Invent. math. (2016) 206:29–55
DOI 10.1007/s00222-016-0648-6

Convergence of the solutions of the discounted Hamilton–Jacobi equation

Convergence of the discounted solutions

Andrea Davini[1] · Albert Fathi[2] · Renato Iturriaga[3] · Maxime Zavidovique[4]

Received: 12 March 2014 / Accepted: 10 January 2016 / Published online: 26 January 2016

Abstract We consider a continuous coercive Hamiltonian H on the cotangent bundle of the compact connected manifold M which is convex in the momentum. If $u_\lambda : M \to \mathbb{R}$ is the viscosity solution of the discounted equation

$$\lambda u_\lambda(x) + H(x, \mathrm{d}_x u_\lambda) = c(H),$$

where $c(H)$ is the critical value, we prove that u_λ converges uniformly, as $\lambda \to 0$, to a specific solution $u_0 : M \to \mathbb{R}$ of the critical equation

Work supported by ANR-07-BLAN-0361-02 KAM faible & ANR-12-BS01-0020 WKBHJ & LAISLA Conacyt-Cnrs.

✉ Albert Fathi
albert.fathi@ens-lyon.fr

Andrea Davini
davini@mat.uniroma1.it

Renato Iturriaga
renato@cimat.mx

Maxime Zavidovique
zavidovique@math.jussieu.fr

[1] Dip. di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, 00185 Roma, Italy

[2] UMPA, ENS-Lyon, 46 allée d'Italie, 69364 Lyon Cedex 7, France

[3] Cimat, Valenciana Guanajuato 36000, Mexico

[4] IMJ-PRG (projet Analyse Algébrique), UPMC, 4, place Jussieu, Case 247, 75252 Paris Cedex 5, France

$$H(x, \mathrm{d}_x u) = c(H).$$

We characterize u_0 in terms of Peierls barrier and projected Mather measures. As a corollary, we infer that the ergodic approximation, as introduced by Lions, Papanicolaou and Varadhan in 1987 in their seminal paper on homogenization of Hamilton–Jacobi equations, selects a specific corrector in the limit.

Mathematics Subject Classification 35B40 · 37J50 · 49L25

1 Introduction

The so called *ergodic approximation* is a technique introduced in [18] to show the existence of viscosity solutions to an equation of the kind

$$H(x, \mathrm{d}_x u) = c \qquad \text{in } \mathbb{T}^m, \tag{1.1}$$

where c is a real constant and H, the *Hamiltonian*, is a continuous function defined on $\mathbb{T}^m \times \mathbb{R}^m$, where $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is the canonical flat torus.

In fact, this technique works as well for a Hamiltonian defined on the cotangent bundle of a compact manifold. Therefore, in the sequel, $H : T^*M \to \mathbb{R}$ will be a given continuous function, called the Hamiltonian, where T^*M is the cotangent bundle of M, a compact connected manifold without boundary.

The method in [18] to find solutions of (1.1) is to perturb the Hamiltonian and to study the *discounted equation*

$$\lambda u(x) + H(x, \mathrm{d}_x u) = 0 \qquad \text{in} \quad M, \tag{1.2}$$

where λ is a positive parameter. This new equation obeys a maximum principle, and therefore it has a unique solution $v_\lambda : M \to \mathbb{R}$. The idea is then to study the behavior of v_λ when the *discount factor* λ tends to zero. When the Hamiltonian $H(x, p)$ is coercive in p, uniformly with respect to x, the functions λv_λ are equi–bounded and the v_λ are equi–Lipschitz. Furthermore, $-\lambda v_\lambda$ uniformly converges on M, as λ tends to 0, to a constant $c(H)$, henceforth termed *critical value*. By setting $\hat{v}_\lambda := v_\lambda - \min_M v_\lambda$, one obtains an equi–bounded and equi–Lipschitz family of functions satisfying, for each $\lambda > 0$,

$$H(x, \mathrm{d}_x \hat{v}_\lambda) = -\lambda \min_M v_\lambda \qquad \text{in } M$$

in the viscosity sense. By the Ascoli–Arzelà Theorem and the stability of the notion of viscosity solution, the functions $\hat{v}_\lambda$ uniformly converge, *along* *subsequences* as λ goes to 0, to viscosity solutions of the *critical equation*

$$H(x, \mathrm{d}_x u) = c(H) \qquad \text{in } M. \tag{1.3}$$

This is also the sole equation of the family (1.1) that admits solutions. Solutions, subsolutions and supersolutions of (1.3) will be termed *critical* in the sequel.

Due to the lack of a uniqueness result for the critical equation, it is not clear at this point that limits of $\hat{v}_\lambda$ along different subsequences yield the same solution of (1.3). In this paper, we address the problem when H is convex in the momentum. The main theorem we prove is the following:

Theorem 1.1 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive and convex in the momentum. For $\lambda > 0$, denote by $u_\lambda : M \to \mathbb{R}$ the unique continuous viscosity solution of*

$$\lambda u_\lambda + H(x, \mathrm{d}_x u_\lambda) = c(H) \qquad \textit{in} \quad M, \tag{1.4}$$

where $c(H)$ is the critical value of H. The family u_λ uniformly converges, as $\lambda \to 0$, to a single critical solution u_0.

Note that we have replaced 0 in the second member of (1.2) by the critical constant $c(H)$. With this choice, the solutions of (1.4) are uniformly bounded independently of λ, see Proposition 2.6. We also remark that the solution of

$$\lambda u + H(x, \mathrm{d}_x u) = c \qquad \text{in } M$$

is $u_\lambda + \big(c - c(H)\big)/\lambda$, where u_λ is the solution of (1.4). In particular, there is no other c for which the family of solutions may be bounded, independently of λ.

A straightforward consequence of this and of Theorem 1.1 is that the family $(\hat{v}_\lambda)_{\lambda>0}$ considered in [18] also converges to a particular critical solution:

Corollary 1.2 *Take H as above. For every $\lambda > 0$, let v_λ be the unique continuous viscosity solution of (1.2), and set $\hat{v}_\lambda := v_\lambda - \min_M v_\lambda$. Then $\hat{v}_\lambda = u_\lambda - \min_M u_\lambda$, in particular the family $\hat{v}_\lambda$ uniformly converges, as $\lambda \to 0$, to $u_0 - \min_M u_0$.*

As we will see, without loss of generality we can assume in Theorem 1.1 that H is superlinear. In that case, by Fenchel's formula, the Hamiltonian H has a conjugated Lagrangian $L : TM \to \mathbb{R}$ which is superlinear and convex in the fibers of the tangent bundle. We can then apply the weak KAM theory—or rather its extension to general Lagrangians—to characterize u_0 in terms of the Peierls barrier and of projected Mather measures, defined respectively by Eq. (5.3) and Definition 5.8 in Sect. 1.1.

Proposition 1.3 *The function $u_0 = \lim_{\lambda \to 0} u_\lambda$, obtained in Theorem 1.1 above, can be characterized in either of the following two ways:*

(i) *it is the largest critical subsolution $u : M \to \mathbb{R}$ such that $\int_M u \, d\mu \le 0$ for every projected Mather measure μ,*
(ii) *it is the infimum over all projected Mather measures μ of the functions h_μ defined by $h_\mu(x) := \int_M h(y, x) \, d\mu(y)$, where h is the Peierls barrier.*

The theorem and proposition above extend results of Renato Iturriaga and Hector Sánchez-Morgado [17], where the convergence is proved for a Tonelli Hamiltonian under the assumption that the Aubry set consists of a finite number of hyperbolic fixed points of the Lagrangian flow. While this work was in progress, the authors were aware that Diogo Gomes [15] found some constraints on the possible accumulation points of u_λ in terms of a concept of generalized Mather measures. In April 2014, after the first version of this work was completed, Diogo Gomes showed to the second author how to deduce the convergence of the u_λ, as $\lambda \to 0$, from [15,16]. Diogo Gomes will publish the details elsewhere.

Our analysis strongly relies on the convexity of the Hamiltonian in the momentum: we clearly take advantage of the rich qualitative analysis for the critical equation made available by weak KAM Theory; moreover, in the study of the asymptotics, we make a crucial use of a suitable representation formula for the solution of the discounted equation, that holds true due to the convexity assumption. It would be interesting to understand whether the selection principle herein pointed out still takes place in the non–convex case. A positive hint in this direction is provided by the following

Proposition 1.4 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, which is coercive in the momentum, uniformly with respect to x. Suppose that the constants are critical subsolutions. Then $u_\lambda \ge 0$ on M for every $\lambda > 0$, and $u_\lambda \nearrow u_0$ uniformly as $\lambda \searrow 0$ for some critical solution u_0.*

We now give some explanations about the content of the paper and the proofs of our results.

In Sect. 2, we fix notations and recall some known facts on solutions of both the stationary and the discounted Hamilton-Jacobi equation. Section 3 is the crux of the work. The starting point of our analysis was to guess the possible limit u_0. A first characterization of u_0 is given in terms of Mather (minimizing) measures, see (3.2). The properties of these measures are recalled in Appendix 1. To prove Theorem 1.1, we use that the family of functions $\{u_\lambda \mid \lambda > 0\}$ is compact in the uniform topology, and we show that every accumulation point u of the u_λ, as $\lambda \to 0$, is necessary equal to u_0. This is done by showing both inequalities $u \le u_0$ and $u_0 \le u$. The proof of $u \le u_0$ is rather simple and is given in Proposition 3.4. The proof of $u_0 \le u$ is more delicate. We first establish in Appendix 2 the pointwise expression of u_λ as an infimum of Laplace transforms, see Eq. (3.4). Moreover, we show that such infimum is attained, see Proposition 3.5. This representation allows us to construct Mather minimizing measures related to accumulation points of the u_λ using a limit of Laplace type averages, see Eq. (3.5), instead of the usual Birkhoff averages, see Eq. (5.5), that are not enough here. In Sect. 4, we give another representation of the limit u_0.

