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MULTIVARIATE FRACTIONAL POISSON PROCESSES AND COMPOUND SUMS

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Abstract

In this paper we present multivariate space-time fractional Poisson processes by considering common random time-changes of a (finite-dimensional) vector of independent classical (nonfractional) Poisson processes. In some cases we also consider compound processes. We obtain some equations in terms of some suitable fractional derivatives and fractional difference operators, which provides the extension of known equations for the univariate processes.

Keywords: Conditional independence; Fox–Wright function; fractional differential equation; random time-change

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1. Introduction

Typically fractional processes are defined by considering some known equations in terms of suitable fractional derivatives. In this paper we deal with fractional Poisson processes which are the main examples among counting processes; here we recall the references [4], [5], [11], [12], [15], and [19] (we also cite [10] and [13] where their representation in terms of randomly time-changed and subordinated processes was studied in detail). Moreover, as pointed out in [20], a class of these processes demonstrate the phenomenon of anomalous diffusion (i.e. the variances of the process increase in time according to a power t^γ, with $\gamma \neq 1$); this aspect was also highlighted in [6] where the authors refer to the long-range dependence property (they also present some applications in ruin theory where the surplus process of an insurance company is modeled by a compound fractional Poisson process).

The aim of this paper is to present m-variate space-time fractional (possibly compound) Poisson processes; in this way we generalize some results in the literature for univariate processes, which can be recovered by setting $m = 1$. Often closed formulae for fractional Poisson processes are given in terms of the Mittag-Leffler function, i.e.

$$E_{\alpha,\beta}(x) := \sum_{r\geq 0} \frac{x^r}{\Gamma(\alpha r + \beta)} \tag{1}$$

(see, e.g. [18, p. 17]).

We start with the simplest case, i.e. the multivariate version of the space-time fractional Poisson process in [15]. In particular we consider the time-change approach in terms of the

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stable subordinator and of its inverse (see [2, Equations (3.18) and (3.1)]; see also [22]). So, for $\nu \in (0,1)$, we consider the following processes:

- let $\{\mathcal{A}^\nu(t)\colon t \ge 0\}$ be the stable subordinator, i.e. the nondecreasing Lévy process with Laplace transform
$$\mathbb{E}[\mathrm{e}^{-sA^\nu(t)}] = \mathrm{e}^{-s^\nu t} \quad \text{for all } s \ge 0$$
(see, e.g. [1, Example 1.3.18]);
- let $\{\mathcal{L}^\nu(t)\colon t \ge 0\}$ be the inverse of $\{\mathcal{A}^\nu(t)\colon t \ge 0\}$, i.e. the process defined by
$$\mathcal{L}^\nu(t) := \inf\{z \ge 0\colon \mathcal{A}^\nu(z) \ge t\}.$$

In what follows we denote the continuous density of $\mathcal{L}^\nu(t)$ by $f_{\mathcal{L}^\nu(t)}$, and the continuous density of $\mathcal{A}^\nu(t)$ by $f_{\mathcal{A}^\nu(t)}$. Stable subordinators are well studied in the references on Lévy processes (see, e.g. [1] and [21]); for the inverse of stable subordinators, we recall [7], [13], and [17].

Definition 1. Let $\{\{N_i(t)\colon t \ge 0\}\colon i \in \{1,\dots,m\}\}$ be m independent Poisson processes with intensities $\lambda_1,\dots,\lambda_m > 0$, respectively, and set
$$N(t) := (N_1(t),\dots,N_m(t)).$$
Then, for $\eta,\nu \in (0,1]$, we consider the m-variate process $\{N^{\eta,\nu}(t)\colon t \ge 0\}$ defined by
$$N^{\eta,\nu}(t) := N(\mathcal{A}^\eta(\mathcal{L}^\nu(t))),$$
where $\{N(t)\colon t \ge 0\}$, $\{\mathcal{A}^\eta(t)\colon t \ge 0\}$, and $\{\mathcal{L}^\nu(t)\colon t \ge 0\}$ are three independent processes. When we consider the cases $\eta = 1$ and/or $\nu = 1$, we are setting $\mathcal{A}^1(t) = t$ and/or $\mathcal{L}^1(t) = t$, respectively; thus, in particular, $\{N^{1,1}(t)\colon t \ge 0\}$ coincides with $\{N(t)\colon t \ge 0\}$.

We remark that $\{\{N_i^{\eta,\nu}(t)\colon t \ge 0\}\colon i \in \{1,\dots,m\}\}$ in Definition 1 are conditionally independent given $\{\mathcal{A}^\eta(\mathcal{L}^\nu(t))\colon t \ge 0\}$ (except for the case $\eta = \nu = 1$ where they are independent).

Throughout this paper we deal with m-variate processes and we use the notation $\boldsymbol{a} = (a_1,\dots,a_m)$ for m-dimensional vectors. For instance, we often write $\boldsymbol{k} \ge \boldsymbol{0}$ where $k_1,\dots,k_m$ are nonnegative integers (because we deal with processes with nonnegative integer-valued components) and $\boldsymbol{0} = (0,\dots,0)$ is the null vector. Moreover, we write $\boldsymbol{a} \le \boldsymbol{b}$ (or $\boldsymbol{a} \ge \boldsymbol{b}$) to mean that $a_i \le b_i$ (or $a_i \ge b_i$) for all $i \in \{1,\dots,m\}$; $\boldsymbol{a} \prec \boldsymbol{b}$ (or $\boldsymbol{a} \succ \boldsymbol{b}$) to mean that $a_i \le b_i$ (or $a_i \ge b_i$) for all $i \in \{1,\dots,m\}$, but $\boldsymbol{a} \ne \boldsymbol{b}$. Finally, we remark that the probability generating functions assume finite values when their arguments $\boldsymbol{u}$ belong to $[0,1]^m$ but, in some cases, the condition $\boldsymbol{u} \in [0,1]^m$ can be neglected or weakened (for instance, when $\eta = 1$, this happens for the probability generating functions in (4) and (5); in the first case the finiteness of $G_1(u_1),\dots,G_m(u_m)$ is also needed).

Our results mainly concern the state probabilities $\{\{p_{\boldsymbol{k}}^{\eta,\nu}(t)\colon \boldsymbol{k} \ge \boldsymbol{0}\}\colon t \ge 0\}$ defined by
$$p_{\boldsymbol{k}}^{\eta,\nu}(t) := \mathbb{P}(N^{\eta,\nu}(t) = \boldsymbol{k}) \quad \text{for all integer } k_1,\dots,k_m \ge 0. \tag{2}$$

We also consider two generalizations of the process $\{N^{\eta,\nu}(t)\colon t \ge 0\}$ in Definition 1: we mean the multivariate space-time fractional compound Poisson process (see Definition 2) and the multivariate version of the process in [16], where we have a general subordinator associated to a Bernštein function f in place of the stable subordinator $\{\mathcal{A}^\eta(t)\colon t \ge 0\}$ (see Definition 3). We start with the first generalization.

Definition 2. For $\eta, \nu \in (0, 1]$, let $\{C^{\eta,\nu}(t) : t \geq 0\}$ be defined by

$$C^{\eta,\nu}(t) := (C_1^{\eta,\nu}(t), \dots, C_m^{\eta,\nu}(t)),$$

where $C_i^{\eta,\nu}(t) := \sum_{j=1}^{N_i^{\eta,\nu}(t)} Y_j^i$ for all $i \in \{1, \dots, m\}$, and $\{\{Y_n^i : n \geq 1\} : i \in \{1, \dots, m\}\}$ are m independent sequences of independent and identically distributed positive integer-valued random variables, independent of $\{N^{\eta,\nu}(t) : t \geq 0\}$ as in Definition 1.

Obviously, the process $\{C^{\eta,\nu}(t) : t \geq 0\}$ in Definition 2 coincides with $\{N^{\eta,\nu}(t) : t \geq 0\}$ in Definition 1 when all the random variables $\{\{Y_n^i : n \geq 1\} : i \in \{1, \dots, m\}\}$ are equal to 1; see also Remark 1 below. In view of what follows it is useful to introduce the following notation. We start with the state probabilities $\{\{q_{\boldsymbol{k}}^{\eta,\nu}(t) : \boldsymbol{k} \geq \boldsymbol{0}\} : t \geq 0\}$ defined by

$$q_{\boldsymbol{k}}^{\eta,\nu}(t) := \mathbb{P}(C^{\eta,\nu}(t) = \boldsymbol{k}) \quad \text{for all integer } k_1, \dots, k_m \geq 0, \tag{3}$$

the probability mass functions

$$\widetilde{q}_j^i := \mathbb{P}(Y_n^i = j) \quad \text{for all integer } j \geq 1,\ i \in \{1, \dots, m\} \text{ and } n \geq 1$$

and the probability generating functions

$$G_i(u) := \sum_{j \geq 0} u^j \widetilde{q}_j^i \quad (i \in \{1, \dots, m\}) \quad \text{and} \quad G_C^{\eta,\nu}(\boldsymbol{u}; t) := \sum_{\boldsymbol{k} \geq \boldsymbol{0}} u_1^{k_1} \cdots u_m^{k_m} q_{\boldsymbol{k}}^{\eta,\nu}(t).$$

We remark that

$$G_C^{\eta,\nu}(\boldsymbol{u}; t) := \mathbb{E}[u_1^{C_1(\mathcal{A}^\eta(\mathcal{L}^\nu(t)))} \dots u_m^{C_m(\mathcal{A}^\eta(\mathcal{L}^\nu(t)))}] = \mathbb{E}[\mathbb{E}[u_1^{C_1(r)} \dots u_m^{C_m(r)}]_{r=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}]$$

and

$$\mathbb{E}[u_1^{C_1(r)} \dots u_m^{C_m(r)}] = \exp\biggl(\sum_{i=1}^m \lambda_i (G_i(u_i) - 1) r\biggr);$$

thus, by taking into account [2, Equation (3.8)], we obtain

$$G_C^{\eta,\nu}(\boldsymbol{u}; t) = E_{\nu,1}\biggl(-\biggl(\sum_{i=1}^m \lambda_i (1 - G_i(u_i))\biggr)^\eta t^\nu\biggr). \tag{4}$$

As a particular case, we can consider the probability generating functions

$$G^{\eta,\nu}(\boldsymbol{u}; t) := \sum_{\boldsymbol{k} \geq \boldsymbol{0}} u_1^{k_1} \cdots u_m^{k_m} p_{\boldsymbol{k}}^{\eta,\nu}(t)$$

and, we have

$$G^{\eta,\nu}(\boldsymbol{u}; t) = \mathbb{E}\biggl[\exp\biggl(\sum_{i=1}^m \lambda_i (u_i - 1) \mathcal{A}^\eta(\mathcal{L}^\nu(t))\biggr)\biggr] = E_{\nu,1}\biggl(-\biggl(\sum_{i=1}^m \lambda_i (1 - u_i)\biggr)^\eta t^\nu\biggr); \tag{5}$$

note that both (4) and (5) can be seen as a generalization of [2, Equation (3.20)]. Finally, we consider the probability mass functions concerning convolutions, i.e.

$$(\widetilde{q}^i)_j^{*h} := \mathbb{P}(Y_1^i + \cdots + Y_h^i = j) \quad \text{for all } j \geq 1,\ i \in \{1, \dots, m\} \text{ and } h \geq 1.$$

We remark that, since the random variables $\{\{Y_n^i : n \geq 1\} : i \in \{1, \dots, m\}\}$ are positive, we have (where $\mathbf{1}$ is the indicator function)

$$(\widetilde{q}^i)_j^{*0} = \mathbf{1}_{\{j=0\}}; \quad \text{if } j < h \quad \text{then } (\widetilde{q}^i)_j^{*h} = 0.$$

Remark 1. Obviously the state probabilities $\{\{q_{\boldsymbol{k}}^{\eta,\nu}(t)\colon \boldsymbol{k} \geq \boldsymbol{0}\}\colon t \geq 0\}$ reduce to $\{\{p_{\boldsymbol{k}}^{\eta,\nu}(t)\colon \boldsymbol{k} \geq \boldsymbol{0}\}\colon t \geq 0\}$ when we have $\widetilde{q}_j^i := \mathbf{1}_{\{j=1\}}$ for all $i \in \{1, \dots, m\}$.

