QUASI-COASSOCIATIVE C^*-QUANTUM GROUPOIDS OF TYPE A AND MODULAR C^*-CATEGORIES

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In memory of Joe Jerome Pérez

ABSTRACT. We construct a new class of finite-dimensional C^*-quantum groupoids at roots of unity $q = e^{i\pi/\ell}$, with limit the discrete dual of the classical $\mathrm{SU}(N)$ for large orders. The representation category of our groupoid turns out to be tensor equivalent to the well known quotient C^*-category of the category of tilting modules of the non-semisimple quantum group $U_q(\mathfrak{sl}_N)$ of Drinfeld, Jimbo and Lusztig.

As an algebra, the C^*-groupoid is a quotient of $U_q(\mathfrak{sl}_N)$. As a coalgebra, it naturally reflects the categorical quotient construction. In particular, it is not coassociative, but satisfies axioms of the weak quasi-Hopf C^*-algebras: quasi-coassociativity and non-unitality of the coproduct. There are also a multiplicative counit, an antipode, and an R-matrix.

For this, we give a general construction of quantum groupoids for complex simple Lie algebras $\mathfrak{g} \neq E_8$ and certain roots of unity. Our main tools here are Drinfeld's coboundary associated to the R-matrix, related to the algebra involution, and certain canonical projections introduced by Wenzl, which yield the coproduct and Drinfeld's associator in an explicit way. Tensorial properties of the negligible modules reflect in a rather special nature of the associator. We next reduce the proof of the categorical equivalence to the problems of establishing semisimplicity and computing dimension of the groupoid. In the case $\mathfrak{g} = \mathfrak{sl}_N$ we construct a (non-positive) Haar-type functional on an associative version of the dual groupoid satisfying key non-degeneracy properties. This enables us to complete the proof.

1. INTRODUCTION

Let $\mathfrak{g}$ be a complex simple Lie algebra, q a primitive root of unity, and let $U_q(\mathfrak{g})$ be Lusztig's restricted quantum group [8]. It is well known that this is a non-semisimple algebra, and so is its representation category. However, a semisimple tensor category, $\mathcal{F}$, called the fusion category, can be obtained as a quotient of the category of tilting modules by the ideal of the negligible ones [27, 28, 3, 4, 17, 47]. These categories play a prominent role in conformal field theory [5], subfactor theory [22], topological quantum field theory and the study of invariants of 3-manifolds [50, 41].

For the roots of unity $q = e^{i\pi/d\ell}$ with d the ratio of the square lengths of a long root to a short root, Kirillov introduced a *–involution and an associated inner product on the arrow spaces of $\mathcal{F}$ and conjectured positivity [24]. Wenzl proved the conjecture, and derived existence of a C^*–structure making the braiding unitary [49]. See also [52] for an independent related result and [45, 42, 43, 44] and references therein for a complete discussion on unitarity and modularity. In particular these results are of interest in the algebraic approach to low dimensional QFT where tensor C^*–categories with unitary braided symmetries arise [18], and constituted our original motivation.

The aim of this paper is to construct C^*–quantum groupoids with representation category tensor equivalent to the fusion C^*–categories of $\mathfrak{sl}_N$. They arise as linear duals of quantised function algebras, and we thus denote them by $\mathcal{C}(\widehat{\mathrm{SU}(N)}, \ell)$. Every such groupoid turns out to be described by a finite dimensional C^*-algebra which is a quotient of $U_q(\mathfrak{sl}_N)$ determined by the irreducible *–representations with dominant weights in the open Weyl alcove. Hence the dimensions of the representations are classical. Our constructions may be interpreted as a natural dual of the quotient construction at the categorical level as recalled above.

The problem of constructing quantum groupoids describing a given tensor category via representation theory originates in the physics literature (see [18] and references therein) and subsequently it has been vastly considered also with different motivations. For the C^*–categories $\mathcal{F}$, the fusion rules become close to those of $\mathfrak{g}$ for ℓ large. Physical considerations lead to look for quantisations of the compact real form G of $\mathfrak{g}$ [16]. The quotient construction defining $\mathcal{F}$ allows a more precise formulation of the problem. Indeed, it may be understood as taking a tilting module T to a maximal non-negligible summand $\overline{T}$ and a morphism to its compression to corresponding summands. Hence objects of $\mathcal{F}$ can be viewed as Hilbert spaces of classical dimension, and arrows as linear maps between them. We may regard this association as an approximation of the usual embedding functor of $\text{Rep}(G)$ into the Hilbert spaces for ℓ large, and ask whether the former can rigorously be interpreted as the embedding functor of a C^*–quantum groupoid. Since the non-negligible modules are after all the tilting representations of $U_q(\mathfrak{g})$ with positive quantum dimension, to construct this groupoid one is led to start with the semisimple quotient of $U_q(\mathfrak{g})$ determined by such representations.

The first construction of this kind has been done in the physics literature by Mack and Schomerus in the early 90s, who were motivated by certain models of rational conformal field theory [29, 30, 31]. Specifically, they started with the quotient of $U_q(\mathfrak{s}l_2)$ alluded to above and showed that truncation of tensor products of non-negligible modules leads to a non-unital and quasi-coassociative coproduct. Furthermore, they introduced the general notion of a weak quasi-Hopf C^*–algebra, as a generalisation to the non-unital case of the quasi-Hopf algebras previously introduced by Drinfeld [12].

To the best of our knowledge in the last two decades there has not been any progress in extending Mack and Scomerus construction to pairs $(\mathfrak{g}, \ell)$ for other Lie algebras. Notice that such a generalisation would not be obvious, as the construction of the coproduct in the $\mathfrak{s}l_2$ case relies on the special property of that Lie algebra that tensor products of all the irreducible representations are multiplicity free. Furthermore, not much is known about the relation between Mack and Schomerus groupoid and the quotient category of $\mathfrak{s}l_2$, although such a relation is certainly expected.

This may be partially explained by the fact that most later studies have focused on the coassociative weak Hopf algebras of [6, 7, 34]. Indeed, soon after their introduction such algebras were shown to cover several physical models, including orbifold models previously described by genuine quasi-Hopf structures, see e.g. [10, 6, 15, 36, 39, 33]. Furthermore, every semisimple fusion category of a rather general kind was shown to correspond to a weak Hopf algebra [20, 46, 35]. We should however remark that when applied to the fusion categories $\mathcal{F}$, the relation between the reconstructed weak Hopf algebra and the original Lie algebra is not apparent. For example, the set of dimensions of the base coefficient algebras is unbounded if the order of the root of unity becomes large. This is due to the fact that Hayashi-Szlachanyi duality relies on the construction of an (associative) embedding functor which does not reproduce the above truncation procedure.

Our groupoids turn out to satisfy axioms of the weak quasi-Hopf C^*-algebras, and we shall interchangeably refer to them in this way or as quasi-coassociative C^*–quantum groupoids. More in detail, they are finite dimensional C^*-algebras endowed with a quasi-coassociative and non-unital coproduct and a multiplicative counit. They also have an invertible antimultiplicative antipode, and an R-matrix. Coassociativity failure is described by a Drinfeld associator explicitly derived from an iterative truncation procedure associated to powers of the vector representation. Tensorial properties of the negligible modules in the tilting category reflect in rather strong properties of the associator that we describe in Sect. 9. For example, it is an idempotent. Counit multiplicativity has the advantage that representations act on vector spaces rather than bimodules. Furthermore, since our groupoids are built from the classical group using only representations of the non-semisimple counterpart of positive quantum dimensions, they naturally approximate the classical Lie group in the sense described above.

We next explain our approach. While there is in general no canonical choice of truncated tensor products, Wenzl was able to make one for tensor products $V_\lambda \otimes V$ of an irreducible V_λ

of $U_q(\mathfrak{g})$ in the open alcove by a suitable generating representation V, specifically chosen for each Lie type (fundamental representation). He then proved Kirillov conjecture by showing that this subspace is a Hilbert space under the restriction of an hermitian form obtained as a deformation of the usual tensor product structure via the R-matrix. The corresponding projection then becomes selfadjoint. We are interested in a coherent iterative choice of a sequence of such projections $p_n : V^{\otimes n} \to V^{\underline{\otimes} n}$ onto truncated tensor powers of V. One thus obtains a natural functor

$$W : \mathcal{F} \to \mathcal{H}$$

from the fusion C^*–category to the Hilbert spaces taking a tensor power $V^{\otimes n}$ to the range of p_n.

We adapt Tannakian methods for quantum groups to the functor W. In other words, we pass to the dual viewpoint and talk of comodules. Notice that Wenzl's functor is not a tensor *–functor into the Hilbert spaces in the usual sense [32]. Even compared to more general Tannakian reconstructions (e.g. for weak Hopf algebras [46], or ergodic actions of compact quantum groups [38]), there are independent new obstructions. First, the usual tensor product of Hilbert space representations is not a representation on the tensor product Hilbert space, already if q is not a root of unity. This is due to the anticomultiplicativity property of the involution of $U_q(\mathfrak{g})$. This means that the construction of the adjoint can not be treated along lines similar, e.g., to those for compact quantum groups [51], as the involution is comultiplicative for the latter. At roots of unity, a second obstacle is associativity failure of W and it is due to the truncation procedure. Indeed, Wenzl's projections p_n fail associativity already for $\mathfrak{sl}_2$ at the smallest root, in agreement with Mack and Schomerus observation, cf. Ex. 5.2. In essence, our contribution may be regarded as the construction of a tensor structure of a weak type on Wenzl's functor. Specifically, it turns out to be a weak quasi-tensor structure in the sense of [19], but of a rather special kind, in that it satisfies suitable weak tensoriality relations which are generalizations of an ordinary notion of weak tensor functor relying on coassociativity relations. This viewpoint will be clarified in [9] where we put the results of this paper in an abstract framework.

To obtain this, we replace associativity failure of W by certain remarkable properties of the negligible modules of the tilting category, due to Andersen [3] and emphasised by Gelfand and Kazhdan [17] in their quotient construction. We show that these properties play a role also in our case: they allow us to construct a coassociative but non-associative bi-algebra $\mathcal{D}(V, \ell)$ acting as a universal algebra of a vector space. We endow it with an involution induced by Drinfeld's coboundary associated to the so called unitarized R-matrix $\overline{R}$ of [12]. We next introduce the 'function algebra' $\mathcal{C}(G, \ell)$ as a quotient of $\mathcal{D}(V, \ell)$ by a coideal induced by the fusion category which, by our coherent choice of the p_n is also an ideal, but only of one-sided type. Hence $\mathcal{C}(G, \ell)$ is naturally only a coalgebra with involution. The problem becomes that of making it into an algebra. Notice that this corresponds to the most delicate point in Mack and Schomerus approach, namely the construction of a coproduct.

We observe that our problem can be reduced to the question of cosemisimplicity of $\mathcal{C}(G, \ell)$. Specifically, we show that if we know that $\mathcal{C}(G, \ell)$ is cosemisimple then one can derive a non-associative algebra structure on it as a pull back of the product of $\mathcal{D}(V, \ell)$. But more is true: the dual groupoid $\widehat{\mathcal{C}(G, \ell)}$ can be made into a weak quasi-Hopf C^*–algebra with R-matrix with representation category equivalent to $\mathcal{F}$. The associator is explicitly described by associativity failure of Wenzl's projections p_n. It has the virtue of being an idempotent. Our second main result is that $\mathcal{C}(G, \ell)$ is indeed cosemisimple for $G = \mathrm{SU}(N)$. This is done via the construction of a suitable filtration of $\mathcal{C}(G, \ell)$ in the general case, for which an (even associative) algebra structure is naturally defined, and of a Haar functional for that filtration in the type A case. This functional is in turn achieved via an analysis on the Weyl filtrations of the negligible summands of the tensor powers $V^{\otimes n}$ of the fundamental up to a specific value of n specified by the conjugation structure of the representation category of $\mathrm{SU}(N)$.

We should remark that because of space limitations, cosemisimplicity of $\mathcal{C}(G,\ell)$ for Lie types other than A has not been considered in this paper, but will be taken up in the future. A positive answer would allow us to extend our main result to such groups provided $\mathfrak{g}\neq E_8$.

The paper is organised as follows. Sect. 2 is dedicated to preliminaries on quantum groups at roots of unity and the associated fusion categories. In Sect. 3 we recall Drinfeld's unitarized R-matrix and the associated coboundary. In Sect. 4 we recall Kirillov-Wenzl's theory mostly following Wenzl's approach, while in Sect. 5 we construct Wenzl's functor. We dedicate Sect. 6 to Tannakian reconstruction of the function algebra quantum group in the generic case (i.e. q is not a root of unity), while in Sect. 7 we perform the construction of the universal non-associative algebra $\mathcal{D}(V,\ell)$ at roots of unity and we show multiplicativity of the coproduct for all $\mathfrak{g}\neq E_8$. Section 8 deals with the construction of the involutive function coalgebra $\mathcal{C}(G,\ell)$ and an associated filtration endowed with an associative product. In Sect. 9 we construct quasi-coassociative dual C^*-groupoids $\widehat{\mathcal{C}(G,\ell)}$ under the assumption of cosemisimplicity of $\mathcal{C}(G,\ell)$, we study Drinfeld's associator, verify the quasi-Hopf C^*-algebra axioms, and show the mentioned equivalence of tensor C^*–categories. Finally, in Sect. 10 we develop a cosemisimplicity condition for $\mathcal{C}(G,\ell)$ involving a Haar functional on the filtration, and we verify its validity for $\mathfrak{g}=\mathfrak{sl}_N$. We include an appendix where we explicit the generator for the fusion C^*–categories of $\mathfrak{sl}_N$ and the conjugation structure.

