mu^2 - z\tau^2} dz\right) d\tau \\
=& \frac{\sigma^2}{\Gamma(\beta)}\int_0^\infty z^{\beta-1}e^{-z\mu^2}\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau h - z\tau^2} d\tau\right) dz \\
=& \frac{\sigma^2}{\Gamma(\beta)}\int_0^\infty z^{\beta-1}e^{-z\mu^2}\frac{e^{-\frac{h^2}{4z}}}{\sqrt{4\pi z}} dz \\
=& \frac{\sigma^2}{2\Gamma(\beta)\Gamma(\frac{1}{2})}\int_0^\infty z^{\beta-\frac{1}{2}-1}e^{-z\mu^2-\frac{h^2}{4z}} dz \\
=& \frac{\sigma^2}{\Gamma(\beta)\Gamma(\frac{1}{2})}\left(\frac{|h|}{2\mu}\right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}\left(\mu|h|\right), \quad h \geq 0
\end{aligned}$$

where K_ν is the modified Bessel function with integral representation given by

$$\int_0^\infty x^{\nu-1}\exp\left\{-\beta x^p - \alpha x^{-p}\right\} dx = \frac{2}{p}\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\alpha\beta}\right), \quad p, \alpha, \beta, \nu > 0 \tag{2.21}$$

(see for example [5], formula 3.478). We observe that $K_\nu = K_{-\nu}$ and $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$. Moreover,

$$K_\nu(x) \approx \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu} \quad \text{for} \quad x \to 0^+ \tag{2.22}$$

([10, pag. 136]) and

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{for} \quad x \to \infty. \tag{2.23}$$

Thus, we get that

$$Cov_X(h) \approx \mu^{1-2\beta}, \quad \text{for} \quad h \to 0^+ \tag{2.24}$$

and

$$Cov_X(h) \approx \left(\frac{h}{\mu}\right)^\beta \frac{1}{h} e^{-\mu h}, \quad \text{for} \quad h \to \infty. \tag{2.25}$$

We now study the covariance of (1.2). Recall that, a symmetric stable process S of order α with density g has the following characteristic function

$$\widehat{g}(\xi, t) = \mathbb{E} e^{i\xi S(t)} = e^{-\sigma^2 |\xi|^\alpha t}, \quad \alpha \in (0, 2].$$

Consider two independent stable processes $S_1(w)$, $S_2(w)$, $w \geq 0$, with $\sigma_1^2 = 1$ and $\sigma_2^2 = 2\mu \cos \frac{\pi\alpha}{2}$. Let $g_1(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ and $g_2(x, w)$, $x \in \mathbb{R}$, $w \geq 0$ be the corresponding density laws. Then, the following result holds true.

Theorem 2.8. *The covariance function of* (1.2) *is*

$$Cov_X(h) = \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w\mu^2} \int_{-\infty}^{+\infty} g_1(h - z, w) g_2(z, w) dz\, dw \tag{2.26}$$

or

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} g_{S_1+S_2}(h, W_\beta) \tag{2.27}$$

and W_β is a gamma r.v. with parameters μ^2, β.

Proof. Notice that

$$f(\tau) = \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w(\mu^2 + 2|\tau|^\alpha \mu \cos \frac{\pi\alpha}{2} + |\tau|^{2\alpha})} dw$$

where

$$e^{-2\mu \cos \frac{\pi\alpha}{2} |\tau|^\alpha w} = \mathbb{E} e^{i\tau S_2(w)} = \widehat{g_2}(\tau, w) \quad \text{and} \quad e^{-|\tau|^{2\alpha} w} = \mathbb{E} e^{i\tau S_1(w)} = \widehat{g_1}(\tau, w).$$

Thus,

$$f(\tau) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}[\widehat{g_1}(\tau, W_\beta)\, \widehat{g_2}(\tau, W_\beta)]$$

from which, we immediately get that

$$\begin{aligned} CovX(h) =& \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} \left[\int_{-\infty}^{+\infty} g_1(h - z, W_\beta) g_2(z, W_\beta) dz \right] \\ =& \frac{\sigma^2}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-w\mu^2} \int_{-\infty}^{+\infty} g_1(h - z, w) g_2(z, w) dz\, dw \end{aligned}$$

□

3 Fractional powers of higher-order operators

We focus our attention on the following equation

$$\left(\mu - \frac{d^2}{dt^2}\right)^\beta X(t) = \mathcal{E}(t), \quad \mu > 0,\ \beta > 0,\ t \in \mathbb{R} \tag{3.1}$$

that is, on the equation (1.4) for $n = 1$.