A further generalization of the process $\{N^{\eta,\nu}(t)\colon t \geq 0\}$ in Definition 1 is the multivariate version of the process in [16]. In view of this we recall that, given a nondecreasing Lévy process (subordinator) $\{\mathcal{H}^f(t)\colon t \geq 0\}$ associated with the Bernštein function f, we have

$$\mathbb{E}[\mathrm{e}^{-\mu\mathcal{H}^f(t)}] = \mathrm{e}^{-tf(\mu)} \quad \text{for all } \mu, t \geq 0;$$

moreover, we have the following integral representation:

$$f(\mu) = \int_0^\infty (1 - \mathrm{e}^{-\mu r})\rho_f(\mathrm{d}r) \quad \text{for all } \mu \geq 0,$$

where ρ_f is the Lévy measure associated with f (we also recall that ρ_f is a nonnegative measure concentrated on $(0, \infty)$ such that $\int_0^\infty (r \wedge 1)\rho_f(\mathrm{d}r) < \infty$).

Definition 3. Let us consider the processes in Definition 1 and an independent subordinator $\{\mathcal{H}^f(t)\colon t \geq 0\}$ associated with a Bernštein function f. Then let $\{N^{f,\nu}(t)\colon t \geq 0\}$ be defined by

$$N^{f,\nu}(t) := N(\mathcal{H}^f(\mathcal{L}^\nu(t))).$$

Remark 2. If $\{\mathcal{H}^f(t)\colon t \geq 0\}$ is the stable subordinator $\{\mathcal{A}^\eta(t)\colon t \geq 0\}$ cited above, we have (see, e.g. [1, Example 1.3.18])

$$f(\mu) := \mu^\eta, \quad \text{or, equivalently,} \quad \rho_f(\mathrm{d}r) = \frac{\eta}{\Gamma(1-\eta)} \frac{1}{r^{\eta+1}} \mathbf{1}_{(0,\infty)}(r)\,\mathrm{d}r.$$

Obviously, in this case $\{N^{f,\nu}(t)\colon t \geq 0\}$ in Definition 3 coincides with $\{N^{\eta,\nu}(t)\colon t \geq 0\}$ in Definition 1.

In what follows all the items concerning the process $\{N^{f,\nu}(t)\colon t \geq 0\}$ will be a modification of the ones for $\{N^{\eta,\nu}(t)\colon t \geq 0\}$ in Definition 1 with f in place of η; thus, for instance, we set

$$p_{\boldsymbol{k}}^{f,\nu}(t) := \mathbb{P}(N^{f,\nu}(t) = \boldsymbol{k}) \quad \text{for all integer } k_1, \dots, k_m \geq 0 \tag{6}$$

and

$$G^{f,\nu}(\boldsymbol{u}; t) := \sum_{\boldsymbol{k} \geq \boldsymbol{0}} u_1^{k_1} \cdots u_m^{k_m} p_{\boldsymbol{k}}^{f,\nu}(t). \tag{7}$$

We conclude with the outline of the paper. We start with some preliminaries in Section 2. The results are presented in Section 3, which is divided into two parts:

(i) the results for the processes in Definitions 1 and 2;

(ii) the results for the process in Definition 3.

Some examples of fractional compound Poisson processes and the generalization of a result in [3] for the fractional Pólya–Aeppli process are presented in Section 4.

2. Preliminaries

We recall some useful special functions. We start with the generalized Mittag-Leffler function which is defined by

$$E_{\alpha,\beta}^{\gamma}(x) := \sum_{j\geq 0} \frac{(\gamma)^{(j)} x^j}{j!\,\Gamma(\alpha j + \beta)}$$

(see, e.g. [8, Equation (1.9.1)]), where

$$(\gamma)^{(j)} := \begin{cases} \gamma(\gamma+1)\dots(\gamma+j-1) & \text{if } j \geq 1, \\ 1 & \text{if } j = 0, \end{cases}$$

is the rising factorial, also called Pochhammer symbol (see, e.g. [8, Equation (1.5.5)]). Note that, we have $E_{\alpha,\beta}^{1}$, i.e. $E_{\alpha,\beta}^{\gamma}$ with $\gamma = 1$ coincides with $E_{\alpha,\beta}$ in (1).

We also recall the Fox–Wright function (see, e.g. [8, Equation (1.11.14)]) defined by

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1)\cdots(a_p,\alpha_p)\\(b_1,\beta_1)\cdots(b_q,\beta_q)\end{matrix}\right](z) := \sum_{j\geq 0} \frac{\prod_{h=1}^{p}\Gamma(a_h+\alpha_h j)}{\prod_{k=1}^{q}\Gamma(b_k+\beta_k j)} \frac{z^j}{j!}, \tag{8}$$

under the convergence condition

$$\sum_{k=1}^{q}\beta_k - \sum_{h=1}^{p}\alpha_h > -1 \tag{9}$$

(see, e.g. [8, Equation (1.11.15)]).

We conclude this section with the definitions of two fractional derivatives and of a fractional difference operator. Firstly, we consider the Caputo fractional derivative of order $\nu \in (0,1]$, i.e. ${}^{C}D_{a+}^{\nu}$ in [8, Equation (2.4.17)] with $a = 0$:

$$ {}^{C}D_{0+}^{\nu} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\nu)} \displaystyle\int_0^t \frac{1}{(t-s)^{\nu}} \frac{\mathrm{d}}{\mathrm{d}s} f(s)\,\mathrm{d}s & \text{if } \nu \in (0,1), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} f(t) & \text{if } \nu = 1. \end{cases} \tag{10}$$

We also consider the (left-sided) Riemann–Liouville fractional derivative $\mathrm{d}^{\nu}/\mathrm{d}(-t)^{\nu}$ of order $\nu \geq 1$ (see, e.g. [8, Equation (2.2.4)]) defined by

$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}(-t)^{\nu}} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\nu)} \left(-\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^m \displaystyle\int_t^{\infty} \frac{f(s)}{(s-t)^{1+\nu-m}}\,\mathrm{d}s & \text{if } \nu \text{ is not integer} \\ & \text{and } m := [\nu]+1, \\ (-1)^{\nu} \dfrac{\mathrm{d}^{\nu}}{\mathrm{d}t^{\nu}} f(t) & \text{if } \nu \text{ is integer.} \end{cases} \tag{11}$$

Moreover, for $\eta \in (0,1]$, we consider the (fractional) difference operator $(I-B)^{\eta}$ in [15]. More precisely, I is the identity operator, B is the backward shift operator defined by

$$Bf(k) = f(k-1) \tag{12}$$

and, if we consider Newton's generalized binomial theorem for operators, we have

$$(I-B)^{\eta} = \sum_{j\geq 0} (-1)^j \binom{\eta}{j} B^j.$$

3. Results

In general we show that the state probabilities (and the probability generating functions) solve suitable fractional differential equations and we provide some explicit expressions. In order to have a simpler presentation of the results, throughout this paper we always set

$$s(\boldsymbol{\lambda}) := \sum_{i=1}^{m} \lambda_i,$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$. Moreover, let $\{B_i : i \in \{1, \dots, m\}\}$ be the operators defined by

$$B_i f(k_1, \dots, k_m) = f(k_1, \dots, k_i - 1, \dots, k_m); \tag{13}$$

these operators play the role of the operator B in (12) for the $m = 1$ case.

3.1. Results for the processes in Definitions 1 and 2

The first result shows that the state probabilities $\{\{p_{\boldsymbol{k}}^{\eta,\nu}(t) : \boldsymbol{k} \geq \boldsymbol{0}\} : t \geq 0\}$ in (2) solve fractional differential equations, and we consider the fractional derivative in (10).

Proposition 1. *For $\eta, \nu \in (0, 1]$, the state probabilities $\{\{p_{\boldsymbol{k}}^{\eta,\nu}(t) : \boldsymbol{k} \geq \boldsymbol{0}\} : t \geq 0\}$ in (2) solve the following fractional differential equation:*

$$^{C}D_{0+}^{\nu} p_{\boldsymbol{k}}^{\eta,\nu}(t) = -(s(\boldsymbol{\lambda}))^{\eta}\left(I - \frac{\sum_{i=1}^{m} \lambda_i B_i}{s(\boldsymbol{\lambda})}\right)^{\eta} p_{\boldsymbol{k}}^{\eta,\nu}(t), \qquad p_{\boldsymbol{k}}^{\eta,\nu}(t) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

Proof. Firstly, by (5), we have

$$^{C}D_{0+}^{\nu} G^{\eta,\nu}(\boldsymbol{u}; t) = -\left(\sum_{i=1}^{m} \lambda_i (1 - u_i)\right)^{\eta} G^{\eta,\nu}(\boldsymbol{u}; t), \qquad G^{\eta,\nu}(\boldsymbol{u}; 0) = 1,$$

by [8, Equation (2.4.58)], and, therefore,

$$^{C}D_{0+}^{\nu} G^{\eta,\nu}(\boldsymbol{u}; t) = -(s(\boldsymbol{\lambda}))^{\eta}\left(1 - \frac{\sum_{i=1}^{m} \lambda_i u_i}{s(\boldsymbol{\lambda})}\right)^{\eta} G^{\eta,\nu}(\boldsymbol{u}; t), \qquad G^{\eta,\nu}(\boldsymbol{u}; 0) = 1. \tag{14}$$

From now on we concentrate our attention on the first equation only (the second one concerning the $t = 0$ case trivially holds). Then, if we use the symbol '$\sum_{r_1,\dots,r_m \in \mathcal{S}_j}$' for the sum over all $r_1, \dots, r_m \geq 0$ such that $r_1 + \cdots + r_m = j$, we have

$$\begin{aligned}\left(1 - \frac{\sum_{i=1}^{m} \lambda_i u_i}{s(\boldsymbol{\lambda})}\right)^{\eta} &= \sum_{j \geq 0} \binom{\eta}{j} (-1)^j \left(\frac{\sum_{i=1}^{m} \lambda_i u_i}{s(\boldsymbol{\lambda})}\right)^{j} \\ &= \sum_{j \geq 0} \binom{\eta}{j} \frac{(-1)^j}{(s(\boldsymbol{\lambda}))^j} \sum_{r_1,\dots,r_m \in \mathcal{S}_j} \frac{j!}{r_1! \cdots r_m!} \lambda_1^{r_m} \cdots \lambda_m^{r_m} \cdot u_1^{r_1} \cdots u_m^{r_m}.\end{aligned}$$