2. Quantum groups at roots of unity

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $\alpha_1,\dots,\alpha_r$ a set of simple roots, and $A=(a_{ij})$ the associated Cartan matrix. Consider the unique invariant symmetric and bilinear form on $\mathfrak{h}^*$ such that $\langle\alpha,\alpha\rangle=2$ for a short root α. Let E be the real vector space generated by the roots endowed with its euclidean structure $\langle x,y\rangle$. Let Λ be the weight lattice of E and Λ^+ the cone of dominant weights.

Consider the complex *–algebra $\mathbb{C}[x,x^{-1}]$ of Laurent polynomials with involution $x^*=x^{-1}$, and let $\mathbb{C}(x)$ be the associated quotient field, endowed with the involution naturally induced from $\mathbb{C}[x,x^{-1}]$. We consider Drinfeld-Jimbo quantum group $U_x(\mathfrak{g})$, i.e. the algebra over $\mathbb{C}(x)$ defined by generators E_i, F_i, K_i, K_i^{-1}, $i=1,\dots,r$, and relations

$$K_iK_j=K_jK_i,\quad K_iK_i^{-1}=K_i^{-1}K_i=1,$$
$$K_iE_jK_i^{-1}=x^{\langle\alpha_i,\alpha_j\rangle}E_j,\quad K_iF_jK_i^{-1}=x^{-\langle\alpha_i,\alpha_j\rangle}F_j,$$
$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{x^{d_i}-x^{-d_i}},$$
$$\sum_0^{1-a_{ij}}(-1)^kE_i^{(1-a_{ij}-k)}E_jE_i^{(k)}=0,\quad \sum_0^{1-a_{ij}}(-1)^kF_i^{(1-a_{ij}-k)}F_jF_i^{(k)}=0,\quad i\neq j,$$

where $d_i=\langle\alpha_i,\alpha_i\rangle/2$, and, for $k\geq 0$, $E_i^{(k)}=E_i^k/[k]_{d_i}!$, $F_i^{(k)}=F_i^k/[k]_{d_i}!$. Note that d_i is an integer, hence so is every inner product $\langle\alpha_i,\alpha_j\rangle$. Quantum integers and factorials are defined in the usual way, $[k]_x=\frac{x^k-x^{-k}}{x-x^{-1}}$; $[k]_x!=[k]_x\dots[2]_x$, $[k]_{d_i}=[k]_{x^{d_i}}$, and result selfadjoint scalars of $\mathbb{C}(x)$. There is a unique *–involution on $U_x(\mathfrak{g})$ making it into a *–algebra over $\mathbb{C}(x)$ such that

$$K_i^*=K_i^{-1},\quad E_i^*=F_i.$$

This algebra becomes a Hopf algebra, i.e. a coassociative coalgebra with coproduct Δ, counit ε and antipode S defined, e.g, as in [49], where his $\tilde{K}_i$ corresponds to our K_i.

One has the following relations between coproduct, antipode and involution for $a\in U_x(\mathfrak{g})$,

$$\Delta(a^*)=\Delta^{\rm op}(a)^*,\tag{2.1}$$
$$\varepsilon(a^*)=\overline{\varepsilon(a)}.\tag{2.2}$$
$$S(a^*)=S(a)^*,\tag{2.3}$$
$$S^2(a)=K_{2\rho}^{-1}aK_{2\rho},\tag{2.4}$$

where $\Delta^{\rm op}$ is the coproduct opposite to Δ, 2ρ the sum of the positive roots, and, for an element $\alpha = \sum_i k_i\alpha_i$ of the root lattice, $K_\alpha := K_1^{k_1} \dots K_r^{k_r}$.

2.1. Remark Notice that $U_x(\mathfrak{g})$ is not a Hopf *–algebra in the sense of, e.g., [8], where (2.1)–(2.3) are replaced by $\Delta(a^*) = \Delta(a)^*$ and $S(a^*) = S^{-1}(a)^*$.

We next consider Lusztig's integral form [27]. In order to construct braided tensor categories, it is well known that we need to embed the original algebra into a larger algebra, via a procedure involving, among other things, extension of scalars, see [45] for details. For the purposes of this paper (see the next section) we shall actually need a further extension, and the correct polynomial ring for the integral form will be

$$\mathcal{A} := \mathbb{C}[x^{1/2L}, x^{-1/2L}],$$

with L the smallest positive integer such that $L\langle\lambda, \mu\rangle \in \mathbb{Z}$ for all dominant weights λ, μ. The explicit values are listed in [45] for all Lie types. For example, $L = N$ for $\mathfrak{g} = \mathfrak{s}l_N$.

We define the integral form $\mathcal{U}_\mathcal{A}$ as the $\mathcal{A}$–subalgebra generated by the elements $E_i^{(k)}$, $F_i^{(k)}$ and K_i. This is known to be a *–invariant Hopf $\mathcal{A}$–algebra with the structure inherited from $U_x(\mathfrak{g})$. Notice that, in connection with the mentioned extension needs, $\mathcal{U}_\mathcal{A}$ is not quasi triangular, even topologically, but the representation categories that we next consider will be braided tensor categories, and this will suffice for our purposes.

We fix $q \in \mathbb{T}$, and consider the *–homomorphism $\mathcal{A} \to \mathbb{C}$ which evaluates every polynomial in q, and form the tensor product *–algebra,

$$U_q(\mathfrak{g}) := \mathcal{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{C},$$

which becomes a complex Hopf algebra with a *–involution, and properties (2.1)–(2.4) still hold for $U_q(\mathfrak{g})$.

Given a dominant weight λ of $\mathfrak{g}$, we can associate various modules $V_\lambda(x)$, $V_\lambda(\mathcal{A})$, and $V_\lambda(q)$ of $U_x(\mathfrak{g})$, $\mathcal{U}_\mathcal{A}$ and $U_q(\mathfrak{g})$, respectively, usually called Weyl modules, and thus form corresponding representation categories as follows. We shall mostly be interested in $V_\lambda(q)$ that we shall eventually simply denote by V_λ as well.

a) The category Rep$(\mathcal{U}_\mathcal{A})$

Let $V_\lambda(x)$ be the irreducible representation of $U_x(\mathfrak{g})$ with highest weight λ and let v_λ be the highest weight vector of $V_\lambda(x)$ and form the cyclic module of $\mathcal{U}_\mathcal{A}$ generated by v_λ,

$$V_\lambda(\mathcal{A}) := u_\lambda(\mathcal{U}_\mathcal{A})v_\lambda.$$

It is known to be a free $\mathcal{A}$–module satisfying

$$V_\lambda(\mathcal{A}) \otimes_\mathcal{A} \mathbb{C}(x) = V_\lambda(x).$$

We denote by Rep$(\mathcal{U}_\mathcal{A})$ the linear category over $\mathcal{A}$ with objects finite tensor products of modules $V_\lambda(\mathcal{A})$. It becomes a tensor $\mathcal{A}$–category in the natural way.

b) The tilting category $\mathcal{T}(\mathfrak{g}, \ell)$

Every module $V_\lambda(\mathcal{A})$ gives rise to the complex $U_q(\mathfrak{g})$-modules via specialisation at a complex number q:

$$V_\lambda(q) := V_\lambda(\mathcal{A}) \otimes_\mathcal{A} \mathbb{C}.$$

This is obviously a cyclic module, but it is not always irreducible if q is a root of unity. The *linkage principle* gives information on irreducibility at primitive roots of unity. Briefly, one needs to consider the *affine* Weyl group, and its *translated* action on the real vector space E spanned by the roots, defined by $w.x = w(x+\rho)-\rho$. The structure of this group depends on the parity and divisibility by $d := \max d_i$ of the order m of q [45]. In this paper we are interested in the case where m is divisible by $2d$, and more specifically we take $q = e^{\pi i/d\ell}$, as in this case one obtains tensor C^*–categories [49, 52]. The affine Weyl group, W_ℓ, is generated by the ordinary

Weyl group W and translations by $\ell\theta$, where θ is the highest root. The translated action admits a fundamental domain, called the principal Weyl alcove, that intersects Λ^+ in the set

$$\overline{\Lambda}_\ell := \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle \leq d\ell\}.$$

The linkage principle then implies that $V_\lambda(q)$ are irreducible for $\lambda \in \overline{\Lambda}_\ell$. Moreover, they are pairwise inequivalent. We refer to [3, 45, 8] for complete explanations.

We follow [49] for the construction of the tilting category. Namely, we fix, for each Lie type, a representation V of $\mathfrak{g}$ taken from a specific list, that we call *fundamental*. For example, in the type A case, V is the vector representation. Each fundamental representation has in particular the property that every irreducible of $\mathfrak{g}$ is contained in some tensor power of V.

One can form the category $\mathcal{T}(\mathfrak{g}, \ell)$, also denoted $\mathcal{T}_\ell$ for brevity, with objects finite tensor powers of $V(q)$ and arrows the intertwining operators, completed with subobjects (i.e. summands) and direct sums.

This is a strict tensor but non-semisimple category. It is known [49] that the objects of $\mathcal{T}_\ell$ are tilting modules in the sense of Andersen [3], and that conversely, for ℓ large enough every tilting module is isomorphic to an object of $\mathcal{T}_\ell$. More precisely, one needs an order ℓ such that $\kappa \in \Lambda_\ell$, where κ is the dominant weight of V and

$$\Lambda_\ell := \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle < d\ell\}.$$

Tilting modules were originally defined as those modules W admitting, together with their dual, a Weyl filtration, i.e. a sequence of modules $\{0\} \subset W_1 \subset \cdots \subset W$ such that W_{i+1}/W_i is isomorphic to a Weyl module $V_{\lambda_i}(q)$ with $\lambda_i \in \Lambda^+$. Weyl filtrations are non-unique, but for all filtrations of W the number of successive factors isomorphic to a given $V_\lambda(q)$ is unique, and it is in fact given by the multiplicity of $V_\lambda(x)$ in $W(x)$ if W is obtained from a specialisation $x \to q$ of a module $W(x)$ of $U_x(\mathfrak{g})$, see Prop. 3 and Remark 2 in [45] for a precise statement. This in particular implies that the multiplicities of the dominant weights of the factors in the Weyl filtrations of tensor products $V_{\lambda_1}(q) \otimes \cdots \otimes V_{\lambda_n}(q)$ with $\lambda_i \in \overline{\Lambda}_\ell$ (or more generally of tilting modules) are the same as those determined by decomposition into irreducibles of the corresponding tensor product in the classical (or generic) case.

c) The quotient category $\mathcal{F}_\ell$

We next briefly recall the construction of the semisimple quotient of $\mathcal{T}_\ell$, henceforth written $\mathcal{F}_\ell$, following the approach of Gelfand and Kazhdan [17]. Every object of $\mathcal{T}_\ell$ decomposes as a direct sum of indecomposable submodules, and this decomposition is unique up to isomorphism. One can form two full linear (non-tensorial) subcategories, $\mathcal{T}^0$, and $\mathcal{T}^\perp$ of $\mathcal{T}_\ell$, with objects, respectively, those representations which can be written as direct sums of $V_\lambda(q)$, with $\lambda \in \Lambda_\ell$ only, and those which have no such $V_\lambda(q)$ as a direct summand.

2.2. Definition The objects of $\mathcal{T}^\perp$ and $\mathcal{T}^0$ are called negligible and non-negligible, respectively. An arrow $T : W \to W'$ of $\mathcal{T}_\ell$ is called negligible if it factors through $W \to N \to W'$ with N negligible.

The category $\mathcal{T}^\perp$ of negligible modules satisfies the following properties, first shown by Andersen [3], and abstracted in [17]

(1) Any object $W \in \mathcal{T}_\ell$ is isomorphic to a direct sum $W \simeq W_0 \oplus N$ with $W_0 \in \mathcal{T}^0$ and $N \in \mathcal{T}^\perp$.

(2) For any pair of arrows $T : W_1 \to N$, $S : N \to W_2$ of $\mathcal{T}_\ell$, with $N \in \mathcal{T}^\perp$, $W_i \in \mathcal{T}^0$, then

$$ST = 0.$$

(3) For any pair of objects $W \in \mathcal{T}_\ell$, $N \in \mathcal{T}^\perp$, then $W \otimes N$ and $N \otimes W \in \mathcal{T}^\perp$.

Property (1) follows immediately from the mentioned decomposition of objects of $\mathcal{T}_\ell$, while property (2) means that no non-negligible module can be a summand of a negligible one (however, it can be a factor of a Weyl filtration of a negligible).

Let Neg(W, W') be the subspace of negligible arrows of (W, W'). Then the quotient category, $\mathcal{F}_\ell$, is the category with the same objects as $\mathcal{T}_\ell$ and arrows between the objects W and W' the quotient space,

$$(W, W')_{\mathcal{F}_\ell} := (W, W')/\text{Neg}(W, W').$$

Gelfand and Kazhdan endow $\mathcal{F}_\ell$ with the unique structure of a tensor category such that the quotient map $\mathcal{T}_\ell \to \mathcal{F}_\ell$ is a tensor functor. The tensor product of objects and arrows of $\mathcal{F}_\ell$, is usually denoted by $W \underline{\otimes} W'$ and $S \underline{\otimes} T$ respectively, and referred to as the truncated tensor product in the physics literature. This is now a semisimple tensor category and $\{V_\lambda, \lambda \in \Lambda_\ell\}$ is a complete set of irreducible objects.