Theorem 3.1. *A generalized m.s. solution to the equation* (3.1) *is*

$$X(t) = \frac{1}{\mu^\beta}\mathbf{E}[\mathcal{E}(t+Y_2(W_\beta))], \quad \beta>0,\ \mu>0 \tag{3.2}$$
$$= \frac{1}{\Gamma(\beta)}\int_{-\infty}^{+\infty}\mathcal{E}(t+x)\int_0^\infty w^{\beta-1}e^{-\mu w}\frac{e^{-\frac{x^2}{4w}}}{\sqrt{4\pi w}}dwdx.$$

Moreover, the spectral density of (3.2) *reads*

$$f(\tau) = \frac{\sigma^2}{(\mu+\tau^2)^{2\beta}} \tag{3.3}$$

and the corresponding covariance function has the form

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[\frac{e^{-\frac{h^2}{4W_{2\beta}}}}{2\sqrt{\pi W_{2\beta}}}\right] = \frac{\sigma^2}{\sqrt{\pi}\Gamma(2\beta)}\left(\frac{|h|}{2\sqrt{\mu}}\right)^{2\beta-\frac{1}{2}}K_{2\beta-\frac{1}{2}}(|h|\sqrt{\mu}). \tag{3.4}$$

Proof. We can formally write

$$e^{w\frac{d^2}{dt^2}} = \int_{-\infty}^\infty e^{x\frac{d}{dt}}\frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}}\,dx \tag{3.5}$$

so that from (3.1) we have that

$$X(t) = \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-\mu w}w^{\beta-1}\,dw\int_{-\infty}^\infty\frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}}e^{x\frac{d}{dt}}\mathcal{E}(t)\,dx$$
$$= \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-\mu w}w^{\beta-1}dw\int_{-\infty}^\infty\frac{e^{-\frac{x^2}{4w}}}{2\sqrt{\pi w}}\mathcal{E}(t+x)\,dx. \tag{3.6}$$

By observing that, from (1.18),

$$\mathbb{E}\mathcal{E}(t+x_1)\mathcal{E}(t+h+x_2) = \sigma^2\delta(h+x_2-x_1)$$

we can write

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty e^{-\mu w_1}w_1^{\beta-1}dw_1\int_0^\infty e^{-\mu w_2}w_2^{\beta-1}dw_2\int_{-\infty}^\infty\frac{e^{-\frac{x_1^2}{4w_1}}}{2\sqrt{\pi w_1}}\frac{e^{-\frac{(h-x_1)^2}{4w_2}}}{2\sqrt{\pi w_2}}dx_1$$
$$= \frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty e^{-\mu w_1}w_1^{\beta-1}dw_1\int_0^\infty e^{-\mu w_2}w_2^{\beta-1}dw_2\frac{e^{-\frac{h^2}{4(w_1+w_2)}}}{2\sqrt{\pi(w_1+w_2)}}$$
$$= \frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[\frac{e^{-\frac{h^2}{4(W_1+W_2)}}}{2\sqrt{\pi(W_1+W_2)}}\right]$$
$$= \frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[\frac{e^{-\frac{h^2}{4W_{2\beta}}}}{2\sqrt{\pi W_{2\beta}}}\right]$$
$$= \frac{\sigma^2}{\Gamma(2\beta)}\int_0^\infty\frac{e^{-\frac{h^2}{4w}}}{2\sqrt{\pi w}}w^{2\beta-1}e^{-\mu w}dw$$
$$= \frac{\sigma^2}{\sqrt{\pi}\Gamma(2\beta)}\left(\frac{h}{2\sqrt{\mu}}\right)^{2\beta-\frac{1}{2}}K_{2\beta-\frac{1}{2}}(h\sqrt{\mu}).$$

We notice that

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}} P(B(W_{2\beta}) \in dh)/dh$$

where $B(W_{2\beta})$ is a Brownian motion with random time $W_{2\beta}$. Thus, we obtain that

$$f(\tau) = \int_{-\infty}^{\infty} e^{i\tau h} Cov_X(h)\, dh = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^{\infty} e^{-w\tau^2} w^{2\beta-1} e^{-\mu w} dw = \frac{\sigma^2}{(\mu+\tau^2)^{2\beta}}. \qquad \square$$