Thus,

$$\begin{aligned}^{C}D_{0+}^{\nu} G^{\eta,\nu}(\boldsymbol{u}; t) = -(s(\boldsymbol{\lambda}))^{\eta} \sum_{j \geq 0} \binom{\eta}{j} \frac{(-1)^j}{(s(\boldsymbol{\lambda}))^j} \sum_{r_1,\dots,r_m \in \mathcal{S}_j} \frac{j!}{r_1! \cdots r_m!} \lambda_1^{r_1} \cdots \lambda_m^{r_m} \\ \times \sum_{\boldsymbol{k} \geq \boldsymbol{0}} u_1^{k_1 + r_1} \cdots u_m^{k_m + r_m} p_{\boldsymbol{k}}^{\eta,\nu}(t),\end{aligned}$$

where, for the last factor in the right-hand side, we have

$$\sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1+r_1}\cdots u_m^{k_m+r_m} p_{\boldsymbol{k}}^{\eta,\nu}(t) = \sum_{\boldsymbol{k}\geq\boldsymbol{r}} u_1^{k_1}\cdots u_m^{k_m} p_{\boldsymbol{k}-\boldsymbol{r}}^{\eta,\nu}(t).$$

Then (in the next equality we should have $r_1 \leq k_1, \dots, r_m \leq k_m$, but this restriction can be neglected)

$$\begin{aligned} &{}^C D_{0+}^{\nu} G^{\eta,\nu}(\boldsymbol{u};t) \\ &\quad = -(s(\boldsymbol{\lambda}))^{\eta} \sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1}\cdots u_m^{k_m} \sum_{j\geq 0}\binom{\eta}{j}\frac{(-1)^j}{(s(\boldsymbol{\lambda}))^j}\sum_{r_1,\dots,r_m\in\mathcal{S}_j}\frac{j!}{r_1!\cdots r_m!}\lambda_1^{r_1}\cdots\lambda_m^{r_m} p_{\boldsymbol{k}-\boldsymbol{r}}^{\eta,\nu}(t). \end{aligned}$$

We conclude the proof noting that, since

$$\sum_{r_1,\dots,r_m\in\mathcal{S}_j}\frac{j!}{r_1!\cdots r_m!}\lambda_1^{r_1}\cdots\lambda_m^{r_m} p_{\boldsymbol{k}-\boldsymbol{r}}^{\eta,\nu}(t) = \bigg(\sum_{i=1}^m \lambda_i B_i\bigg)^j p_{\boldsymbol{k}}^{\eta,\nu}(t),$$

where $B_1, \dots, B_m$ are the shift operators in (13), we have

$$\begin{aligned} {}^C D_{0+}^{\nu} G^{\eta,\nu}(\boldsymbol{u};t) &= -(s(\boldsymbol{\lambda}))^{\eta} \sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1}\cdots u_m^{k_m} \sum_{j\geq 0}\binom{\eta}{j}\frac{(-1)^j}{(s(\boldsymbol{\lambda}))^j}\bigg(\sum_{i=1}^m \lambda_i B_i\bigg)^j p_{\boldsymbol{k}}^{\eta,\nu}(t) \\ &= -(s(\boldsymbol{\lambda}))^{\eta} \sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1}\cdots u_m^{k_m}\bigg(I - \frac{\sum_{i=1}^m \lambda_i B_i}{s(\boldsymbol{\lambda})}\bigg)^{\eta} p_{\boldsymbol{k}}^{\eta,\nu}(t), \end{aligned}$$

which yields the desired equation. □

The second result concerns the state probabilities of the fractional compound Poisson process, i.e. $\{\{q_{\boldsymbol{k}}^{\eta,\nu}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ in (3). More precisely, we mean the probabilities $\{\{q_{\boldsymbol{k}}^{1,\nu}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ (time fractional case) and $\{\{q_{\boldsymbol{k}}^{\eta,1}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ (space fractional case). We show that they solve two fractional differential equations: the first one is a generalization of Proposition 1 with $\eta = 1$; in the second one we have the fractional derivative (11).

Proposition 2. *For $\nu \in (0, 1]$, the state probabilities $\{\{q_{\boldsymbol{k}}^{1,\nu}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ in (3) solve the following fractional differential equations:*

$${}^C D_{0+}^{\nu} q_{\boldsymbol{k}}^{1,\nu}(t) = -s(\boldsymbol{\lambda}) q_{\boldsymbol{k}}^{1,\nu}(t) + \sum_{i=1}^m \lambda_i \sum_{j_i=1}^{k_i} \widetilde{q}_{j_i}^{i} q_{k_1,\dots,k_i-j_i,\dots,k_m}^{1,\nu}(t), \qquad q_{\boldsymbol{k}}^{1,\nu}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

For $\eta \in (0, 1]$, the state probabilities $\{\{q_{\boldsymbol{k}}^{\eta,1}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ in (3) solve the following fractional differential equations:

$$\frac{\mathrm{d}^{1/\eta}}{\mathrm{d}(-t)^{1/\eta}} q_{\boldsymbol{k}}^{\eta,1}(t) = s(\boldsymbol{\lambda}) q_{\boldsymbol{k}}^{\eta,1}(t) - \sum_{i=1}^m \lambda_i \sum_{j_i=1}^{k_i} \widetilde{q}_{j_i}^{i} q_{k_1,\dots,k_i-j_i,\dots,k_m}^{\eta,1}(t), \qquad q_{\boldsymbol{k}}^{\eta,1}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

Proof. Firstly, by (4), we have

$$ {}^{C}D^{\nu}_{0+}G^{1,\nu}_{C}(\boldsymbol{u};t) = -\sum_{i=1}^{m}\lambda_i(1-G_i(u_i))G^{1,\nu}_{C}(\boldsymbol{u};t), \qquad G^{1,\nu}_{C}(\boldsymbol{u};0)=1, $$

by [8, Equation (2.4.58)], and

$$ \frac{\mathrm{d}^{1/\eta}}{\mathrm{d}(-t)^{1/\eta}}G^{\eta,1}_{C}(\boldsymbol{u};t) = \sum_{i=1}^{m}\lambda_i(1-G_i(u_i))G^{\eta,1}_{C}(\boldsymbol{u};t), \qquad G^{\eta,1}_{C}(\boldsymbol{u};0)=1, $$

by [8, Equation (2.2.15)]. In both cases the second equation (concerning the $t=0$ case) is trivial, and therefore we concentrate the attention on the first equation. So, if we compare the equations above and the ones in the statement of the proposition, we have to check that

$$ \begin{aligned} &-\sum_{i=1}^{m}\lambda_i(1-G_i(u_i))G^{1,\nu}_{C}(\boldsymbol{u};t)\\ &\qquad = \sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}\bigg(-s(\boldsymbol{\lambda})q^{1,\nu}_{\boldsymbol{k}}(t)+\sum_{i=1}^{m}\lambda_i\sum_{j_i=1}^{k_i}\widetilde{q}^{\,i}_{j_i}q^{1,\nu}_{k_1,\dots,k_i-j_i,\dots,k_m}(t)\bigg), \end{aligned} $$

and

$$ \begin{aligned} &\sum_{i=1}^{m}\lambda_i(1-G_i(u_i))G^{\eta,1}_{C}(\boldsymbol{u};t)\\ &\qquad = \sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}\bigg(s(\boldsymbol{\lambda})q^{\eta,1}_{\boldsymbol{k}}(t)-\sum_{i=1}^{m}\lambda_i\sum_{j_i=1}^{k_i}\widetilde{q}^{\,i}_{j_i}q^{\eta,1}_{k_1,\dots,k_i-j_i,\dots,k_m}(t)\bigg); \end{aligned} $$

moreover, after some easy manipulation, the above equalities are equivalent to

$$ \sum_{i=1}^{m}\lambda_iG_i(u_i)G^{1,\nu}_{C}(\boldsymbol{u};t) = \sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}\sum_{i=1}^{m}\lambda_i\sum_{j_i=1}^{k_i}\widetilde{q}^{\,i}_{j_i}q^{1,\nu}_{k_1,\dots,k_i-j_i,\dots,k_m}(t) $$

and

$$ \sum_{i=1}^{m}\lambda_iG_i(u_i)G^{\eta,1}_{C}(\boldsymbol{u};t) = \sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}\sum_{i=1}^{m}\lambda_i\sum_{j_i=1}^{k_i}\widetilde{q}^{\,i}_{j_i}q^{\eta,1}_{k_1,\dots,k_i-j_i,\dots,k_m}(t), $$

respectively. In the first case, we have

$$ \begin{aligned} \sum_{i=1}^{m}\lambda_iG_i(u_i)G^{1,\nu}_{C}(\boldsymbol{u};t) &= \sum_{i=1}^{m}\lambda_i\sum_{j_i\geq1}u_i^{j_i}\widetilde{q}^{\,i}_{j_i}\sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}q^{1,\nu}_{\boldsymbol{k}}(t)\\ &= \sum_{i=1}^{m}\lambda_i\sum_{j_i\geq1}\widetilde{q}^{\,i}_{j_i}\sum_{\boldsymbol{k}\geq\boldsymbol{0}}u_1^{k_1}\cdots u_m^{k_m}q^{1,\nu}_{k_1,\dots,k_i-j_i,\dots,k_m}(t), \end{aligned} $$

and the desired equality holds because the sums and the factors in the last expression can be rearranged in a different order and $q^{1,\nu}_{k_1,\dots,k_i-j_i,\dots,k_m}(t)=0$ when $j_i>k_i$. The other case can be treated in the same way (we have to consider $G^{\eta,1}_{C}$ and $\{\{q^{\eta,1}_{\boldsymbol{k}}(t)\colon \boldsymbol{k}\geq\boldsymbol{0}\}\colon t\geq0\}$ in place of $G^{1,\nu}_{C}$ and $\{\{q^{1,\nu}_{\boldsymbol{k}}(t)\colon \boldsymbol{k}\geq\boldsymbol{0}\}\colon t\geq0\}$). □

As a special case we give a version of the equations in Proposition 2 for the state probabilities $\{\{p^{\eta,\nu}_{\boldsymbol{k}}(t)\colon \boldsymbol{k}\geq\boldsymbol{0}\}\colon t\geq0\}$ in (2) for the multivariate fractional Poisson process in Definition 1.

The first equation (where $\eta = 1$) meets Proposition 1; the second equation (where $\nu = 1$) with $\eta = 1$ meets the well-known equations for the nonfractional case (i.e. Proposition 1 with $\eta = \nu = 1$).

Corollary 1. *For $\nu \in (0, 1]$, the state probabilities $\{\{p_{\boldsymbol{k}}^{1,\nu}(t)\colon \boldsymbol{k} \geq \boldsymbol{0}\}\colon t \geq 0\}$ in (2) solve the following fractional differential equations:*

$$^{C}D_{0+}^{\nu} p_{\boldsymbol{k}}^{1,\nu}(t) = -s(\boldsymbol{\lambda}) p_{\boldsymbol{k}}^{1,\nu}(t) + \sum_{i=1}^{m} \lambda_i p_{k_1,\ldots,k_i-1,\ldots,k_m}^{1,\nu}(t), \qquad p_{\boldsymbol{k}}^{1,\nu}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

For $\eta \in (0, 1]$, the state probabilities $\{\{p_{\boldsymbol{k}}^{\eta,1}(t)\colon \boldsymbol{k} \geq \boldsymbol{0}\}\colon t \geq 0\}$ in (2) solve the following fractional differential equations:

$$\frac{\mathrm{d}^{1/\eta}}{\mathrm{d}(-t)^{1/\eta}} p_{\boldsymbol{k}}^{\eta,1}(t) = s(\boldsymbol{\lambda}) p_{\boldsymbol{k}}^{\eta,1}(t) - \sum_{i=1}^{m} \lambda_i p_{k_1,\ldots,k_i-1,\ldots,k_m}^{\eta,1}(t), \qquad p_{\boldsymbol{k}}^{\eta,1}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

Proof. The proof is an immediate consequence of Proposition 2 and Remark 1. □

Now we give some expressions of the state probabilities $\{\{p_{\boldsymbol{k}}^{\eta,\nu}(t)\colon \boldsymbol{k} \geq \boldsymbol{0}\}\colon t \geq 0\}$ in (2). We start with an implicit expression which generalizes [2, Equation (3.19)] (note that we use the notation ∂_{λ_i} in place of $\partial/\partial_{\lambda_i}$). The most explicit formulae are given in Proposition 4.

