Asymptotic Expansion for Harmonic Functions in the Half-Space with a Pressurized Cavity

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Abstract. In this paper, we address a simplified version of a problem arising from volcanology. Specifically, as reduced form of the boundary value problem for the Lamé system, we consider a Neumann problem for harmonic functions in the half-space with a cavity C. Zero normal derivative is assumed at the boundary of the half-space; differently, at ∂C, the normal derivative of the function is required to be given by an external datum g, corresponding to a pressure term exerted on the medium at ∂C. Under the assumption that the (pressurized) cavity is small with respect to the distance from the boundary of the half-space, we establish an asymptotic formula for the solution of the problem. Main ingredients are integral equation formulations of the harmonic solution of the Neumann problem and a spectral analysis of the integral operators involved in the problem. In the special case of a datum g which describes a constant pressure at ∂C, we recover a simplified representation based on a polarization tensor.

Keywords. Asymptotic expansions; harmonic functions in the half-space; single and double layer potentials.

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1. Introduction

The aim of the paper is to provide a detailed mathematical study of a simplified version of a problem arising from volcanology. The analysis can be considered as a blueprint, useful to address the original problem in a forthcoming research; at the same time, the result has an interest on its own, entering in the stream of the asymptotic analysis for the conductivity equation, see [1, 2, 3, 4, 10] and references therein, with the principal novelties of dealing with a case of an unbounded domain with unbounded boundary and of a different choice of boundary datum (homogeneous on the boundary of the half-space and non-homogeneous on the boundary of the cavity).

The geological problem is the detection of geometrical and physical features of magmatic reservoirs from changes within calderas. The starting evidence is that the magma exerts

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a force aside the surrounding crust when migrates toward the earth's surface, producing appreciable horizontal and vertical ground displacements, which can be detected by a variety of modern techniques (see the detailed description provided in [17]). As stated in [6], *the main questions that emerge when monitoring volcanoes are how to constrain the source of unrest*, that is to estimate the parameters related to its depth, dimension, volume and pressure, *and how to better asses hazards associated with the unrest.* For this purpose, the measurements of crust deformations are a useful tool to study magmatic processes since they are sensitive to changes in the source pressure and volume.

From a modelling point of view, the displacements are described using the theory of linear elasticity, by replacing caldera with a half-space having a stress-free flat boundary, and the magma chamber by a cavity subjected to an internal pressure; in more detail, let v be the displacement vector $v(x) = (v_1(x), v_2(x), v_3(x))$ and $\widehat{\nabla} v = \big(\nabla v + (\nabla v)^T\big)/2$ the strain tensor, then the elastic model is defined by the linear system of equations

$$\operatorname{div}(\mathbb{C}\widehat{\nabla} v) = 0 \quad \text{in } \mathbb{R}^3_- \setminus C, \tag{1}$$

with boundary conditions

$$\begin{array}{ll} (\mathbb{C}\widehat{\nabla} v) n \cdot n = g \quad \text{on } \partial C & \qquad (\mathbb{C}\widehat{\nabla} v) n \cdot \tau = 0 \quad \text{on } \partial C \\ (\mathbb{C}\widehat{\nabla} v) n = 0 \quad \text{on } \mathbb{R}^2 & \qquad v(x) \to 0 \quad \text{as } |x| \to +\infty, \end{array} \tag{2}$$

where C is the pressurized cavity, g is a vector-valued function represents the pressure, n the unit outer normal vector and τ the tangential one; $\mathbb{C}$ is the isotropic elasticity tensor with Lamé parameters λ and μ defined as

$$\mathbb{C} := \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{\text{sym}},$$

with I_3 the identity matrix of order 3 and $\mathbb{I}_{\text{sym}}$ the fourth order tensor such that $\mathbb{I}_{\text{sym}} A = \widehat{A}$. The goal is to determine g and geometrical features of C (position, volume...) from surface measurements of the displacement field v.

In applications, to handle more easily the inverse problem related to (1)–(2), the function g is taken constant and equal to a vector $p \in \mathbb{R}^3$. Additionally, from a geological point of view, sometimes it is reasonable to consider the magma chamber small compared to the distance from the boundary of the half-space, see [6, 15, 17]; by adding these hypotheses and fixing the geometry of the cavity, some efforts have been done during the last decades to find some explicit or approximate solutions to the mathematical model. The simplest approximate solution obtained by asymptotic expansions is due to McTigue when the cavity is a sphere, see [15]. The other few solutions available concern ellipsoidal shapes, dike and faults, see [6, 17].

With the ultimate goal to study in detail the elastic problem (1)–(2), establishing an asymptotic expansion in the presence of a pressurized cavity of generic shape, in this paper we analyse a simplified scalar version of this model so as to shed light and mark the path to treat the elastic case. Denoting by $\mathbb{R}^d_-$ the half-space and $\mathbb{R}^{d-1}$ its boundary, we consider the Laplace equation

$$\Delta u = 0 \quad \text{in } \mathbb{R}^d_- \setminus C_\varepsilon$$

with boundary conditions

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial C_\varepsilon, \qquad \frac{\partial u}{\partial x_d} = 0 \quad \text{on } \mathbb{R}^{d-1}, \qquad u(x) \to 0 \quad \text{as } |x| \to +\infty$$

where C_ε is the analogous of the pressurized cavity in the elastic case, with ε a small parameter controlling its size, g is a function defined on ∂C_ε and $d \geq 3$. Obviously, the choice to focus the attention on dimensions greater than two comes from the application we have in mind.

Presentation of the main result. The main goals of this paper are first to analyse the well-posedness of the scalar problem, find a representation formula of the solution and then determine an asymptotic expansion with respect to the parameter ε of the solution. For the first two steps we do not need to assume the cavity to be small.

It is worth noticing that in terms of well-posedness the case of the half-space and, in general, of unbounded domain with unbounded boundary, is more difficult to treat compared to bounded or exterior domains since both the control of the solution decay and integrability on the boundary are needed. Indeed, it is typical to treat these problems by means of weighted Sobolev spaces, see for example [5]. In our case, we choose to use the particular symmetry of the half-space to prove the well-posedness in order to mantain a simple mathematical interpretation of the results. Therefore, we bring the problem back to an exterior domain in the whole space for which there is a vast literature on the well-posedness, see [9].

To trackle the issue of the asymptotic expansion, we consider the approach developed by Ammari and Kang based on single and double layer potentials for harmonic functions, see for example [3, 4]. This is the reason why we search an integral representation formula of the solution. To do this, we take advantage of the explicit expression of the Neumann function for the half-space

$$N(x, y) = \Gamma(x - y) + \Gamma(\widetilde{x} - y),$$

where Γ is the fundamental solution of the Laplacian and $\widetilde{x}$ is the symmetric point of x with respect to the x_d-plane, in order to get a representation formula containing only integral contributions on the boundary of C. In detail, we find that

$$u(x)=\int_{\partial C}\left[N(x,y)g(y)-\frac{\partial}{\partial n_y}N(x,y)f(y)\right]d\sigma(y),\quad x\in\mathbb{R}^d_-\setminus C,$$

where f is the trace of the solution u on ∂C. From the point of view of the inverse problem we are interested in evaluating the solution u on the boundary of the half-space; since $\Gamma(x-y)=\Gamma(\widetilde{x}-y)$, for $x\in\mathbb{R}^{d-1}$, the integral formula becomes

$$\frac{1}{2}u(x)=\int_{\partial C}\left[\Gamma(x-y)g(y)-\frac{\partial}{\partial n_y}\Gamma(x-y)f(y)\right]d\sigma(y),\quad x\in\mathbb{R}^{d-1}.$$

Taking B a bounded Lipschitz domain containing the origin and $z\in\mathbb{R}^d_-$ we consider $C_\varepsilon := C = z+\varepsilon B$ with the assumption that $\mathrm{dist}(z,\mathbb{R}^{d-1})\geq\delta_0>0$. Therefore, defining $\widehat{g}(\zeta;\varepsilon)=g(z+\varepsilon\zeta)$, with $\zeta\in B$, and $S_B\widehat{g}$ the single layer potential, the main result holds

Theorem 1.1. *For any $\varepsilon>0$, let $g\in L^2(\partial C_\varepsilon)$ such that $\widehat{g}$ is independent on ε. Then, at any $x\in\mathbb{R}^{d-1}$, we have*

$$\begin{aligned}u_\varepsilon(x)&=2\varepsilon^{d-1}\Gamma(x-z)\int_{\partial B}\widehat{g}(\zeta)d\sigma(\zeta)\\&\quad+2\varepsilon^d\nabla\Gamma(x-z)\cdot\int_{\partial B}\left\{n_\zeta\left(\tfrac{1}{2}I+K_B\right)^{-1}S_B\widehat{g}(\zeta)-\zeta\widehat{g}(\zeta)\right\}d\sigma(\zeta)+O(\varepsilon^{d+1})\end{aligned}$$

where $O(\varepsilon^{d+1})$ denotes a quantity uniformly bounded by $C\varepsilon^{d+1}$ with $C=C(\delta_0)$ which tends to infinity when δ_0 goes to zero.