An alternative representation of the process (3.2) can be also given in terms of the Bessel function K_ν. In particular, we observe that

$$X(t) = \frac{1}{\sqrt{\pi}\Gamma(\beta)} \int_{-\infty}^{+\infty} \mathcal{E}(t+x) \left(\frac{|x|}{2\sqrt{\mu}} \right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(|x|\sqrt{\mu})\, dx$$

The covariance function of (3.2) can be alternatively written as

$$\begin{aligned}\mathbb{E}X(t)X(t+h) =& \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^{\infty} e^{-\mu w_1} w_1^{\beta-1} dw_1 \int_0^{\infty} e^{-\mu w_2} w_2^{\beta-1} dw_2 \int_{-\infty}^{\infty} \frac{e^{-\frac{x_1^2}{4w_1}}}{2\sqrt{\pi w_1}} \frac{e^{-\frac{(x_1-h)^2}{4w_2}}}{2\sqrt{\pi w_2}} dx_1 \\ =& \frac{\sigma^2}{\pi\Gamma^2(\beta)} \int_{-\infty}^{+\infty} \left(\frac{|x_1||x_1-h|}{4\mu} \right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(\sqrt{\mu}|x_1|)\, K_{\beta-\frac{1}{2}}(\sqrt{\mu}|x_1-h|)\, dx_1\end{aligned}$$

where, in the last step we applied formula (2.21).

We now pass to the general even-order fractional equation (1.4).

Theorem 3.2. *A generalized m.s. solution to the equation* (1.4) *is*

$$\begin{aligned}X(t) =& \frac{1}{\mu^\beta} \mathbf{E}[\mathcal{E}(t+Y_{2n}(W_\beta))], \quad \beta>0,\ \mu>0 \\ =& \frac{1}{\Gamma(\beta)} \int_0^{\infty} w^{\beta-1} e^{-\mu w} \int_{-\infty}^{+\infty} u_{2n}(x,w) \mathcal{E}(t+x)\, dx\, dw.\end{aligned} \tag{3.7}$$

Moreover, the spectral density of (3.7) *reads*

$$f(\tau) = \frac{\sigma^2}{(\mu+\tau^{2n})^{2\beta}} \tag{3.8}$$

and the related covariance function becomes

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}\left[u_{2n}(h, W_{2\beta})\right]. \tag{3.9}$$

Proof. The solution $u_{2n}(x,t)$ to

$$\frac{\partial}{\partial t} u_{2n} \;=\; (-1)^{n+1} \frac{\partial^{2n}}{\partial x^{2n}} u_{2n} \tag{3.10}$$

has Fourier transform

$$U(\beta,t) \;=\; e^{(-1)^{n+1}(-i\beta)^{2n} t} \;=\; e^{-\beta^{2n} t}. \tag{3.11}$$

We write

$$e^{-w\frac{\partial^{2n}}{\partial t^{2n}}} \;=\; \int_{-\infty}^{\infty} e^{ix\frac{\partial}{\partial t}} u_{2n}(x,w)\, dx. \tag{3.12}$$

Since

$$U(-i\beta,t) \;=\; e^{-(-1)^n \beta^{2n} t}, \tag{3.13}$$

we also write

$$e^{-w(-1)^n \frac{\partial^{2n}}{\partial t^{2n}}} \; = \int_{-\infty}^{\infty} e^{x\frac{\partial}{\partial t}} u_{2n}(x,w)\, dx. \tag{3.14}$$

In conclusion, we have that

$$\begin{aligned} X(t) &= \left(\mu + (-1)^n \frac{\partial^{2n}}{\partial t^{2n}}\right)^{-\beta} \mathcal{E}(t) \qquad (3.15)\\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty dw\, e^{-\mu w} w^{\beta-1} \left(\int_{-\infty}^{+\infty} dx\, u_{2n}(x,w)\, e^{x\frac{\partial}{\partial t}} \mathcal{E}(t)\right)\\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty dw\, e^{-\mu w} w^{\beta-1} \int_{-\infty}^{+\infty} dx\, u_{2n}\left(x,w\right)\, \mathcal{E}(t+x) \qquad (3.16)\end{aligned}$$

and this confirms (3.7).