Proposition 3. *Let $\eta, \nu \in (0, 1]$ be arbitrarily fixed. Then, for all integer $k_1, \ldots, k_m \geq 0$, we have*

$$p_{\boldsymbol{k}}^{\eta,\nu}(t) = \prod_{i=1}^{m} (-\lambda_i \partial_{\lambda_i})^{k_i} E_{\nu,1}(-(s(\boldsymbol{\lambda}))^{\eta} t^{\nu}).$$

Proof. By construction, we have

$$\begin{aligned} p_{\boldsymbol{k}}^{\eta,\nu}(t) &= \mathbb{E}\bigg[\prod_{i=1}^{m} \bigg\{\frac{(\lambda_i z)^{k_i}}{k_i!} \mathrm{e}^{-\lambda_i z}\bigg\}\bigg|_{z=\mathcal{A}^{\eta}(\mathcal{L}^{\nu}(t))}\bigg] \\ &= \frac{1}{k_1! \cdots k_m!} \mathbb{E}\bigg[\prod_{i=1}^{m} \{(-\lambda_i \partial_{\lambda_i})^{k_i}\} \mathrm{e}^{-s(\boldsymbol{\lambda})\mathcal{A}^{\eta}(\mathcal{L}^{\nu}(t))}\bigg]; \end{aligned}$$

then we can conclude by following the same lines of the proof of [2, Equation (3.19)], where we take into account that $\mathbb{E}[\mathrm{e}^{-s(\boldsymbol{\lambda})\mathcal{A}^{\eta}(\mathcal{L}^{\nu}(t))}] = E_{\nu,1}(-(s(\boldsymbol{\lambda}))^{\eta} t^{\nu})$ by [2, Equation (3.8)]. □

Proposition 4. *Let $\eta, \nu \in (0, 1]$ be arbitrarily fixed. Then, for all integer $k_1, \ldots, k_m \geq 0$, we have*

$$\begin{aligned} p_{\boldsymbol{k}}^{\eta,\nu}(t) &= \frac{\lambda_1^{k_1} \cdots \lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}} \frac{(-1)^{k_1+\cdots+k_m}}{k_1! \cdots k_m!} \\ &\quad \times \sum_{r \geq 0} \frac{(-(s(\boldsymbol{\lambda}))^{\eta} t^{\nu})^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\eta r + 1)}{\Gamma(\eta r - (k_1 + \cdots + k_m) + 1)}, \end{aligned} \tag{15}$$

or, equivalently,

$$\begin{aligned} p_{\boldsymbol{k}}^{\eta,\nu}(t) &= \frac{\lambda_1^{k_1} \cdots \lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}} \frac{(-1)^{k_1+\cdots+k_m}}{k_1! \cdots k_m!} \\ &\quad \times {}_2\Psi_2 \left[\begin{matrix} (1, \eta) & (1, 1) \\ (1, \nu) & (1 - (k_1 + \cdots + k_m), \eta) \end{matrix}\right] (-(s(\boldsymbol{\lambda}))^{\eta} t^{\nu}). \end{aligned} \tag{16}$$

Proof. Equation (16) follows from (15). In fact, by taking into account (8), it suffices to multiply the terms of the series in the right-hand side of (15) by $\Gamma(r+1)/r! = 1$ (note that the convergence condition (9) holds because $\nu + \eta - (\eta + 1) > -1$). So from now on we can concentrate our attention on (15) only.

Firstly, we have

$$\begin{aligned} p_{\boldsymbol{k}}^{\eta,\nu}(t) &= \mathbb{P}\bigg(\{N^{\eta,\nu}(t) = \boldsymbol{k}\} \cap \bigg\{\sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg\}\bigg) \\ &= \mathbb{P}\bigg(N^{\eta,\nu}(t) = \boldsymbol{k} \;\bigg|\; \sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg)\mathbb{P}\bigg(\sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg). \end{aligned} \tag{17}$$

We start with the conditional probability in (17). We have

$$\mathbb{P}\bigg(N^{\eta,\nu}(t) = \boldsymbol{k} \;\bigg|\; \sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg) = \frac{\mathbb{P}(N^{\eta,\nu}(t) = \boldsymbol{k})}{\mathbb{P}(\sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i)}$$

and, if we consider the conditional distributions given $\mathcal{A}^\eta(\mathcal{L}^\nu(t))$, we obtain

$$\begin{aligned} &\mathbb{P}\bigg(N^{\eta,\nu}(t) = \boldsymbol{k} \;\bigg|\; \sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg) \\ &\quad = \mathbb{E}\bigg[\prod_{i=1}^m \frac{(\lambda_i r)^{k_i}}{k_i!}\mathrm{e}^{-\lambda_i r}\bigg|_{r=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}\bigg]\bigg(\mathbb{E}\bigg[\frac{(s(\boldsymbol{\lambda})r)^{\sum_{i=1}^m k_i}}{(\sum_{i=1}^m k_i)!}\mathrm{e}^{-s(\boldsymbol{\lambda})r}\bigg|_{r=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}\bigg]\bigg)^{-1} \\ &\quad = \frac{(k_1 + \cdots + k_m)!}{k_1! \cdots k_m!}\frac{\lambda_1^{k_1} \cdots \lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}} \end{aligned}$$

after some computation, where there is a factor equal to 1 given by

$$\mathbb{E}[(\mathcal{A}^\eta(\mathcal{L}^\nu(t)))^{\sum_{i=1}^m k_i}\mathrm{e}^{-s(\boldsymbol{\lambda})\mathcal{A}^\eta(\mathcal{L}^\nu(t))}]$$

divided by itself. For the second factor in (17), we consider again the conditional distributions given $\mathcal{A}^\eta(\mathcal{L}^\nu(t))$ and we have

$$\begin{aligned} \mathbb{P}\bigg(\sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg) &= \mathbb{E}\bigg[\mathbb{P}\bigg(\sum_{i=1}^m N_i^{1,1}(r) = \sum_{i=1}^m k_i\bigg)\bigg|_{r=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}\bigg] \\ &= \mathbb{E}\bigg[\frac{(s(\boldsymbol{\lambda})r)^{\sum_{i=1}^m k_i}}{(\sum_{i=1}^m k_i)!}\mathrm{e}^{-s(\boldsymbol{\lambda})r}\bigg|_{r=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}\bigg]; \end{aligned}$$

then we have

$$\begin{aligned} &\mathbb{P}\bigg(\sum_{i=1}^m N_i^{\eta,\nu}(t) = \sum_{i=1}^m k_i\bigg) \\ &\quad = \frac{(-1)^{k_1+\cdots+k_m}}{(k_1 + \cdots + k_m)!}\sum_{r\geq 0}\frac{(-(s(\boldsymbol{\lambda}))^\eta t^\nu)^r}{\Gamma(\nu r + 1)}\frac{\Gamma(\eta r + 1)}{\Gamma(\eta r - (k_1 + \cdots + k_m) + 1)} \end{aligned}$$

by taking into account the known formula for the $m = 1$ case (see [2, Equation (3.24)], where the formula is given in terms of a binomial coefficient, with a typographical error; see also [15, Equation (1.8)]). Finally, (15) can be easily checked. □

Here we present some remarks on Proposition 4. Firstly, (15) with $m = 1$ meets known formulae in the literature (see, e.g. [15, Equation (1.8)]). Moreover, for $\nu = 1$, we have

$$p_{\boldsymbol{k}}^{\eta,1}(t) = \frac{\lambda_1^{k_1}\cdots\lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}}\frac{(-1)^{k_1+\cdots+k_m}}{k_1!\cdots k_m!}\sum_{r\ge 0}\frac{(-(s(\boldsymbol{\lambda}))^{\eta}t)^r}{r!}\frac{\Gamma(\eta r+1)}{\Gamma(\eta r-(k_1+\cdots+k_m)+1)},$$

$$p_{\boldsymbol{k}}^{\eta,1}(t) = \frac{\lambda_1^{k_1}\cdots\lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}}\frac{(-1)^{k_1+\cdots+k_m}}{k_1!\cdots k_m!}\,{}_1\Psi_1\left[\begin{matrix}(1,\eta)\\(1-(k_1+\cdots+k_m),\eta)\end{matrix}\right](-(s(\boldsymbol{\lambda}))^{\eta}t);$$

both formulae reduce to those in [15, Theorem 2.2] concerning the $m = 1$ case. Finally, for $\eta = 1$, (15) can be expressed as

$$p_{\boldsymbol{k}}^{1,\nu}(t) = \frac{\lambda_1^{k_1}\cdots\lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}}\frac{(-1)^{k_1+\cdots+k_m}}{k_1!\cdots k_m!}\sum_{r\ge k_1+\cdots+k_m}\frac{(-s(\boldsymbol{\lambda})t^{\nu})^r}{\Gamma(\nu r+1)}\frac{r!}{(r-(k_1+\cdots+k_m))!}$$

(because the summands with $r < k_1+\cdots+k_m$ are equal to 0), and, therefore,

$$\begin{aligned}p_{\boldsymbol{k}}^{1,\nu}(t) &= \frac{\lambda_1^{k_1}\cdots\lambda_m^{k_m}}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m}}\frac{(-1)^{k_1+\cdots+k_m}}{k_1!\cdots k_m!}\\ &\quad\times\sum_{r\ge 0}\frac{(-s(\boldsymbol{\lambda})t^{\nu})^{r+k_1+\cdots+k_m}}{\Gamma(\nu r+\nu(k_1+\cdots+k_m)+1)}\frac{(r+k_1+\cdots+k_m)!}{r!}\\ &= \frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\,t^{\nu(k_1+\cdots+k_m)}\\ &\quad\times\sum_{r\ge 0}\frac{(k_1+\cdots+k_m+1)^{(r)}(-s(\boldsymbol{\lambda})t^{\nu})^r}{r!\,\Gamma(\nu r+\nu(k_1+\cdots+k_m)+1)}\\ &= \frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}\lambda_1^{k_1}\cdots\lambda_m^{k_m}\,t^{\nu(k_1+\cdots+k_m)}E_{\nu,\nu(k_1+\cdots+k_m)+1}^{(k_1+\cdots+k_m)+1}(-s(\boldsymbol{\lambda})t^{\nu});\end{aligned}$$

the last expression meets [5, Equation (2.5)] concerning the $m = 1$ case.