Finally, with the asymptotic expansion in hand, we consider the Neumann boundary datum $g=-p\cdot n$ where p is a constant vector in $\mathbb{R}^d$. This particular choice has a double purpose: to reconnect this problem with the constant boundary conditions of the elastic model and to make more explicit the integrals in the asymptotic formula. The result we get contains the same polarization tensor obtained by Friedman and Vogelius in [10] for cavities in a bounded domain.

The organization of the paper is the following. Section 2 is divided into three parts: in the first one we recall some well-known results about harmonic functions and layer potentials; in the second one we examine the well-posedness of the scalar problem; in the third one we get the representation formula of the solution. In Section 3, we state and prove a spectral result crucial for the derivation of our main asymptotic expansion. In Section 4 we present and prove the theorem on the asymptotic expansion and finally we illustrate the result for the particular choice $g=-p\cdot n$.

Notation. All the analysis is performed in $\mathbb{R}^d$, with $d \geq 3$; $B_r(x) \subset \mathbb{R}^d$ denotes the ball with centre x and radius $r > 0$, and ω_d the area of the $(d-1)$-dimensional unit sphere. The scalar product between two vectors is represented by $x \cdot y$ and n_x indicates the unit outward normal vector in x on the boundary of some specified domain. The fundamental solution Γ of the Laplace operator in $\mathbb{R}^d$, with $d \geq 3$, is given by $\Gamma(x) = \kappa_d |x|^{2-d}$ with $\kappa_d := 1/\omega_d(2-d)$.

Points $x \in \mathbb{R}^d$, $d \geq 3$, are decomposed as $x = (x', x_d)$ where $x' = (x_1, \cdots, x_{d-1})$. We denote by $\mathbb{R}^d_-$ the half-space $\{x \in \mathbb{R}^d : \ x_d < 0\}$ and by $\mathbb{R}^{d-1}$ its boundary. Given a point $x \in \mathbb{R}^d_-$, its reflected point $(x', -x_d)$ with respect to the plane $x_d = 0$ is represented by $\widetilde{x}$.

2. The direct problem

In this Section, we analyse the boundary value problem

$$
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^d_- \setminus C \\
\dfrac{\partial u}{\partial n} = g & \text{on } \partial C \\
\dfrac{\partial u}{\partial x_d} = 0 & \text{on } \mathbb{R}^{d-1} \\
u \to 0 & \text{as } |x| \to +\infty
\end{cases}
\tag{3}
$$

where C is the cavity, with a twofold aim: to establish well-posedness of the problem and to provide a representation formula.

Preliminaries. The specific symmetry of the half-space permits to show well-posedness by extending the problem to an exterior domain, viz. the complementary set of a bounded set. Hence, it is useful to recall the classical results on the asymptotic behaviour of harmonic functions in exterior domains. Given a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 3$, if v is harmonic in $\mathbb{R}^d \setminus \Omega$ then v is harmonic at infinity if and only if v tends to 0 as $|x| \to \infty$. Moreover, there exist $r_0, M > 0$, such that if $|x| \geq r_0$, the following estimates hold

$$
|v(x)| \leq M|x|^{2-d}, \qquad |v_{x_i}(x)| \leq M|x|^{1-d} \qquad \left| v_{x_j x_k}(x) \right| \leq M|x|^{-d}
\tag{4}
$$

for $i, j, k = 1, \dots, d$. The proof can be found in [9] (see also [8, 14]).

The representation formula makes use of layer potentials whose definitions we now recall; see [3, 9, 14]. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and a function $\varphi \in L^2(\partial\Omega)$, we introduce the integral operators

$$
\begin{aligned}
S_\Omega \varphi(x) &:= \int_{\partial\Omega} \Gamma(x-y)\varphi(y) d\sigma(y), && x \in \mathbb{R}^d, \\
D_\Omega \varphi(x) &:= \int_{\partial\Omega} \frac{\partial}{\partial n_y} \Gamma(x-y)\varphi(y) d\sigma(y), && x \in \mathbb{R}^d \setminus \partial\Omega,
\end{aligned}
\tag{5}
$$

which are called, respectively, *single and double layer potential* relative to the set Ω. By definition, $S_\Omega\varphi$ and $D_\Omega\varphi$ are well-defined and harmonic in $\mathbb{R}^d \setminus \partial\Omega$. Further, still for $d \geq 3$, we have

$$S_\Omega\varphi = O(|x|^{2-d}) \qquad \text{and} \qquad D_\Omega\varphi = O(|x|^{1-d})$$

as $|x| \to +\infty$. In addition, if φ has zero mean on $\partial\Omega$, the decay rate of the single layer potential S_Ω is increased, precisely,

$$\text{if} \quad \int_{\partial\Omega} \varphi(x)d\sigma(x) = 0, \quad \text{then} \quad S_\Omega\varphi = O(|x|^{1-d})$$

as $|x| \to +\infty$ (such property holds for $d \geq 2$).

Next, we introduce the compact operator $K_\Omega : L^2(\partial\Omega) \to L^2(\partial\Omega)$

$$K_\Omega\varphi(x) := \frac{1}{\omega_d}\text{p.v.}\int_{\partial\Omega} \frac{(y-x)\cdot n_y}{|x-y|^d}\varphi(y)d\sigma(y) \tag{6}$$

and its L^2-adjoint

$$K_\Omega^*\varphi(x) = \frac{1}{\omega_d}\text{p.v.}\int_{\partial\Omega} \frac{(x-y)\cdot n_x}{|x-y|^d}\varphi(y)d\sigma(y). \tag{7}$$

Given a function v defined in a neighbourhood of $\partial\Omega$, set

$$v(x)\big|_\pm := \lim_{h\to 0^+} v(x \pm hn_x), \quad x \in \partial\Omega.$$

The following relations hold, a.e. in $\partial\Omega$,

$$S_\Omega\varphi\big|_+ = S_\Omega\varphi\big|_-, \quad \frac{\partial S_\Omega\varphi}{\partial n_x}\Big|_\pm = \left(\pm\tfrac{1}{2}I + K_\Omega^*\right)\varphi, \quad D_\Omega\varphi\Big|_\pm = \left(\mp\tfrac{1}{2}I + K_\Omega\right)\varphi. \tag{8}$$

For the proof see [3, 9].

Well-posedness. Proving existence and uniqueness results for unbounded domains with unbounded boundary is, in general, much more difficult with respect to the case of exterior domains. The main obstacle is the control of both solution decay and integrability on the boundary and usual approach is based on the use of weighted Sobolev spaces [5]. Here, we take advantage of the symmetry property of the half-space to extend the problem to the whole space and to establish well-posedness resorting in a standard Sobolev setting.

Given a bounded Lipschitz domain $C \subset \mathbb{R}^d_-$ and the function $g : \partial C \to \mathbb{R}$, we define

$$\widetilde{C} := \{(x', x_d) : \ (x', -x_d) \in C\}$$

and $G : \partial C \cup \partial\widetilde{C} \to \mathbb{R}$ as

$$G(x) := \begin{cases} g(x) & \text{if } x \in \partial C \\ g(\tilde{x}) & \text{if } x \in \partial\widetilde{C}. \end{cases}$$

Theorem 2.1. *The problem (3) has a unique solution. This solution coincides with the restriction to the half-space $\mathbb{R}^d_-$ of the solution to*

$$\Delta U = 0 \quad \text{in } \mathbb{R}^d \setminus \left(C \cup \widetilde{C}\right), \qquad \frac{\partial U}{\partial n} = G \quad \text{on } \partial C \cup \partial\widetilde{C}, \qquad U \to 0 \quad \text{as } |x| \to +\infty. \tag{9}$$

Proof. The proof is divided into three steps: uniqueness for (9), existence for (9), equivalence between (3) and (9).