2.3. Remark By Lemma 1.1 in [17], composition of inclusion $\mathcal{T}^0 \to \mathcal{T}_\ell$ with projection $\mathcal{T}_\ell \to \mathcal{F}_\ell$ is an equivalence of linear categories. Hence $\mathcal{T}^0$ becomes a semisimple tensor category as well tensor equivalent to $\mathcal{F}_\ell$. This procedure may be understood as a categorification of the following visualisation of truncated tensor products at the level of Grothendieck rings. For λ, $\mu \in \Lambda_\ell$, one can decompose $V_\lambda \otimes V_\mu$ uniquely up to isomorphism in $\mathcal{T}_\ell$,

$$V_\lambda \otimes V_\mu \simeq \oplus_{\nu \in \Lambda_\ell} m^\nu_{\lambda,\mu} V_\nu \oplus N,$$

with N negligible. Then in $\mathcal{F}_\ell$,

$$V_\lambda \underline{\otimes} V_\mu \simeq \oplus_{\nu \in \Lambda_\ell} m^\nu_{\lambda,\mu} V_\nu.$$

Notice that, although unique up to isomorphism, the decomposition of $V_\lambda \otimes V_\nu$ described in $\mathcal{T}_\ell$ is not canonical (cf. also Sect. 11.3C in [8] and references therein.)

3. Ribbon and coboundary structures

The main topic of this section is the coboundary structure of the category Rep$(\mathcal{U}_{\mathcal{A}})$ associated to the ribbon structure. It was first introduced by Drinfeld [12] for different purposes and used by Wenzl to obtain a variant of Kirillov inner product [49]. We shall later need it as well for the construction of the *–involution of the groupoid.

In order to obtain ribbon Hopf algebras, one needs to pass to a suitable extension of $\mathcal{U}_{\mathcal{A}}$. We refrain from explicitly recalling the construction of R and we refer to [45] for a detailed treatment, but our notation will agree with [49]. The R matrix then lies in a suitable topological and algebraic completion of the of the tensor product algebra by itself. As anticipated in the previous section, we slightly modify this construction in that we further extend the scalars to $\mathcal{A} = \mathbb{C}[x^{1/2L}, x^{-1/2L}]$. This will allow the construction of a central square root of the ribbon element in the completed algebra which will be useful later for the hermitian structure.

We recall the main algebraic properties of R [13],

$$\Delta^{\text{op}}(a) = R\Delta(a)R^{-1}, \tag{3.1}$$

$$1 \otimes \Delta(R) = R_{13}R_{12}, \tag{3.2}$$

$$\Delta \otimes 1(R) = R_{13}R_{23}. \tag{3.3}$$

Relations (3.1)–(3.3) mean that for any pair of representations u, v of Rep$(\mathcal{U}_A)$, the operators $\varepsilon_{u,v} := \Sigma u \otimes v(R)$, with Σ the flip map, are intertwiners of the category satisfying naturality in u and v. Hence $\varepsilon_{u,v}$ is a braided symmetry for the category. In particular, the Yang-Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

follows, see e.g. [47] for details.

There is an associated ribbon element which is a central invertible element v in the completed Hopf algebra such that

$$R_{21}R = v \otimes v\Delta(v^{-1}),$$

and it is given by $v = K_{2\rho}u$, where $u = m(S \otimes 1(R_{21}))$, and often called the quantum Casimir operator. The element u was originally introduced in [13] the notion of ribbon Hopf algebras and ribbon tensor category is due to Reshetikhin and Turaev [40, 41]. For the following important result, see Theorem 3 in [45] and references therein.

3.1. Theorem *The category Rep*$(\mathcal{U}_\mathcal{A})$ *is a ribbon tensor category. Hence so are* $\mathcal{T}_\ell$ *and* $\mathcal{F}_\ell$.

We next consider the relation between the R matrix and the *–involution. Wenzl proves the following [49],

$$R^* = {R_{21}}^{-1}.$$

On the other hand it is known that the inverse of u can be computed as

$$u^{-1} = m(S^{-1} \otimes 1((R^{-1})_{21})).$$

It follows that

$$u^* = m(1 \otimes S(R^*)) = m(S^{-1} \otimes 1(R^*)) = u^{-1},$$

for the second equality, see Lemma 2.1.1 in [47]. Therefore

$$v^* = v^{-1}$$

as well. The action of v on an irreducible module is derived from the expression of the R matrix, and it is given by scalar multiplication by $x^{-\langle\lambda,\lambda+2\rho\rangle}$ on a highest weight module of weight λ, [13], see also see [49, 8]. By [12] one can construct a central square root w lying in the topological completion of the extension of $\mathcal{U}_\mathcal{A}$. It acts as the scalar $x^{-\frac{1}{2}\langle\lambda,\lambda+2\rho\rangle}$ on the same module. One thus has:

$$w^2 = v,$$
$$w^* = w^{-1}.$$

It follows that one can compute a square root of $R_{21}R$,

$$(R_{21}R)^{1/2} = w \otimes w\Delta(w^{-1}).$$

Drinfeld used the element w to construct the so called unitarized R matrix $\overline{R}$ as follows. Set

$$\Theta := (R_{21}R)^{-1/2} = w^{-1} \otimes w^{-1}\Delta(w),$$

and define

$$\overline{R} := R\Theta = Rw^{-1} \otimes w^{-1}\Delta(w).$$

We recall the following relations.

3.2. Lemma

a) $\Theta^* = (\Theta)_{21}^{-1}$,
b) $\Theta_{21}R = R\Theta$.

proof a) $\Theta^* = \Delta(w)^* w \otimes w = \Delta^{\mathrm{op}}(w^*)w \otimes w = w \otimes w\Delta^{\mathrm{op}}(w^{-1}) = \Theta_{21}^{-1}$. b) By (3.1) and centrality of w,

$$\Theta_{21}R = w^{-1} \otimes w^{-1}\Delta^{\mathrm{op}}(w)R = w^{-1} \otimes w^{-1}R\Delta(w) = Rw^{-1} \otimes w^{-1}\Delta(w) = R\Theta.$$

3.3. Proposition *One has:*

$$\overline{R}^* = (\overline{R})_{21}^{-1} = \overline{R}.$$

In particular, $\overline{R}$ *is selfadjoint.*

Proof By a) of the previous lemma,

$$\overline{R}^* = (R\Theta)^* = \Theta^* R^* = (\Theta)_{21}^{-1} R_{21}^{-1} = (R_{21}\Theta_{21})^{-1} = (\overline{R})_{21}^{-1}.$$

Furthermore, by b) of the same lemma,

$$\overline{R}_{21}\overline{R} = (R\Theta)_{21}R\Theta = R_{21}\Theta_{21}R\Theta = R_{21}R\Theta^2 = 1.$$

The Yang-Baxter equation can equivalently be read as an associativity property for the R matrix:

$$R_{12}\Delta \otimes 1(R) = R_{23}1 \otimes \Delta(R).$$

One can thus unambiguously define elements in the iterated tensor powers of the ribbon Hopf algebra,

$$R^{(n+1)} := R^{(n)} \otimes 1\Delta^{(n-1)} \otimes 1(R) = 1 \otimes R^{(n)} 1 \otimes \Delta^{(n-1)}(R)$$

where $R^{(2)} := R$. We set, for $n > 2$,

$$\Theta^{(n)} := w^{-1} \otimes \cdots \otimes w^{-1}\Delta^{(n-1)}(w)$$

and note that it is associative as well:

$$\Theta^{(n)} \otimes 1\Delta^{(n-1)} \otimes 1(\Theta) = w^{-1} \otimes \cdots \otimes w^{-1} \otimes 1\Delta^{(n-1)}(w) \otimes 1\Delta^{(n-1)} \otimes 1(w^{-1} \otimes w^{-1}\Delta(w)) =$$
$$\Theta^{(n+1)} = 1 \otimes \Theta^{(n)} 1 \otimes \Delta^{(n-1)}(\Theta).$$

We also set:

$$\overline{R}^{(n)} := R^{(n)}\Theta^{(n)}.$$

3.4. Lemma *The following pairs of operators commute:*

a) $\Delta^{(n)} \otimes 1(R)$ *and* Θ_{12},
b) $1 \otimes \Delta^{(n)}(R)$ *and* Θ_{23},
c) $\Delta^{(n-1)} \otimes 1(R)$ *and* $\Theta^{(n)} \otimes 1$,
d) $1 \otimes \Delta^{(n-1)}(R)$ *and* $1 \otimes \Theta^{(n)}$.

proof a) we need to show that $\Delta^{(n)} \otimes 1(R)$ commutes with $R_{21}R = \Theta^{-2}$. We explicit $\Delta^{(n)}$ by leaving Δ always on the left: $\Delta^{(2)} = \Delta \otimes 1 \circ \Delta$, $\Delta^{(3)} = \Delta \otimes 1 \otimes 1 \circ \Delta \otimes 1 \circ \Delta$, and so on. An iteration of (3.3) gives

$$\Delta^{(n)} \otimes 1(R) = R_{1\,n+2}R_{2\,n+2} \dots R_{n+1\,n+2}.$$

$R_{21}R_{12}$ obviously commutes with all the factors in this expression except the first two. For them, we use the twice the Yang-Baxter equation, the first time with 1 and 2 exchanged, in both cases 3 is replaced by $n+2$:

$$(R_{1\,n+2}R_{2\,n+2})(R_{21}R) = (R_{1\,n+2}R_{2\,n+2}R_{21})R = (R_{21}R_{2\,n+2}R_{1\,n+2})R_{12} =$$
$$R_{21}(R_{2\,n+2}R_{1\,n+2}R_{12}) = R_{21}(R_{12}R_{1\,n+2}R_{2\,n+2}) = (R_{21}R)(R_{1\,n+2}R_{2\,n+2})$$

c) follows from a) and the fact that

$$\Theta^{(n)} = \Theta\Delta \otimes 1(\Theta) \dots \Delta^{(n-2)} \otimes 1(\Theta).$$

b) and d) can be proved in a similar way.

3.5. Proposition $\overline{R}^{(n)}$ *is associative:*

$$\overline{R}^{(n+1)} = \overline{R}^{(n)} \otimes 1\Delta^{(n-1)} \otimes 1(\overline{R}) = 1 \otimes \overline{R}^{(n)} 1 \otimes \Delta^{(n-1)}(\overline{R}).$$

Proof This follows from properties c) and d) of the previous lemma.

3.6. Proposition $\overline{R}^{(n)}$ *is selfadjoint for all* n.

Proof For all n,

$$\Delta^{(n)}(a)^* = \Delta^{\mathrm{op}(n)}(a^*).$$

Furthermore, being w central, we may replace R with $\overline{R}$ in (2.6):

$$\Delta^{\mathrm{op}}(a)\overline{R} = \overline{R}\Delta(a).$$

It follows that, for all n,

$$\Delta^{\mathrm{op}(n)}(a)\overline{R}_{n+1} = \overline{R}_{n+1}\Delta^{(n)}(a),$$

where $\overline{R}_{n+1} := \Delta^{\mathrm{op}(n-1)} \otimes 1(\overline{R}) \dots \Delta^{\mathrm{op}} \otimes 1(\overline{R})\overline{R}$. We thus have, being $\overline{R}$ selfadjoint,

$$(\overline{R}^{(n)})^* = (\overline{R}\Delta \otimes 1(\overline{R}) \dots \Delta^{(n-2)} \otimes 1(\overline{R}))^* =$$

$$\Delta^{\mathrm{op}(n-2)}\otimes 1(\overline{R})\dots\Delta^{\mathrm{op}}\otimes 1(\overline{R})\overline{R}=\overline{R}_n=$$

$$\Delta^{\mathrm{op}(n-2)}\otimes 1(\overline{R})\overline{R}_{n-1}=\overline{R}_{n-1}\Delta^{(n-2)}\otimes 1(\overline{R}).$$

By induction on n, $\overline{R}^{(n-1)}=(\overline{R}^{(n-1)})^*=\overline{R}_{n-1}$. Inserting this information in the above computation gives

$$(\overline{R}^{(n)})^*=\overline{R}^{(n-1)}\Delta^{(n-2)}\otimes 1(\overline{R})=\overline{R}^{(n)}.$$

We next introduce Drinfeld's coboundary of $\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$. First consider, for any pair of objects u, $v\in\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$, the braiding operators:

$$\varepsilon(u,v)=\Sigma u\otimes v(R)\in(u\otimes v,v\otimes u).$$

As recalled above, these are natural isomorphisms in u and v and define a braided symmetry, in that

$$\varepsilon(u,w)\otimes 1_v\circ 1_u\otimes\varepsilon(v,w)=\varepsilon(u\otimes v,w),$$

$$1_v\otimes\varepsilon(u,w)\circ\varepsilon(u,w)\otimes 1_w=\varepsilon(u,v\otimes w).$$

Correspondingly, we consider the associated modified form

$$\sigma(u,v)=\Sigma u\otimes v(\overline{R})\in(u\otimes v,v\otimes u),$$

obviously still arrows of the category, However, they do not define a braided symmetry. Rather, one has the following properties, which are immediate consequences of the previous propositions.