The authors would like to thank the referees for their careful reading, and their useful suggestions.

2 Preliminaries

2.1 Generalities

In this work, we will denote by M a compact connected smooth manifold without boundary of dimension m. It will be convenient to endow M with an auxiliary C^∞ Riemannian metric. The associated Riemannian distance on M will be denoted by d. We denote by TM the tangent bundle and by $\pi : TM \to M$ the canonical projection. A point of TM will be denoted by (x, v) with $x \in M$ and $v \in T_x M = \pi^{-1}(x)$. In the same way, a point of the cotangent bundle T^*M will be denoted by (x, p), with $x \in M$ and $p \in T_x^* M$ a linear form on the vector space $T_x M$. We will denote by $p(v)$ the value of the linear form $p \in T_x^* M$ evaluated at $v \in T_x M$, and by $\|v\|_x$ the norm of v at the point x. We will use the same notation $\|p\|_x$ for the dual norm of a form $p \in T_x^* M$.

On a smooth manifold like M, there is an intrinsic notion of measure zero set: a subset Z of M is said to be of measure zero, if for every smooth coordinate patch $\varphi : U \to \mathbb{R}^m$, the image $\varphi(U \cap Z)$ has Lebesgue measure 0 in $\mathbb{R}^m$. Note that Z has measure zero in this sense if and only if it has measure 0 for the *Riemannian volume measure* associated to the Riemannian metric. We say that a property holds *almost everywhere* (*a.e.* for short) on M if it holds up to a set of measure zero as defined above.

Since M is compact, we can endow the space $C(M)$ of continuous real valued functions on M with the *sup–norm*

$$\|u\|_\infty := \sup_{x \in M} |u(x)|, \qquad u \in C(M).$$

We will say that κ is a *Lipschitz constant* for $u \in C(M)$ if it satisfies $u(x) - u(y) \leq \kappa\, d(x, y)$ for every $x, y \in M$. Any such a function will be termed *Lipschitz*, or κ*–Lipschitz* if we want to specify the Lipschitz constant.

If the function $u : M \to \mathbb{R}$ is differentiable at a point $x \in M$, we will denote by $d_x u$ its derivative (called also differential). Rademacher's theorem states that a (locally) Lipschitz function has a derivative almost everywhere.

We will consider Hamilton–Jacobi equations of the form

$$G(u(x), x, \mathrm{d}_x u) = 0 \qquad \text{in } M, \tag{2.1}$$

where $G \in \mathrm{C}(\mathbb{R} \times T^*M)$. The notion of *solution, subsolution* and *supersolution* of (2.1) adopted in this paper is the one in the *viscosity sense*, see [2,3,10]. Solutions, subsolutions and supersolutions will be implicitly assumed continuous and the adjective *viscosity* will be often omitted, with no further specification.

A Lipschitz–continuous subsolution u is also an almost everywhere subsolution, i.e. $G(u(x), x, d_x u) \le 0$ for a.e. $x \in M$. This is a straightforward consequence of the notion of viscosity subsolution and of Rademacher's theorem. The converse is true when G is convex in p, see [2,3,10,22].

We will also use the following results, see for instance [3,10,22]:

Proposition 2.1 *Assume $G \in \mathrm{C}(T^*M)$ such that $G(x, \cdot)$ is convex in T_x^*M for every $x \in M$, and let $u \in \mathrm{C}(M)$. The following properties hold:*

- (i) *if u is the pointwise supremum (respectively, infimum) of a family of subsolutions (resp., supersolutions) to* (2.1)*, then u is a subsolution (resp., supersolution) of* (2.1)*;*
- (ii) *if u is the pointwise infimum of a family of equi–Lipschitz subsolutions to* (2.1)*, then u is a Lipschitz subsolution of* (2.1)*;*
- (iii) *if u is a convex combination of a family of equi–Lipschitz subsolutions to* (2.1)*, then u is a Lipschitz subsolution of* (2.1).

More precisely, items (ii) and (iii) above require the convexity of G in the momentum, while item (i) is actually independent of this.

We conclude this section by a standard approximation result that we shall repeatedly use in our analysis. The proof is done either by mollifications and partitions of unity or by a direct application of [12, Theorem 8.1] and is therefore omitted.

Lemma 2.2 *Assume $G \in \mathrm{C}(T^*M)$ such that $G(x, \cdot)$ is convex in T_x^*M for every $x \in M$, and let u be a Lipschitz subsolution of equation* (2.1)*. Then, for all $\varepsilon > 0$, there exists a smooth function $u_\varepsilon : M \to \mathbb{R}$ such that $\|u - u_\varepsilon\|_\infty \le \varepsilon$ and $G(x, \mathrm{d}_x u_\varepsilon) \le \varepsilon$ for all $x \in M$.*

2.2 Critical and discounted Hamilton–Jacobi equations

In the sequel, we will call *Hamiltonian* a continuous function $H : T^*M \to \mathbb{R}$. If not otherwise stated, we will always assume that H satisfies the following assumptions:

(H1) (Convexity) For every $x \in M$, the map $p \mapsto H(x, p)$ is convex on $T_x^* M$.
(H2) (Coercivity) $H(x, p) \to +\infty$ as $\|p\|_x \to +\infty$ uniformly in $x \in M$.

The coercivity condition will be actually reinforced as follows:

(H2′) (Superlinearity) $H(x, p)/\|p\|_x \to +\infty$ as $\|p\|_x \to +\infty$ uniformly in $x \in M$.

We will show below that, for our study, we can always reduce to the case of a superlinear Hamiltonian, without any loss of generality.

Conditions (H2) and (H2′) are given in terms of the norm $\|\cdot\|_x$ associated with the Riemannian metric, but they do not actually depend on the particular choice of it for all Riemannian metrics are equivalent on a compact manifold.

Next, we recall some preliminary facts about stationary Hamilton–Jacobi equations we will use in the sequel. In the remainder of this section, we assume that H satisfies condition (H2) only, without requiring convexity in the momentum.

For $c \in \mathbb{R}$, we consider the Hamilton–Jacobi equation

$$H(x, \mathrm{d}_x u) = c. \tag{2.2}$$

Notice that any given C^1 function $u : M \to \mathbb{R}$ is a subsolution (resp. supersolution) of $H(x, \mathrm{d}_x u) = c$ provided $c \geq \max_{x \in M} H(x, \mathrm{d}_x u)$ (resp. $c \leq \min_{x \in M} H(x, \mathrm{d}_x u)$). Moreover, the coercivity of H allow to give the following property of viscosity subsolutions of (2.2), see for instance [3,10].

Proposition 2.3 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying (H2) and let $c \in \mathbb{R}$. Then any viscosity subsolution u of* (2.2) *is Lipschitz continuous, and satisfies*

$$H(x, \mathrm{d}_x u) \leq c \qquad \textit{for a.e. } x \in M.$$

Moreover, the set of viscosity subsolutions of (2.2) *is equi–Lipschitz, with a common Lipschitz constant κ_c given by*

$$\kappa_c = \sup\{\|p\|_x \mid H(x, p) \leq c\}. \tag{2.3}$$

We define the *critical value* $c(H)$ as

$$c(H) = \inf\{c \in \mathbb{R} \mid \text{equation (2.2) admits subsolutions}\}. \tag{2.4}$$

Since H is bounded from below, such an infimum is finite. By the Ascoli–Arzelà Theorem and the stability of the notion of viscosity subsolution, it can be easily proved that such an infimum is attained, meaning that there are subsolutions also at the critical level. Moreover, $c(H)$ is the only real value c for which Eq. (2.2) admits solutions.

We will refer to

$$H(x, \mathrm{d}_x u) = c(H) \tag{2.5}$$

as the *critical equation*. Correspondingly, solutions, subsolutions and supersolutions to (2.5) will be termed *critical* in the sequel.

We will be also interested in the discounted version of (2.5), that is the equation

$$\lambda u(x) + H(x, \mathrm{d}_x u) = c(H) \quad \text{in } M, \tag{2.6}$$

where $\lambda > 0$. The following holds:

Proposition 2.4 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying* (H2) *and $\lambda \geq 0$. Then any subsolution w of* (2.6) *is Lipschitz–continuous and satisfies*

$$\lambda w(x) + H(x, \mathrm{d}_x w) \leq c(H) \quad \textit{for a.e. } x \in M. \tag{2.7}$$

Proof A subsolution w of (2.6) satisfies

$$H(x, \mathrm{d}_x w) \leq c(H) + \|\lambda w\|_\infty \qquad \text{in } M$$

in the viscosity sense, hence it is Lipschitz continuous by the coercivity of H, see [3]. In particular, it satisfies the inequality (2.7) at every differentiability point, i.e. almost everywhere by Rademacher's theorem. □

The crucial difference between the critical equation (2.5) and the discounted equation (2.6) with $\lambda > 0$ is that the latter satisfies a strong comparison principle. In fact, we have the following well known theorem, see for instance [3, Théorème 2.4].

Theorem 2.5 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying* (H2). *If v, u are, respectively, a sub and a supersolution of* (2.6), *with $\lambda > 0$, then $v \leq u$ in M. Moreover, there exists a unique solution of* (2.6).