1. For $\Lambda := C \cup \widetilde{C}$, let $R > 0$ be such that $\Lambda \subset B_R(0)$ and set $\Omega_R := B_R \setminus \Lambda$. Given two solutions, U_1 and U_2, to problem (9), the difference $W := U_1 - U_2$ solves the corresponding homogeneous problem. Multiplying equation $\Delta W = 0$ by W and integrating over the domain $\Omega_R = B_R \setminus \Lambda$, we infer

$$\begin{aligned} 0 &= \int_{\Omega_R} W(x)\Delta W(x)dx \\ &= \int_{\partial B_R(0)} W(x)\frac{\partial}{\partial n}W(x)d\sigma(x) - \int_{\Omega_R} \left|\nabla W(x)\right|^2 dx, \end{aligned}$$

using integration by parts and boundary conditions. Exploiting the behaviour of the harmonic functions in exterior domains, as described in (4), we get

$$\Big|\int_{\partial B_R(0)} W(x)\frac{\partial}{\partial n}W(x)\, d\sigma(x)\Big| \leq \frac{C}{R^{d-2}}.$$

Then, as $R \to \infty$, we find

$$\int_{\mathbb{R}^d\setminus\Lambda} \left|\nabla W(x)\right|^2 dx = 0$$

which implies $W = 0$.

2. We represent the solution of (9) by means of single layer potential

$$S_\Lambda\psi(x) = \int_{\partial\Lambda} \Gamma(x-y)\psi(y)d\sigma(y), \qquad x \in \mathbb{R}^d \setminus \Lambda, \tag{10}$$

with function ψ to be determined. By the properties of single layer potential, $S_\Lambda\psi$ is harmonic in $\mathbb{R}^d \setminus \partial\Lambda$, $S_\Lambda\psi(x){=}O(|x|^{2-d})$ as $|x| \to \infty$ and we have

$$\frac{\partial S_\Lambda\psi}{\partial n}(x)\Big|_+ = \tfrac{1}{2}\psi + K^*_\Lambda\psi, \qquad x \in \partial\Lambda.$$

We now prove that there exists a function ψ such that

$$\left(\tfrac{1}{2}I + K^*_\Lambda\right)\psi(x) = G(x), \qquad x \in \partial\Lambda. \tag{11}$$

To this aim we prove that the operator $\frac{1}{2}I{+}K^*_\Lambda$ is injective. Indeed, given $\zeta \in \ker(\frac{1}{2}I{+}K^*_\Lambda)$, define $V := S_\Lambda\zeta$. Then, from the properties of layer potentials, W solves

$$\Delta V = 0 \quad \text{in } \mathbb{R}^d \setminus \Lambda, \qquad \frac{\partial}{\partial n}V = 0 \quad \text{on } \partial\Lambda, \qquad V \to 0 \quad \text{as } |x| \to \infty,$$

hence, from Step 1., the function V is identically zero; then it follows that $\zeta \equiv 0$. Finally, observing that K^*_Λ is a compact operator and $G \in L^2(\partial\Lambda)$, equation (11) admits a unique solution.

3. To show that $u := U\big|_{x_d<0}$, we have to verify that the normal derivative is null on the boundary of the half-space. This is an immediate consequence of the symmetry property

$$U(x', -x_d) = U(x', x_d), \tag{12}$$

which follows from the observation that, by definition of the boundary datum G, the function $\bar{u}(x', x_d) := U(x', -x_d)$ solves (9) and the solution to such problem is unique.

As a consequence of (12), we obtain

$$\frac{\partial \bar{u}}{\partial x_d}(x', x_d) = \frac{\partial U}{\partial x_d}(x', x_d) = -\frac{\partial U}{\partial x_d}(x', -x_d).$$

Thus the derivative of U with respect to x_d computed at any point with $x_d = 0$ is zero. □

Representation formula. Next, we derive an integral representation formula for the solution u to problem (3). This makes use of the single and double layer potentials defined in (5) and of contributions relative to the image cavity $\widetilde{C}$, given by

$$\begin{aligned} \widetilde{S}_C\varphi(x) &:= \int_{\partial C} \Gamma(\widetilde{x} - y)\varphi(y)d\sigma(y), & x \in \mathbb{R}^d, \\ \widetilde{D}_C\varphi(x) &:= \int_{\partial C} \frac{\partial}{\partial n_y}\Gamma(\widetilde{x} - y)\varphi(y)d\sigma(y) & x \in \mathbb{R}^d \setminus \partial\widetilde{C}. \end{aligned} \tag{13}$$

These operators, referred to as *image layer potentials*, can be read as single and double layer potentials on $\widetilde{C}$ applied to the reflection of the function φ with respect to x_d coordinate.

Theorem 2.2. *The solution u to problem (3) is such that*

$$u(x) = S_C g(x) - D_C f(x) + \widetilde{S}_C g(x) - \widetilde{D}_C f(x), \qquad x \in \mathbb{R}^d_- \setminus C, \tag{14}$$

where S_C, D_C are defined in (5), $\widetilde{S}_C, \widetilde{D}_C$ in (13), g is the Neumann boundary condition in (3) and f is the trace of u on ∂C.

Using properties of layer potentials, from equation (14), we infer

$$f(x) = S_C g(x) - \left(-\tfrac{1}{2}I + K_C\right) f(x) - \widetilde{D}_C f(x) + \widetilde{S}_C g(x), \quad x \in \partial C,$$

where K_C is defined in (6). Thus, the trace f satisfies the relation

$$\left(\tfrac{1}{2}I + K_C + \widetilde{D}_C\right) f = S_C g + \widetilde{S}_C g,$$

which will turn out to be useful in the sequel.

Before proving Theorem 2.2, we first recall the definition of the Neumann function, see [11], that is the solution $N = N(x, y)$ to

$$\Delta_y N(x,y) = \delta_x(y) \quad \text{in } \mathbb{R}^d_-, \qquad \frac{\partial N}{\partial y_d}(x,y) = 0 \quad \text{on } \mathbb{R}^{d-1},$$

where $\delta_x(y)$ is the delta function centred in a fixed point $x \in \mathbb{R}^d$ and $\partial N/\partial y_d$ represents the normal derivative on the boundary of the half-space $\mathbb{R}^d_-$. The classical method of images provides the explicit expression

$$N(x,y) = \frac{\kappa_d}{|x-y|^{d-2}} + \frac{\kappa_d}{|\widetilde{x}-y|^{d-2}}.$$

With the function N at hand, the representation formula (14) can be equivalently written as

$$\begin{aligned} u(x) &= \mathcal{N}(f,g)(x) \\ &:= \int_{\partial C} \left[N(x,y)g(y) - \frac{\partial}{\partial n_y} N(x,y) f(y) \right] d\sigma(y), \quad x \in \mathbb{R}^d_- \setminus C, \end{aligned} \tag{15}$$

which we now prove.