3.7. Proposition *The arrows $\sigma(u,v)\in(u\otimes v,v\otimes u)$ of Rep$(\mathcal{U}_{\mathcal{A}})$ are natural isomorphisms satisfying*

$$\sigma(v,w)\circ\sigma(w,v)=1_{w\otimes v},\tag{3.4}$$

$$\sigma(v\otimes u,w)\circ\sigma(u,v)\otimes 1_w=\sigma(u,w\otimes v)\circ 1_u\otimes\sigma(v,w).\tag{3.5}$$

An abstract tensor category admitting arrows $\sigma(u,v)$ satisfying the properties stated in the last proposition, is called a coboundary category [12].

Notice that both sides of (3.5) define an intertwiner of the category which reverses the order in triple tensor products,

$$\sigma_3: v_1\otimes v_2\otimes v_3\to v_3\otimes v_2\otimes v_1.$$

More generally, we consider arrows

$$\sigma_n\in(v_1\otimes\cdots\otimes v_n,v_n\otimes\cdots\otimes v_1)$$

inductively defined by

$$\sigma_n:=\sigma(v_{n-1}\otimes\cdots\otimes v_1,v_n)\circ\sigma_{n-1}\otimes 1_{v_n}.$$

We explicit σ_n dropping the involved representations in the notation. Let Σ_n denote the permutation that reverses the order in a tensor product space with n factors.

3.8. Proposition

a) $\sigma_n=\Sigma_n\overline{R}_n$,
b) $\sigma_n=\sigma_{n-1}\otimes 1_{v_1}\circ\sigma(v_1,v_2\otimes\cdots\otimes v_n)=$
$\sigma(v_1,v_n\otimes\cdots\otimes v_2)\circ 1_{v_1}\otimes\sigma_{n-1}=$
$1_{v_n}\otimes\sigma_{n-1}\circ\sigma(v_1\otimes\cdots\otimes v_{n-1},v_n)$,
c) $\sigma_n^2=1$.

Proof a) By induction on n,

$$\sigma_n = \Sigma_{n-1,1}\Delta^{(n-1)} \otimes 1(\overline{R})\Sigma_{n-1}\overline{R}_{n-1} = \Sigma_{n-1,1}\Sigma_{n-1}\Delta^{\mathrm{op}(n-1)} \otimes 1(\overline{R})\overline{R}_{n-1} \otimes 1 =$$

$$\Sigma_n\Delta^{\mathrm{op}(n-1)} \otimes 1(\overline{R})\overline{R}_{n-1} \otimes 1 = \Sigma_n\overline{R}_n,$$

where $\Sigma_{m,n}$ is the permutation on a tensor product of $m+n$ factors that exchanges the first m factors with the remaining n, and Σ_n is the permutation that reverses the order in a tensor product space with n factors. b) One similarly shows that the right hand side of b) equals $\Sigma_n\overline{R}_n$. c) follows from b) and induction on n.

4. Kirillov-Wenzl theory

a) Hermitian structures on Rep$(\mathcal{U}_{\mathcal{A}})$

In this subsection we recall Kirillov's *–involution in Rep$(\mathcal{U}_{\mathcal{A}})$ making it into a tensor *–category, following Wenzl's approach.

By an hermitian space we shall mean a free finitely generated module H over the involutive algebra $\mathcal{A}$ endowed with a non degenerate, sesquilinear $\mathcal{A}$-valued form (ξ, η) on H, hermitian with respect to the involution of $\mathcal{A}$:

$$(\xi, \eta)^* = (\eta, \xi).$$

Given a linear map $T : H \to H'$ between hermitian spaces, one can define the adjoint $T^* : H' \to H$. The category $\mathcal{H}$ of hermitian spaces over $\mathcal{A}$ is a *–category, in the sense of [11].

By a *–representation of $\mathcal{U}_{\mathcal{A}}$ we mean a representation u of it on an hermitian space which preserves the involutions, $u(x)^* = u(x^*)$. Weyl modules can be made into *–representations as follows. Given a dominant weight λ, and the associated Weyl module $V_\lambda(x)$ at the generic level, we can form both the dual module $V_\lambda(x)^*$ and the conjugate module $\overline{V_\lambda(x)}$. In the former case, $U_x(\mathfrak{g})$ acts by transposition of the right action $\xi a := S^{-1}(a)\xi$ while in the latter the action is given on the conjugate vector space by $a\overline{\xi} := \overline{S^{-1}(a^*)\xi}$. These representations are still irreducible and have the same lowest weight, $-\lambda$. Hence a unique invertible intertwiner $\Phi : \overline{W_\lambda} \to W_\lambda^*$ arises from the identification of their natural lowest weight vectors, thereby defining the form:

$$(\xi, \eta) := \Phi(\overline{\xi})(\eta).$$

This form turns out hermitian and, when restricted to $V_\lambda(\mathcal{A})$, takes values in the corresponding base ring, hence it does make that module into a *–representation , see Sect. 2 in [49].

The main crux here is that if u and v are *–representations of $\mathcal{U}_{\mathcal{A}}$ then their tensor product representation $u \otimes v$ defined in the usual way by means of the coproduct, $u \otimes v := u \otimes v \circ \Delta$, is *not* a *–representation with respect to the natural product form,

$$(\xi \otimes \eta, \xi' \otimes \eta')_p := (\xi, \xi')(\eta, \eta').$$

This is due to the fact that the Δ is not *–preserving. However, the ribbon structure fixes the problem. Not only this, but it is the first key step towards the construction of unitary braid group representations, in the sense that unitarity is first achieved at the algebraic level, as we next recall.

4.1. Proposition *For any pair of *–representations u and v of $\mathcal{U}_{\mathcal{A}}$, the following form:*

$$(\xi \otimes \eta, \xi' \otimes \eta') := (\xi \otimes \eta, u \otimes v(\overline{R})\xi' \otimes \eta')_p \tag{4.1}$$

is hermitian and makes makes $u \otimes v$ into a *–representation.

Proof Recall that $\overline{R}$ is an invertible and selfadjoint element of the completed tensor product of the extended algebra by itself, and this implies that $u \otimes v(\overline{R})$ is a selfadjoint operator with respect to the product form. Hence the right hand side does define an hermitian form on the tensor product space. Moreover, the adjoints A^* and A^+ of an operator $A : U \otimes V \to U \otimes V$, with U and V the spaces corresponding to u and v, and computed with respect to the modified and product forms respectively, are related by

$$A^* = u\otimes v(\overline{R})^{-1}A^+ u\otimes v(\overline{R}).$$

It follows that for $a\in\mathcal{U}_{\mathcal{A}}$,

$$u\otimes v(a)^* = u\otimes v(\overline{R})^{-1}(u\otimes v\circ\Delta(a))^+ u\otimes v(\overline{R}) =$$

$$u\otimes v(\overline{R})^{-1}u\otimes v\circ\Delta^{\mathrm{op}}(a^*)u\otimes v(\overline{R}) = u\otimes v(a^*).$$

In order to simplify notation, we shall switch from representation to module notation when this will cause no confusion. Hence we identify a representation u with its space U, and simply denote by $a\xi$ the action of the operator $u(a)$ on a vector ξ.

We now make all objects of $\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$ into *–representations in the way just explained. In other words, we endow a tensor product module $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ with the form defined by the action of the matrix $\overline{R}^{(n)}$,

$$(\xi,\eta) := (\xi,\overline{R}^{(n)}\eta)_p,$$

which is indeed hermitian by selfadjointness of $\overline{R}^{(n)}$, cf. Proposition 3.6. We can thus compute the adjoint of every arrow of $\mathrm{Rep}(\mathcal{U}_{\mathcal{A}})$.

4.2. Theorem *The hermitian forms so defined on objects of Rep($\mathcal{U}_{\mathcal{A}}$) make it into a ribbon tensor *–category. Furthermore, both the braiding operators $\varepsilon(U,V)=\Sigma R\in(U\otimes V,V\otimes U)$ and the coboundary operators $\sigma(U,V)=\Sigma\overline{R}\in(U\otimes V,V\otimes U)$ are unitary arrows of Rep($\mathcal{U}_{\mathcal{A}}$).*

Proof We need to verify that the tensor product makes the set of objects into a (unital) associative semigroup and that the tensor product of arrows satisfies $(S\otimes T)^*=S^*\otimes T^*$. In regards to the former, a computation shows that the hermitian form of $(U\otimes V)\otimes W$ is defined by the action of $\overline{R}\otimes 1\Delta\otimes 1(\overline{R})$, while that for $U\otimes(V\otimes W)$ corresponds to $1\otimes\overline{R}1\otimes\Delta(\overline{R})$, and these two coincide by Proposition 3.5. Let now $S\in(U,U')$, $T\in(V,V')$. Using (4.2) we get

$$(S\otimes T)^* = \Theta^{-1}R^{-1}S^*\otimes T^*R\Theta = S^*\otimes T^*.$$

We next check unitarity of $\varepsilon(U,V)$,

$$\varepsilon(U,V)^* = \overline{R}^{-1}(\Sigma R)^+\overline{R} = \Theta^{-1}R^{-1}R^+\Sigma R\Theta =$$

$$\Theta^{-1}R^{-1}R_{21}^{-1}\Sigma R\Theta = \Theta^{-1}R^{-1}\Sigma\Theta = (R\Theta)^{-1}\Theta_{21}\Sigma =$$

$$(\Theta_{21}R)^{-1}\Theta_{21}\Sigma = \varepsilon(U,V)^{-1}$$

by Lemma 3.2 b). Finally, since $\sigma(U,V)=\varepsilon(U,V)\Theta$, it suffices to show that $\Theta\in(U\otimes V,U\otimes V)$ is unitary. By Lemma 3.2 again,

$$\Theta^* = \overline{R}^{-1}(\Theta)^+\overline{R} = \overline{R}^{-1}(\Theta)_{21}^{-1}\overline{R} =$$

$$\Theta^{-1}(\Theta_{21}R)^{-1}R\Theta = \Theta^{-1}(R\Theta)^{-1}R\Theta = \Theta^{-1}.$$

b) C^*–structures at roots of unity

We next recall the main results of [49]. The hermitian form previously considered on Weyl modules $V_\lambda(\mathcal{A})$ specialises to any $q\in\mathbb{T}$, hence induces a complex sesquilinear and hermitian form on $V_\lambda(q)$.

4.3. Theorem *Let $q\in\mathbb{T}$, then the hermitian form of $V_\lambda(\mathcal{A})$ specialises to a positive definite form on $V_\lambda(q)$ in the following cases,*

a) *for $\lambda\in\Lambda^+$ satisfying $\langle\lambda+\rho,\theta\rangle<d+\frac{1}{|t|}$ if $q=e^{i\pi t}$ is not a root of unity,*
b) *for $\lambda\in\overline{\Lambda}_\ell$ if $q=e^{\pi i/d\ell}$.*

Let $V = V_\kappa$ denote the fundamental representation of $\mathfrak{g}$ as in [49], with κ its dominant weight. For example, if $\mathfrak{g}$ is of type A or C, V is the vector representation, for types B is the spin representation, for type D is one of the two spin representations, V_{κ_1}, V_{κ_2}. The order of the root of unity is chosen in such a way that $V(q)$ is non-negligible, i.e., $\kappa \in \Lambda_\ell$. This representation ($V_{\kappa_1} \oplus V_{\kappa_2}$ in type D) has the property that its tensor powers contain every irreducible of $\mathfrak{g}$.

We summarise properties concerning fusion of tensor products with the fundamental. We add a few remarks in the proof aiming to connect various results known in the literature.

4.4. Theorem *Let V be the fundamental representation of $\mathfrak{g} \neq E_8$, ($V = V_{\kappa_i}$ in the type D case) and pick $\lambda \in \Lambda_\ell$. Then*

a) *all irreducible submodules V_μ of $V_\lambda \otimes V$ in Rep$(\mathfrak{g})$ have weights $\mu \in \overline{\Lambda_\ell}$,*
b) *the maximal negligible and non-negligible summands $V_\lambda(q) \otimes V(q)$ in $\mathcal{T}_\ell$ are unique and given by*

$$N_\lambda = \bigoplus_{\mu \in \overline{\Lambda_\ell} \setminus \Lambda_\ell} m_\mu V_\mu, \qquad V_\lambda \underline{\otimes} V = \bigoplus_{\mu \in \Lambda_\ell} m_\mu V_\mu,$$

with multiplicities as in the classical case. Specifically, both decompositions are multiplicity free for $\mathfrak{g} \neq F_4$; while N_λ is so for $\mathfrak{g} = F_4$,
c) *if $\mathfrak{g}$ is a classical Lie algebra, V is minuscule hence fusion rules are given by*

$$m_\mu = 1 \iff \mu \in E_\lambda := \{\lambda + \gamma : \gamma \text{ weight of } V\} \cap \Lambda^+.$$

Proof a) and b) are Theorem 3.5. in [49]. Notice that on one hand, the dominant weights of the Weyl modules appearing as factors in the Weyl filtration of an indecomposable submodule T_μ with maximal weight μ of $V_\lambda(q) \otimes V(q)$ must belong to the set of dominant weights arising from the classical decomposition of $V_\lambda \otimes V$ see e.g. Prop. 3 in [45]. On the other, the Weyl modules $V_\mu(q)$ with $\mu \in \overline{\Lambda_\ell}$ are irreducible and tilting [3, 4, 49]. Since decomposition into isotypic components of irreducibles is unique, this shows b). c) Every summand V_μ of $V_\lambda \otimes V$ has weight of the form $\mu = \lambda + \gamma$, where γ is a weight of V. But if in addition $\mathfrak{g}$ is of type $ABCD$, then V is minuscule by, e.g., table A. 2.3 in [25]. This implies the previous statement has a converse: for any weight γ of V such that $\lambda + \gamma$ is dominant, $V_{\lambda+\gamma}$ does appear in $V_\lambda \otimes V$, see, e.g., Lemma 3.1 in [26] .