In Proposition 5, we compute the covariance

$$\begin{aligned}&\operatorname{cov}(N_j^{1,\nu}(t),N_h^{1,\nu}(t))\\ &\quad:= \mathbb{E}[N_j^{1,\nu}(t)N_h^{1,\nu}(t)]-\mathbb{E}[N_j^{1,\nu}(t)]\mathbb{E}[N_h^{1,\nu}(t)]\quad\text{for } j,h\in\{1,\dots,m\};\end{aligned}$$

note that we take $\eta = 1$, otherwise the covariance would not be finite. In what follows we refer to

$$Z(\nu) := \frac{1}{\nu}\left(\frac{1}{\Gamma(2\nu)}-\frac{1}{\nu\Gamma^2(\nu)}\right), \tag{18}$$

where, as shown in [3, Subsection 3.1], $Z(\nu) \ge 0$ for $\nu \in (0,1]$ and $Z(\nu) = 0$ if and only if $\nu = 1$. The codifference $\tau(X_1, X_2)$ has been studied in the literature (see, e.g. [9, Equation (1.7)]) when the random variables X_1 and X_2 have infinite variance and it is known that it reduces to $\operatorname{cov}(X_1, X_2)$ when (X_1, X_2) forms a Gaussian vector (see the displayed equality just after [9, Equation (1.7)]). So in Proposition 5 we also compute the codifference

$$\begin{aligned}\tau(N_j^{\eta,\nu}(t),N_h^{\eta,\nu}(t)) &:= \log\mathbb{E}[\mathrm{e}^{\mathrm{i}(N_j^{\eta,\nu}(t)-N_h^{\eta,\nu}(t))}]\\ &\qquad-\log\mathbb{E}[\mathrm{e}^{\mathrm{i}N_j^{\eta,\nu}(t)}]-\log\mathbb{E}[\mathrm{e}^{-\mathrm{i}N_h^{\eta,\nu}(t)}]\quad\text{for } j,h\in\{1,\dots,m\},\end{aligned}$$

where i is the imaginary unit.

Proposition 5. *Let $\eta, \nu \in (0,1]$ be arbitrarily fixed. Then, for $j, h \in \{1, \dots, m\}$, we have*

$$\operatorname{cov}(N_j^{1,\nu}(t), N_h^{1,\nu}(t)) = \mathbf{1}_{\{j=h\}} \frac{\lambda_j t^\nu}{\Gamma(\nu+1)} + \lambda_j \lambda_h t^{2\nu} Z(\nu),$$

where $Z(\nu)$ is as in (18);

$$\begin{aligned}\tau(N_j^{\eta,\nu}(t), N_h^{\eta,\nu}(t)) = \mathbf{1}_{\{j\neq h\}} \log E_{\nu,1}(-(\lambda_j(1-\mathrm{e}^{\mathrm{i}}) + \lambda_h(1-\mathrm{e}^{-\mathrm{i}}))^\eta t^\nu) \\ - \log E_{\nu,1}(-(\lambda_j(1-\mathrm{e}^{\mathrm{i}}))^\eta t^\nu) - \log E_{\nu,1}(-(\lambda_h(1-\mathrm{e}^{-\mathrm{i}}))^\eta t^\nu).\end{aligned}$$

Proof. Firstly, it is useful to recall the following formulae:

$$\mathbb{E}[N_k^{1,\nu}(t)] = \frac{\lambda_k t^\nu}{\Gamma(\nu+1)} \quad \text{for all } k \in \{1, \dots, m\} \tag{19}$$

(see, e.g. [4, Equation (2.7)]);

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}uN_k^{\eta,\nu}(t)}] = E_{\nu,1}(-(\lambda_k(1-\mathrm{e}^{\mathrm{i}u}))^\eta t^\nu) \quad \text{for all } u \in \mathbb{R} \text{ and } k \in \{1, \dots, m\} \tag{20}$$

which can be obtained by adapting the computation in [15] for the generating functions.

We start with the $j = h$ case. The formula for the covariance holds noting that

$$\operatorname{cov}(N_j^{1,\nu}(t), N_j^{1,\nu}(t)) = \operatorname{var}[N_j^{1,\nu}(t)]$$

and by taking into account [4, Equation (2.8)]. The formula for the codifference holds noting that $\mathbb{E}[\mathrm{e}^{\mathrm{i}(N_j^{\eta,\nu}(t) - N_j^{\eta,\nu}(t))}] = 1$ and by taking into account (20).

We conclude with the $j \neq h$ case. Firstly, we have

$$\mathbb{E}[N_j^{1,\nu}(t) N_h^{1,\nu}(t)] = \mathbb{E}[\mathbb{E}[N_j^{1,1}(s)]\mathbb{E}[N_h^{1,1}(s)]|_{s=\mathcal{L}^\nu(t)}] = \lambda_j\lambda_h \int_0^\infty s^2 f_{\mathcal{L}^\nu(t)}(s)\,\mathrm{d}s$$

and, since

$$\int_0^\infty s^k f_{\mathcal{L}^\nu(t)}(s)\,\mathrm{d}s = \frac{k!\,t^{\nu k}}{\Gamma(\nu k+1)} \quad \text{for all } k \geq 0$$

by combining [17, Equations (2.4) and (2.7)], we have

$$\mathbb{E}[N_j^{1,\nu}(t) N_h^{1,\nu}(t)] = \lambda_j\lambda_h \frac{2t^{2\nu}}{\Gamma(2\nu+1)};$$

then, by taking into account (19), we obtain

$$\begin{aligned}\operatorname{cov}(N_j^{1,\nu}(t), N_h^{1,\nu}(t)) &= \lambda_j\lambda_h \frac{2t^{2\nu}}{\Gamma(2\nu+1)} - \frac{\lambda_j t^\nu}{\Gamma(\nu+1)} \frac{\lambda_h t^\nu}{\Gamma(\nu+1)} \\ &= \lambda_j\lambda_h t^{2\nu}\bigg(\frac{2}{\Gamma(2\nu+1)} - \frac{1}{\Gamma^2(\nu+1)}\bigg) \\ &= \lambda_j\lambda_h t^{2\nu}\bigg(\frac{2}{2\nu\Gamma(2\nu)} - \frac{1}{\nu^2\Gamma^2(\nu)}\bigg) \\ &= \lambda_j\lambda_h t^{2\nu} Z(\nu)\end{aligned}$$

and the formula for the covariance is proved. Furthermore, since we have

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}(N_j^{\eta,\nu}(t)-N_h^{\eta,\nu}(t))}] &= \mathbb{E}[\mathbb{E}[\mathrm{e}^{\mathrm{i}N_j^{1,1}(s)}]\mathbb{E}[\mathrm{e}^{-\mathrm{i}N_h^{1,1}(s)}]|_{s=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}]\\ &= \mathbb{E}[\mathrm{e}^{\lambda_j s(\mathrm{e}^{\mathrm{i}}-1)+\lambda_h s(\mathrm{e}^{-\mathrm{i}}-1)}|_{s=\mathcal{A}^\eta(\mathcal{L}^\nu(t))}]\\ &= E_{\nu,1}(-(\lambda_j(1-\mathrm{e}^{\mathrm{i}})+\lambda_h(1-\mathrm{e}^{-\mathrm{i}}))^\eta t^\nu),\end{aligned}$$

the formula for the codifference can be easily obtained by taking into account (20). □

It is known that $\{C^{\eta,1}(t)\colon t\geq 0\}$ and $\{N^{\eta,1}(t)\colon t\geq 0\}$ are Lévy processes and, moreover, when $\eta=1$ their Lévy measures ρ_C^1 and ρ_N^1 are defined by

$$\rho_C^1(A_1\times\cdots\times A_m)=\sum_{i=1}^m\lambda_i\widetilde{q}^i(A_i) \tag{21}$$

and

$$\rho_N^1(A_1\times\cdots\times A_m)=\sum_{i=1}^m\lambda_i\mathbf{1}_{\{1\in A_i\}}. \tag{22}$$

In the next proposition we present the Lévy measures ρ_C^η and ρ_N^η when $\eta\in(0,1)$.

Proposition 6. *Let $\eta\in(0,1)$ be arbitrarily fixed. Then we define the Lévy measure ρ_C^η of $\{C^{\eta,1}(t)\colon t\geq 0\}$ by*

$$\begin{aligned}&\rho_C^\eta(A_1\times\cdots\times A_m)\\ &\quad=\frac{\eta}{\Gamma(1-\eta)}\sum_{\boldsymbol{k}\succ\boldsymbol{0}}\int_0^\infty\prod_{i=1}^m\bigg\{\sum_{n_i\geq 0}\bigg\{(\widetilde{q}^i)_{k_i}^{*n_i}\frac{(\lambda_i z)^{n_i}}{n_i!}\bigg\}\mathbf{1}_{\{k_i\in A_i\}}\bigg\}\frac{\mathrm{e}^{-s(\boldsymbol{\lambda})z}}{z^{\eta+1}}\,\mathrm{d}z.\end{aligned} \tag{23}$$

Moreover the Lévy measure ρ_N^η of $\{N^{\eta,1}(t)\colon t\geq 0\}$ is defined by

$$\begin{aligned}&\rho_N^\eta(A_1\times\cdots\times A_m)\\ &\quad=\frac{\eta}{\Gamma(1-\eta)}\sum_{\boldsymbol{k}\succ\boldsymbol{0}}\frac{\Gamma(k_1+\cdots+k_m-\eta)}{(s(\boldsymbol{\lambda}))^{k_1+\cdots+k_m-\eta}}\prod_{i=1}^m\bigg\{\frac{\lambda_i^{k_i}}{k_i!}\mathbf{1}_{\{k_i\in A_i\}}\bigg\}.\end{aligned} \tag{24}$$

Proof. Firstly, by [21, Equation (30.8)] and the Lévy measure ρ_f for the stable subordinator $\{\mathcal{A}^\nu(t)\colon t\geq 0\}$ in Remark 2, we have

$$\begin{aligned}&\rho_C^\eta(A_1\times\cdots\times A_m)\\ &\quad=\sum_{\boldsymbol{k}\succ\boldsymbol{0}}\int_0^\infty\prod_{i=1}^m\bigg\{\sum_{n_i\geq 0}\bigg\{(\widetilde{q}^i)_{k_i}^{*n_i}\frac{(\lambda_i z)^{n_i}}{n_i!}\mathrm{e}^{-\lambda_i z}\bigg\}\mathbf{1}_{\{k_i\in A_i\}}\bigg\}\frac{\eta}{\Gamma(1-\eta)}\frac{1}{z^{\eta+1}}\,\mathrm{d}z.\end{aligned}$$