Proof of Theorem 2.2. Given $R, \varepsilon > 0$ such that $C \subset B_R(0)$ and $B_\varepsilon(x) \subset \mathbb{R}^d_- \setminus C$, let

$$\Omega_{R,\varepsilon} := \Big(\mathbb{R}^d_- \cap B_R(0)\Big) \setminus \Big(C \cup B_\varepsilon(x)\Big).$$

We also define $\partial B^h_R(0)$ as the intersection of the hemisphere with the boundary of the half-space, and with $\partial B^b_R(0)$ the spherical cap (see Figure 1). Applying second Green's

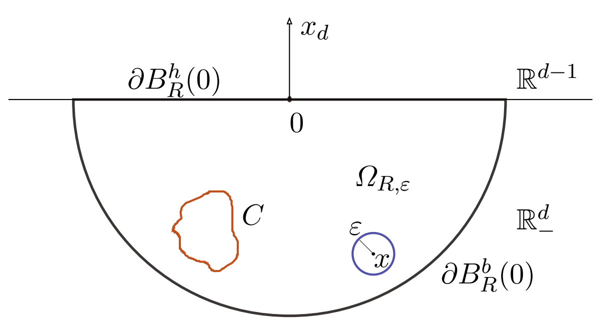

Figure 1. Domain $\Omega_{R,\varepsilon}$ used to obtain the integral representation formula (14).

identity to $N(x,\cdot)$ and u in $\Omega_{R,\varepsilon}$, we get

$$\begin{aligned}
0 =& \int_{\Omega_{R,\varepsilon}} [N(x,y)\Delta u(y) - u(y)\Delta_y N(x,y)]\,dy \\
=& \int_{\partial B^h_R(0)} \left[N(x,y)\frac{\partial u}{\partial y_d}(y) - \frac{\partial}{\partial y_d}N(x,y)u(y)\right] d\sigma(y) \\
&+ \int_{\partial B^b_R(0)} \left[N(x,y)\frac{\partial u}{\partial n_y}(y) - \frac{\partial}{\partial n_y}N(x,y)u(y)\right] d\sigma(y) \\
&+ \int_{\partial B_\varepsilon(x)} \left[\frac{\partial}{\partial n_y}N(x,y)u(y) - N(x,y)\frac{\partial u}{\partial n_y}(y)\right] d\sigma(y) \\
&- \int_{\partial C} \left[N(x,y)\frac{\partial u}{\partial n_y}(y) - \frac{\partial}{\partial n_y}N(x,y)u(y)\right] d\sigma(y) \\
=& I_1 + I_2 + I_3 - \mathcal{N}(f,g)(x).
\end{aligned}$$

The term I_1 is zero since both the normal derivative of the function N and u are zero above the boundary of the half-space.

Next, taking into account the behaviour of harmonic functions in exterior domains, formulas (4), we deduce

$$\begin{aligned}
\Big|\int_{\partial B^b_R(0)} \frac{\partial}{\partial n_y}N(x,y)u(y)\,d\sigma(y)\Big| &\le \frac{C}{R^{2d-3}}\int_{\partial B^b_R(0)} d\sigma(y) = \frac{C}{R^{d-2}}, \\
\Big|\int_{\partial B^b_R(0)} N(x,y)\frac{\partial u(y)}{\partial n_y}\,d\sigma(y)\Big| &\le \frac{C}{R^{2d-3}}\int_{\partial B^b_R(0)} d\sigma(y) = \frac{C}{R^{d-2}},
\end{aligned}$$

where C denotes a generic positive constant. As $R \to +\infty$, I_2 tends to zero.

Finally, we decompose I_3 as

$$I_3 = I_{31} - I_{32} = \int_{\partial B_\varepsilon(x)} \frac{\partial}{\partial n_y}N(x,y)u(y)\,d\sigma(y) - \int_{\partial B_\varepsilon(x)} N(x,y)\frac{\partial u}{\partial n_y}(y)d\sigma(y).$$

Using the expression of N and the continuity of u, we derive

$$\begin{aligned}
I_{31} = \int_{\partial B_\varepsilon(x)} \frac{\partial}{\partial n_y}N(x,y)u(y)d\sigma(y) =& u(x)\int_{\partial B_\varepsilon(x)} \frac{\partial}{\partial n_y}N(x,y)d\sigma(y) \\
&+ \int_{\partial B_\varepsilon(x)} [u(y)-u(x)]\,\frac{\partial}{\partial n_y}N(x,y)d\sigma(y),
\end{aligned}$$

which tends to $u(x)$ as $\varepsilon \to 0$. Moreover, we infer

$$\begin{aligned}
\big|I_{32}\big| &\le C \sup_{y\in\partial B_\varepsilon(x)} \Big|\frac{\partial u}{\partial n_y}\Big| \int_{\partial B_\varepsilon(x)} \Big|N(x,y)\Big| d\sigma(y) \\
&\le C' \sup_{y\in\partial B_\varepsilon(x)} \Big|\frac{\partial u}{\partial n_y}\Big| \left[\int_{\partial B_\varepsilon(x)} \frac{1}{\varepsilon^{d-2}}d\sigma(y) + \int_{\partial B_\varepsilon(x)} \frac{1}{|\widetilde{x}-y|^{d-2}}d\sigma(y)\right].
\end{aligned}$$

Observing that both the integrals tend to zero when ε goes to zero because the second one has a continuous kernel while the first one behaves as $O(\varepsilon)$, we infer that $I_{32} \to 0$ as $\varepsilon \to 0$. Putting together all the results, we obtain (15). □

3. Spectral analysis

Following the approach of Ammari and Kang, see [3, 4], in this Section, we prove the invertibility of the operator $\frac{1}{2} I + K_C + \widetilde{D}_C$ showing that, under suitable assumptions, the following inclusion holds

$$\sigma(K_C + \widetilde{D}_C) \subset (-1/2, 1/2]\,.$$

Such task is accomplished by determining the spectrum of the adjoint operator $K_C^* + \widetilde{D}_C^*$ in $L^2(\partial C)$, relying on the fact that the two spectra are conjugate.

The explicit expression of K_C^* is in (7). Computing the L^2-adjoint of $\widetilde{D}_C$ is straightforward: indeed, given $\psi \in L^2(\partial C)$, we have

$$\begin{aligned}\int_{\partial C} \psi(x)\widetilde{D}_C\varphi(x)d\sigma(x) &= \int_{\partial C} \psi(x)\left(\frac{1}{\omega_d}\int_{\partial C} \frac{(y-\widetilde{x})\cdot n_y}{|\widetilde{x}-y|^d}\varphi(y)d\sigma(y)\right)d\sigma(x)\\ &= \int_{\partial C} \varphi(y)\left(\frac{1}{\omega_d}\int_{\partial C} \frac{(y-\widetilde{x})\cdot n_y}{|\widetilde{x}-y|^d}\psi(x)d\sigma(x)\right)d\sigma(y)\end{aligned}$$

and thus

$$\widetilde{D}_C^*\varphi(x) = \frac{1}{\omega_d}\int_{\partial C} \frac{(x-\widetilde{y})\cdot n_x}{|\widetilde{y}-x|^d}\varphi(y)d\sigma(y). \tag{16}$$

Note that the kernel of the integral operator $\widetilde{D}_C^*$ is smooth on ∂C.

As proved in [13], for smooth domains the eigenvalues of K_C^* on $L^2(\partial C)$ lie in $(-1/2, 1/2]$; the same result it is also true for Lipschitz domain (see [3, 7]). With the same approach, it can be shown that the same property holds true for $K_C^* + \widetilde{D}_C^*$.

Theorem 3.1. *Let C be an open bounded domain with Lipschitz boundary. Then*

$$\sigma(K_C^* + \widetilde{D}_C^*) \subset (-1/2, 1/2]\,.$$

For completeness, we provide here a complete proof of such fact.

Firstly, we observe that the regular operator $\widetilde{D}_C^*$ on the boundary of the cavity can be seen as the normal derivative of an appropriate single layer potential.

Lemma 3.2. *Given $\varphi \in L^2(\partial C)$ we have that*

$$\widetilde{D}_C^*\varphi(x) = \frac{\partial}{\partial n_x}\left(S_{\widetilde{C}}\widetilde{\varphi}(x)\right), \quad x \in \partial C,$$

where $\widetilde{\varphi} \in L^2(\partial\widetilde{C})$ is defined by $\widetilde{\varphi}(x) := \varphi(\widetilde{x})$.