Notation. From now on we shall mostly work in the tilting category $\mathcal{T}_\ell$, or its quotient $\mathcal{F}_\ell$, hence we shall simply write V_λ for the Weyl module $V_\lambda(q)$.

The specialised coboundary matrices $\overline{R}^{(n)}$ are still invertible and selfadjoint, hence the corresponding hermitian form of $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r}$ is non degenerate, provided $\lambda_i \in \overline{\Lambda}_\ell$, by Theorem 4.3. However, it may degenerate on subspaces. Wenzl used the above properties of the fundamental representation to show the following important result.

4.5. Theorem *Under the same assumptions of the previous theorem, for $q = e^{\pi i / d\ell}$ and $\lambda \in \Lambda_\ell$, Wenzl's hermitian form is positive definite on the submodule $V_\lambda \underline{\otimes} V$. Furthermore, for any $\gamma \in \Lambda_\ell$, the canonical projection*

$$p_{\lambda,\gamma} : V_\lambda \otimes V \to m_\gamma V_\gamma$$

is selfadjoint under the same form and one has $p_{\lambda,\gamma} p_{\lambda,\mu} = 0$ for $\gamma \neq \mu$.

4.6. Remark Notice that Wenzl includes the E_8 case, which we exclude in this paper, as, beyond being the most delicate case, it may give rise to indecomposable but reducible summands of $V_\lambda \otimes V$, cf. Theorem 3.5 c) of [49], while our later constructions need complete reducibility of such tensor products.

5. Wenzl's functor

In this section we construct a sequence of projections p_n on powers $V^{\otimes n}$ onto Hilbert subspaces and describe their main properties.

Consider, for each $\lambda \in \Lambda_\ell$, the (selfadjoint) projection $p_\lambda : V_\lambda \otimes V \to V_\lambda \underline{\otimes} V$, given by $p_\lambda = \sum p_{\lambda,\gamma}$, where the sum is made over all possible summands V_γ of $V_\lambda \otimes V$ with $\gamma \in \Lambda_\ell$. Let p_0 and p_1 denote the identity maps on the trivial module $\mathbb{C}$, and V respectively. Set, for $\mathfrak{g} \neq D_n$, $p_2 = p_\kappa$, with κ the dominant weight of V. In the type D_n case, if κ_1 and κ_2 denote the dominant weights of the two half-spin irreducible subrepresentations of V, we set $p_2 = p_{\kappa_1} + p_{\kappa_2}$. Define inductively canonical projectons p_n on $V^{\otimes n}$, as follows. Given

$$p_n : V^{\otimes n} \to \oplus_{\alpha \in A_n} V_{\alpha,n}$$

iteratively projecting onto the non-negligible part of a canonical orthogonal decomposition into submodules,

$$V^{\otimes n} = \bigoplus_{\alpha \in A_n} V_{\alpha,n} \bigoplus N_n,$$

with N_n negligible, consider unitaries $U_{\alpha,\mu} : V_\mu \to V_{\alpha,n}$ with $\mu \in \Lambda_\ell$, and then set

$$p_{n+1} = \sum_{\alpha \in A_n} U_{\alpha,\mu} \otimes 1_V \circ p_\mu \circ U^{-1}_{\alpha,\mu} \otimes 1_V \circ p_n \otimes 1_V.$$

The next lemmas will often turn out useful.

5.1. Lemma

a) $p_n \circ p_m \otimes 1_{V^{\otimes r}} = p_n = p_m \otimes 1_{V^{\otimes r}} \circ p_n$,

b) $A \otimes 1_{V^{\otimes r}} \circ p_{m+r} = p_{n+r} \circ A \otimes 1_{V^{\otimes r}} = A \underline{\otimes} 1_{V^{\underline{\otimes} r}}$, *for* $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$.

Proof The case $r = 1$ holds by construction, a simple iteration shows the general case. b) It suffices to show the first equality. Assume $r = 1$. For $A = U_{\beta,\mu} U^{-1}_{\alpha,\mu} : V_{\alpha,m} \to V_\mu \to V_{\beta,n}$

$$A \otimes 1_V \circ p_{m+1} = U_{\beta,\mu} \otimes 1_V \circ p_\mu \circ U^{-1}_{\alpha,\mu} \otimes 1_V = p_{n+1} \circ A \otimes 1_V = A \underline{\otimes} 1_V.$$

The equality also holds for $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$, by complete reducibility of $V^{\underline{\otimes} m}$ and $V^{\underline{\otimes} n}$. For $r > 1$, by a) and induction on r,

$$A \otimes 1_{V^{\otimes r}} \circ p_{m+r} = (A \otimes 1_{V^{\otimes r-1}} \circ p_{m+r-1}) \otimes 1_V \circ p_{m+r} =$$

$$(p_{n+r-1} \circ A \otimes 1_{V^{\otimes r-1}}) \otimes 1_V \circ p_{m+r} = p_{n+r} \circ (p_{n+r-1} \circ A \otimes 1_{V^{\otimes r-1}}) \otimes 1_V =$$

$$p_{n+r} \circ A \otimes 1_{V^{\otimes r}}.$$

One inductively derives from a) that p_n is selfadjoint w.r.t. Kirillov-Wenzl form, for all n. In general, the projections p_n do not satisfy right associativity, i.e. $p_n \circ 1_{V^{\otimes r}} \otimes p_m \neq p_n$. We see this with an example.

5.2. Example Set $\mathfrak{g} = \mathfrak{sl}_2$, $\ell = 3$. We denote by α the unique simple root, so $d = 1$, $\theta = \alpha$, $\rho = \frac{1}{2}\alpha$, and $\langle \alpha, \alpha \rangle = 2$. These data imply that the open Weyl alcove contains two non-negligible irreducible representations V_0 and V_1, with V_0 the trivial and $V_1 = V$ the vector representation, respectively. Furthermore, V_2 is a negligible irreducible representation. We have the following decompositions $V \otimes V \simeq V_0 \oplus V_2$, implying $V \underline{\otimes} V \simeq V_0$, and $V_0 \otimes V \simeq V \simeq V_0 \underline{\otimes} V$. The quantum determinant element $S = \psi_1 \otimes \psi_2 - q\psi_2 \otimes \psi_1$ is a morphism in $(V_0, V \otimes V)$ satisfying $S^* \otimes 1_V \circ 1_V \otimes S = -1_V$, $S^* S = [2]_q = q + q^{-1}$, see Prop. A.4 and A.5 in the appendix for details. Now $q + q^{-1} = 1$ for $\ell = 3$, hence S is an isometry, so we have $p_2 = SS^*$. There is no further truncation at the next power, so $p_3 = p_2 \otimes 1_V = SS^* \otimes 1_V$. We next compute

$$p_3 \circ 1_V \otimes p_2 = p_2 \otimes 1_V \circ 1_V \otimes p_2 =$$

$$SS^* \otimes 1_V \circ 1_V \otimes SS^* = -S \otimes 1_V \circ 1_V \otimes S^*$$

and it is now easy to see that it differs from p_3.

However, the following property will function as a replacement of right associativity failure of the p_n. It is the way how Andersen-Gelfand-Kazhdan properties of Sect. 2 will often manifest themselves in this paper.

5.3. Lemma *For any pair of morphisms $S \in (V^{\otimes m}, V^{\otimes n})$, $T \in (V^{\otimes n}, V^{\otimes r})$ of the tilting category,*

$$p_r \circ T \circ 1_{V^{\otimes s}} \otimes p_t \otimes 1_{V^{\otimes u}} \circ S \circ p_m = p_r \circ T \circ S \circ p_m$$

Proof The range of $1_{V^{\otimes s}} \otimes (1 - p_t) \otimes 1_{V^{\otimes u}}$ is a negligible module by (3), while p_m and p_r have support on non-negligible modules, hence $p_r \circ T \circ 1_{V^{\otimes s}} \otimes (1 - p_t) \otimes 1_{V^{\otimes u}} \circ S \circ p_m = 0$ by (2).

As an illustration, in our example 5.2,

$$p_3 \circ 1 \otimes p_2 \circ p_3 = -S \otimes 1_V \circ 1_V \otimes S^* \circ SS^* \otimes 1_V =$$
$$-S \otimes 1_V \circ (S^* \otimes 1_V \circ 1_V \otimes S)^* \circ S^* \otimes 1_V = SS^* \otimes 1_V = p_3.$$

We next describe the first important consequence. Consider the category $\mathcal{G}_\ell$ with objects

$$V^{\underline{\otimes} n} := p_n V^{\otimes n}$$

and morphisms

$$(V^{\underline{\otimes} n}, V^{\underline{\otimes} m}) = \{S \in (V^{\otimes n}, V^{\otimes m}) : Sp_n = p_m S = S\}.$$

We introduce a tensor product in $\mathcal{G}_\ell$,

$$V^{\underline{\otimes} m} \underline{\otimes} V^{\underline{\otimes} n} := V^{\underline{\otimes} m+n}, \qquad S \underline{\otimes} T := p_{m'+n'} \circ S \otimes T \circ p_{m+n},$$

where $S \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} m'})$, $T \in (V^{\underline{\otimes} n}, V^{\underline{\otimes} n'})$.

The following theorem is due to [49]. We briefly comment on those aspects that will play a role in the sequel.

5.4. Theorem *This tensor product makes $\mathcal{G}_\ell$ into a strict tensor C^*–category with a unitary braided symmetry given by*

$$\underline{\varepsilon}(b) := p_n \varepsilon(b) p_n, \quad b \in \mathbb{B}_n.$$

Furthermore the composition of the inclusion $\mathcal{G}_\ell \to \mathcal{T}_\ell$ with the quotient $\mathcal{T}_\ell \to \mathcal{F}_\ell$ is an equivalence of braided tensor *–categories.

Proof The statement about the equivalence of the two categories can be proved in a way similar to the proof of Lemma 1.1 in [17], once the tensor structure of $\mathcal{G}_\ell$ has been verified. The identity arrow on $V^{\underline{\otimes} n}$ is p_n. We need to verify the following properties: $p_n \underline{\otimes} p_m = p_{n+m}$; $(S \underline{\otimes} T)^* = S^* \underline{\otimes} T^*$, associativity, $(S \underline{\otimes} T) \underline{\otimes} U = S \underline{\otimes} (T \underline{\otimes} U)$, and the exchange rule between composition and tensor product, $(S \circ T) \underline{\otimes} (S' \circ T') = (S \underline{\otimes} S') \circ (T \underline{\otimes} T')$. The second property follows from the fact that the projections p_n are selfadjoint arrows of $\mathcal{T}_\ell$, while the remaining properties follow from Lemma 5.3. For example,

$$(S \underline{\otimes} T) \underline{\otimes} U = p_{m'+n'+p'} \circ p_{m'+n'} \otimes 1_{p'} \circ S \otimes T \otimes U \circ p_{m+n} \otimes 1_p \circ p_{m+n+p} =$$
$$p_{m'+n'+p'} \circ S \otimes T \otimes U \circ p_{m+n+p} = p_{m'+n'+p'} \circ 1_{m'} \otimes p_{n'+m'} \circ S \otimes T \otimes U \circ 1_m \otimes p_{n+m} \circ p_{m+n+p} =$$
$$S \underline{\otimes} (T \underline{\otimes} U).$$

Similar computations show that $\underline{\varepsilon}$ is a braided symmetry for $\mathcal{G}_\ell$. The C^*–structure and unitarity of the braided symmetry are consequences of the results recalled in the previous section.

This realisation of the quotient category is useful to obtain an analogue of a fibre functor, the starting point of our reconstruction, $W : \mathcal{G}_\ell \to \mathcal{H}$ taking $V^{\underline{\otimes} n}$ to its Hilbert space, $p_n V^{\otimes n}$, and acting identically on arrows. In the following we shall identify the abstract quotient category $\mathcal{F}_\ell$ with its concrete realisation $\mathcal{G}_\ell$ and we shall not distinguish the notation.

6. Algebraic compact quantum groups $\mathcal{C}(G_q)$ for $q \in \mathbb{T}$ generic

In this section we pause on roots of unity and Wenzl's projections. Thus we fix $q \in \mathbb{T}$ not a root of unity, and $\mathfrak{g}$ is a complex simple Lie algebra. We apply Tannaka-Krein duality to the tensor category of *–representations of $U_q(\mathfrak{g})$ and construct a quantum group, denoted $\mathcal{C} = \mathcal{C}(G_q)$, which may be viewed as a quantization of the function algebra over the compact group G corresponding to the compact real form of $\mathfrak{g}$. This construction conveys a general strategy that we will extend at roots of unity in later sections.