Then we easily obtain (23) with some manipulation. Finally, as far as (24) is concerned, we have to consider (23) with $\widetilde{q}_j^i:=\mathbf{1}_{\{j=1\}}$ for all $i\in\{1,\dots,m\}$; therefore, we have $(\widetilde{q}^i)_{k_i}^{*n_i}=\mathbf{1}_{\{k_i=n_i\}}$ and we obtain

$$\begin{aligned}\rho_N^\eta(A_1\times\cdots\times A_m)&=\frac{\eta}{\Gamma(1-\eta)}\sum_{\boldsymbol{k}\succ\boldsymbol{0}}\int_0^\infty\prod_{i=1}^m\bigg\{\frac{(\lambda_i z)^{k_i}}{k_i!}\mathbf{1}_{\{k_i\in A_i\}}\bigg\}\frac{\mathrm{e}^{-s(\boldsymbol{\lambda})z}}{z^{\eta+1}}\,\mathrm{d}z\\ &=\frac{\eta}{\Gamma(1-\eta)}\sum_{\boldsymbol{k}\succ\boldsymbol{0}}\int_0^\infty z^{k_1+\cdots+k_m-\eta-1}\mathrm{e}^{-s(\boldsymbol{\lambda})z}\,\mathrm{d}z\prod_{i=1}^m\bigg\{\frac{\lambda_i^{k_i}}{k_i!}\mathbf{1}_{\{k_i\in A_i\}}\bigg\},\end{aligned}$$

which yields (24). □

We remark that ρ_C^1 in (23) meets (21). In fact, if we set $\Gamma(1-1)/\Gamma(1-1)=1$, we have a nonnull contribution if and only if $(n_1,\dots,n_m)$ belongs to the set

$$\{(1,0,\dots,0),\dots,(0,\dots,0,1)\};$$

thus, (23) yields

$$\begin{aligned}\rho_C^{\eta}(A_1\times\cdots\times A_m)&=\frac{1}{\Gamma(1-1)}\int_0^\infty z^{1-1-1}\mathrm{e}^{-s(\boldsymbol{\lambda})z}\,\mathrm{d}z\sum_{i=1}^m\sum_{k_i\geq 1}\{\lambda_i\widetilde{q}^i_{k_i}\mathbf{1}_{\{k_i\in A_i\}}\}\\&=\frac{1}{\Gamma(1-1)}\frac{\Gamma(1-1)}{(s(\boldsymbol{\lambda}))^0}\sum_{i=1}^m\lambda_i\sum_{k_i\geq 1}\{\widetilde{q}^i_{k_i}\mathbf{1}_{\{k_i\in A_i\}}\}\\&=\sum_{i=1}^m\lambda_i\widetilde{q}^i(A_i).\end{aligned}$$

Similarly, ρ_N^1 in (24) meets (22). In fact we have a nonnull contribution if and only if $(k_1,\dots,k_m)$ belongs to the set $\{(1,0,\dots,0),\dots,(0,\dots,0,1)\}$, and (24) yields

$$\rho_N^{\eta}(A_1\times\cdots\times A_m)=\frac{1}{\Gamma(1-1)}\sum_{i=1}^m\frac{\Gamma(1-1)}{(s(\boldsymbol{\lambda}))^0}\lambda_i\mathbf{1}_{\{1\in A_i\}}=\sum_{i=1}^m\lambda_i\mathbf{1}_{\{1\in A_i\}}.$$

3.2. Results for the process in Definition 3

Here we give a multivariate version of [16, Theorem 2.1, Remark 2.3, and Remark 2.5]. In particular we recover those results and remarks by setting $m=1$. In view of what follows we consider the analogue of [16, Equation (1.1)], i.e.

$$\begin{aligned}&\mathbb{P}(N^{f,1}(t+\mathrm{d}t)-N^{f,1}(t)=\boldsymbol{k})\\&=\begin{cases}\displaystyle\int_0^\infty\bigg(\prod_{i=1}^m\frac{(\lambda_i r)^{k_i}}{k_i!}\mathrm{e}^{-\lambda_i r}\bigg)\rho_f(\mathrm{d}r)\,\mathrm{d}t+o(\mathrm{d}t) & \text{for } \boldsymbol{k}\succ\mathbf{0},\\ \displaystyle 1-\int_0^\infty\bigg(\prod_{i=1}^m\mathrm{e}^{-\lambda_i r}\bigg)\rho_f(\mathrm{d}r)\,\mathrm{d}t+o(\mathrm{d}t) & \text{for } \boldsymbol{k}=\mathbf{0},\end{cases}\\&=\begin{cases}\displaystyle\prod_{i=1}^m\frac{\lambda_i^{k_i}}{k_i!}\int_0^\infty r^{\sum_{i=1}^m k_i}\mathrm{e}^{-s(\boldsymbol{\lambda})r}\rho_f(\mathrm{d}r)\,\mathrm{d}t+o(\mathrm{d}t) & \text{for } \boldsymbol{k}\succ\mathbf{0},\\ \displaystyle 1-\int_0^\infty\mathrm{e}^{-s(\boldsymbol{\lambda})r}\rho_f(\mathrm{d}r)\,\mathrm{d}t+o(\mathrm{d}t) & \text{for } \boldsymbol{k}=\mathbf{0},\end{cases}\end{aligned}$$

and we consider the function $\widetilde{f}_m$ defined by

$$\widetilde{f}_m(\boldsymbol{\lambda};\boldsymbol{u}):=\int_0^\infty\bigg(1-\mathrm{e}^{-s(\boldsymbol{\lambda})r}\sum_{\boldsymbol{j}\geq\mathbf{0}}\prod_{i=1}^m\frac{(\lambda_i u_i r)^{j_i}}{j_i!}\bigg)\rho_f(\mathrm{d}r);$$

in particular, we have

$$\widetilde{f}_m(\boldsymbol{\lambda};\mathbf{0})=\int_0^\infty(1-\mathrm{e}^{-s(\boldsymbol{\lambda})r})\rho_f(\mathrm{d}r)=f(s(\boldsymbol{\lambda}))\quad\text{for }\boldsymbol{u}=\mathbf{0},$$

and

$$\widetilde{f}_1(\lambda_1; u_1) = \int_0^\infty (1 - \mathrm{e}^{-\lambda_1 r + \lambda_1 u_1 r}) \rho_f(\mathrm{d}r) = f(\lambda_1(1 - u_1))$$

for the univariate case $m = 1$.

Proposition 7. *Let f be a Bernštein function. Then we have the following results.*

(i) *The state probabilities $\{\{p_{\boldsymbol{k}}^{f,1}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ in (6) solve the following fractional differential equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\boldsymbol{k}}^{f,1}(t) = \sum_{\boldsymbol{0} \prec \boldsymbol{j} \leq \boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t) \prod_{i=1}^m \frac{\lambda_i^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i} \mathrm{e}^{-rs(\boldsymbol{\lambda})} \rho_f(\mathrm{d}r) - f(s(\boldsymbol{\lambda})) p_{\boldsymbol{k}}^{f,1}(t),$$
$$p_{\boldsymbol{k}}^{f,1}(t) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.$$

(ii) *The probability generating functions $\{G^{f,1}(\cdot; t) \colon t \geq 0\}$ in (7) solve the following fractional differential equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t} G^{f,1}(\boldsymbol{u}; t) = -\widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{u}) G^{f,1}(\boldsymbol{u}; t), \qquad G^{f,1}(\boldsymbol{u}; 0) = 1,$$

and, therefore, we have $G^{f,1}(\boldsymbol{u}; t) = \mathrm{e}^{-t \widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{u})}$.

Proof. (i) The initial condition trivially holds. Then, since $\{N^{f,1}(t) \colon t \geq 0\}$ has independent increments, by taking into account the distribution of the jumps given above, we have

$$\begin{aligned} p_{\boldsymbol{k}}^{f,1}(t + \mathrm{d}t) &= \sum_{\boldsymbol{0} \leq \boldsymbol{j} \leq \boldsymbol{k}} \mathbb{P}(N^{f,1}(t) = \boldsymbol{j},\ N^{f,1}(t + \mathrm{d}t) - N^{f,1}(t) = \boldsymbol{k} - \boldsymbol{j}) \\ &= \sum_{\boldsymbol{0} \leq \boldsymbol{j} \prec \boldsymbol{k}} p_{\boldsymbol{j}}^{f,1}(t) \biggl(\int_0^\infty \biggl(\prod_{i=1}^m \frac{(\lambda_i r)^{k_i - j_i}}{(k_i - j_i)!} \mathrm{e}^{-\lambda_i r} \biggr) \rho_f(\mathrm{d}r)\, \mathrm{d}t + o(\mathrm{d}t) \biggr) \\ &\quad + p_{\boldsymbol{k}}^{f,1}(t) \biggl(1 - \int_0^\infty \mathrm{e}^{-s(\boldsymbol{\lambda}) r} \rho_f(\mathrm{d}r)\, \mathrm{d}t + o(\mathrm{d}t) \biggr), \end{aligned}$$

and, therefore, we consider a suitable change of summation indices in the last equality

$$\begin{aligned} &p_{\boldsymbol{k}}^{f,1}(t + \mathrm{d}t) - p_{\boldsymbol{k}}^{f,1}(t) \\ &\quad = \sum_{\boldsymbol{0} \leq \boldsymbol{j} \prec \boldsymbol{k}} p_{\boldsymbol{j}}^{f,1}(t) \biggl(\prod_{i=1}^m \frac{\lambda_i^{k_i - j_i}}{(k_i - j_i)!} \int_0^\infty r^{\sum_{i=1}^m (k_i - j_i)} \mathrm{e}^{-s(\boldsymbol{\lambda}) r} \rho_f(\mathrm{d}r)\, \mathrm{d}t + o(\mathrm{d}t) \biggr) \\ &\qquad - p_{\boldsymbol{k}}^{f,1}(t)(f(s(\boldsymbol{\lambda}))\, \mathrm{d}t + o(\mathrm{d}t)) \\ &\quad = \sum_{\boldsymbol{0} \prec \boldsymbol{j} \leq \boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t) \biggl(\prod_{i=1}^m \frac{\lambda_i^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i} \mathrm{e}^{-s(\boldsymbol{\lambda}) r} \rho_f(\mathrm{d}r)\, \mathrm{d}t + o(\mathrm{d}t) \biggr) \\ &\qquad - p_{\boldsymbol{k}}^{f,1}(t)(f(s(\boldsymbol{\lambda}))\, \mathrm{d}t + o(\mathrm{d}t)). \end{aligned}$$

We conclude by dividing by $\mathrm{d}t$ and taking the limit as $\mathrm{d}t$ goes to 0.

(ii) The initial condition trivially holds. Then, if we take into account the differential equation obtained for the proof of (i), after some manipulation we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G^{f,1}(\boldsymbol{u};t) &= \sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1}\cdots u_m^{k_m}\frac{\mathrm{d}}{\mathrm{d}t}p_{\boldsymbol{k}}^{f,1}(t)\\ &= \sum_{\boldsymbol{k}\geq\boldsymbol{0}} u_1^{k_1}\cdots u_m^{k_m}\\ &\quad\times\bigg(\sum_{\boldsymbol{0}\prec\boldsymbol{j}\leq\boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t)\prod_{i=1}^m\frac{\lambda_i^{j_i}}{j_i!}\int_0^\infty r^{\sum_{i=1}^m j_i}\mathrm{e}^{-rs(\boldsymbol{\lambda})}\rho_f(\mathrm{d}r)-f(s(\boldsymbol{\lambda}))p_{\boldsymbol{k}}^{f,1}(t)\bigg)\\ &= -f(s(\boldsymbol{\lambda}))G^{f,1}(\boldsymbol{u};t)\\ &\quad+\sum_{\boldsymbol{k}\geq\boldsymbol{0}}\prod_{i=1}^m u_i^{k_i}\bigg(\sum_{\boldsymbol{0}\prec\boldsymbol{j}\leq\boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t)\prod_{i=1}^m\frac{\lambda_i^{j_i}}{j_i!}\int_0^\infty r^{\sum_{i=1}^m j_i}\mathrm{e}^{-rs(\boldsymbol{\lambda})}\rho_f(\mathrm{d}r)\bigg);\end{aligned}$$

moreover, if we rearrange the summands in a different order, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G^{f,1}(\boldsymbol{u};t) &= -f(s(\boldsymbol{\lambda}))G^{f,1}(\boldsymbol{u};t)\\ &\quad+\sum_{\boldsymbol{j}\succ\boldsymbol{0}}\sum_{\boldsymbol{k}\geq\boldsymbol{j}}\prod_{i=1}^m u_i^{k_i}\bigg(p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t)\prod_{i=1}^m\frac{\lambda_i^{j_i}}{j_i!}\int_0^\infty r^{\sum_{i=1}^m j_i}\mathrm{e}^{-rs(\boldsymbol{\lambda})}\rho_f(\mathrm{d}r)\bigg)\\ &= -f(s(\boldsymbol{\lambda}))G^{f,1}(\boldsymbol{u};t)\\ &\quad+\sum_{\boldsymbol{j}\succ\boldsymbol{0}}\int_0^\infty \mathrm{e}^{-rs(\boldsymbol{\lambda})}\prod_{i=1}^m\frac{(\lambda_i u_i r)^{j_i}}{j_i!}\rho_f(\mathrm{d}r)\sum_{\boldsymbol{k}\geq\boldsymbol{j}}\prod_{i=1}^m u_i^{k_i-j_i}p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t)\\ &= \bigg(-f(s(\boldsymbol{\lambda}))+\sum_{\boldsymbol{j}\succ\boldsymbol{0}}\int_0^\infty \mathrm{e}^{-rs(\boldsymbol{\lambda})}\prod_{i=1}^m\frac{(\lambda_i u_i r)^{j_i}}{j_i!}\rho_f(\mathrm{d}r)\bigg)G^{f,1}(\boldsymbol{u};t).\end{aligned}$$