From (3.7), in view of (2.17), we obtain

$$\begin{aligned} \mathbb{E}X(t)X(t+h) =& \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1\, w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2\, w_2^{\beta-1} e^{-\mu w_2}\\ &\cdot \int_{-\infty}^{+\infty} dx_1\, u_{2n}\left(x_1,w_1\right) \int_{-\infty}^{+\infty} dx_2\, u_{2n}\left(x_2,w_2\right)\, \delta(x_2 - x_1 + h)\\ =& \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1\, w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2\, w_2^{\beta-1} e^{-\mu w_2}\\ &\cdot \int_{-\infty}^{+\infty} dx_1\, u_{2n}\left(x_1,w_1\right)\, u_{2n}\left(x_1 - h, w_2\right)\\ =& \frac{\sigma^2}{\Gamma^2(\beta)} \int_0^\infty dw_1\, w_1^{\beta-1} e^{-\mu w_1} \int_0^\infty dw_2\, w_2^{\beta-1} e^{-\mu w_2}\, u_{2n}(h, w_1 + w_2)\\ =& \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n}(h, W_1 + W_2).\end{aligned}$$

By following the same arguments as in the previous proof, we get that

$$\mathbb{E}X(t)X(t+h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E} u_{2n}(h, W_{2\beta}) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw\, w^{2\beta-1} e^{-\mu w}\, u_{2n}(h,w)$$

The spectral density of $X(t)$ is therefore

$$f(\tau) = \frac{\sigma^2}{\Gamma(2\beta)} \int_0^\infty dw\, w^{2\beta-1} e^{-\mu w - \tau^{2n} w} = \frac{\sigma^2}{(\mu + \tau^{2n})^{2\beta}}. \qquad \square$$

Theorem 3.2 extends the results of Theorem 3.1 when even-order heat-type equations are involved.

We now pass to the study of the equation (1.9) for $n = 1$ and $\kappa = \mp 1$,

$$\left(\mu + \kappa \frac{d^3}{dt^3}\right)^{\beta} X(t) = \mathcal{E}(t), \quad \mu > 0,\ \beta > 0,\ t \in \mathbb{R}. \tag{3.17}$$

Theorem 3.3. *A generalized solution to the equation* (3.17) *is*

$$\begin{aligned} X(t) =& \frac{1}{\mu^\beta} \mathbf{E}[\mathcal{E}(t + Y_3(W_\beta))], \quad \beta > 0,\ \mu > 0 \qquad (3.18)\\ =& \frac{1}{\Gamma\left(\beta\right)} \int_{-\infty}^{\infty} \mathcal{E}(t+x) \int_0^\infty w^{\beta-1} e^{-\mu w} \frac{1}{\sqrt[3]{3w}} \boldsymbol{Ai}\left(\frac{\kappa x}{\sqrt[3]{3w}}\right) dw dx.\end{aligned}$$

Moreover, the covariance function

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}} \mathbb{E}\left[\frac{\sigma^2}{\sqrt[3]{3W_{2\beta}}} \mathit{Ai} \left(\frac{-\kappa h}{\sqrt[3]{3W_{2\beta}}} \right) \right] \tag{3.19}$$

where $\mathit{Ai}(x)$ *is the Airy function has Fourier transform*

$$f(\tau) = \frac{\sigma^2}{(\mu + i\kappa\tau^3)^{2\beta}}. \tag{3.20}$$

Proof. By following the approach adopted above, after some calculation, we can write that

$$X^-(t) \; = \; \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w + w \frac{d^3}{dt^3}} \, \mathcal{E}(t) \, dw \tag{3.21}$$

is the solution to

$$\left(\mu - \frac{d^3}{dt^3} \right)^\beta X(t) = \mathcal{E}(t) \tag{3.22}$$

whereas

$$X^+(t) \; = \; \frac{1}{\Gamma(\beta)} \int_0^\infty w^{\beta-1} e^{-\mu w - w \frac{d^3}{dt^3}} \, \mathcal{E}(t) \, dw \tag{3.23}$$

is the solution to

$$\left(\mu + \frac{d^3}{dt^3} \right)^\beta X(t) = \mathcal{E}(t) \tag{3.24}$$

The third-order heat type equation

$$\frac{\partial}{\partial t} u \; = \; \kappa \, \frac{\partial^3}{\partial x^3} u, \qquad u(x,0) \; = \; 0, \tag{3.25}$$

has solution, for $\kappa = -1$,

$$u(x,t) \; = \; \frac{1}{\sqrt[3]{3t}} \, \mathrm{Ai} \left(\frac{x}{\sqrt[3]{3t}} \right), \qquad x \in \mathbb{R}, t > 0, \tag{3.26}$$

with Fourier transform

$$\int_{-\infty}^\infty e^{i\beta x} u(x,t) \, dx \; = \; e^{-it\beta^3}. \tag{3.27}$$