The main novelty of our construction, in comparison with the corresponding construction for the compact quantum groups G_q for $q \in \mathbb{R}$ [51], is the algebra involution, which now relies on the coboundary. The main result of this section is cosemisimplicity of the associated function algebra, which is derived from existence of a Haar functional, which, however, is not positive.

Let $\mathcal{V}_{\overline{R}}$ denote the category with objects the hermitian spaces arising as representation spaces of the objects of $\mathcal{T}_\ell$ and arrows all the linear maps. The adjoint still makes sense for the arrows of this category, and one easily verifies that in this way $\mathcal{V}_{\overline{R}}$ satisfies all the axioms of a tensor *–category, except the rule of the adjoint of a tensor product arrow, which, unless S and T are morphisms of representations, it is replaced by the property

$$(S \otimes T)^* = \overline{R}^{-1} \circ S^* \otimes T^* \circ \overline{R}. \tag{6.1}$$

We consider the spaces of the tensor powers of the fundamental representation of $U_q(\mathfrak{g})$, simply denoted by V^n, $n \in \mathbb{N}$. Same notation applies to simple tensors $\phi_1 \dots \phi_n$ of V^n. Elements of V^n are regarded as arrows of (ι, V^n) in $\mathcal{V}_{\overline{R}}$ in the natural way. Similarly, elements of the dual space $(V^n)^*$ will be regarded as arrows of (V^n, ι).

We form the diagonal subalgebra $\mathcal{D} = \mathcal{D}(V)$ of the mixed tensor algebra associated to V,

$$\mathcal{D} := \oplus_{n\in\mathbb{N}}(V^n)^* \otimes V^n.$$

In other words, the multiplication in $\mathcal{D}$ is given by

$$(\varphi \otimes \psi)(\varphi' \otimes \psi') := \varphi\varphi' \otimes \psi\psi', \tag{6.2}$$

$\varphi \in (V^n)^*$, $\varphi' \in (V^m)^*$, $\psi \in V^n$, $\psi' \in V^m$. Notice that the subset of simple tensors

$$\{v_{\phi,\psi} := \phi \otimes \psi;\ \psi \in V, \phi \in V^*\},$$

generates $\mathcal{D}$ as an algebra. A non-involutive quantum group can be reconstructed via usual Tannaka-Krein duality. Specifically, let $\mathcal{I}$ be the linear subspace of $\mathcal{D}$ generated by

$$\mathcal{I} := \text{l.s.}\{\varphi \otimes A \circ \psi - \varphi \circ A \otimes \psi;\ \psi \in V^n, \varphi \in V^{*m}, A \in (V^n, V^m)\}.$$

6.1. Proposition *$\mathcal{I$ is a two-sided ideal of $\mathcal{D}$, hence the quotient space*

$$\mathcal{C} := \mathcal{D}/\mathcal{I}$$

is an associative and unital algebra.

Proof It suffices to show that $\mathcal{I}$ is stable under left and right multiplication by the generators $v_{\phi,\psi}$. This follows from the fact that the arrows $A \in (V^n, V^m)$ are stable under left and right tensoring by 1_V.

We next make $\mathcal{C}$ into an involutive Hopf algebra. The adjoint can equivalently be introduced in two different ways, one relying on the n-th tensor power of Hermitian structure of V and the other on Kirillov-Wenzl hermitian structure of V^n (i.e. a deformation of the former via $\overline{R}_n$). Both turn out useful.

For $\psi \in V$, ψ^* is the linear functional on V defined by $\psi^*(\psi') = (\psi, \psi')$. We can thus identify the dual space $(V^n)^*$ with V^{*n}, denoting by $\psi_1^* \dots \psi_n^*$ the tensor product functional $\psi_1^* \otimes \dots \otimes \psi_n^*$. We introduce the *–involution as follows

$$(\phi_1^* \dots \phi_n^* \otimes \psi_1 \dots \psi_n)^* = \psi_n^* \dots \psi_1^* \otimes \phi_n \dots \phi_1, \tag{6.3}$$

for ϕ_i, $\psi_i \in V$. This is clearly a well defined map on $\mathcal{D}$. We next express this adjoint in terms of the hermitian structure of V^n and the (specialised) coboundary operators $\sigma_n = \Sigma_n \overline{R}^{(n)} \in (V^n, V^n)$ of Sect. 3,

6.2. Lemma *For* $\psi_1, \dots \psi_n \in V$,

$$(\psi_1 \dots \psi_n)^* = \psi_n^* \dots \psi_1^* \sigma_n.$$

Proof Taking into account (6.1) for arrows $(\psi_1 \dots \psi_n)^*$ and $\psi_1^* \dots \psi_n^*$,

$$(\psi_1 \dots \psi_n)^* = \psi_1^* \dots \psi_n^* \circ \overline{R}^{(n)} = \psi_n^* \dots \psi_1^* \sigma_n.$$

We can thus alternatively represent simple tensors of $\mathcal{D}$ in the form $\psi^* \sigma_n^{-1} \otimes \phi$, and define the adjoint of $\mathcal{D}$ by

$$(\psi^* \sigma_n^{-1} \otimes \phi)^* = \phi^* \sigma_n^{-1} \otimes \psi, \qquad \phi, \psi \in V^n. \tag{6.4}$$

6.3. Proposition *This involution makes $\mathcal{D}$ into a *–algebra and $\mathcal{I}$ into a *–ideal. Hence $\mathcal{C}$ is a *–algebra.*

Proof If $A \in (V^m, V^n)$, $\psi \in V^n$, $\phi \in V^m$, we compute the following adjoints by means of (6.4),

$$(\psi^* \sigma_n^{-1} A \otimes \phi)^* = ((\sigma_n^{-1} A^* \sigma_n \psi)^* \sigma_n^{-1} \otimes \phi)^* = \phi^* \sigma_n^{-1} \otimes \sigma_n^{-1} A^* \sigma_n \psi,$$

since σ_n is unitary. Similarly,

$$(\psi^* \sigma_n^{-1} \otimes A\phi)^* = (A\phi)^* \sigma_n^{-1} \otimes \psi = \phi^* \sigma_n^{-1} (\sigma_n A^* \sigma_n^{-1}) \otimes \psi.$$

Taking into account the important relation $\sigma_n^2 = 1$, this computation shows that the adjoint of an element of the form $\psi^* \sigma_n^{-1} A \otimes \phi - \psi^* \sigma_n^{-1} \otimes A\phi$ is of the same form.

6.4. Remark Notice that, for class elements of $\mathcal{C}$, for $\psi, \psi' \in V^n$, $[\psi^* \otimes \psi']^* = [\psi'^* \otimes \psi]$, where the adjoints of ψ and ψ' refer to the hermitian form of V^n. On the other hand, up to a scalar multiple, this form restricts to the hermitian form of V_α if V_α is a submodule of V^n. Hence, for $\psi, \psi' \in V_\alpha$, we may regard ψ^* and ψ'^* as adjoints relative to the hermitian form of V_α.

We next introduce in $\mathcal{C}$ the structure of a Hopf algebra by first endowing $\mathcal{D}$ with coproduct defined by:

$$\Delta(\phi \otimes \psi) = (\phi \otimes \eta_r) \otimes (\eta^r \otimes \psi),$$

where $\eta_r \in V^n$ is a basis and $\eta^r \in (V^n)^*$ is a dual basis. A routine computation shows that Δ does not depend on the choice of the basis.

6.5. Proposition *The coproduct Δ is unital and coassociative, and satisfies $\Delta(a^*) = \Delta^{\mathrm{op}}(a)^*$. Furthermore $\mathcal{I}$ is also a coideal, hence Δ induces a coproduct on $\mathcal{C}$, still denoted Δ, satisfying the same properties.*

Proof It is straightforward to check that Δ is unital, multiplicative and coassociative. Since $a \to \Delta(a^*)$ and $a \to \Delta^{\mathrm{op}}(a)^*$ are both antimultiplicative maps on $\mathcal{D}$, it suffices, and it is easy, to check that $\Delta(a^*) = \Delta^{\mathrm{op}}(a)^*$ for $a = \phi^* \otimes \psi$, ϕ, $\psi \in V$. Finally, if $A \in (V^n, V^m)$ is an intertwiner, and $\eta_i \in V^n$, $\xi_j \in V^m$ are linear bases with dual bases η^i, ξ^j respectively, for $\psi \in V^n$, $\phi \in (V^m)^*$,

$$\Delta(\phi \circ A \otimes \psi) = \phi \circ A \otimes \eta_i \otimes \eta^i \otimes \psi, \quad \Delta(\phi \otimes A \circ \psi) = \phi \otimes \xi_j \otimes \xi^j \otimes A \circ \psi.$$

An easy computation gives

$$\Delta(\phi A \otimes \psi - \phi \otimes A\psi) = (\phi A \otimes \eta_i - \phi \otimes A\eta_i) \otimes \eta^i \otimes \psi - \phi \otimes \xi_j \otimes (\xi^j A \otimes \psi - \xi^j \otimes A\psi),$$

implying $\Delta(\mathcal{I}) \subset \mathcal{I} \otimes \mathcal{D} + \mathcal{D} \otimes \mathcal{I}$.

We next introduce the functional

$$h : \mathcal{C} \to \mathbb{C},$$

which corresponding to the Haar measure in the classical case by,

$$h(\phi \otimes \psi) = \phi(e_n \psi), \quad \psi \in V^n, \phi \in V^{n*},$$

where e_n is the projection onto the isotypical component of the trivial subrepresentation of V^n. Every $A \in (V^n, V^m)$ satisfies $Ae_n = e_m A$, and this shows that h annihilates $\mathcal{I}$, thus defining a linear functional on $\mathcal{C}$.

This functional turns out useful to show cosemisimplicity of $\mathcal{C}$. Consider a complete set, V_λ, $\lambda \in \Lambda^+$, of irreducible representations. An isometric intertwiner $S \in (V_\lambda, V^{\otimes n})$ induces a linear inclusion

$$V_\lambda^* \otimes_{\mathbb{C}} V_\lambda \to \mathcal{C}$$

which takes a simple tensor $\phi \otimes \psi$ to the class of $\phi S^* \otimes S\psi$.

6.6. Proposition *The above inclusion $V_\lambda^* \otimes_{\mathbb{C}} V_\lambda \to \mathcal{C}$ does not depend on n and $S \in (V_\lambda, V^{\otimes n})$. Furthermore, the image is a subcoalgebra.*

Proof If $T \in (V_\lambda, V^{\otimes m})$ is another isometry then in the quotient,

$$[\phi S^* \otimes S\psi] = [\phi S^* \otimes ST^*T\psi] = [\phi S^* ST \otimes T\psi] = [\phi T^* \otimes T\psi].$$

For the last statement we notice that a similar computation shows that if ψ and ψ' lie in orthogonal invariant subspaces of some $V^{\otimes n}$ then $[\psi^* \otimes \psi'] = 0$. Therefore if ϕ and ψ lie in the same irreducible component of $V^{\otimes n}$ then $\Delta([\phi^* \otimes \psi])$ can be simply expressed by means of an orthonormal basis of that submodule, rather than of the whole $V^{\otimes n}$.

6.7. Theorem *For every $\lambda \in \Lambda^+$, the natural inclusion $V_\lambda^* \otimes_{\mathbb{C}} V_\lambda \to \mathcal{C}$ is faithful. Therefore, as a coalgebra,*

$$\mathcal{C} = \bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes_{\mathbb{C}} V_\lambda.$$

and $\mathcal{C}$ is cosemisimple.