The following holds:

Proposition 2.6 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying (H2). Then the solutions $\{u_\lambda \mid \lambda > 0\}$ of* (2.6) *are equi–Lipschitz and equi–bounded. In particular, $\|\lambda u_\lambda\|_\infty \to 0$ as $\lambda \to 0$.*

Proof We already know that u_λ is Lipschitz. We want to prove that its Lipschitz constant can be chosen independent of λ. Let us set $\beta = \max_{x \in M} H(x, 0)$. The function $w \equiv -(\beta - c(H))/\lambda$ is obviously a subsolution of (2.6). By Theorem 2.5, we must have $\lambda u_\lambda(x) \geq -\beta + c(H)$ for every $x \in M$. Hence, we get

$$H(x, \mathrm{d}_x u_\lambda) \leq -\lambda u_\lambda(x) + c(H) \leq \beta \quad \text{for a.e. } x \in M,$$

and u_λ is κ_β–Lipschitz by the coercivity of H, with κ_β given by (2.3).

To see they are equi–bounded, take a solution u of (2.5). By addition of suitable constants, we obtain two critical solutions $\underline{u}, \overline{u}$ of Eq. (2.5) such that $\underline{u} \leq 0 \leq \overline{u}$ in M. It is easily seen that, for every fixed $\lambda > 0$, $\underline{u}$ and $\overline{u}$ are, respectively, a sub and a supersolution of (2.6). By the comparison principle stated in Theorem 2.5 we derive

$$\underline{u} \leq u_\lambda \leq \overline{u} \quad \text{in } M \qquad \text{for every } \lambda > 0,$$

as it was to be shown. □

Remark 2.7 Note that $\beta := \max_{x \in M} H(x, 0) \geq c(H)$. In fact, we know that there exists a solution $u : M \to \mathbb{R}$ of equation (2.5). At a minimum x_0 of u the constant function $w \equiv u(x_0)$ is a subtangent, therefore $H(x_0, 0) \geq c(H)$, which implies $\beta \geq c(H)$.

3 Convergence of the solutions of the discounted equation

In this section we will prove our main result, namely that the solutions u_λ of the discounted equation converge, as $\lambda \to 0$, to a particular solution u_0 of the critical equation (2.5). As a warm–up, we prove a result that does not need any convexity assumption on the Hamiltonian.

Proposition 3.1 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying (H2). Suppose that the constants are critical subsolutions. Then $u_\lambda \geq 0$, and $u_\lambda \nearrow u_0$ uniformly as $\lambda \searrow 0$ for some solution u_0 of the critical equation $H(x, \mathrm{d}_x u) = c(H)$.*

Proof The fact that $u_\lambda \geq 0$ is a direct consequence of the Comparison Principle (Theorem 2.5) and of the fact that the function identically equal to 0 is a subsolution of the discounted equation (2.6) for any $\lambda > 0$. Using this fact, for $\lambda' < \lambda$ we get

$$\lambda' u_\lambda + H(x, \mathrm{d}_x u_\lambda) \leq \lambda u_\lambda + H(x, \mathrm{d}_x u_\lambda) = c(H) \qquad \text{in } M$$

in the viscosity sense, i.e. u_λ is a subsolution of the discounted equation for λ'. By the Comparison principle again we infer that $u_{\lambda'} \geq u_\lambda$. The assertion easily follows from this in view of Proposition 2.6. □

Remark 3.2 It is worth noticing that, due to their nonnegative character, the functions u_λ in the above proposition are also subsolutions of the critical equation.

Let us now pass to the case when H is both coercive and convex. Our preliminary remark is that, in view of Propositions 2.3 and 2.6, not all of H is relevant in order to study the discounted and critical equations. To see this, let us take a constant c larger than $\max_{x\in M} H(x, 0)$. We can modify H outside the compact set $\{(x, p) \in T^*M \mid \|p\|_x \leq \kappa\}$, with $\kappa = \kappa_c$ defined according to (2.3), to obtain a new Hamiltonian $\widetilde{H}$ which is still continuous and convex, and satisfies the stronger growth condition (H2′). According to Proposition 2.3, to the proof of Proposition 2.6 and to Remark 2.7, the solutions of the critical and discounted equations corresponding to H and $\widetilde{H}$ are κ-Lipschitz continuous. Since the κ–sublevel of the two Hamiltonians coincide, this means that $c(H) = c(\widetilde{H})$ and the solutions of the corresponding critical and discounted equations are the same.

In the remainder of the paper, we will therefore assume that H is convex and superlinear in p, i.e. it satisfies hypothesis (H1) and (H2′). This allows us to introduce the associated Lagrangian $L : TM \to \mathbb{R}$ and to make use of the tools of Aubry–Mather Theory recalled in Appendices 1 and 2.

The first step consists in identifying a good candidate u_0 for the limit of the solutions u_λ of the discounted equations. To this aim, we consider the family $\mathcal{F}_-$ of subsolutions $u : M \to \mathbb{R}$ of the critical equation (2.5) satisfying the following condition

$$\int_M u(y)\,\mathrm{d}\mu(y) \leq 0 \qquad \text{for every } \mu \in \mathfrak{M}(L), \tag{3.1}$$

where $\mathfrak{M}(L)$ denotes the set of projected Mather measures, see Appendix 1.

Note that, given any critical subsolution u, the function $u - \|u\|_\infty$ is in $\mathcal{F}_-$. Therefore $\mathcal{F}_-$ is not empty.

Lemma 3.3 *The family $\mathcal{F}_-$ is uniformly bounded from above, i.e.*

$$\sup\{u(x) \mid x \in M, u \in \mathcal{F}_-\} < +\infty.$$

Proof The family of critical subsolutions is equi–Lipschitz. Call κ a common Lipschitz constant. Since the set of projected Mather measure μ is not empty, picking such a probability measure μ, for $u \in \mathcal{F}_-$, we have $\min u = \int_M \min u\,\mathrm{d}\mu \leq \int_M u\,\mathrm{d}\mu \leq 0$. Hence $\max u \leq \max u - \min u$. Since u is κ-Lipschitz, we also have $\max u - \min u \leq \kappa\,\mathrm{diam}(M) < +\infty$. □

Therefore we can define $u_0 : M \to \mathbb{R}$ by

$$u_0 := \sup_{\mathcal{F}_-} u. \tag{3.2}$$

As the supremum of a family of viscosity subsolutions, we know that u_0 is itself a critical subsolution, see Proposition 2.1. We will obtain later that u_0 is an element of $\mathcal{F}_-$ and a critical solution, see Theorem 3.8 below.

We now start to study the asymptotic behavior of the discounted value functions u_λ as $\lambda \to 0$ and the relation with u_0. We begin with the following result:

Proposition 3.4 *For $\lambda > 0$, we have $\int_M u_\lambda(x)\,\mathrm{d}\mu(x) \leq 0$, for every $\mu \in \mathfrak{M}(L)$. In particular, if the functions u_{λ_n} uniformly converge to u for some sequence $\lambda_n \to 0$, then $u \leq u_0$ on M.*

Proof By applying Theorem 2.2 with $G(x, p) := \lambda u_\lambda(x) + H(x, p) - c(H)$, we infer that there exists a sequence $(w_n)_n$ of functions in $\mathrm{C}^1(M)$ such that $\|u_\lambda - w_n\|_\infty \leq 1/n$ and

$$\lambda u_\lambda(x) + H(x, \mathrm{d}_x w_n) \leq c(H) + 1/n \quad \text{for every } x \in M.$$

By the Fenchel inequality

$$L(x, v) + H(x, \mathrm{d}_x w_n) \geq \mathrm{d}_x w_n(v) \quad \text{for every } (x, v) \in TM.$$

Combining these two inequalities yields

$$\lambda u_\lambda(x) + \mathrm{d}_x w_n(v) \leq L(x, v) + c(H) + \frac{1}{n} \quad \text{for every } (x, v) \in TM. \tag{3.3}$$

Let us fix some $\tilde{\mu} \in \tilde{\mathfrak{M}}(L)$, and set $\mu = \pi_\# \tilde{\mu} \in \mathfrak{M}(L)$. Since μ is closed and minimizing, we have $\int_{TM} \mathrm{d}_x w_n(v)\,\mathrm{d}\tilde{\mu}(x, v) = 0$, and $\int_{TM} L(x, v)\,\mathrm{d}\tilde{\mu}(x, v) = -c(H)$. Therefore if we integrate (3.3), we obtain

$$\lambda \int_M u_\lambda(x)\,\mathrm{d}\mu(x) \leq \frac{1}{n}.$$

Since $\lambda > 0$, letting $n \to \infty$, yields $\int_M u_\lambda(x)\,\mathrm{d}\mu(x) \leq 0$. If u is the uniform limit of $\left(u_{\lambda_n}\right)_n$ for some $\lambda_n \to 0$, we know that it is a solution of the critical equation (2.5). Moreover, it also has to satisfy $\int_M u(x)\,\mathrm{d}\mu(x) \leq 0$ for every projected Mather measure μ. Therefore $u \in \mathcal{F}_-$ and $u \leq u_0$. □

The next (and final) step is to show that $u \geq u_0$ in M whenever u is the uniform limit of $\left(u_{\lambda_n}\right)_n$ for some $\lambda_n \to 0$. For this, we will exploit the following representation formula for the solution u_λ of the discounted equation (2.6):

$$u_\lambda(x) = \inf_\gamma \int_{-\infty}^0 \mathrm{e}^{\lambda s}\left[L\big(\gamma(s), \dot\gamma(s)\big) + c(H)\right] \mathrm{d}s, \tag{3.4}$$

for every $x \in M$, where the infimum is taken over all absolutely continuous curves $\gamma : (-\infty, 0] \to M$, with $\gamma(0) = x$. We refer the reader to Appendix 2 for more details. Moreover, we will need the following property concerning minimizing curves, whose proof is also given in Appendix 2:

Proposition 3.5 *Let $\lambda > 0$ and $x \in M$. Then there exists a curve $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$ such that*

$$u_\lambda(x) = \mathrm{e}^{-\lambda t} u_\lambda\big(\gamma_x^\lambda(-t)\big) + \int_{-t}^0 \mathrm{e}^{\lambda s}\left[L\big(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s)\big) + c(H)\right] \mathrm{d}s$$

for every $t > 0$. Moreover, there exists a constant $\alpha > 0$, independent of λ and x, such that $\|\dot\gamma_x^\lambda\|_\infty \leq \alpha$. In particular

$$u_\lambda(x) = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\left[L\big(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s)\big) + c(H)\right] \mathrm{d}s.$$

Let us now fix $x \in M$. For every $\lambda > 0$, we choose $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$ as in Proposition 3.5, and we define a measure $\tilde\mu_x^\lambda = \tilde\mu_{\gamma_x^\lambda}^\lambda$ on TM by setting

$$\begin{aligned}\int_{TM} f(x, v)\, \mathrm{d}\tilde\mu_x^\lambda(x, v) :&= \lambda \int_{-\infty}^0 \mathrm{e}^{\lambda s} f\big(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s)\big)\, \mathrm{d}s \\ &= \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\lambda s}) f\big(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s)\big)\, \mathrm{d}s,\end{aligned} \tag{3.5}$$

for every $f \in \mathrm{C}_c(TM)$. The following holds:

Proposition 3.6 *The measures $\{\tilde\mu_x^\lambda \mid \lambda > 0\}$ defined above are probability measures, whose supports are all contained in a common compact subset of TM. In particular, they are relatively compact in the space of probability measures on TM with respect to the weak convergence. Furthermore, if $\left(\tilde\mu_x^{\lambda_n}\right)_n$ is weakly converging to $\tilde\mu$ for some sequence $\lambda_n \to 0$, then $\tilde\mu$ is a (closed) Mather measure.*

Proof Call α a common Lipschitz constant for the family of curves $\{\gamma_x^\lambda \mid \lambda > 0\}$, according to Proposition 3.5. Then the measures $\tilde{\mu}_x^\lambda$ have all support in the compact set $K := \{(x, v) \in TM \mid \|v\|_x \leq \alpha\}$, and they are all probability measures, as can be easily checked by their definition. This readily implies the asserted relative compactness of $\{\tilde{\mu}_x^\lambda \mid \lambda > 0\}$. Let now assume that $\left(\tilde{\mu}_x^{\lambda_n}\right)_n$ is weakly converging to $\tilde{\mu}$ for some $\lambda_n \to 0$. Then $\tilde{\mu}$ is a probability measure with support contained in K, in particular $\int_{TM} \|v\|_x \, \mathrm{d}\tilde{\mu}(x, v) \leq \alpha < +\infty$. Moreover, if $\varphi : M \to \mathbb{R}$ is C^1, then the function $s \mapsto \mathrm{e}^{\lambda s}\varphi\big(\gamma_x^\lambda(s)\big)$ is Lipschitz on $(-\infty, 0]$ with derivative

$$s \mapsto \lambda \mathrm{e}^{\lambda s}\varphi\big(\gamma_x^\lambda(s)\big) + \mathrm{e}^{\lambda s}\mathrm{d}_{\gamma_x^\lambda(s)}\varphi\big(\dot{\gamma}_x^\lambda(s)\big).$$

Hence

$$\varphi\big(\gamma_x^\lambda(0)\big) = \int_{-\infty}^0 \mathrm{e}^{\lambda s}\mathrm{d}_{\gamma_x^\lambda(s)}\varphi\big(\dot{\gamma}_x^\lambda(s)\big)\,\mathrm{d}s + \int_{-\infty}^0 \lambda \mathrm{e}^{\lambda s}\varphi\big(\gamma_x^\lambda(s)\big)\,\mathrm{d}s.$$

Note that the left hand side is bounded by $\|\varphi\|_\infty$, and also

$$\left|\int_{-\infty}^0 \lambda \mathrm{e}^{\lambda s}\varphi\big(\gamma_x^\lambda(s)\big)\,\mathrm{d}s\right| \leq \int_{-\infty}^0 \lambda \mathrm{e}^{\lambda s}\|\varphi\|_\infty\,\mathrm{d}s \leq \|\varphi\|_\infty.$$

It follows that

$$\left|\int_{TM} \mathrm{d}_x\varphi(v)\,\mathrm{d}\tilde{\mu}_{\gamma_x^{\lambda_n}}^{\lambda_n}(x, v)\right| = \left|\lambda_n \int_{-\infty}^0 \mathrm{e}^{\lambda_n s}\mathrm{d}_{\gamma_x^{\lambda_n}(s)}\varphi\big(\dot{\gamma}_x^{\lambda_n}(s)\big)\,\mathrm{d}s\right|$$
$$\leq 2\lambda_n\|\varphi\|_\infty \to 0,$$

as $\lambda_n \to 0$. Therefore we obtain

$$\int_{TM} \mathrm{d}_x\varphi(v)\,\mathrm{d}\tilde{\mu} = \lim_{n\to+\infty}\int_{TM} \mathrm{d}_x\varphi(v)\,\mathrm{d}\tilde{\mu}_{\gamma_x^{\lambda_n}}^{\lambda_n} = 0.$$

It remains to show that $\int_{TM} L(x, v)\,\mathrm{d}\tilde{\mu}(x, v) = -c(H)$. We have

$$\begin{aligned}\int_{TM} L(x, v)\,\mathrm{d}\tilde{\mu}(x, v) &= \lim_{n\to\infty}\int_{TM} L(x, v)\,\mathrm{d}\tilde{\mu}_x^{\lambda_n}(x, v)\\ &= \lim_{n\to\infty}\int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\lambda_n s})L\big(\gamma_x^{\lambda_n}(s), \dot{\gamma}_x^{\lambda_n}(s)\big)\,\mathrm{d}s\\ &= \lim_{n\to\infty}\lambda_n u_{\lambda_n}(x) - c(H) = -c(H),\end{aligned}$$

where the last equality follows from the fact that $\lambda u_\lambda \to 0$, see Proposition 2.6 □

The following lemma will be crucial for the proof of our main result, see Theorem 3.8 below.

Lemma 3.7 *Let w be any critical subsolution. For every $\lambda > 0$ and $x \in M$*

$$u_\lambda(x) \geq w(x) - \int_{TM} w(y)\,\mathrm{d}\tilde{\mu}_x^\lambda(y,v). \tag{3.6}$$

Proof Let $\varepsilon > 0$. According to Theorem 2.2, there exists a smooth function w_ε such that

$$\|w - w_\varepsilon\|_\infty < \varepsilon \quad \text{and} \quad H(x, \mathrm{d}_x w_\varepsilon) < c(H) + \varepsilon \quad \text{for every} \quad x \in M.$$

By the Fenchel inequality, we have

$$\begin{aligned} L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) &\geq \mathrm{d}_{\gamma_x^\lambda(s)} w_\varepsilon\big(\dot{\gamma}_x^\lambda(s)\big) \\ -H(\gamma_x^\lambda(s), \mathrm{d}_{\gamma_x^\lambda(s)} w_\varepsilon) &\geq \mathrm{d}_{\gamma_x^\lambda(s)} w_\varepsilon\big(\dot{\gamma}_x^\lambda(s)\big) - \varepsilon - c(H) \end{aligned}$$

for every $s < 0$. Using the definition of the curve γ_x^λ, see Proposition 3.5, we get

$$\begin{aligned} u_\lambda(x) &= \mathrm{e}^{-\lambda t} u_\lambda\big(\gamma_x^\lambda(-t)\big) + \int_{-t}^0 \mathrm{e}^{\lambda s}\big[L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) + c(H)\big]\mathrm{d}s \\ &\geq \mathrm{e}^{-\lambda t} u_\lambda\big(\gamma_x^\lambda(-t)\big) + \int_{-t}^0 \mathrm{e}^{\lambda s} \mathrm{d}_{\gamma_x^\lambda(s)} w_\varepsilon\big(\dot{\gamma}_x^\lambda(s)\big)\,\mathrm{d}s - \varepsilon \int_{-t}^0 \mathrm{e}^{\lambda s}\mathrm{d}s \\ &= w_\varepsilon(x) - \int_{-t}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\lambda s}) w_\varepsilon\big(\gamma_x^\lambda(s)\big)\mathrm{d}s \\ &\quad + \mathrm{e}^{-\lambda t}\Big(u_\lambda\big(\gamma_x^\lambda(-t)\big) - w_\varepsilon\big(\gamma_x^\lambda(-t)\big)\Big) - \frac{\varepsilon}{\lambda}(1 - \mathrm{e}^{-\lambda t}), \end{aligned}$$

where, for the last equality, we have used an integration by parts and the fact that $\mathrm{d}_{\gamma_x^\lambda(s)} w_\varepsilon\big(\dot{\gamma}_x^\lambda(s)\big) = \frac{\mathrm{d}}{\mathrm{d}s} w_\varepsilon\big(\gamma_x^\lambda(s)\big)$. Sending now $t \to +\infty$ we infer

$$\begin{aligned} u_\lambda(x) &\geq w_\varepsilon(x) - \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\lambda s})\, w_\varepsilon\big(\gamma_x^\lambda(s)\big)\,\mathrm{d}s - \frac{\varepsilon}{\lambda} \\ &= w_\varepsilon(x) - \int_{TM} w_\varepsilon(y)\,\mathrm{d}\tilde{\mu}_x^\lambda(y,v) - \frac{\varepsilon}{\lambda}. \end{aligned}$$

The assertion follows by letting $\varepsilon \to 0$. □

We are now ready to prove our main theorem:

Theorem 3.8 *The functions u_λ uniformly converge to u_0 on M as $\lambda \to 0$. In particular, as an accumulation point of u_λ, as $\lambda \to 0$, the function u_0 is a viscosity solution of* (2.5).