Formula (3.27) leads to the integral

$$\int_{-\infty}^\infty e^{\theta x} u(x,t) \, dx \; = \; e^{t\theta^3}, \quad \theta \in \mathbb{R}$$

because of the asymptotic behaviour of the Airy function (see [1] and [11]). The solution to (1.9) with $n = 1$ (that is $\kappa = -1$) is therefore (3.21).

The equation (3.25) has solution, for $\kappa = +1$, given by

$$u(x,t) \; = \; \frac{1}{\sqrt[3]{3t}} \, \mathrm{Ai} \left(\frac{-x}{\sqrt[3]{3t}} \right), \qquad x \in \mathbb{R}, t > 0. \tag{3.28}$$

Thus, by following the same reasoning as before, we arrive at

$$\int_{-\infty}^\infty e^{\theta x} u(x,t) \, dx \; = \; e^{-t\theta^3}, \quad \theta \in \mathbb{R}$$

and we obtain that (3.23) solves (3.17) with $\kappa = +1$ is (3.23).

In light of (2.17) we get

$$\begin{aligned}\mathbb{E}[X^-(t)\,X^-(t+h)] =&\frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty e^{-\mu w_1}dw_1\,w_1^{\beta-1}\int_0^\infty e^{-\mu w_2}dw_2\,w_2^{\beta-1}\\ &\cdot\int_{-\infty}^\infty \frac{1}{\sqrt[3]{3w_1}}\mathrm{Ai}\left(\frac{x_1}{\sqrt[3]{3w_1}}\right)\frac{1}{\sqrt[3]{3w_2}}\mathrm{Ai}\left(\frac{x_1-h}{\sqrt[3]{3w_2}}\right)dx_1\\ =&\frac{\sigma^2}{\Gamma^2(\beta)}\int_0^\infty e^{-\mu w_1}dw_1\,w_1^{\beta-1}\int_0^\infty e^{-\mu w_2}dw_2\,w_2^{\beta-1}\\ &\cdot\frac{1}{\sqrt[3]{3(w_1+w_2)}}\mathrm{Ai}\left(\frac{h}{\sqrt[3]{3(w_1+w_2)}}\right)\\ =&\frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[\frac{1}{\sqrt[3]{3W_{2\beta}}}\mathrm{Ai}\left(\frac{h}{\sqrt[3]{3W_{2\beta}}}\right)\right].\end{aligned}$$

From the Fourier transform (3.27), we get that

$$\begin{aligned}f^-(\tau) =&\frac{\sigma^2}{\mu^{2\beta}}\int_{\mathbb{R}} e^{i\tau h}\,\mathbb{E}\left[\frac{1}{\sqrt[3]{3W_{2\beta}}}\mathrm{Ai}\left(\frac{h}{\sqrt[3]{3W_{2\beta}}}\right)\right]dh\\ =&\frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[e^{-i\tau^3 W_{2\beta}}\right]\\ =&\frac{\sigma^2}{(\mu+i\tau^3)^{2\beta}}\\ =&\frac{\sigma^2 e^{-i2\beta\arctan\frac{\tau^3}{\mu}}}{(\mu^2+\tau^6)^\beta}.\end{aligned}$$

Also, we obtain that

$$\mathbb{E}[X^+(t)\,X^+(t+h)] =\frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}\left[\frac{1}{\sqrt[3]{3W_{2\beta}}}\mathrm{Ai}\left(\frac{-h}{\sqrt[3]{3W_{2\beta}}}\right)\right].$$

with Fourier transform

$$f^+(\tau) = \frac{\sigma^2}{(\mu-i\tau^3)^{2\beta}} = \frac{\sigma^2 e^{+i2\beta\arctan\frac{\tau^3}{\mu}}}{(\mu^2+\tau^6)^\beta}. \tag{3.29}$$