Proof Consider isometries $S_{\lambda,i} \in (V_\lambda, V^n)$ such that $\sum_{\lambda,i} S_{\lambda,i} S_{\lambda,i}^* = 1_{V^n}$. An element $\phi \otimes \psi \in (V^n)^* \otimes V^n$, regarded as an element of $\mathcal{C}$, can be written in the form

$$\phi \otimes \psi = \phi \otimes \sum S_{\lambda,i} S_{\lambda,i}^* \psi = \sum \phi S_{\lambda,i} \otimes S_{\lambda,i}^* \psi.$$

Hence $\mathcal{C}$ is linearly generated by the various $V_\lambda^* \otimes V_\lambda$. We next show that the linear inclusion of $V_\lambda^* \otimes_{\mathbb{C}} V_\lambda$ in $\mathcal{C}$ is faithful. We evaluate $h(ab)$, for $a = \psi^* \otimes \psi' \in V_\lambda^* \otimes V_\lambda$, $b = \phi^* \otimes \phi' \in V_\mu^* \otimes V_\mu$. If V_λ is not conjugate to V_μ, $h(ab) = 0$, as $V_\lambda \otimes V_\mu$ does not contain the trivial representation. If V_λ is conjugate to V_μ, we identify V_λ with the conjugate hermitian representation $\overline{V_\mu}$. If ψ_i is an orthonormal basis of V_μ, then $r := \sum_i \overline{\psi_i} \otimes K_{-2\rho} \psi_i \in (\iota, V_\lambda \otimes V_\mu)$. Lemma 3.4 of [49] easily implies that the adjoint of r with respect to the hermitian form is given by $r^*(\overline{\psi} \otimes \phi) := (\psi, \phi)$. Hence for $a = \overline{\phi}^* \otimes \overline{\psi}$, $b = \xi^* \otimes \eta$, and a non-zero scalar $d(\mu)$, the quantum dimension of V_μ,

$$d(\mu)h(ab) = \overline{\phi}^* \xi^* \otimes rr^* \overline{\psi}\eta = \overline{\phi}^* \xi^* r \otimes r^* \overline{\psi}\eta =$$

$$\sum_i (\psi_i, \phi)(\xi, K_{-2\rho}\psi_i)(\psi, \eta) = (\xi, K_{-2\rho}\phi)(\psi, \eta).$$

Let us fix a complete set of irreducibles parametrised by Λ^+. Thus for any $\mu \in \Lambda^+$, the conjugate of V_μ is V_λ, with $\lambda = -w_0\mu$. The composition of the complex conjugation $J_\mu : V_\mu \to \overline{V_\mu}$ with a unitary intertwiner $U_\mu : \overline{V_\mu} \to V_\lambda$ is an antiunitary map $j_\mu : V_\mu \to V_\lambda$, unique up to scalar multiples by $z \in \mathbb{C}$ with $|z| = 1$. We can thus define a linear map, the antipode, $S : \mathcal{C} \to \mathcal{C}$, by

$$S(\phi^* \otimes \psi) = (j_\mu \psi)^* \otimes j_\mu \phi, \quad \phi^* \otimes \psi \in V_\mu^* \otimes V_\mu.$$

Notice that S does not depend on the choice of j_μ or of the set of irreducibles. We also define the counit

$$\varepsilon : \mathcal{C} \to \mathbb{C}, \quad \varepsilon(\phi^* \otimes \psi) = (\phi, \psi).$$

6.8. Proposition *Antipode S and counit ε make $\mathcal{C}$ into a Hopf algebra. Furthermore, S and ε commute with the adjoint map and $S^2(\phi^* \otimes \psi) = \phi^* K_{2\rho} \otimes K_{2\rho}^{-1}\psi$.*

Proof The relations $S(a)^* = S(a^*)$ and $\overline{\varepsilon(a)} = \varepsilon(a^*)$ are easy to check. If η_r is an orthonormal basis of V_μ, with μ and j as above, then $\overline{r} = \sum_r \eta_r \otimes j\eta_r \in (\iota, V_\mu \otimes V_\lambda)$. We verify the relation $m \circ 1 \otimes S \circ \Delta(a) = \varepsilon(a)$ for $a = \phi \otimes \psi \in V_\mu^* \otimes V_\mu$.

$$m(1 \otimes S(\Delta(\phi \otimes \psi)) =$$

$$m(1 \otimes S(\phi \otimes \eta_r \otimes \eta_r^* \otimes \psi)) =$$

$$m((\phi \otimes \eta_r) \otimes ((j_\mu \psi)^* \otimes j_\mu \eta_r)) =$$

$$\phi(j_\mu\psi)^* \otimes \overline{r}(1_{\mathbb{C}}) = \phi(j_\mu\psi)^* \circ \overline{r} \otimes 1_{\mathbb{C}} =$$

$$\sum_r \phi(\eta_r)(j_\mu\psi, j_\mu\eta_r) = \sum_r \phi(\eta_r)(\eta_r, \psi) =$$

$$\phi(\psi) = \varepsilon(\phi \otimes \psi).$$

The analogous relation with S on the left follows from this after taking the adjoint. We finally compute S^2. Keeping the same notation used in the definition of S, we have that $S^2(\phi^* \otimes \psi) = (j_\lambda j_\mu \phi)^* \otimes j_\lambda j_\mu \psi$. We identify $\overline{\overline{V_\mu}}$ with V_μ as an hermitian space. In other words, the corresponding conjugation identifies with $J_\mu{}^{-1}$. Observe that $J_\mu{}^{-1} U_\mu^* J_\lambda^{-1}$ is a unitary intertwiner of $(\overline{V_\lambda}, \overline{\overline{V_\mu}})$. The action of $U_q(\mathfrak{g})$ on $\overline{\overline{V_\mu}}$ is $a\xi = S^{-2}(a)\xi$, and we know that $S^{-2}(a) = K_{2\rho} a K_{2\rho}^{-1}$, hence $K_{2\rho}^{-1} \in (\overline{\overline{V_\mu}}, V_\mu)$. We can thus choose $U_\lambda = K_{2\rho}^{-1} J_\mu{}^{-1} U_\mu^* J_\lambda^{-1}$, implying $j_\lambda = K_{2\rho}^{-1} J_\mu{}^{-1} U_\mu^*$, which together with $j_\mu = U_\mu J_\mu$ gives $j_\lambda j_\mu = K_{2\rho}^{-1}$.

We finally pass to the dual space, $\mathcal{C}'$ and we endow it with the usual dual algebra structure given by

$$\omega\omega' := \omega \otimes \omega' \circ \Delta$$

coproduct $\Delta : \mathcal{C}' \to (\mathcal{C} \otimes \mathcal{C})' \supset \mathcal{C}' \otimes \mathcal{C}'$,

$$\Delta(\omega)(a, b) = \omega(ab), \qquad a, b \in \mathcal{C}, \quad \omega \in \mathcal{C}',$$

and involution given by duality with the involution of $\mathcal{C}$,

$$\omega^*(a) := \overline{\omega(a^*)}, \qquad a \in \mathcal{C}, \quad \omega \in \mathcal{C}'.$$

Notice that duality for the involution differs from the case of a ordinary Hopf *–algebra (which is defined via duality with $a \to S(a^*)$.) The counit $\varepsilon : \mathcal{C}' \to \mathbb{C}$ and antipode $S : \mathcal{C}' \to \mathcal{C}'$ are defined as usual by

$$\varepsilon(\omega) = \omega(I), \qquad S(\omega) = \omega \circ S.$$

6.9. Theorem *$\mathcal{C}'$ is isomorphic, as a *-algebra, to the direct product of full matrix algebras*

$$\mathcal{C}' \simeq \prod M_{n_\lambda}(\mathbb{C}), \quad n_\lambda = \mathit{dim}(V_\lambda).$$

The coproduct Δ is a homomorphism satisfying again $\Delta(\omega^*) = \Delta^{op}(\omega)^*$.

Counit and antipode of $\mathcal{C}'$ satisfy the same properties as for $U_q(\mathfrak{g})$. We identify the dual space of $\mathcal{C} \otimes \mathcal{C}$ with $\prod_{\lambda,\mu} M_{n_\lambda} \otimes M_{n_\mu}$ in the natural way, and in this sense we understand the stated homomorphism property of Δ. Furthermore, the R matrix can be found as an element of this algebra, see Sect. 1 in [45], or Sect. 2.5 in [32].

7. The universal algebra $\mathcal{D}(V, \ell)$

In this section the deformation parameter is a fixed root of unity of the form $q = e^{i\pi/d\ell}$. Our aim is to construct non-associative bi-algebras $\mathcal{D}(V, \ell)$ endowed with an involution, for $\mathfrak{g} \neq E_8$, playing a role similar to that of $\mathcal{D}(V)$ of the generic case.

Let V be Wenzl's fundamental representation of $\mathfrak{g}$ ($V = V_{\kappa_1} \oplus V_{\kappa_2}$ in the type D case.) Consider the (infinite dimensional) linear space,

$$\mathcal{D}(V, \ell) = \bigoplus_{n=0}^{\infty} (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}.$$

Notice that $\mathcal{D} = \mathcal{D}(V, \ell)$ depends not only on V but also on the root of unity. We define a multiplication on $\mathcal{D}$ by

$$\alpha\beta := \phi\phi' p_{m+n} \otimes p_{m+n} \psi\psi',$$

for

$$\alpha = \phi \otimes \psi \in V^{m*} p_m \otimes p_m V^m, \quad \beta = \phi' \otimes \psi' \in V^{n*} p_n \otimes p_n V^n.$$

In this way $\mathcal{D}$ becomes a unital but not associative algebra, as, if we pick a third element $\gamma = \phi'' \otimes \psi'' \in V^{r*} p_r \otimes p_r V^r$, and we take into account the relation $p_{m+n+r} \circ p_{m+n} \otimes 1_r = p_{m+n+r}$, of Lemma 5.1, we see that

$$(\alpha\beta)\gamma = (\phi\phi'\phi'') p_{m+n+r} \otimes p_{m+n+r} (\psi\psi'\psi'')$$

but

$$\alpha(\beta\gamma) = (\phi\phi'\phi'') 1_m \otimes p_{n+r} \circ p_{m+n+r} \otimes p_{m+n+r} \circ 1_m \otimes p_{n+r} (\psi\psi'\psi'')$$

that differs from the previous one as in general $p_{m+n+r} \circ 1_m \otimes p_{n+r} \neq p_{m+n+r}$. Notice however that the elements

$$v_{\xi,\eta} := \xi^* \otimes \eta, \qquad \xi, \eta \in V$$

still generate $\mathcal{D}$ as an algebra,

$$\xi_1 \dots \xi_n p_n \otimes p_n \eta_1 \dots \eta_n = (\dots((v_{\xi_1,\eta_1} v_{\xi_2,\eta_2}) v_{\xi_3,\eta_3}) \dots v_{\xi_n,\eta_n}).$$

We next introduce an involution in $\mathcal{D}$ as suggested by the generic case. Specifically, we replace the boundary operators σ_n by their truncated version:

$$\tau_n := p_n \sigma_n p_n \in (V^{\underline{\otimes} n}, V^{\underline{\otimes} n}),$$

which satisfy properties similar to those of σ_n (except associativity).

7.1. Proposition $\tau_n^* = \tau_n$, $\tau_n^2 = p_n$.

Proof Lemma 5.3 gives $p_n \sigma_n (1 - p_n) \sigma_n p_n = 0$, implying $\tau_n^2 = p_n \sigma_n p_n \sigma_n p_n = p_n \sigma_n^2 p_n = p_n$. Furthermore, $\tau_n^* = p_n \sigma_n^* p_n = p_n \sigma_n p_n = \tau_n$.

We can write any element $\phi \in (V^{\otimes n})^* p_n$ uniquely in the form $\phi = \psi^*$ with $\psi \in V^{\underline{\otimes} n}$, the adjoint of ψ being computed with respect to Wenzl's inner product of $V^{\underline{\otimes} n}$. We set for $\psi, \psi' \in V^{\underline{\otimes} n}$,

$$(\psi^* \otimes \psi')^* := (\tau_n \psi')^* \otimes \tau_n^{-1} \psi.$$

7.2. Proposition *The following properties hold for the involution of $\mathcal{D}$. For any $a \in \mathcal{D}$ and $v_{\xi,\eta} \in V^* \otimes V$,*

a) $a \to a^*$ *is antilinear,*
b) $a^{**} = a$,
c) ${v_{\xi,\eta}}^* = v_{\eta,\xi}$,
d) $(v_{\xi\eta} a)^* = a^* v_{\xi,\eta}^*$.

Proof a) is obvious, b) is a consequence of involutivity of τ_n, and c) of $\tau_1 = 1$. We show d) for $a = \phi^* \otimes \psi \in (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}$. We have

$$a^* v_{\xi,\eta}^* = (\psi^* {\tau_n}^{-1} \otimes \tau_n^{-1}\phi)(\eta^* \otimes \xi) =$$
$$[\psi^*\eta^* \circ \tau_n^{-1} \otimes 1_V \circ p_{n+1}] \otimes [p_{n+1} \circ \tau_n^{-1} \otimes 1_V \circ \phi\xi] =$$
$$[(\eta\psi)^* \circ \sigma(V, V^{\underline{\otimes} n})^{-1} \circ \tau_n^{-1} \otimes 1_V \circ p_{n+1}] \otimes [p_{n+1} \circ \tau_n^{-1} \otimes 1_V \circ \phi\xi].$$

Notice that $1_V \otimes p_n \circ (1 - p_{n+1}) \circ \sigma(V, V^{\underline{\otimes} n})^{-1} \circ \tau_n^{-1} \otimes 1_V \circ p_{n+1} = 0$ by the following lemma. Hence we can insert the projection p_{n+1} after $(\eta\psi)^*$ in the above computation, use once again Lemma 5.3 and the iterative definition of σ_n to conclude that

$$a^* v_{\xi,\eta}^* = (\eta\psi)^* \circ \tau_{n+1}^{-1} \otimes [p_{n+1} \circ \tau_n^{-1} \otimes 1_V \circ \phi\xi].$$

Similar arguments show that $(v_{\xi,\eta} a)^*$ yields the same expression.

7.3. Lemma *Assume $\mathfrak{g} \neq E_8$, and let $T \in (V^{\otimes m}, V^{\otimes n})$ be a negligible arrow of the tilting category $\mathcal{T}_\ell$. Then*

$$p_n \circ T \circ 1_V \otimes p_{m-1} = 0.$$

Proof Set $Y = p_n \circ T \circ 1_V \otimes p_{m-1}$. We take into account the basic property of Wenzl's fundamental representation V for $\mathfrak{g} \neq E_8$ recalled in Theorem 4.4 a) according to which for any $\lambda \in \Lambda_\ell$, $V \otimes V_\lambda$ is completely reducible (in a way that, although with same multiplicities, may differ from the decomposition of $V_\lambda \otimes V$) and the dominant weights μ of the irreducible components V'_μ all lie in $\overline{\Lambda_\ell}$. Thus the space of such a component with $\mu \in \Lambda_\ell$ must be in the kernel of Y by Lemma 5.3. On the other hand, if $\mu \in \overline{\Lambda_\ell} \setminus \Lambda_\ell$ then $YV'_\mu = \{0\}$ as otherwise it would be an irreducible submodule of $p_n V^{\otimes n}$ of weight μ.