Proof By Proposition 2.6 we know that the functions u_λ are equi–Lipschitz and equi–bounded, hence it is enough, by the Ascoli–Arzelà theorem, to prove that any converging subsequence has u_0 as limit.

Let $\lambda_n \to 0$ be such that u_{λ_n} uniformly converges to some $u \in \mathrm{C}(M)$. We have seen in Proposition 3.4 that

$$u(x) \le u_0(x) \qquad \text{for every} \quad x \in M.$$

To prove the opposite inequality, let us fix $x \in M$. Let w be a critical subsolution. By Lemma 3.7, we have

$$u_{\lambda_n}(x) \ge w(x) - \int_{TM} w(y)\,\mathrm{d}\tilde{\mu}_x^{\lambda_n}(y, v).$$

By Proposition 3.6, extracting a further subsequence, we can assume that $\tilde{\mu}_x^{\lambda_n}$ converges weakly to a Mather measure $\tilde{\mu}$ whose projection on M is denoted by μ. Passing to the limit in the last inequality, we get

$$u(x) \ge w(x) - \int_M w(y)\,\mathrm{d}\mu(y),$$

where μ is a projected Mather measure. If we furthermore assume that $w \in \mathcal{F}_-$, the set of subsolutions satisfying (3.1), we obtain $\int_M w(y)\,\mathrm{d}\mu(y) \le 0$, and $u \ge w$. Hence $u \ge u_0 = \sup\limits_{w \in \mathcal{F}_-} w$. □

4 Another formula for the limit of the discounted value functions

In this section, we will give a characterization of u_0 as an infimum, using the Peierls barrier h, and the projected Mather measures. We define $\hat{u}_0 : M \to \mathbb{R}$ by

$$\hat{u}_0(x) = \min_{\mu \in \mathfrak{M}(L)} \int_M h(y, x)\,\mathrm{d}\mu(y), \qquad \text{for every } x \in M, \tag{4.1}$$

where $\mathfrak{M}(L)$ is the set of projected Mather measures, see Definition 5.8. We establish some properties of $\hat{u}_0$.

Lemma 4.1 *The function $\hat{u}_0$ is a critical subsolution.*

Proof We first remark that $\hat{u}_0 \geq \min_{M\times M} h > -\infty$, where the last strict inequality comes from the continuity of h. We then observe that the function $h_\mu : M \to \mathbb{R},\ x \mapsto \int_{TM} h(y,x)\,\mathrm{d}\mu(y)$ is a convex combination of the family of critical solutions $(h_y)_{y\in M}$, where $h_y(x) = h(y,x)$. By the convexity of H in the momentum and the equi–Lipschitz character of the critical subsolutions, see Propositions 2.1 and 2.3, it follows that each h_μ is a critical subsolution. By Proposition 2.1 again, we infer that a finite valued infimum of critical subsolutions is itself a critical subsolution. Therefore $\hat{u}_0$ is a critical subsolution. □

Lemma 4.2 *We have $u_0 \leq \hat{u}_0$ everywhere on M.*

Proof By the definitions of u_0, and $\hat{u}_0$, it suffices to show that $u \leq h_\mu$, for every critical subsolution u satisfying $\int_M u\,\mathrm{d}\mu \leq 0$, where μ is a projected Mather measure on M. In fact, by Proposition 5.3, we have $u(x) \leq u(y) + h(y,x)$, for every $x, y \in M$. If we integrate with respect to y, we get $u(x) \leq \int_M u\,\mathrm{d}\mu + h_\mu(x)$. But $\int_M u\,\mathrm{d}\mu \leq 0$ by assumption. □

Theorem 4.3 *We have $u_0 = \hat{u}_0$ everywhere on M. In particular, the function $\hat{u}_0$ is a critical solution.*

Proof Since, by Lemma 4.2, we already know that $u_0 \leq \hat{u}_0$, we have to show the reverse inequality $u_0 \geq \hat{u}_0$. By Lemma 4.1, the function $\hat{u}_0$ is a subsolution of the critical Hamilton–Jacobi equation (2.5). Moreover, the function u_0 is a solution of (2.5). Therefore by Theorem 5.5, it suffices to show that $\hat{u}_0(y) \leq u_0(y)$ for every y in the projected Aubry set $\mathcal{A}$. Fix $y \in \mathcal{A}$, by part (d) of Proposition 5.3, the function $x \mapsto -h(x,y)$ is a critical subsolution. Hence the function $x \mapsto w(x) = -h(x,y) + \inf_{\mu\in\mathfrak{M}(L)} \int_M h(z,y)\,\mathrm{d}\mu(z)$ is also a critical subsolution which satisfies condition (3.1). This implies $u_0 \geq w$ everywhere. In particular

$$u_0(y) \geq -h(y,y) + \inf_{\mu\in\mathfrak{M}(L)} \int_M h(z,y)\,\mathrm{d}\mu(z).$$

Using $h(y,y) = 0$ for $y \in \mathcal{A}$, we get $u(y) \geq \inf\limits_{\mu\in\mathfrak{M}(L)} \int_M h(z,y)\,\mathrm{d}\mu(z) = \hat{u}_0(y)$. □

We end this section by considering the case when the constant functions are critical subsolutions. In this case, the limit function u_0 can be identified by a simpler formula. Notice that the proof is actually independent of the results of this section.

Proposition 4.4 *Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian satisfying (H1)– (H2′), and assume that the constant functions are critical subsolutions.*

Then

$$u_0(x) = \min_{y \in \mathcal{A}} h(y, x) \qquad \textit{for every } x \in M.$$

Proof From Proposition 3.1 we already know that $u_\lambda \geq 0$ on M for every $\lambda > 0$. Let us set $\overline{u}(x) := \min\{h(y, x) \mid y \in \mathcal{A}\}$. Since the constant functions are critical subsolutions, from Proposition 5.3-(b) we infer that $h \geq 0$ on $M \times M$. Then $\overline{u}$ is a minimum of non-negative critical solutions, see Proposition 5.3-(c). By Proposition 2.1, we infer that $\overline{u}$ is itself a non–negative critical solution, and also a supersolution of the discounted equation (2.6). That implies $0 \leq u_\lambda \leq \overline{u}$ in M by the Comparison Principle (Theorem 2.5), in particular $0 = u_\lambda = \overline{u}$ on $\mathcal{A}$ by definition of $\overline{u}$ and of projected Aubry set. The critical solution u_0 obtained as the limit of the u_λ is therefore equal to 0 on $\mathcal{A}$, hence it coincides with $\overline{u}$ by Theorem 5.5. □

Appendix 1: Aubry–Mather Theory for non–smooth Lagrangians

In this appendix, we present the main results of weak KAM Theory we use. This material is well known in the case of a Tonelli Hamiltonian, see [4,7,11]. The lack of Hamiltonian and Lagrangian flows requires some different arguments, see [8–10,14]. Although given specifically for the torus, the results of [8,9,14] can be easily rephrased in our setting and proved along the same lines.

Weak KAM Theory

Let H be a continuous Hamiltonian satisfying (H1)–(H2′). We can associate with it a *Lagrangian* $L : TM \to \mathbb{R}$ through the *Fenchel transform* by setting

$$L(x, v) := \sup_{p \in T_x^* M} p(v) - H(x, p) \qquad \text{for every } (x, v) \in TM. \tag{5.1}$$

The function L is continuous on TM and satisfies properties analogous to (H1) and (H2′), see Appendix 1 in [6]. In particular, $L(x, \cdot)$ is superlinear in $T_x M$ for every fixed $x \in M$. The following facts are well known, see [21, Theorem 23.5].

Proposition 5.1 *Let H and L be as above. The following inequality, called* Fenchel inequality, *holds*

$$L(x, v) + H(x, p) \geq p(v), \quad \textit{for every } (x, v) \in TM \textit{ and } (x, p) \in T^*M, \tag{5.2}$$

and

$$H(x, p) = \sup_{v \in T_x M} \{p(v) - L(x, v)\} \qquad \text{for every } (x, p) \in T^*M.$$

For every $t > 0$, we define a function $h_t : M \times M \to \mathbb{R}$ by setting

$$h_t(x, y) := \inf \left\{ \int_{-t}^{0} \big[L(\gamma, \dot\gamma) + c(H)\big] \, \mathrm{d}s \;\middle|\; \gamma \in \mathrm{AC}([-t, 0];\ M),\ \gamma(-t) = x,\ \gamma(0) = y \right\},$$

where we have denoted by $\mathrm{AC}([-t, 0];\ M)$ the family of absolutely continuous curves from $[-t, 0]$ to M.