□

Theorem 3.4. *A generalized m.s. solution to the equation* (1.9) *is*

$$\begin{aligned}X(t) =&\frac{1}{\mu^\beta}\mathbf{E}[\mathcal{E}(t+Y_{2n+1}(W_\beta))],\quad \beta>0,\ \mu>0\\ =&\frac{1}{\Gamma(\beta)}\int_0^\infty w^{\beta-1}e^{-\mu w}\int_{-\infty}^{+\infty} u_{2n+1}(\kappa x, w)\mathcal{E}(t+x)dwdx.\end{aligned}$$

Moreover, the covariance function

$$Cov_X(h) = \frac{\sigma^2}{\mu^{2\beta}}\mathbb{E}u_{2n+1}(\kappa h, W_{2\beta})$$

has Fourier transform

$$f(\tau) = \frac{\sigma^2}{(\mu+i\kappa\tau^{2n+1})^{2\beta}} = \frac{\sigma^2 e^{-i2\beta\kappa\arctan\frac{\tau^{2n+1}}{\mu}}}{(\mu^2+\tau^{2(2n+1)})^\beta}.$$

Proof. The proof follows the same lines as in the previous theorem. □

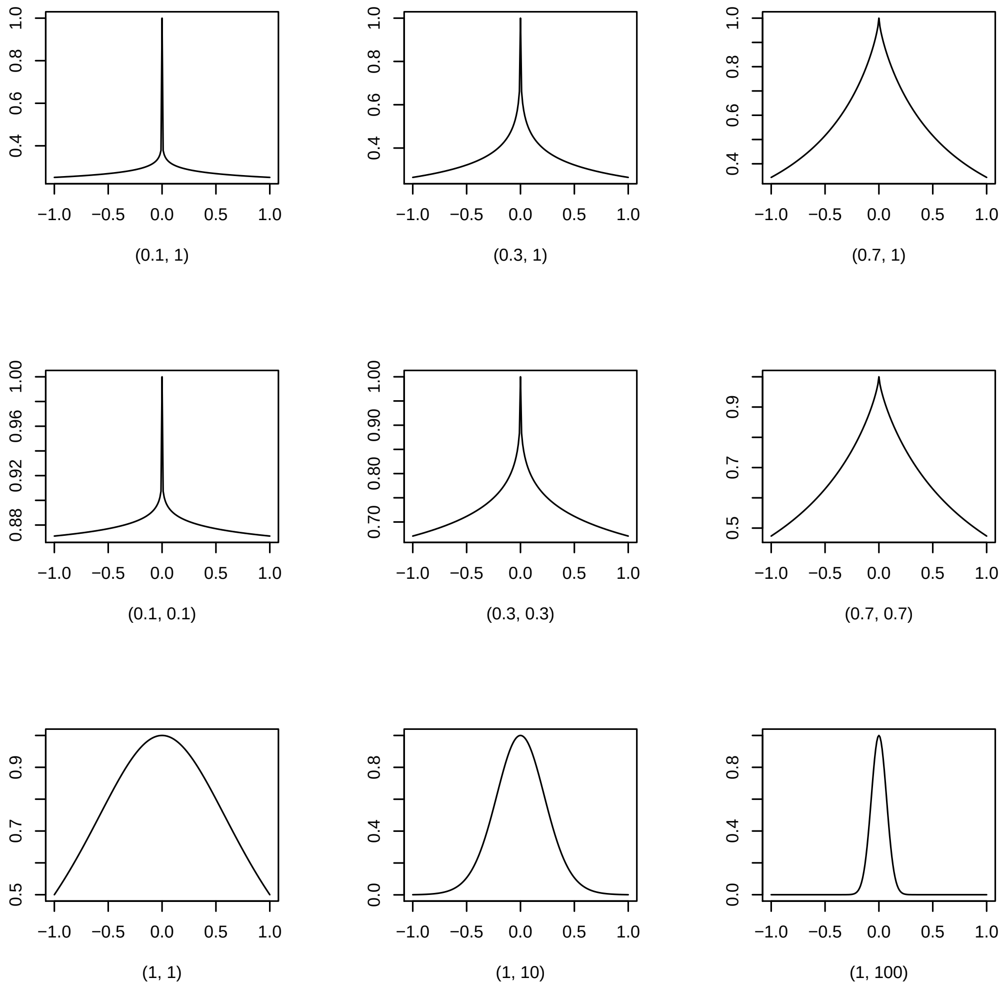

Figure 1: The spectral density (1.3) with different values for the parameters (α, β).