Proof. Using the expression (16) of $\widetilde{D}_C^*$ and the identity

$$\nabla_x \left(\frac{1}{(2-d)|x-y|^{d-2}} \right) = \frac{x-y}{|x-y|^d},$$

we find that

$$\widetilde{D}_C^* \varphi(x) = \nabla_x \left(\int_{\partial C} \frac{\kappa_d\, \varphi(y)}{|\widetilde{y}-x|^{d-2}} d\sigma(y) \right) \cdot n_x.$$

Given $\varphi \in L^2(\partial C)$ and $\widetilde{\varphi} \in L^2(\partial \widetilde{C})$ as previously defined, we have

$$\begin{aligned}\int_{\partial C} \frac{\varphi(y)}{|\widetilde{y}-x|^{d-2}} d\sigma(y) &= \int_{\partial \widetilde{C}} \frac{\varphi(\widetilde{z})}{|\widetilde{\widetilde{z}}-x|^{d-2}} d\sigma(z) \\ &= \int_{\partial \widetilde{C}} \frac{\varphi(\widetilde{z})}{|z-x|^{d-2}} d\sigma(z) = \int_{\partial \widetilde{C}} \frac{\widetilde{\varphi}(z)}{|z-x|^{d-2}} d\sigma(z),\end{aligned}$$

which gives the conclusion. □

We are now ready to prove the main result of this Section.

Proof of Theorem (3.1). Given $\varphi \in L^2(\partial C)$, let ψ be defined by $\psi := S_C \varphi + S_{\widetilde{C}} \widetilde{\varphi}$. By the known properties of single layer potentials, we derive on ∂C

$$\frac{\partial \psi}{\partial n}\Big|_{\pm} = \left(\pm \tfrac{1}{2} I + K_C^* + \widetilde{D}_C^* \right) \varphi$$

and, as a consequence,

$$\frac{\partial \psi}{\partial n}\Big|_{+} + \frac{\partial \psi}{\partial n}\Big|_{-} = 2 \left(K_C^* + \widetilde{D}_C^* \right) \varphi, \qquad \frac{\partial \psi}{\partial n}\Big|_{+} - \frac{\partial \psi}{\partial n}\Big|_{-} = \varphi. \tag{17}$$

Taking a linear combination of the two relations in (17), we deduce

$$\begin{aligned}\left(\lambda I - K_C^* - \widetilde{D}_C^* \right) \varphi &= \lambda \left(\frac{\partial \psi}{\partial n}\Big|_{+} - \frac{\partial \psi}{\partial n}\Big|_{-} \right) - \frac{1}{2} \left(\frac{\partial \psi}{\partial n}\Big|_{+} + \frac{\partial \psi}{\partial n}\Big|_{-} \right) \\ &= \left(\lambda - \frac{1}{2} \right) \frac{\partial \psi}{\partial n}\Big|_{+} - \left(\lambda + \frac{1}{2} \right) \frac{\partial \psi}{\partial n}\Big|_{-}.\end{aligned}$$

If λ is an eigenvalue of $K_C^* + \widetilde{D}_C^*$ with eigenfunction φ, then

$$\left(\lambda - \frac{1}{2} \right) \frac{\partial \psi}{\partial n}\Big|_{+} - \left(\lambda + \frac{1}{2} \right) \frac{\partial \psi}{\partial n}\Big|_{-} = 0, \qquad \text{on} \quad \partial C.$$

Multiplying such relation by the function ψ and integrating over ∂C, we get

$$\left(\lambda - \frac{1}{2} \right) \int_{\partial C} \psi(x) \frac{\partial \psi}{\partial n}(x)\Big|_{+} d\sigma(x) - \left(\lambda + \frac{1}{2} \right) \int_{\partial C} \psi(x) \frac{\partial \psi}{\partial n}(x)\Big|_{-} d\sigma(x) = 0. \tag{18}$$

Integrating by parts we have

$$\text{(19)}\qquad \begin{aligned}\int_{\partial C} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{-} d\sigma(x) &= \int_C \psi(x)\Delta\psi(x)dx + \int_C |\nabla\psi(x)|^2\, dx\\ &= \int_C |\nabla\psi(x)|^2\, dx.\end{aligned}$$

The first integral in (18) can be dealt with as done in the proof of Theorem 2.2. Precisely, given large $R > 0$, applying the Green's formula in $\Omega_R := \left(\mathbb{R}^d_- \cap B_R(0)\right) \setminus C$, we get

$$\begin{aligned}\int_{\partial C} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{+} d\sigma(x) = &\int_{\partial B^h_R(0)} \psi(x)\frac{\partial\psi}{\partial x_d}(x)d\sigma(x) + \int_{\partial B^b_R(0)} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{+} d\sigma(x)\\ &- \int_{\Omega_R} \psi(x)\Delta\psi(x)dx - \int_{\Omega_R} |\nabla\psi(x)|^2 dx,\end{aligned}$$

where $\partial B^h_R(0)$ is the intersection of the hemisphere with the half-space and $\partial B^b_R(0)$ is the spherical cap. The quantity $\partial\psi/\partial n$ is identically zero on the boundary of the half-space since the kernel of the operator is the normal derivative of the Neumann function which, by hypothesis, is null on $\mathbb{R}^{d-1}$. Moreover, ψ is harmonic in Ω_R, so we infer

$$\int_{\partial C} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{+} d\sigma(x) = \int_{\partial B^b_R(0)} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{+} d\sigma(x) - \int_{\Omega_R} |\nabla\psi(x)|^2 dx.$$

Recalling the asymptotic behaviour of simple layer potential,

$$|S_C\varphi| + |S_{\widetilde{C}}\varphi| = O(|x|^{2-d}), \qquad |\nabla S_C\varphi| + |\nabla S_{\widetilde{C}}\varphi| = O(|x|^{1-d}) \qquad \text{as } |x| \to \infty.$$

we obtain, for some $C > 0$,

$$\begin{aligned}\left|\int_{\partial B^b_R(0)} \psi(x)\frac{\partial\psi}{\partial n}(x)\Big|_{+} d\sigma(x)\right| &\le \int_{\partial B^b_R(0)} |\psi(x)|\left|\frac{\partial\psi}{\partial n}(x)\Big|_{+}\right| d\sigma(x)\\ &\le \frac{C}{R^{2d-3}}\int_{\partial B^b_R(0)} d\sigma(x) = \frac{1}{R^{d-2}}.\end{aligned}$$

Passing to the limit $R \to +\infty$, we find

$$\text{(20)}\qquad \int_{\partial C} \psi(x)\frac{\partial\psi}{\partial n}\Big|_{+} d\sigma(x) = -\int_{\mathbb{R}^d_- \setminus \overline{C}} |\nabla\psi(x)|^2 dx.$$

Plugging (19) and (20) into (18), we infer the identity

$$\left(\lambda - \frac{1}{2}\right)\int_{\mathbb{R}^d_- \setminus C} |\nabla\psi(x)|^2 dx + \left(\lambda + \frac{1}{2}\right)\int_C |\nabla\psi(x)|^2 dx = 0,$$

that is

$$(A + B)\lambda = \frac{1}{2}(A - B)$$

with

$$A := \int_{\mathbb{R}^d_- \setminus C} |\nabla\psi(x)|^2 dx \qquad \text{and} \qquad B := \int_C |\nabla\psi(x)|^2 dx.$$

The coefficient of λ is non-zero. On the contrary, if $A+B=0$ then $\nabla\psi=0$ in $\mathbb{R}^d_-$ which means that $\psi\equiv 0$, hence, from the second equation in (17), we get $\varphi=0$ in ∂C.

Therefore, solving with respect to λ, we finally get

$$\lambda=\frac{1}{2}\cdot\frac{A-B}{A+B}\in\left[-\frac{1}{2},\frac{1}{2}\right]. \tag{21}$$

The value $\lambda=-1/2$ is not an eigenvalue for the operator $K^*_C+\widetilde{D}^*_C$. Indeed, in such a case, we would have

$$A=\int_{\mathbb{R}^d_-\setminus C}\left|\nabla\psi(x)\right|^2 dx=0,$$

and thus $\psi=0$ in $\mathbb{R}^d_-\setminus C$. By definition of ψ, we deduce that $\psi=0$ on ∂C and since ψ is harmonic in C, we get that $\psi=0$ also in C. As before, by (17), this would imply that $\varphi=0$ in ∂C. □

For completeness, let us observe that the value $\lambda=1/2$ is an eigenvalue with geometric multiplicity equal to one. Indeed, identity (21) implies that, for such value of λ,

$$B=\int_C\left|\nabla\psi(x)\right|^2 dx=0,$$

hence ψ is constant in C. Normalizing $\psi=1$ in C, the function ψ in $\mathbb{R}^d_-\setminus C$ is given by the restriction of the solution U to the Dirichlet problem in the exterior domain $\mathbb{R}^d\setminus\left(C\cup\widetilde{C}\right)$ with boundary data equal to 1. Then, by the second equation in (17), the function φ is the normal derivative of U at ∂C.