Finally, we can check that (in the first equality we take into account the integral representation of f)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G^{f,1}(\boldsymbol{u};t) &= -\bigg(\int_0^\infty\Big(1-\mathrm{e}^{-rs(\boldsymbol{\lambda})}\Big)\rho_f(\mathrm{d}r)\\ &\qquad-\sum_{\boldsymbol{j}\succ\boldsymbol{0}}\int_0^\infty \mathrm{e}^{-rs(\boldsymbol{\lambda})}\prod_{i=1}^m\frac{(\lambda_i u_i r)^{j_i}}{j_i!}\rho_f(\mathrm{d}r)\bigg)G^{f,1}(\boldsymbol{u};t)\\ &= -\bigg(\int_0^\infty\bigg(1-\mathrm{e}^{-rs(\boldsymbol{\lambda})}\sum_{\boldsymbol{j}\geq\boldsymbol{0}}\prod_{i=1}^m\frac{(\lambda_i u_i r)^{j_i}}{j_i!}\bigg)\rho_f(\mathrm{d}r)\bigg)G^{f,1}(\boldsymbol{u};t)\\ &= -\widetilde{f}_m(\boldsymbol{\lambda};\boldsymbol{u})G^{f,1}(\boldsymbol{u};t),\end{aligned}$$

and this completes the proof. □

Remark 3. The equation in Proposition 7(i) can, alternatively, be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{\boldsymbol{k}}^{f,1}(t) = -\widetilde{f}_m(\boldsymbol{\lambda};\boldsymbol{B})p_{\boldsymbol{k}}^{f,1}(t),$$

where $\boldsymbol{B} = (B_1, \dots, B_m)$. In fact, we have

$$\begin{aligned}
-\widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{B}) p_{\boldsymbol{k}}^{f,1}(t) &= -\int_0^\infty \bigg(1 - \mathrm{e}^{-s(\boldsymbol{\lambda})r} \sum_{\boldsymbol{j} \geq \boldsymbol{0}} \prod_{i=1}^m \frac{(\lambda_i B_i r)^{j_i}}{j_i!}\bigg) \rho_f(\mathrm{d}r) \\
&= -f(s(\boldsymbol{\lambda})) p_{\boldsymbol{k}}^{f,1}(t) + \int_0^\infty \mathrm{e}^{-s(\boldsymbol{\lambda})r} \sum_{\boldsymbol{j} \succ \boldsymbol{0}} \prod_{i=1}^m \frac{(\lambda_i B_i r)^{j_i}}{j_i!} \rho_f(\mathrm{d}r) p_{\boldsymbol{k}}^{f,1}(t) \\
&= \sum_{\boldsymbol{j} \succ \boldsymbol{0}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t) \prod_{i=1}^m \frac{\lambda_i^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i} \mathrm{e}^{-s(\boldsymbol{\lambda})r} \rho_f(\mathrm{d}r) - f(s(\boldsymbol{\lambda})) p_{\boldsymbol{k}}^{f,1}(t) \\
&= \sum_{\boldsymbol{0} \prec \boldsymbol{j} \leq \boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,1}(t) \prod_{i=1}^m \frac{\lambda_i^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i} \mathrm{e}^{-rs(\boldsymbol{\lambda})} \rho_f(\mathrm{d}r) - f(s(\boldsymbol{\lambda})) p_{\boldsymbol{k}}^{f,1}(t).
\end{aligned}$$

Remark 4. If we follow the same lines as [16, Remark 2.5], for $\nu \in (0, 1)$ the state probabilities $\{\{p_{\boldsymbol{k}}^{f,\nu}(t) \colon \boldsymbol{k} \geq \boldsymbol{0}\} \colon t \geq 0\}$ in (6) solve the fractional differential equation

$$\begin{aligned}
{}^C D_{0+}^\nu p_{\boldsymbol{k}}^{f,\nu}(t) &= \sum_{\boldsymbol{0} \prec \boldsymbol{j} \leq \boldsymbol{k}} p_{\boldsymbol{k}-\boldsymbol{j}}^{f,\nu}(t) \prod_{i=1}^m \frac{\lambda_i^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i} \mathrm{e}^{-rs(\boldsymbol{\lambda})} \rho_f(\mathrm{d}r) - f(s(\boldsymbol{\lambda})) p_{\boldsymbol{k}}^{f,\nu}(t), \\
p_{\boldsymbol{k}}^{f,\nu}(t) &= \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}},
\end{aligned}$$

or, equivalently,

$$ {}^C D_{0+}^\nu p_{\boldsymbol{k}}^{f,\nu}(t) = -\widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{B}) p_{\boldsymbol{k}}^{f,\nu}(t), \qquad p_{\boldsymbol{k}}^{f,\nu}(t) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}. \tag{25}$$

Moreover, the probability generating functions $\{G^{f,\nu}(\cdot; t) \colon t \geq 0\}$ in (7) solve the fractional differential equation

$$ {}^C D_{0+}^\nu G^{f,\nu}(\boldsymbol{u}; t) = -\widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{u}) G^{f,\nu}(\boldsymbol{u}; t), \qquad G^{f,\nu}(\boldsymbol{u}; 0) = 1, \tag{26}$$

and, therefore, we have $G^{f,\nu}(\boldsymbol{u}; t) = E_{\nu,1}(-t^\nu \widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{u}))$.

In particular, considering the Bernštein function f for the stable subordinator $\{\mathcal{A}^\eta(t) \colon t \geq 0\}$ and the corresponding Lévy measure ρ_f (see Remark 2), we have

$$\begin{aligned}
\widetilde{f}_m(\boldsymbol{\lambda}; \boldsymbol{u}) &= \int_0^\infty \bigg(1 - \mathrm{e}^{-s(\boldsymbol{\lambda})r} \sum_{\boldsymbol{j} \geq \boldsymbol{0}} \prod_{i=1}^m \frac{(\lambda_i u_i r)^{j_i}}{j_i!}\bigg) \frac{\eta}{\Gamma(1-\eta)} \frac{1}{r^{\eta+1}} \,\mathrm{d}r \\
&= (s(\boldsymbol{\lambda}))^\eta - \frac{\eta}{-\eta\Gamma(-\eta)} \sum_{\boldsymbol{j} \succ \boldsymbol{0}} \prod_{i=1}^m \frac{(\lambda_i u_i)^{j_i}}{j_i!} \int_0^\infty r^{\sum_{i=1}^m j_i - \eta - 1} \mathrm{e}^{-s(\boldsymbol{\lambda})r} \,\mathrm{d}r \\
&= (s(\boldsymbol{\lambda}))^\eta + \frac{1}{\Gamma(-\eta)} \sum_{\boldsymbol{j} \succ \boldsymbol{0}} \frac{\Gamma(\sum_{i=1}^m j_i - \eta)}{(s(\boldsymbol{\lambda}))^{\sum_{i=1}^m j_i - \eta}} \prod_{i=1}^m \frac{(\lambda_i u_i)^{j_i}}{j_i!} \\
&= (s(\boldsymbol{\lambda}))^\eta \bigg(1 + \frac{1}{\Gamma(-\eta)} \sum_{\boldsymbol{j} \succ \boldsymbol{0}} \Gamma\bigg(\sum_{i=1}^m j_i - \eta\bigg) \prod_{i=1}^m \frac{1}{j_i!} \bigg(\frac{\lambda_i u_i}{s(\boldsymbol{\lambda})}\bigg)^{j_i}\bigg) \\
&= (s(\boldsymbol{\lambda}))^\eta \sum_{\boldsymbol{j} \geq \boldsymbol{0}} \frac{\Gamma(\sum_{i=1}^m j_i - \eta)}{\Gamma(-\eta)} \prod_{i=1}^m \frac{1}{j_i!} \bigg(\frac{\lambda_i u_i}{s(\boldsymbol{\lambda})}\bigg)^{j_i};
\end{aligned}$$

moreover, if we use the symbol '$\sum_{j_1,\dots,j_m\in\mathcal{S}_h}$' for the sum over all $j_1,\dots,j_m\ge 0$ such that $j_1+\cdots+j_m=h$ (as in the proof of Proposition 1), we obtain

$$\begin{aligned}\widetilde{f}_m(\boldsymbol{\lambda};\boldsymbol{u}) &= (s(\boldsymbol{\lambda}))^{\eta}\sum_{h\ge 0}\frac{\Gamma(h-\eta)}{\Gamma(-\eta)h!}\sum_{j_1,\dots,j_m\in\mathcal{S}_h}\prod_{i=1}^{m}\frac{h!}{j_i!}\bigg(\frac{\lambda_i u_i}{s(\boldsymbol{\lambda})}\bigg)^{j_i}\\ &= (s(\boldsymbol{\lambda}))^{\eta}\sum_{h\ge 0}\frac{\Gamma(h-\eta)}{\Gamma(-\eta)h!}\bigg(\sum_{i=1}^{m}\frac{\lambda_i u_i}{s(\boldsymbol{\lambda})}\bigg)^{h}\\ &= (s(\boldsymbol{\lambda}))^{\eta}\bigg(1-\sum_{i=1}^{m}\frac{\lambda_i u_i}{s(\boldsymbol{\lambda})}\bigg)^{\eta}\end{aligned}$$

(for the last equality, see, e.g. [23, Equation (15)] with $\alpha=-\eta-1$ and $\beta=0$; in fact t and ζ in that reference satisfy $\zeta=t(1+\zeta)$, and, therefore, $\zeta=t/(1-t)$ and $1+\zeta=1/(1-t)$; obviously here we consider $u_1,\dots,u_m\in[0,1]$ and, therefore, $t=\sum_{i=1}^{m}\lambda_i u_i/s(\boldsymbol{\lambda})\in[0,1]$). Thus, (25) meets the equation in the statement of Proposition 1 (with $p_{\boldsymbol{k}}^{\eta,\nu}(t)$ in place of $p_{\boldsymbol{k}}^{f,\nu}(t)$) and, similarly, (26) meets (14) (with $G^{\eta,\nu}(\boldsymbol{u};t)$ in place of $G^{f,\nu}(\boldsymbol{u};t)$).