The following characterization holds, see [9,11,14]:

Proposition 5.2 *Let $w \in \mathrm{C}(M)$. Then w is a critical subsolution if and only if*

$$w(x) - w(y) \le h_t(y, x) \qquad \text{for every} \quad x, y \in M \text{ and } t > 0.$$

The *Peierls barrier* is the function $h : M \times M \to \mathbb{R}$ defined by

$$h(x, y) := \liminf_{t \to +\infty} h_t(x, y). \tag{5.3}$$

It satisfies the following properties, see for instance [9]:

Proposition 5.3 (a) *The Peierls barrier h is finite valued and Lipschitz continuous.*

(b) *If w is a critical subsolution, then*

$$w(x) - w(y) \le h(y, x) \quad \text{for every } x, y \in M.$$

(c) *For every fixed $y \in M$, the function $h(y, \cdot)$ is a critical solution.*

(d) *For every fixed $y \in M$, the function $-h(\cdot, y)$ is a critical subsolution.*

The projected *Aubry set* $\mathcal{A}$ is the closed set defined by

$$\mathcal{A} := \{y \in M \mid h(y, y) = 0\}.$$

The following holds, see [10,14]:

Theorem 5.4 *There exists a critical subsolution w which is both strict and of class C^1 in $M \setminus \mathcal{A}$, i.e. satisfies*

$$H(x, \mathrm{d}_x w) < c(H) \quad \text{for every } x \in M \setminus \mathcal{A}.$$

In particular, the projected Aubry set $\mathcal{A}$ is nonempty.

The last assertion directly follows from the definition of $c(H)$, see (2.4). A consequence of Theorem 5.4 is that $\mathcal{A}$ is a uniqueness set for the critical equation. In fact, we have, see [10,14]:

Theorem 5.5 *Let w, u be a critical sub and supersolution, respectively. If $w \leq u$ on $\mathcal{A}$, then $w \leq u$ on M. In particular, if two critical solutions coincide on the projected Aubry set $\mathcal{A}$, then they coincide on the whole manifold M.*

Mather measures

Let X be a metric separable space. A *probability measure* on X is a nonnegative, countably additive set function μ defined on the σ–algebra $\mathscr{B}(X)$ of Borel subsets of X such that $\mu(X) = 1$. In this paper, we deal with probability measures defined either on the compact manifold M or on its tangent bundle TM. A measure on TM is denoted by $\tilde{\mu}$, where the tilde on the top is to keep track of the fact that the measure is on the space TM. We say that a sequence $(\tilde{\mu}_n)_n$ of probability measures on TM (weakly) converges to a probability measure $\tilde{\mu}$ on TM if

$$\lim_{n\to+\infty} \int_{TM} f(x,v)\, \mathrm{d}\tilde{\mu}_n(x,v) = \int_{TM} f(x,v)\, \mathrm{d}\tilde{\mu}(x,v) \quad \text{for every } f \in \mathrm{C}_c(TM), \tag{5.4}$$

where $\mathrm{C}_c(TM)$ denotes the family of continuous real functions with compact support on TM. If $\tilde{\mu}$ is a probability measure on TM, we denote by μ its projection $\pi_\#\tilde{\mu}$ on M, i.e. the probability measure on M defined as

$$\pi_\#\tilde{\mu}(B) := \tilde{\mu}\big(\pi^{-1}(B)\big) \qquad \text{for every } B \in \mathscr{B}(M).$$

Note that

$$\int_M f(x)\, \pi_\#\tilde{\mu}(x) = \int_{TM} (f \circ \pi)(x,v)\, \mathrm{d}\tilde{\mu}(x,v),$$

for every $f \in \mathrm{C}(M)$.

Let H be a continuous Hamiltonian satisfying (H1)–(H2′) and let us denote by L the associated Lagrangian. Mather theory states that the constant $-c(H)$, where $c(H)$ is the critical value, can be also obtained by minimizing the integral of the Lagrangian over TM with respect to a suitable family of probability measures on TM. In the case of a Tonelli Hamiltonian, it is customary to choose this family as the one made up by probability measures on TM that are invariant by the Euler–Lagrange flow, see [20]. It was shown that this minimization problem yields the same result if it is done on the set of closed measures [1, 13, 14, 19] and the minimizing measures are the same. This is a set that does not depend on the Hamiltonian, and therefore this is the approach that can be adapted to the case when a Hamiltonian flow cannot be defined. The definition of closed measure is the following:

Definition 5.6 A probability measure $\tilde{\mu}$ on TM is called *closed* if it satisfies the following properties:

(a) $\displaystyle\int_{TM} \|v\|_x \,\mathrm{d}\tilde{\mu}(x, v) < +\infty;$

(b) $\displaystyle\int_{TM} \mathrm{d}_x\varphi(v)\,\mathrm{d}\tilde{\mu}(x, v) = 0 \quad \text{for every } \varphi \in \mathrm{C}^1(M).$

A way to construct a closed measure is the following: if $\gamma : [a, b] \to M$ is an absolutely continuous curve, define the probability measure $\tilde{\mu}_\gamma$ on TM, by

$$\int_{TM} f(x, v)\, d\tilde{\mu}_\gamma(x, v) := \frac{1}{b-a}\int_a^b f\big(\gamma(t), \dot{\gamma}(t)\big)\, dt, \tag{5.5}$$

for every $f \in \mathrm{C}_c(TM)$. It is easily seen that $\tilde{\mu}_\gamma$ is a closed measure whenever γ is a loop.

The relation linking closed probability measures to the critical value is clarified by the next theorem.

Theorem 5.7 *The following holds:*

$$\min_{\tilde{\mu}} \int_{TM} L(x, v)\, d\tilde{\mu}(x, v) = -c(H) \tag{5.6}$$

where $\tilde{\mu}$ varies in the set of closed measures and $c(H)$ is the critical value for H.

Proof Let us first show that the integral appearing at the right–hand side of (5.6) is greater than or equal to $-c(H)$. Using Theorem 2.2, we can construct a sequence of C^1 function $u_n : M \to \mathbb{R}$ such that $H(x, \mathrm{d}_x u_n) \le c(H) + 1/n$ for every $x \in M$. For every $(x, v) \in TM$, by Fenchel's inequality, we have

$$\mathrm{d}_x u_n(v) \le L(x, v) + H(x, \mathrm{d}_x u_n) \le L(x, v) + c(H) + \frac{1}{n}.$$

By integrating this inequality with respect to a closed measure $\tilde{\mu}$, we get

$$\int_{TM} L(x, v)\,\mathrm{d}\tilde{\mu}(x, v) \ge -c(H) - \frac{1}{n}.$$

The asserted inequality follows by letting $n \to +\infty$. Let us show that the infimum in (5.6) is achieved by a closed measure. Let $(\tilde{\mu}_n)_n$ a minimizing sequence of closed measure for the problem (5.6). According to Theorem 2-4.1-(3) in [7], up to extraction of a subsequence, there exists a probability measure $\tilde{\mu}$ on TM satisfying item (a) in Definition 5.6 and such that (5.4) holds for every $f \in \mathrm{C}(TM)$ enjoying the following property:

$$\sup_{(x,v)\in TM} \frac{|f(x,v)|}{1+\|v\|_x} < +\infty.$$

This readily implies that $\tilde{\mu}$ is closed. Moreover, according to the same Theorem 2-4.1-(3) in [7], we know that $\int_{TM} L\, \mathrm{d}\tilde{\mu} \leq -c(H)$, showing that $\tilde{\mu}$ is a solution of the minimization problem (5.6). □

Note also that Proposition 3.6 provides some examples of minimizing measures.

Definition 5.8 A Mather measure for the Lagrangian L is a closed probability measure $\tilde{\mu}$ on TM such that $\int_{TM} L(x,v)\, d\tilde{\mu}(x,v) = -c(H)$. The set of Mather measures will be denoted by $\tilde{\mathfrak{M}}(L)$. A projected Mather measure is a Borel probability measure in μ on M of the form $\mu = \pi_\# \tilde{\mu}$, where $\tilde{\mu} \in \tilde{\mathfrak{M}}(L)$. The set of projected Mather measures is denoted by $\mathfrak{M}(L)$.

Sometimes the terminology Mather minimizing measure, rather than Mather measure, is used to emphasize that a Mather measure is solving the miminization problem (5.6).

Appendix 2: Representation formulae for the discounted equation

For every $\lambda > 0$ and $x \in M$, we define the *discounted value function* as

$$\hat{u}_\lambda(x) := \inf_{\gamma(0)=x} \left\{ \int_{-\infty}^{0} \mathrm{e}^{\lambda s}\big[L\big(\gamma(s), \dot{\gamma}(s)\big) + c(H)\big]\, \mathrm{d}s \,\Big|\, \gamma \in \mathrm{AC}\,([-t,0];\ M) \right\}, \tag{6.1}$$

where we have denoted by AC $([-t,0];\ M)$ the family of absolutely continuous curves from $[-t,0]$ to M. The following holds:

Theorem 6.1 *For every $\lambda > 0$, the function $\hat{u}_\lambda$ given by* (6.1) *is the unique continuous viscosity solution of* (2.6).

Therefore $\hat{u}_\lambda$ is equal to the function u_λ used in Sect. 3. The uniqueness part in the above statement is a consequence of the Comparison Principle stated in Theorem 2.5. The fact that the discounted value function is a continuous viscosity solution of (2.6) is usually proved in Optimal Control Theory under the assumption that the speed of admissible curves is bounded by a constant independent of $x \in M$, see for instance [2, Chapter III] or [3, Chapter 3]. These bounds are not known *a priori* here, but are actually a consequence of the fact that the functions $\hat{u}_\lambda$ are equi–Lipschitz. Indeed, the following holds:

Proposition 6.2 *Let $\lambda > 0$ and $x \in M$. Then there exists a curve $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$ such that*

$$\hat{u}_\lambda(x) = \mathrm{e}^{-\lambda t}\hat{u}_\lambda(\gamma_x^\lambda(-t)) + \int_{-t}^{0} \mathrm{e}^{\lambda s}\big[L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) + c(H)\big]\,\mathrm{d}s \quad (6.2)$$

for every $t > 0$. Moreover, there exists a constant $\alpha > 0$, independent of λ and x, such that $\|\dot{\gamma}_x^\lambda\|_\infty \leq \alpha$. In particular

$$\hat{u}_\lambda(x) = \int_{-\infty}^{0} \mathrm{e}^{\lambda s}\big[L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) + c(H)\big]\,\mathrm{d}s. \quad (6.3)$$

A proof for Theorem 6.1 can be easily recovered from this by arguing, for instance, as in [2, Chapter III, Prop. 2.8], to which we refer for the details. The remainder of this appendix is therefore devoted to give a proof of Proposition 6.2, that we also need to prove Theorem 1.1.