4. Asymptotic expansion

In this Section, we derive an asymptotic formula for the solution of problem (3) when the cavity C is small compared to the distance from the half-space $\mathbb{R}^{d-1}$. For the reader's convenience, we recall that the cavity C has the structure

$$C_\varepsilon:=C=z+\varepsilon B$$

where B is a bounded Lipschitz set containing the origin. Moreover, we assume that

$$\text{dist}(z,\mathbb{R}^{d-1})\geq\delta_0>0 \tag{22}$$

otherwise, for the application we have in mind, the problem does not have a real physical meaning. To emphasize the dependence of the solution to the direct problem by the parameter ε we denote it by u_ε. For brevity, we denote the layer potentials relative to C_ε by the index ε, *viz.*

$$S_\varepsilon=S_{C_\varepsilon},\quad D_\varepsilon=D_{C_\varepsilon},\quad \widetilde{S}_\varepsilon=\widetilde{S}_{C_\varepsilon},\quad \widetilde{D}_\varepsilon=\widetilde{D}_{C_\varepsilon},\quad K_\varepsilon=K_{C_\varepsilon}$$

and the trace of the solution u_ε on ∂C_ε by f_ε. In this way the representation formula (14) reads as

$$u_\varepsilon = S_\varepsilon g - D_\varepsilon f_\varepsilon - \widetilde{D}_\varepsilon f_\varepsilon + \widetilde{S}_\varepsilon g.$$

At $x \in \mathbb{R}^{d-1}$, taking into account that $x = \widetilde{x}$, it follows that

$$S_\varepsilon g(x) = \int_{\partial C_\varepsilon} \Gamma(x-y) g(y)\, d\sigma(y) = \int_{\partial C_\varepsilon} \Gamma(\widetilde{x}-y) g(y)\, d\sigma(y) = \widetilde{S}_\varepsilon g(x)$$

$$D_\varepsilon f_\varepsilon(x) = \int_{\partial C_\varepsilon} \frac{\partial}{\partial n_y}\Gamma(x-y) f_\varepsilon(y)\, d\sigma(y) = \int_{\partial C_\varepsilon} \frac{\partial}{\partial n_y}\Gamma(\widetilde{x}-y) f_\varepsilon(y)\, d\sigma(y) = \widetilde{D}_\varepsilon f_\varepsilon(x)$$

Hence, we obtain the equality

$$\tfrac{1}{2}\, u_\varepsilon(x) = S_\varepsilon g(x) - D_\varepsilon f_\varepsilon(x), \qquad x \in \mathbb{R}^{d-1}.$$

Associating with the relation at the boundary ∂C_ε and by (3.1)]

$$f_\varepsilon(x) = \left(\tfrac{1}{2} I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(x), \qquad x \in \partial C_\varepsilon,$$

we get the identity

$$\tfrac{1}{2} u_\varepsilon(x) = J_1(x) + J_2(x), \qquad x \in \mathbb{R}^{d-1}, \tag{23}$$

where

$$J_1(x) := \int_{\partial C_\varepsilon} \Gamma(x-y) g(y)\, d\sigma(y),$$

$$J_2(x) := -\int_{\partial C_\varepsilon} \frac{\partial}{\partial n_y}\Gamma(x-y) \left(\tfrac{1}{2} I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(y)\, d\sigma(y).$$

Analyzing in details the dependence with respect to ε of such relation, we obtain an explicit expression for the first two terms in the asymptotic expansion of u_ε at $\mathbb{R}^{d-1}$.

In what follows, for any fixed value of $\varepsilon > 0$, given $h : \partial C_\varepsilon \to \mathbb{R}$, we introduce the function $\widehat{h} : \partial B \to \mathbb{R}$ defined by

$$\widehat{h}(\zeta; \varepsilon) := h(z + \varepsilon\, \zeta), \qquad \zeta \in \partial B.$$

This definition is useful to consider integrals over a set that is independent on ε.

Theorem 4.1. *Let us assume (22). For any $\varepsilon > 0$, let $g \in L^2(\partial C_\varepsilon)$ such that $\widehat{g}$ is independent on ε. Then, at any $x \in \mathbb{R}^{d-1}$, the following expansion holds*

$$\begin{aligned} u_\varepsilon(x) &= 2\varepsilon^{d-1}\Gamma(x-z) \int_{\partial B} \widehat{g}(\zeta) d\sigma(\zeta) \\ &\quad + 2\varepsilon^d \nabla\Gamma(x-z) \cdot \int_{\partial B} \left\{ n_\zeta \left(\tfrac{1}{2} I + K_B\right)^{-1} S_B \widehat{g}(\zeta) - \zeta \widehat{g}(\zeta) \right\} d\sigma(\zeta) + O(\varepsilon^{d+1}), \end{aligned} \tag{24}$$

where $O(\varepsilon^{d+1})$ denotes a quantity uniformly bounded by $C\varepsilon^{d+1}$ with $C = C(\delta_0)$ which tends to infinity when δ_0 goes to zero.

To prove the theorem we first show the following expansion for the operator $\left(\frac{1}{2}I+K_\varepsilon+\widetilde{D}_\varepsilon\right)^{-1}$

Lemma 4.2. *We have*

$$\left(\tfrac{1}{2}I+K_\varepsilon+\widetilde{D}_\varepsilon\right)^{-1}\left(S_\varepsilon g+\widetilde{S}_\varepsilon g\right)(z+\varepsilon\zeta)=\varepsilon\left(\tfrac{1}{2}I+K_B\right)^{-1}S_B\widehat{g}(\zeta)+O(\varepsilon^{d-1}) \tag{25}$$

Proof. We analyse, separetely, the terms $\left(\frac{1}{2}I+K_\varepsilon+\widetilde{D}_\varepsilon\right)^{-1}$ and $S_\varepsilon+\widetilde{S}_\varepsilon$, collecting, at the very end, the corresponding expansions.

Since $K_\varepsilon+\widetilde{D}_\varepsilon$ is compact and its spectrum is contained in $(-1/2,1/2]$ there exists $\delta>0$ such that

$$\sigma\left(K_\varepsilon+\widetilde{D}_\varepsilon\right)\subset(-1/2+\delta,1/2].$$

Then, the operator

$$A_\varepsilon:=\tfrac{1}{2}I-K_\varepsilon-\widetilde{D}_\varepsilon$$

is such that $\sigma(A_\varepsilon)\subset[0,1-\delta)$ and thus has spectral radius strictly smaller than 1. As a consequence, taking the powers of the operator A_ε one finds

$$\|A_\varepsilon^h\|\le 1\quad\forall h\quad\text{and}\quad\|A_\varepsilon^{h_0}\|<1\quad\text{for some } h_0. \tag{26}$$

The inverse operator of $I-A_\varepsilon=\frac{1}{2}I+K_\varepsilon+\widetilde{D}_\varepsilon$ can be represented by the Neumann series that is

$$(I-A_\varepsilon)^{-1}=\sum_{h=0}^{+\infty}A_\varepsilon^h=\sum_{h=0}^{+\infty}\left(\tfrac{1}{2}I-K_\varepsilon-\widetilde{D}_\varepsilon\right)^h.$$