References

[1] G. Accetta, E. Orsingher, Asymptotic expansion of fundamental solutions of higher order heat equations. *Random Oper. Stochastic Equations*, 5 (1997) 217–226. MR-1483009

[2] J. Angulo, M. Kelbert, N. Leonenko, M.D. Ruiz-Medina, Spatiotemporal random fields associated with stochastic fractional Helmholtz and heat equations. *Stochastic Environmental Research & Risk Assessment* **22** (2008) suppl. 1, 3–13. MR-2416949

[3] Ju. L. Daletsky, V. Ju. Krylov, R. A. Minlos and V. N. Sudakov. Functional integration and measures in function spaces. (Russian) 1964 *Proc. Fourth All-Union Math. Congr.* (Leningrad, 1961) (Russian), Vol. II pp. 282–292 Izdat. "Nauka", Leningrad. MR-0219693

[4] R. Gay, C. C. Heyde, On a class of random field models which allows long range dependence. *Biometrika*, 77 (1990) 401–403. MR-1064814

[5] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, Series and Products*, Accademic Press, Boston, (1994). MR-1243179

[6] M.Y. Kelbert, N.N. Leonenko, M.D. Ruiz-Medina, Fractional random fields associated with stochastic fractional heat equations. *Advances in Applied Probability* 37(1), (2005) 108–133. MR-2135156

[7] V. Ju. Krylov, Some properties of the distribution corresponding to the equation $\partial u/\partial t = (-1)^{p+1}\partial^{2q}u/\partial x^{2q}$. *Dokl. Akad. Nauk SSSR 132 1254–1257* (Russian); translated as *Soviet Math. Dokl.* 1 (1960) 760–763. MR-0118953

[8] A. Lachal, Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations. *Electron. J. Probab.* 8 (2003), no. 20, 1–53. MR-2041821

[9] V.I. Ladokhin, On the "measure" in functional space corresponding to a complex diffusion coefficient. *Kazan. Gos. Univ. Učen. Zap.* 123(6), 1963, 36–42 (in Russian). MR-0192546

[10] N. N. Lebedev, *Special functions and their applications*, Dover, New York (1972). MR-0350075

[11] X. Li, R. Wong, Asymptotic behaviour of the fundamental solution to $\partial u/\partial t = -(-\triangle)^m u$. *Proceedings: Mathematical and Physical Sciences*, 441 (1993) 423–432. MR-1219264

[12] Miyamoto, M., An extension of certain quasi-measure. *Proc. Japan Acad.* 42, (1966) 70–74. MR-0198545

[13] Orsingher, E., Processes governed by signed measures connected with third-order "heat-type" equations. *Lith. Math. J.* 31, (1991) 321–334. MR-1161372

[14] E. Orsingher, M. D'Ovidio, Probabilistic representation of fundamental solutions to $\frac{\partial u}{\partial t} = \kappa_m \frac{\partial^m u}{\partial x^m}$. *Electronic Communications in Probability*, 17, (2012), 1–12. MR-2965747

[15] M. Renardy, R. C. Rogers, *An Introduction to Partial Differential Equations* (Texts in Applied Mathematics) 2nd Edition, Springer-Verlag New York, Inc (2004). MR-2028503

[16] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives. *Gordon and Breach Science Publishers*, Yverdon (1993). MR-1347689

[17] N. V. Smorodina, M. M. Faddeev, The probabilistic representation of the solutions of a class of evolution equations. *Zap. Nauchn. Sem. POMI*, 384, (2010) 238–266. MR-2749364

[18] E. Wong, B. Hajek, Stochastic Processes in Engineering Systems *Springer Texts in Electrical Engineering*, New York Inc (1985). MR-0787046

Acknowledgments. We are grateful to the referee for her/his remarks which certainly improved the quality of the paper.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS[1])
- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1]LOCKSS: Lots of Copies Keep Stuff Safe `http://www.lockss.org/`
[2]EJMS: Electronic Journal Management System `http://www.vtex.lt/en/ejms.html`
[3]IMS: Institute of Mathematical Statistics `http://www.imstat.org/`
[4]BS: Bernoulli Society `http://www.bernoulli-society.org/`
[5]Project Euclid: `https://projecteuclid.org/`
[6]IMS Open Access Fund: `http://www.imstat.org/publications/open.htm`