4. Examples of fractional compound Poisson processes

In this section we study the multivariate fractional version of well-known counting processes which can be obtained as a particular multivariate space-time fractional compound Poisson process $\{C^{\eta,\nu}(t)\colon t\ge 0\}$ as in Definition 2. In particular, the univariate processes (i.e. the $m=1$ case) has been studied in [3, Section 4]. For each example we specify the probability mass functions $\{\{\widetilde{q}^{\,i}_j\colon j\ge 1\}\colon i\in\{1,\dots,m\}\}$ and the values $\lambda_1,\dots,\lambda_m$; we remark that the values $\lambda_1,\dots,\lambda_m$ in Example 1 can be chosen without any restriction.

Example 1. (*Multivariate fractional Pólya–Aeppli process.*) We set

$$\widetilde{q}^{\,i}_j := (1-\widetilde{\alpha}_i)^{j-1}\widetilde{\alpha}_i \quad \text{for some } \widetilde{\alpha}_1,\dots,\widetilde{\alpha}_m\in(0,1];$$

in particular, if $\widetilde{\alpha}_i=1$, we have $C_i^{\eta,\nu}(t)=N_i^{\eta,\nu}(t)$. We recall that in some references the $m=1$ case is presented with ρ in place of $1-\alpha$; see, e.g. [14, Equation (1.3)].

Example 2. (*Multivariate fractional Poisson inverse Gaussian process.*) We set

$$\widetilde{q}^{\,i}_j := \binom{j-3/2}{j}\bigg(\frac{2\widetilde{\beta}_i}{2\widetilde{\beta}_i+1}\bigg)^{j}\bigg[\bigg(\frac{1}{2\widetilde{\beta}_i+1}\bigg)^{-1/2}-1\bigg]^{-1} \quad\text{and}\quad \lambda_i := \frac{\widetilde{\mu}_i}{\widetilde{\beta}_i}((1+2\widetilde{\beta}_i)^{1/2}-1),$$

for some $\widetilde{\beta}_1,\widetilde{\mu}_1,\dots,\widetilde{\beta}_m,\widetilde{\mu}_m>0$.

Example 3. (*Multivariate fractional negative binomial process.*) We set

$$\widetilde{q}^{\,i}_j := -\frac{(1-\widetilde{\alpha}_i)^j}{j\log\widetilde{\alpha}_i} \quad\text{and}\quad \lambda_i := -\log\widetilde{\alpha}_i,$$

for some $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_m\in(0,1)$.

We also present an extension of [3, Proposition 2] concerning Example 1.

Proposition 8. *We assume the same situation as in Example 1. Then, for* $\nu \in (0, 1]$,

$$
\begin{aligned}
&{}^C D^{\nu}_{0+} q^{1,\nu}_{\boldsymbol{k}}(t) - \sum_{i=1}^{m}(1-\widetilde{\alpha}_i)\, {}^C D^{\nu}_{0+} q^{1,\nu}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\qquad = -s(\boldsymbol{\lambda}) q^{1,\nu}_{\boldsymbol{k}}(t) + \sum_{i=1}^{m}(\lambda_i\widetilde{\alpha}_i + s(\boldsymbol{\lambda})(1-\widetilde{\alpha}_i)) q^{1,\nu}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\qquad\quad - \sum_{i=1}^{m}(1-\widetilde{\alpha}_i) \sum_{h=1,\, h\neq i}^{m} \lambda_h \sum_{j_h=1}^{k_h}(1-\widetilde{\alpha}_h)^{j_h-1}\widetilde{\alpha}_h q^{1,\nu}_{k_1,\dots,k_h-j_h,\dots,k_m}(t), \\
&q^{1,\nu}_{\boldsymbol{k}}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}};
\end{aligned}
$$

for $\eta \in (0, 1]$,

$$
\begin{aligned}
&\frac{\mathrm{d}^{1/\eta}}{\mathrm{d}(-t)^{1/\eta}} q^{\eta,1}_{\boldsymbol{k}}(t) - \sum_{i=1}^{m}(1-\widetilde{\alpha}_i) \frac{\mathrm{d}^{1/\eta}}{\mathrm{d}(-t)^{1/\eta}} q^{\eta,1}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\qquad = s(\boldsymbol{\lambda}) q^{\eta,1}_{\boldsymbol{k}}(t) - \sum_{i=1}^{m}(\lambda_i\widetilde{\alpha}_i + s(\boldsymbol{\lambda})(1-\widetilde{\alpha}_i)) q^{\eta,1}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\qquad\quad + \sum_{i=1}^{m}(1-\widetilde{\alpha}_i) \sum_{h=1,\, h\neq i}^{m} \lambda_h \sum_{j_h=1}^{k_h}(1-\widetilde{\alpha}_h)^{j_h-1}\widetilde{\alpha}_h q^{\eta,1}_{k_1,\dots,k_h-j_h,\dots,k_m}(t), \\
&q^{\eta,1}_{\boldsymbol{k}}(0) = \mathbf{1}_{\{\boldsymbol{k}=\boldsymbol{0}\}}.
\end{aligned}
$$

Proof. The initial conditions trivially hold. We start with the proof of the first equation in the statement. By the first equation in Proposition 2, we have

$$
\begin{aligned}
&{}^C D^{\nu}_{0+} q^{1,\nu}_{\boldsymbol{k}}(t) - \sum_{i=1}^{m}(1-\widetilde{\alpha}_i)\, {}^C D^{\nu}_{0+} q^{1,\nu}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\quad = -s(\boldsymbol{\lambda}) q^{1,\nu}_{\boldsymbol{k}}(t) + \sum_{h=1}^{m} \lambda_h \sum_{j_h=1}^{k_h}(1-\widetilde{\alpha}_h)^{j_h-1}\widetilde{\alpha}_h q^{1,\nu}_{k_1,\dots,k_h-j_h,\dots,k_m}(t) \\
&\qquad - \sum_{i=1}^{m}(1-\widetilde{\alpha}_i)\bigg[-s(\boldsymbol{\lambda}) q^{1,\nu}_{k_1,\dots,k_i-1,\dots,k_m}(t) \\
&\qquad\qquad + \sum_{h=1,\, h\neq i}^{m} \lambda_h \sum_{j_h=1}^{k_h}(1-\widetilde{\alpha}_h)^{j_h-1}\widetilde{\alpha}_h q^{1,\nu}_{k_1,\dots,k_h-j_h,\dots,k_m}(t) \\
&\qquad\qquad + \lambda_i \sum_{j_i=1}^{k_i}(1-\widetilde{\alpha}_i)^{j_i-1}\widetilde{\alpha}_i q^{1,\nu}_{k_1,\dots,k_i-1-j_i,\dots,k_m}(t)\bigg].
\end{aligned}
$$

Moreover, if we split in two parts the sum $\sum_{j_h=1}^{k_h}(1-\widetilde{\alpha}_h)^{j_h-1}\widetilde{\alpha}_h q^{1,\nu}_{k_1,\dots,k_h-j_h,\dots,k_m}(t)$ on the right-hand side, i.e. the summand with $j_h = 1$ and the other summands with $j_h \in \{2, \dots, k_h\}$, after some computation, we obtain

$$
\begin{aligned}
&{}^C D_{0+}^{\nu} q_{\boldsymbol{k}}^{1,\nu}(t) - \sum_{i=1}^{m} (1-\widetilde{\alpha}_i)\, {}^C D_{0+}^{\nu} q_{k_1,\dots,k_i-1,\dots,k_m}^{1,\nu}(t) \\
&\quad = -s(\boldsymbol{\lambda}) q_{\boldsymbol{k}}^{1,\nu}(t) + \sum_{h=1}^{m} \lambda_h \widetilde{\alpha}_h q_{k_1,\dots,k_h-1,\dots,k_m}^{1,\nu}(t) \\
&\qquad + \sum_{h=1}^{m} \lambda_h \sum_{j_h=2}^{k_h} (1-\widetilde{\alpha}_h)^{j_h-1} \widetilde{\alpha}_h q_{k_1,\dots,k_h-j_h,\dots,k_m}^{1,\nu}(t) \\
&\qquad + \sum_{i=1}^{m} s(\boldsymbol{\lambda})(1-\widetilde{\alpha}_i) q_{k_1,\dots,k_i-1,\dots,k_m}^{1,\nu}(t) \\
&\qquad - \sum_{i=1}^{m} (1-\widetilde{\alpha}_i) \sum_{h=1,\, h\neq i}^{m} \lambda_h \sum_{j_h=1}^{k_h} (1-\widetilde{\alpha}_h)^{j_h-1} \widetilde{\alpha}_h q_{k_1,\dots,k_h-j_h,\dots,k_m}^{1,\nu}(t) \\
&\qquad - \sum_{i=1}^{m} \lambda_i \sum_{j_i=1}^{k_i} (1-\widetilde{\alpha}_i)^{j_i} \widetilde{\alpha}_i q_{k_1,\dots,k_i-1-j_i,\dots,k_m}^{1,\nu}(t).
\end{aligned}
$$

Finally, after some other computation (in particular we combine two sums and we consider $j_i \in \{2, \dots, k_i+1\}$ in place of $j_i \in \{1, \dots, k_i\}$ in the last sum, with a suitable modification of the summands), we have

$$
\begin{aligned}
&{}^C D_{0+}^{\nu} q_{\boldsymbol{k}}^{1,\nu}(t) - \sum_{i=1}^{m} (1-\widetilde{\alpha}_i) {}^C D_{0+}^{\nu} q_{k_1,\dots,k_i-1,\dots,k_m}^{1,\nu}(t) \\
&\quad = -s(\boldsymbol{\lambda}) q_{\boldsymbol{k}}^{1,\nu}(t) + \sum_{i=1}^{m} (\lambda_i \widetilde{\alpha}_i + s(\boldsymbol{\lambda})(1-\widetilde{\alpha}_i)) q_{k_1,\dots,k_i-1,\dots,k_m}^{1,\nu}(t) \\
&\qquad + \sum_{h=1}^{m} \lambda_h \sum_{j_h=2}^{k_h} (1-\widetilde{\alpha}_h)^{j_h-1} \widetilde{\alpha}_h q_{k_1,\dots,k_h-j_h,\dots,k_m}^{1,\nu}(t) \\
&\qquad - \sum_{i=1}^{m} (1-\widetilde{\alpha}_i) \sum_{h=1,\, h\neq i}^{m} \lambda_h \sum_{j_h=1}^{k_h} (1-\widetilde{\alpha}_h)^{j_h-1} \widetilde{\alpha}_h q_{k_1,\dots,k_h-j_h,\dots,k_m}^{1,\nu}(t) \\
&\qquad - \sum_{i=1}^{m} \lambda_i \sum_{j_i=2}^{k_i+1} (1-\widetilde{\alpha}_i)^{j_i-1} \widetilde{\alpha}_i q_{k_1,\dots,k_i-j_i,\dots,k_m}^{1,\nu}(t).
\end{aligned}
$$

Then the first desired equation is checked because $\widetilde{\alpha}_i q_{k_1,\dots,k_i-(k_i+1),\dots,k_m}^{1,\nu}(t) = 0$ and two sums can be canceled. The second desired equation can be obtained similarly; we have to consider the second equation in Proposition 2 (instead of the first one) and we have the same kind of computation with suitable changes of sign. □

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