At the point $z+\varepsilon\zeta$, we obtain

$$\begin{aligned}
&\left(K_\varepsilon+\widetilde{D}_\varepsilon\right)\varphi(z+\varepsilon\zeta)\\
&\quad=\frac{1}{\omega_d}\text{p.v.}\int_{\partial C_\varepsilon}\frac{(y-z-\varepsilon\zeta)\cdot n_y}{|z+\varepsilon\zeta-y|^d}\varphi(y)\,d\sigma(y)+\int_{\partial C_\varepsilon}\frac{\partial}{\partial n_y}\Gamma(\tilde{z}+\varepsilon\tilde{\zeta}-y)\varphi(y)\,d\sigma(y)\\
&\quad=\frac{1}{\omega_d}\text{p.v.}\int_{\partial B}\frac{(\eta-\zeta)\cdot n_\eta}{|\zeta-\eta|^d}\widehat{\varphi}(\eta)\,d\sigma(\eta)+\varepsilon^{d-1}\int_{\partial B}\frac{\partial}{\partial n_\eta}\Gamma(\tilde{z}+\varepsilon\tilde{\zeta}-z-\varepsilon\eta)\hat{\varphi}(\eta)\,d\sigma(\eta)\\
&\quad=K_B\widehat{\varphi}(\zeta)+\varepsilon^{d-1}R_\varepsilon\widehat{\varphi}(\zeta),
\end{aligned}$$

where

$$R_\varepsilon\widehat{\varphi}(\zeta):=\int_{\partial B}\frac{\partial}{\partial n_\eta}\Gamma\left(\widetilde{z}-z+\varepsilon(\widetilde{\zeta}-\eta)\right)\widehat{\varphi}(\eta)d\sigma(\eta)$$

is uniformly bounded in ε. Using these results, we calculate A_ε^h highlighting the term that do not contain ε and the one of order $d-1$, that is

$$A_\varepsilon^h=\left(\tfrac{1}{2}I-K_B\right)^h-\varepsilon^{d-1}E_{h,\varepsilon}$$

where

$$E_{h,\varepsilon}=\sum_{j=1}^{h}A_\varepsilon\cdots A_\varepsilon\underbrace{R_\varepsilon}_{j-th}A_\varepsilon\cdots A_\varepsilon.$$

For h_0 as in (26) and $h > h_0$ we have

$$\|E_{h,\varepsilon}\| \leq \|R_\varepsilon\| \|A_\varepsilon\|^{2h_0} \|A_\varepsilon^{h_0}\|^{[h/h_0]-1} \leq \|R_\varepsilon\| \|A_\varepsilon\|^{2h_0} \|A_\varepsilon^{h_0}\|^{h/h_0-1},$$

where $[\,\cdot\,]$ denotes the integer part, and thus

$$\sum_{h=0}^{+\infty} \|E_{h,\varepsilon}\| \leq C \sum_{h=0}^{+\infty} \|A_\varepsilon^{h_0}\|^{h/h_0}$$

giving the absolute convergence of $\sum E_{h,\varepsilon}$. Summarizing we conclude that

$$(I - A_\varepsilon)^{-1} = \left(\tfrac{1}{2} I + K_B\right)^{-1} + O(\varepsilon^{d-1}). \tag{27}$$

Let us evaluate the term $S_\varepsilon + \widetilde{S}_\varepsilon$. We have

$$\begin{aligned} S_\varepsilon g(z + \varepsilon\zeta) &= \int_{\partial C_\varepsilon} \Gamma(z + \varepsilon\zeta - y) g(y) d\sigma(y) \\ &= \varepsilon \int_{\partial B} \Gamma(\zeta - \theta) \widehat{g}(\theta) d\sigma(\theta) = \varepsilon S_B \widehat{g}(\zeta) \end{aligned}$$

and

$$\begin{aligned} \widetilde{S}_\varepsilon g(z + \varepsilon\zeta) &= \int_{\partial C_\varepsilon} \Gamma\left(\widetilde{z} + \varepsilon\widetilde{\zeta} - y\right) g(y) d\sigma(y) \\ &= \varepsilon^{d-1} \int_{\partial B} \Gamma\left(\widetilde{z} - z + \varepsilon(\widetilde{\zeta} - \theta)\right) \widehat{g}(\theta) d\sigma(\theta) \\ &= \varepsilon^{d-1} \Gamma(\widetilde{z} - z) \int_{\partial B} \widehat{g}(\theta) d\sigma(\theta) + O(\varepsilon^d) \end{aligned}$$

where we have used the zero order expansion for Γ.

Collecting we infer

$$\left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(z + \varepsilon\zeta) = \varepsilon S_B \widehat{g}(\zeta) + O(\varepsilon^{d-1})$$

and combining with (27) we obtain the conclusion. □

Proof of Theorem 4.1. To prove (24), we analyse the two integrals J_1 and J_2 in (23).

For $x, \zeta \in \mathbb{R}^d$ with $x \neq 0$ and ε sufficiently small, we have

$$\Gamma(x - \varepsilon\zeta) = \Gamma(x) - \varepsilon \nabla\Gamma(x) \cdot \zeta + O(\varepsilon^2).$$

Hence, for $x \in \mathbb{R}^{d-1}$, we get

$$\begin{aligned} J_1 &= \varepsilon^{d-1} \int_{\partial B} \Gamma(x - z - \varepsilon\zeta) \, \widehat{g}(\zeta) \, d\sigma(\zeta) \\ &= \varepsilon^{d-1} \Gamma(x - z) \int_{\partial B} \widehat{g}(\zeta) \, d\sigma(\zeta) - \varepsilon^d \nabla\Gamma(x - z) \cdot \int_{\partial B} \zeta \, \widehat{g}(\zeta) \, d\sigma(\zeta) + O(\varepsilon^{d+1}). \end{aligned} \tag{28}$$

Next we consider the second integral in (23), written as

$$J_2 = -\varepsilon^{d-1} \int_{\partial B} \frac{\partial}{\partial n_\zeta} \Gamma(x - z - \varepsilon\zeta) \widehat{h}_\varepsilon(\zeta) \, d\sigma(\zeta),$$

where the function $\widehat{h}_\varepsilon$ is given by

$$\widehat{h}_\varepsilon(\zeta) = \left(\tfrac{1}{2}I + K_\varepsilon + \widetilde{D}_\varepsilon\right)^{-1} \left(S_\varepsilon g + \widetilde{S}_\varepsilon g\right)(z + \varepsilon\zeta) \tag{29}$$

For $x, \zeta \in \mathbb{R}^d$ with $x \neq 0$ and ε sufficiently small, there holds

$$\nabla_x \Gamma(x + \varepsilon\zeta) = \nabla_x \Gamma(x) + O(\varepsilon), \tag{30}$$

therefore, taking advantage of the expansion (25),

$$\begin{aligned} J_2 &= \varepsilon^{d-1} \int_{\partial B} \frac{\partial}{\partial n_\zeta} \Gamma(x - z) \widehat{h}_\varepsilon(\zeta)\, d\sigma(\zeta) + O(\varepsilon^d) \\ &= \varepsilon^d \int_{\partial B} \frac{\partial}{\partial n_\zeta} \Gamma(x - z) \left(\tfrac{1}{2}I + K_B\right)^{-1} S_B \widehat{g}(\zeta)\, d\sigma(\zeta) + O(\varepsilon^{d+1}). \end{aligned}$$

Collecting the expansions for J_1 and J_2, we deduce (24). □

We show that the term $\left(\frac{1}{2}I + K_B\right)^{-1} S_B g(x)$, for $x \in \partial B$, represents the trace of the solution of the external domain related to the set B and with Neumann boundary condition given by g. To this aim, we consider the problem

$$\Delta U = 0 \quad \text{in } \mathbb{R}^d \setminus B, \quad \frac{\partial U}{\partial n} = g \quad \text{on } \partial B, \quad U \longrightarrow 0 \quad \text{as } |x| \to +\infty, \tag{31}$$

where the cavity B is such that $0 \in B$.

Proposition 4.3. *Let us define $f(x) := U(x)\big|_{x\in\partial B}$, then*

$$\left(\frac{1}{2}I + K_B\right)^{-1} S_B g(x) = f(x).$$

Proof. As done in the proof of Theorem 2.2, that is, applying second Green's identity to the fundamental solution Γ and U in the domain $B_R(0) \setminus B$, with R sufficiently large, it can be proven that the representation formula for U is

$$U(x) = S_B g(x) - D_B f(x), \quad x \in \mathbb{R}^d \setminus B. \tag{32}$$

Therefore, from single and double layer potentials properties

$$f(x) = S_B g(x) - \left(-\frac{1}{2}I + K_B\right) f(x), \quad x \in \partial B,$$

hence

$$f(x) = \left(\frac{1}{2}I + K_B\right)^{-1} S_B g(x), \quad x \in \partial B,$$

that is the assertion. □

Now, we want to consider a specific case of the Neumann condition on the boundary of the cavity C_ε so to get an explicit expression of the asymptotic expansion in terms of the polarization tensor and the fundamental solution.