We begin by deriving some crucial information for $\hat{u}_\lambda$.

Proposition 6.3 *The function $\hat{u}_\lambda$ defined by (6.1) satisfies the following properties:*

(i) *For every $\lambda > 0$*

$$\frac{\min_{TM} L + c(H)}{\lambda} \leq \hat{u}_\lambda(x) \leq \frac{L(x, 0) + c(H)}{\lambda} \qquad \text{for every } x \in M.$$

In particular, $\|\lambda \hat{u}_\lambda\|_\infty \leq C_0$ for some positive constant C_0 independent of $\lambda > 0$.

(ii) *For every absolutely continuous curve $\gamma : [a, b] \to M$, we have*

$$\mathrm{e}^{\lambda b}\hat{u}_\lambda\big(\gamma(b)\big) - \mathrm{e}^{\lambda a}\hat{u}_\lambda\big(\gamma(a)\big) \leq \int_a^b \mathrm{e}^{\lambda s}\big[L\big(\gamma(s), \dot{\gamma}(s)\big) + c(H)\big]\,\mathrm{d}s. \quad (6.4)$$

(iii) *There exists a positive constant κ, independent of $\lambda > 0$, such that*

$$\hat{u}_\lambda(x) - \hat{u}_\lambda(y) \leq \kappa\, d(x, y) \qquad \text{for every } x, y \in M \text{ and } \lambda > 0,$$

that is, the functions $\{\hat{u}_\lambda \mid \lambda > 0\}$ are equi–Lipschitz.

Proof In (i), the first inequality comes from the fact that every absolutely continuous curve $\gamma : (-\infty, 0] \to M$ satisfies

$$\int_{-\infty}^{0} \mathrm{e}^{\lambda s}\big[L\big(\gamma(s), \dot\gamma(s)\big) + c(H)\big]\,\mathrm{d}s \geq \Big(\min_{TM} L + c(H)\Big)\int_{-\infty}^{0} \mathrm{e}^{\lambda s}\,\mathrm{d}s = \frac{\min_{TM} L + c(H)}{\lambda}.$$

The second inequality follows by choosing, as a competitor, the steady curve identically equal to the point x.

To prove (ii), we first note that we can assume $b = 0$, since we can always reduce to this case by replacing γ with the curve $\gamma_{-b}(\cdot) := \gamma(\cdot + b)$ defined on the interval $[a - b, 0]$ and by dividing (6.4) by $\mathrm{e}^{\lambda b}$. Note that a change of variables gives

$$\int_{a}^{b} \mathrm{e}^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s = \mathrm{e}^{\lambda b}\int_{a-b}^{0} \mathrm{e}^{\lambda s} L\big(\gamma_{-b}(s), \dot\gamma_{-b}(s)\big)\,\mathrm{d}s.$$

So, let $\gamma \in \mathrm{AC}\,([a, 0];\ M)$ be fixed. For every absolutely continuous curve $\xi : (-\infty, 0] \to M$ with $\xi(0) = \gamma(a)$, we define a curve $\xi_a : (-\infty, a] \to M$ by setting $\xi_a(\cdot) := \xi(\cdot - a)$ and a curve $\eta := \xi_a \star \gamma : (-\infty, 0] \to M$ obtained by concatenation of ξ_a and γ. By definition of $\hat u_\lambda$ and arguing as above we get:

$$\begin{aligned}\hat u_\lambda\big(\gamma(0)\big) &\leq \int_{-\infty}^{0} \mathrm{e}^{\lambda s}\big[L(\eta, \dot\eta) + c(H)\big]\,\mathrm{d}s\\ &= \int_{-\infty}^{a} \mathrm{e}^{\lambda s}\big[L(\xi_a, \dot\xi_a) + c(H)\big]\,\mathrm{d}s + \int_{a}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma, \dot\gamma) + c(H)\big]\,\mathrm{d}s\\ &= \mathrm{e}^{\lambda a}\int_{-\infty}^{0} \mathrm{e}^{\lambda s}\big[L(\xi, \dot\xi) + c(H)\big]\,\mathrm{d}s + \int_{a}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma, \dot\gamma) + c(H)\big]\,\mathrm{d}s.\end{aligned}$$

By minimizing with respect to all $\xi \in \mathrm{AC}\big((-\infty, 0];\ M\big)$ with $\xi(0) = \gamma(a)$ we get the assertion by definition of $\hat u_\lambda\big(\gamma(a)\big)$.

To prove (iii), pick $x, y \in M$ and let $\gamma : [-d(x, y), 0] \to M$ be the geodesic joining y to x parameterized by the arc–length. According to item (ii), we have

$$\begin{aligned}\hat u_\lambda(x) - \hat u_\lambda(y) &\leq -\hat u_\lambda(y)\big(1 - \mathrm{e}^{-\lambda d(x,y)}\big)\\ &\quad + \int_{-d(x,y)}^{0} \mathrm{e}^{\lambda s}\big[L\big(\gamma(s), \dot\gamma(s)\big) + c(H)\big]\,\mathrm{d}s.\end{aligned}$$

Let $C_1 := \max\{L(z,v) \,:\, z \in M,\ \|v\|_z \leq 1\}$ and C_0 the constant given by item (i). We get

$$\hat{u}_\lambda(x) - \hat{u}_\lambda(y) \leq \big(\|\lambda \hat{u}_\lambda\|_\infty + C_1 + c(H)\big)\,\frac{1 - \mathrm{e}^{-\lambda d(x,y)}}{\lambda}$$
$$\leq \big(C_0 + C_1 + c(H)\big) d(x,y),$$

where, for the last inequality, we have used the fact that, by concavity, $1 - \mathrm{e}^{-h} \leq h$ for every $h \in \mathbb{R}$. □

In the sequel, we will use the following result:

Theorem 6.4 *Let* $[a,b]$ *be a compact interval in* $\mathbb{R}$ *and* $\lambda > 0$. *Let* $(\gamma_n)_n$ *be a sequence in* $\mathrm{AC}([a,b];\ M)$ *such that*

$$\sup_{n \in \mathbb{N}} \int_a^b \mathrm{e}^{\lambda s} L\big(\gamma_n(s), \dot{\gamma}_n(s)\big)\, \mathrm{d}s < +\infty.$$

Then there exists a subsequence $(\gamma_{n_k})_k$ *uniformly converging to a curve* $\gamma \in \mathrm{AC}([a,b];\ M)$. *Moreover*

$$\int_a^b \mathrm{e}^{\lambda s} L\big(\gamma(s), \dot{\gamma}(s)\big)\, \mathrm{d}s \leq \liminf_{k \to +\infty} \int_a^b \mathrm{e}^{\lambda s} L\big(\gamma_{n_k}(s), \dot{\gamma}_{n_k}(s)\big)\, \mathrm{d}s.$$

When M is contained in $\mathbb{R}^k$, the above theorem follows by making use of the Dunford–Pettis theorem, see [5, Theorems 2.11 and 2.12], and of standard semicontinuity results in the Calculus of Variations, see [5, Theorem 3.6]. To get the result on an abstract compact manifold, it suffices to show that we can always reduce to this case by localizing the argument and by reasoning in local charts, see for instance [11].

The following holds:

Proposition 6.5 *Let* $\lambda > 0$. *For every* $x \in M$ *and* $t > 0$

$$\hat{u}_\lambda(x) = \inf_{\gamma(0)=x} \left\{ \mathrm{e}^{-\lambda t}\, \hat{u}_\lambda\big(\gamma(-t)\big) + \int_{-t}^0 \mathrm{e}^{\lambda s}\big[L(\gamma, \dot{\gamma}) + c(H)\big]\, \mathrm{d}s \,\Big|\, \gamma \in \mathrm{AC}\,([-t, 0];\ M) \right\}. \tag{6.5}$$

Moreover, the above infimum is attained.