Corollary 4.4. *Given $p \in \mathbb{R}^d$, let the boundary datum given by*

$$g = -p \cdot n.$$

Then, the following expansion holds

$$u_\varepsilon(x) = 2\,\varepsilon^d \nabla\Gamma(x-z)\cdot Mp + O(\varepsilon^{d+1}), \qquad x \in \mathbb{R}^{d-1}, \tag{33}$$

where M is the symmetric positive definite tensor given by

$$M := \int_{\partial B} n_\zeta \otimes (\zeta + \Psi(\zeta))\, d\sigma(\zeta) \tag{34}$$

and the auxiliary function Ψ has components Ψ_i, $i = 1, \dots, d$, solving

$$\Delta\Psi_i = 0 \quad \text{in } \mathbb{R}^d \setminus B, \qquad \frac{\partial\Psi_i}{\partial n} = -n_i \quad \text{on } \partial B, \qquad \Psi_i \longrightarrow 0 \quad \text{as } |x| \to +\infty.$$

Proof. Let us set

$$\begin{aligned} J_1 &:= \nabla\Gamma(x-z)\cdot \int_{\partial B} n_\zeta \left(\frac{1}{2}I + K_B\right)^{-1} S_B[-p\cdot n](\zeta)\, d\sigma(\zeta), \\ J_2 &:= \nabla\Gamma(x-z)\cdot \int_{\partial B} \zeta\, p\cdot n_\zeta\, d\sigma(\zeta). \end{aligned}$$

Then, expansion (24) with $g = -p \cdot n$ gives

$$\begin{aligned} \tfrac{1}{2} u_\varepsilon(x) &= -\varepsilon^{d-1}\Gamma(x-z)\int_{\partial B} p\cdot n_\zeta\, d\sigma(\zeta) + J_1 + J_2 + O(\varepsilon^{d+1}) \\ &= J_1 + J_2 + O(\varepsilon^{d+1}) \end{aligned} \tag{35}$$

since divergence Theorem guarantees that the first term in the expansion for u_ε is null.

From the equation (31), with $g = -p \cdot n$, since the problem for U is linear, we can decompose U as $U = \sum_i U_i$ where the functions U_i, for $i = 1, \cdots, d$, solve

$$\Delta U_i = 0 \quad \text{in } \mathbb{R}^d \setminus B, \quad \frac{\partial U_i}{\partial n} = -p_i n_i \quad \text{on } \partial B, \quad U_i \longrightarrow 0 \quad \text{as } |x| \to +\infty.$$

From the definition of the functions Ψ_i, we deduce $U = p \cdot \Psi$. Using Proposition 4.3, the term J_1 can be rewritten as

$$J_1 = \nabla\Gamma(x-z)\cdot \int_{\partial B} (\Psi(\zeta)\cdot p)\, n_\zeta\, d\sigma(\zeta) = \nabla\Gamma(x-z)\cdot \int_{\partial B} (n_\zeta \otimes \Psi(\zeta))\, p\, d\sigma(\zeta).$$

To deal with the term J_2, we preliminarly observe that

$$\int_{\partial B} (\zeta \otimes n_\zeta)\; d\sigma(\zeta) = \int_{\partial B} (n_\zeta \otimes \zeta)\; d\sigma(\zeta).$$

Indeed, for $n_\zeta = (n_{\zeta,1}, \dots n_{\zeta,d})$, for any $i,j \in \{1,\dots,d\}$, it follows

$$\begin{aligned}\int_{\partial B} \zeta_i\, n_{\zeta,j}\, d\sigma(\zeta) &= \int_{\partial B} n_\zeta \cdot \zeta_i e_j\, d\sigma(\zeta) = \int_B \mathrm{div}\,(\zeta_i e_j)\, d\zeta = \int_B e_j \cdot e_i\, d\zeta \\ &= \int_B \nabla(\zeta_j)\cdot e_i\, d\zeta = \int_B \mathrm{div}\,(\zeta_j e_i)\; d\zeta = \int_{\partial B} n_\zeta \cdot \zeta_j e_i\, d\sigma(\zeta) \\ &= \int_{\partial B} n_{\zeta,i}\, \zeta_j\, d\sigma(\zeta)\end{aligned}$$

where e_j is the j-th unit vector of $\mathbb{R}^d$. Thus, we get

$$J_2 = \nabla\Gamma(x-z)\cdot \int_{\partial B} (\zeta \otimes n_\zeta)\, p\, d\sigma(\zeta) = \nabla\Gamma(x-z)\cdot \int_{\partial B} (n_\zeta \otimes \zeta)\; p\, d\sigma(\zeta).$$

Collecting the expressions for J_1 and J_2, we obtain formula (33).

Symmetry of the tensor M, defined in (34), follows from

$$\begin{aligned}\int_{\partial B} \Psi_i(\zeta) n_{\zeta,j}\, d\sigma(\zeta) &= -\int_{\partial B} \Psi_i(\zeta) \frac{\partial \Psi_j}{\partial n}(\zeta)\, d\sigma(\zeta) \\ &= \int_{\mathbb{R}^d\setminus B} \mathrm{div}\,(\Psi_i(\zeta)\nabla\Psi_j(\zeta))\; d\zeta = \int_{\mathbb{R}^d\setminus B} \nabla\Psi_i(\zeta)\cdot\nabla\Psi_j(\zeta)\, d\zeta\end{aligned}$$

where the last term is obviously symmetric. Taking $\eta \in \mathbb{R}^d$, we consider

$$\eta \cdot M\eta = \int_{\partial B} (n_\varepsilon \cdot \eta)(\zeta\cdot\eta)\, d\sigma(\zeta) + \int_{\partial B} (n_\varepsilon\cdot\eta)(\Psi(\zeta)\cdot\eta)\, d\sigma(\zeta) = I_1 + I_2.$$

The positivity of the tensor follows from the divergence theorem and integration by parts, in fact

$$I_1 = \int_B \mathrm{div}((\zeta\cdot\eta)\eta)\, d\zeta = \int_B \eta\cdot\nabla(\zeta\cdot\eta)\, d\zeta = \int_B |\eta|^2\, d\zeta = |\eta|^2|B|$$

In the same way, by the definition of the function Ψ

$$\begin{aligned}I_2 = -\int_{\partial B} \frac{\partial}{\partial n}(\Psi(\zeta)\cdot\eta)(\Psi(\zeta)\cdot\eta)\, d\sigma(\eta) &= \int_{\mathbb{R}^d\setminus B} \mathrm{div}\,((\Psi(\zeta)\cdot\eta)\nabla(\Psi(\zeta)\cdot\eta))\; d\sigma(\zeta) \\ &= \int_{\mathbb{R}^d\setminus B} |\nabla(\Psi(\zeta)\cdot\eta)|^2\, d\zeta\end{aligned}$$

The sum of I_1 and I_2 gives the positivity. □

For specific forms of the cavity C, the auxiliary function Ψ can be determined explicitly, providing a corresponding explicit formula for the polarization tensor M. The basic case is the one of a spherical cavity (see [10]). If $B = \{x \in \mathbb{R}^d : |x| < 1\}$, then a direct calculation shows that, for $i = 1,2,3$, there holds $\Psi_i(x) = x_i/((d-1)|x|^n)$, and thus

$$\Psi_i(\zeta) = \frac{1}{d-1}\,\zeta_i, \qquad \zeta \in \partial B.$$

As a consequence, the polarization tensor is a multiple of the identity and, precisely,

$$M = \frac{3}{2}|B| I = 2\pi I.$$

Then, the asymptotic expansion (33) becomes

$$u_\varepsilon(x) = 4\pi\varepsilon^d \nabla\Gamma(x-z)\cdot p + O(\varepsilon^{d+1}), \qquad x\in\mathbb{R}^{d-1}.$$

Explicit formulas can be provided also in the case of ellipsoidal cavities (see [1, 3, 4]).

In general, for given shapes of the cavity C, such auxiliary function can be numerically approximated and, thus, the first term in the expansion (33) can be considered as known in practical cases.

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