Proof The fact that the discounted value function satisfies the Dynamical Programming Principle (6.5) is standard, see for instance [2, Chapter III, Prop. 2.5]. To prove that the infimum is actually a minimum, we take minimizing sequence $\gamma_n : [-t, 0] \to M$ with $\gamma_n(0) = x$, i.e. such that

$$\lim_{n\to+\infty} \mathrm{e}^{-\lambda t}\, \hat{u}_\lambda\big(\gamma_n(-t)\big) + \int_{-t}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma_n, \dot{\gamma}_n) + c(H)\big]\,\mathrm{d}s = \hat{u}_\lambda(x).$$

For n large enough, we have:

$$\int_{-t}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma_n, \dot{\gamma}_n) + c(H)\big]\mathrm{d}s \le 1 + \hat{u}_\lambda\big(\gamma_n(0)\big) - \mathrm{e}^{-\lambda t}\hat{u}_\lambda\big(\gamma_n(-t)\big) \le 1 + 2\|\hat{u}_\lambda\|_\infty.$$

According to Theorem 6.4, the curves γ_n uniformly converge, up to subsequences, to an absolutely continuous curve $\gamma : [-t, 0] \to M$ with $\gamma(0) = x$ and satisfying

$$\int_{-t}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma, \dot{\gamma}) + c(H)\big]\mathrm{d}s \le \liminf_{n\to+\infty} \int_{-t}^{0} \mathrm{e}^{\lambda s}\big[L(\gamma_n, \dot{\gamma}_n) + c(H)\big]\mathrm{d}s.$$

This readily implies that γ is a minimizer of (6.5). □

Proof of Proposition 6.2 According to Proposition 6.5 we know that, for every $n \in \mathbb{N}$, there exists a curve $\xi_n : [-n, 0] \to M$ with $\xi_n(0) = x$ such that

$$\hat{u}_\lambda(x) = \mathrm{e}^{-\lambda n}\hat{u}_\lambda\big(\xi_n(-n)\big) + \int_{-n}^{0} \mathrm{e}^{\lambda s}\big[L\big(\xi_n(s), \dot{\xi}_n(s)\big) + c(H)\big]\,\mathrm{d}s.$$

It is then standard that, for every $[a, b] \subset [-n, 0]$,

$$\mathrm{e}^{\lambda b}\hat{u}_\lambda\big(\xi_n(b)\big) - \mathrm{e}^{\lambda a}\hat{u}_\lambda\big(\xi_n(a)\big) = \int_a^b \mathrm{e}^{\lambda s}\big[L\big(\xi_n(s), \dot{\xi}_n(s)\big) + c(H)\big]\,\mathrm{d}s. \tag{6.6}$$

By reasoning as in the proof of Proposition 6.5 and using a diagonal argument, we derive from Theorem 6.4 that there exists an absolutely continuous curve $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$ which is, up to extraction of a subsequence, the uniform limit of the curves ξ_n over compact subsets of $(-\infty, 0]$. Such curve satisfies

$$\mathrm{e}^{\lambda b}\hat{u}_\lambda\big(\gamma_x^\lambda(b)\big) - \mathrm{e}^{\lambda a}\hat{u}_\lambda\big(\gamma_x^\lambda(a)\big) = \int_a^b \mathrm{e}^{\lambda s}\big[L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) + c(H)\big]\,\mathrm{d}s. \tag{6.7}$$

for every $[a, b] \subset (-\infty, 0]$. To see this, it suffices to pass to the limit in (6.6). The equality holds also for the limit curve γ_x^λ by the lower semicontinuity of the integral functional stated in Theorem 6.4 and by Proposition 6.3–(ii). In particular, this proves assertion (6.2).

The fact that the curves γ_x^λ are equi–Lipschitz is a consequence of the fact that the functions $\hat{u}_\lambda$ are equi–Lipschitz, say κ–Lipschitz, according to Proposition 6.3. Indeed, by superlinearity of L, there exists a constant A_κ, depending on κ, such that

$$L(x, v) + c(H) \geq (\kappa + 1)\|v\|_x - A_\kappa \qquad \text{for every } (x, v) \in TM.$$

For every $a \in (-\infty, 0)$ and $h > 0$ small enough, from (6.7) we get

$$\begin{aligned}
&\mathrm{e}^{\lambda(a+h)}\hat{u}_\lambda\big(\gamma_x^\lambda(a+h)\big) - \mathrm{e}^{\lambda a}\hat{u}_\lambda\big(\gamma_x^\lambda(a)\big) \\
&\quad = \int_a^{a+h} \mathrm{e}^{\lambda s}\big[L\big(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s)\big) + c(H)\big]\,\mathrm{d}s \\
&\quad \geq \mathrm{e}^{\lambda a}\,(\kappa+1)\int_a^{a+h} \|\dot{\gamma}_x^\lambda(s)\|_{\gamma_x^\lambda(s)}\,\mathrm{d}s - A_\kappa \int_a^{a+h} \mathrm{e}^{\lambda s}\,\mathrm{d}s \\
&\quad \geq \mathrm{e}^{\lambda a}\left(\kappa\, d\big(\gamma_x^\lambda(a), \gamma_x^\lambda(a+h)\big) + \int_a^{a+h} \|\dot{\gamma}_x^\lambda(s)\|_{\gamma_x^\lambda(s)}\,\mathrm{d}s - A_\kappa\,\frac{\mathrm{e}^{\lambda h}-1}{\lambda}\right)
\end{aligned} \tag{6.8}$$

On the other hand

$$\begin{aligned}
&\mathrm{e}^{\lambda(a+h)}\hat{u}_\lambda\big(\gamma_x^\lambda(a+h)\big) - \mathrm{e}^{\lambda a}\hat{u}_\lambda\big(\gamma_x^\lambda(a)\big) \\
&\quad \leq (\mathrm{e}^{\lambda(a+h)} - \mathrm{e}^{\lambda a})\hat{u}_\lambda\big(\gamma_x^\lambda(a+h)\big) + \mathrm{e}^{\lambda a}\,\kappa\, d\big(\gamma_x^\lambda(a), \gamma_x^\lambda(a+h)\big) \\
&\quad \leq \mathrm{e}^{\lambda a}\left(C_0\,\frac{\mathrm{e}^{\lambda h}-1}{\lambda} + \kappa\, d\big(\gamma_x^\lambda(a), \gamma_x^\lambda(a+h)\big)\right),
\end{aligned} \tag{6.9}$$

where C_0 is the constant given by Proposition 6.3–(i). Plugging (6.9) into (6.8) and dividing by $h\,\mathrm{e}^{\lambda a}$ we end up with

$$\frac{1}{h}\int_a^{a+h} \|\dot{\gamma}_x^\lambda(s)\|_{\gamma_x^\lambda(s)}\,\mathrm{d}s \leq (A_\kappa + C_0)\,\frac{\mathrm{e}^{\lambda h}-1}{\lambda\, h}.$$

Sending $h \to 0$ we infer

$$\|\dot{\gamma}_x^\lambda(a)\|_{\gamma_x^\lambda(a)} \leq \alpha := (A_\kappa + C_0) \qquad \text{for a.e. } a \in (-\infty, 0],$$

as it was to be shown. In particular, by sending $t \to +\infty$ in (6.2) we get (6.3) by the Dominated Convergence Theorem. □

References

1. Bangert, V.: Minimal measures and minimizing closed normal one-currents. Geom. Funct. Anal. **9**, 413–427 (1999)
2. Bardi, M., Capuzzo-Dolcetta, I.: Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Systems and Control: Foundations & Applications, Birkhäuser Boston Inc., Boston With appendices by Maurizio Falcone and Pierpaolo Soravia (1997)
3. Barles, G.: Solutions de viscosité des équations de Hamilton-Jacobi, vol. 17 of Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Paris (1994)
4. Bernard, P.: The Lax-Oleinik semi-group: a Hamiltonian point of view. Proc. Roy. Soc. Edinburgh Sect. A **142**, 1131–1177 (2012)
5. Buttazzo, G., Giaquinta, M., Hildebrandt, S.: One-dimensional variational problems: an introduction of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford, vol. **15**, University Press, New York. (1998)
6. Cannarsa, P., Sinestrari, C.: Semiconcave functions, Hamilton-Jacobi equations, and Optimal Control, Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston Inc., Boston (2004)
7. Contreras, G., Iturriaga, R.: Global minimizers of autonomous Lagrangians, 22º Colóquio Brasileiro de Matemática. In: 22nd Brazilian Mathematics Colloquium, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro (1999)
8. Davini, A., Siconolfi, A.: A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations. SIAM J. Math. Anal **38**, 478–502 (2006). (electronic)
9. Davini, A., Zavidovique, M.: Weak KAM theory for nonregular commuting Hamiltonians. Discrete Contin. Dyn. Syst. Ser. B **18**, 57–94 (2013)
10. Fathi, A.: Weak KAM from a PDE point of view: viscosity solutions of the Hamilton-Jacobi equation and Aubry set. Proc. Roy. Soc. Edinburgh Sect. A **142**, 1193–1236 (2012)
11. Fathi, A.: Weak KAM Theorem in Lagrangian Dynamics, preliminary version 10, Lyon. unpublished (2008)
12. Fathi, A., Maderna, E.: Weak KAM theorem on non compact manifolds. NoDEA Nonlinear Differ. Equ. Appl. **14**, 1–27 (2007)
13. Fathi, A., Siconolfi, A.: Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. Invent. Math. **155**, 363–388 (2004)
14. Fathi, A., Siconolfi, A.: PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians. Calc. Var. Partial Differen. Equ. **22**, 185–228 (2005)
15. Gomes, D.A.: Generalized Mather problem and selection principles for viscosity solutions and Mather measures. Adv. Calc. Var. **1**, 291–307 (2008)
16. Gomes, D.A., Oliveira, E.R.: Mather problem and viscosity solutions in the stationary setting. São Paulo J. Math. Sci. **6**, 301–334 (2012)
17. Iturriaga, R., Sánchez-Morgado, H.: Limit of the infinite horizon discounted Hamilton-Jacobi equation. Discrete Contin. Dyn. Syst. Ser. B **15**, 623–635 (2011)
18. Lions, P.-L., Papanicolaou, G., Varadhan, S.: Homogenization of Hamilton-Jacobi equation. unpublished preprint, (1987)
19. Mañé, R.: Generic properties and problems of minimizing measures of Lagrangian systems. Nonlinearity **9**, 273–310 (1996)
20. Mather, J.N.: Action minimizing invariant measures for positive definite Lagrangian systems. Math. Z. **207**, 169–207 (1991)
21. Rockafellar, R.T.: Convex analysis, Princeton Mathematical Series. Princeton University Press, Princeton (1970)
22. Siconolfi, A.: Hamilton-Jacobi equations and weak KAM theory, In: Mathematics of complexity and dynamical systems. vols. 1–3, pp. 683–703, Springer, New York (2012)