We introduce a coproduct

$$\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$$

in a way similar to the generic case, i.e. by means of a pair $\eta_r \in p_n V^n$, $\eta^r \in (V^n)^* p_n \simeq (p_n V^n)^*$ of dual bases:

$$\Delta(\phi \otimes \psi) = (\phi \otimes \eta_r) \otimes (\eta^r \otimes \psi), \quad \phi \otimes \psi \in (p_n V^n)^* \otimes (p_n V^n),$$

again independent on the choice of the basis.

7.4. Theorem *The coproduct Δ is unital, coassociative, and satisfies for $a, b \in \mathcal{D}$,*

a) $\Delta(a^*) = \Delta^{\rm op}(a)^*$,
b) $\Delta(ab) = \Delta(a)\Delta(b)$.

Proof a) For $a = \psi^* \otimes \psi' \in (p_n V^n)^* \otimes (p_n V^n)$, let ψ_r be an orthonormal basis of $p_n V^n$ with respect to the Wenzl's inner product, and consider the dual basis $\xi^r := \psi_r^*$. We compute $\Delta(a^*)$ with respect to the dual pair $\tau_n^{-1}\psi_r$, $(\tau_n\psi_r)^*$,

$$\Delta(a^*) = \Delta((\tau_n\psi')^* \otimes \tau_n^{-1}\psi) = \sum_r ((\tau_n\psi')^* \otimes \tau_n^{-1}\psi_r) \otimes ((\tau_n\psi_r)^* \otimes \tau_n^{-1}\psi) =$$
$$\sum_r (\psi_r^* \otimes \psi')^* \otimes (\psi^* \otimes \psi_r)^* = \Delta^{\rm op}(a)^*.$$

b) Let η_s be an orthonormal basis of $p_m V^m$ and $b = \xi^* \otimes \eta \in (p_m V^m)^* \otimes (p_m V^m)$. We have

$$\Delta(a)\Delta(b) =$$
$$\sum_{r,s} (\psi^*\xi^* p_{n+m} \otimes p_{n+m}\psi_r\eta_s) \otimes (\psi_r^*\eta_s^* p_{n+m} \otimes p_{n+m}\psi'\eta).$$

We explicit the middle term

$$\sum_{r,s} p_{n+m}\psi_r\eta_s \otimes \psi_r^*\eta_s^* p_{n+m} = \sum_h \zeta_h \otimes \zeta^h$$

using an orthonormal basis ζ_h of $p_{n+m}V^{n+m}$, where

$$\zeta^h := (\zeta_h, p_{n+m}\psi_r\eta_s)\psi_r^*\eta_s^*p_{n+m}.$$

We adopt the same notation as before to distinguish between a tensor product of inner products $(\zeta, \zeta')_p$ and Wenzl's inner product (ζ, ζ') on the subspace $(p_nV^n) \otimes p_mV^m$ of V^{n+m}. We have,

$$(\zeta^h)^* = \sum_{r,s}(p_{n+m}\psi_r\eta_s, \zeta_h)(\psi_r^*\eta_s^*p_{n+m})^* = \sum_{r,s}(p_{n+m}\psi_r\eta_s, \zeta_h)p_{n+m}\overline{R}^{-1}\psi_r\eta_s =$$

$$\sum_{r,s}(\psi_r\eta_s, \zeta_h)p_{n+m}\overline{R}^{-1}\psi_r\eta_s = \sum_{r,s}p_{n+m}\overline{R}^{-1}(\psi_r\eta_s, \overline{R}\zeta_h)_p\psi_r\eta_s =$$

$$p_{n+m}\overline{R}^{-1} \circ p_n \otimes p_m \circ \overline{R}\zeta_h = p_{n+m}\varepsilon^{-1} \circ p_m \otimes p_n \circ \varepsilon \circ p_{n+m}\zeta_h = \zeta_h,$$

by Lemma 5.3, where

$$\varepsilon = \varepsilon(p_nV^n, p_mV^m).$$

This shows that

$$\Delta(a)\Delta(b) = \sum_h \psi^*\xi^*p_{n+m} \otimes \zeta_h \otimes \zeta_h^* \otimes p_{n+m}\psi'\eta = \Delta(ab).$$

7.5. Proposition *The linear map* $\varepsilon : \phi^*\otimes\psi \in \mathcal{D}(V, \ell) \to (\phi, \psi) \in \mathbb{C}$ *is a counit for the coproduct* Δ *making* $\mathcal{D}(V, \ell)$ *into a coalgebra. It is compatible with the involution,* $\varepsilon(a^*) = \overline{\varepsilon(a)}$.

Proof The former statement is easy to check. The latter follows from Prop. 7.1.

7.6. Remark It is important to notice that, unlike the generic case, ε is *not* multiplicative on $\mathcal{D}(V, \ell)$, as a consequence of $p_n \neq 1$ in general.

8. THE QUANTUM GROUPOID $\mathcal{C}(G, \ell)$ AND A CORRESPONDING ASSOCIATIVE FILTRATION

In analogy to the generic case, we follow a Tannakian reconstruction from the quotient category $\mathcal{F}_\ell$ for which $\mathcal{D} = \mathcal{D}(V, \ell)$ plays the role of a universal algebra and we obtain a quantum groupoid $\mathcal{C}(G, \ell)$. The new main difference with the generic case (beyond associativity failure of $\mathcal{D}(V, \ell)$) is the fact that the ideal of $\mathcal{D}$ defining $\mathcal{C}(G, \ell)$ is only a right ideal. Correspondingly, $\mathcal{C}(G, \ell)$ is naturally only a *-coalgebra, and new effort is needed to construct an algebra structure in $\mathcal{C}(G, \ell)$ compatible with the coproduct, which we consider in the next sections.

Specifically, in this section we define $\mathcal{C}(G, \ell)$ and establish the main properties. We then introduce an associated non-trivial, possibly nilpotent, but associative structure, described by a finite sequence $\widetilde{\mathcal{C}}_k$ of *-coalgebras, which we regard as a generalised algebra filtration. We shall eventually be able to give positive answers to the above questions for $\mathcal{C}(G, \ell)$ in the type A case by analysing this filtration.

a) The coalgebra $\mathcal{C}(G, \ell)$

We introduce identifications in $\mathcal{D}$ arising from arrows of $\mathcal{F}_\ell$ as follows. Consider the linear span $\mathcal{J}$ of $\mathcal{D}$ of elements of the form

$$[\phi, A, \psi] := \phi^* \otimes A\psi - \phi^* \circ A \otimes \psi,$$

where $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$ and set

$$\mathcal{C}(G, \ell) = \mathcal{D}(V, \ell)/\mathcal{J}.$$

We start summarising the structure that $\mathcal{C}$ inherits from $\mathcal{D}$.

8.1. Proposition $\mathcal{C}(G, \ell)$ *is a finite dimensional, coassociative, counital coalgebra with involution. More precisely,*

a) $\mathcal{J}$ *is a right ideal,*
b) $\mathcal{C}$ *is finite dimensional and linearly spanned by class tensors* $v^\lambda_{\phi,\psi} = [\phi^* \otimes \psi]$, $\phi, \psi \in V_\lambda$, $\lambda \in \Lambda_\ell$,

c) $\mathcal{J}$ *is a coideal annihilated by* ε,

d) $\mathcal{J}$ *is* **-invariant, hence the involution of* $\mathcal{D}$ *factors through* $\mathcal{C}$ *and satisfies*

$$\Delta(a^*) = \Delta^{\rm op}(a)^*, \quad \varepsilon(a^*) = \overline{\varepsilon(a)}, \quad a \in \mathcal{C},$$
$$\Delta(v^\lambda_{\phi,\psi}) = \sum_r v^\lambda_{\phi,\eta_r} \otimes v^\lambda_{\eta_r,\psi}, \quad (v^\lambda_{\phi,\psi})^* = v^\lambda_{\psi,\phi}.$$

Proof a) follows from Lemma 5.1. Specifically, for

$$x = \phi \otimes A \circ \psi - \phi \circ A \otimes \psi \in \mathcal{J}$$

and any simple tensor

$$\zeta := \xi \otimes \eta \in (V^h)^* p_h \otimes p_h V^h,$$

we have

$$x\zeta = \phi\xi p_{n+h} \otimes p_{n+h}(A\psi)\eta - (\phi A)\xi p_{m+h} \otimes p_{m+h}\psi\eta =$$
$$\phi\xi p_{n+h} \otimes p_{n+h} \circ A \otimes 1_{V^h}(\psi\eta) - (\phi\xi)A \otimes 1_{V^h} \circ p_{m+h} \otimes p_{m+h}\psi\eta =$$
$$\phi\xi p_{n+h} \otimes (A \underline{\otimes} 1) p_{m+h}\psi\eta - \phi\xi p_{n+h}(A\underline{\otimes}1) \otimes p_{m+h}\psi\eta \in \mathcal{J}.$$

To show b) we may argue as in the generic case but now with a choice of isometries $S_i \in (V_{\lambda_i}, V^{\underline{\otimes} n})$, with $\lambda_i \in \Lambda_\ell$, in the C^*–category $\mathcal{F}_\ell$ satisfying $\sum_i S_i {S_i}^* = p_n$. The remaining statements can be proved in analogy with the generic case, taking into account the results of the previous section.

b) The filtration $\widetilde{\mathcal{C}}_k$

We filter $\mathcal{D}$ by the size of tensor products. Set

$$\mathcal{D}_k = \bigoplus_{n \leq k} (V^{\otimes n})^* p_n \otimes p_n V^{\otimes n}.$$

8.2. Proposition

a) $\mathcal{D}_k$ *is a filtration of* $\mathcal{D}$*, i.e. it is an increasing sequence of subspaces satisfying*

$$\mathcal{D}_0 = \mathbb{C}, \quad \mathcal{D}_h \mathcal{D}_k \subset \mathcal{D}_{h+k}, \quad \bigcup_{k=0}^\infty \mathcal{D}_k = \mathcal{D}.$$

b) $\mathcal{D}_k$ *are* **-invariant subcoalgebras:* $(\mathcal{D}_k)^* = \mathcal{D}_k, \quad \Delta(\mathcal{D}_k) \subset \mathcal{D}_k \otimes \mathcal{D}_k$.

Set

$$\mathcal{J}_k := \text{l.s.}\{[\phi, A, \psi], \quad A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n}), \psi \in V^{\underline{\otimes} m}, \phi \in V^{\underline{\otimes} n}, m, n \leq k\}$$

and notice that $\mathcal{J}_k \subset \mathcal{J}$ and $\mathcal{J}_k \subset \mathcal{J}_{k+1}$. Set $\mathcal{C}_k = \mathcal{D}_k/\mathcal{J}_k$. Hence there are obvious maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$, and $\mathcal{C}_k \to \mathcal{C}$.

8.3. Lemma $\mathcal{J} \cap \mathcal{D}_k = \mathcal{J}_k$. *Furthermore,* $\mathcal{J}_{k+1} \cap \mathcal{D}_k = \mathcal{J}_k$.

Proof It suffices to show the first relation. The inclusion $\mathcal{J}_k \subset \mathcal{J} \cap \mathcal{D}_k$ is obvious. Let now $X \in \mathcal{J} \cap \mathcal{D}_k$ be written as a finite sum of the spanning set of $\mathcal{J}$. We can assume that the indices m, n appearing in it satisfy $\min\{m, n\} \leq k$. After subtracting to X some element of $\mathcal{J}_k$, we may assume that X takes the form

$$X = \sum(\phi \otimes A \circ \psi - \phi \circ A \otimes \psi) + \sum \xi \otimes A' \circ \eta - \xi \circ A' \otimes \eta$$

where $A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n})$, $A' \in (V^{\underline{\otimes} q}, V^{\underline{\otimes} r})$, $n, q \leq k$, $m, r > k$. We reduce m and r as follows. Consider isometries $S \in (V_\lambda, V^{\underline{\otimes} m})$, where for simplicity we have dropped indices to S and λ, which have pairwise orthogonal ranges which sum up to p_m, and similarly consider $T \in (V_\mu, V^{\underline{\otimes} q})$. Notice that the indices of the domains of S and T satisfy the required bounds. We write the first sum in X as

$$\sum(\phi \otimes [AS] \circ [S^*\psi] - \phi \circ [AS] \otimes [S^*\psi]) + \sum(\phi \circ [AS] \otimes [S^*\psi] - \phi A \otimes \psi)$$

and similarly for the second sum. The first sum above now lies in the span of the claimed spanning set, hence we are left to show that the remaining part, now written as

$$Y = \sum([\phi A]S \otimes S^*\psi - \phi A \otimes \psi) + \sum(\xi \otimes A'\eta - \xi T \otimes T^*[A'\eta])$$

vanishes. But $Y \in \mathcal{J} \cap \mathcal{D}_k$, and the domain of A and the range of A' have large indices, hence the sum of the terms in second and third position must vanish. We are thus reduced to show the general statement that given elements ϕ_i and ψ_i of a Hilbert space H and operators $Y_i : K_i \to H$, on another Hilbert spaces K_i such that $\sum_i \phi_i^* \otimes \psi_i = 0$ then $\sum_{i,r} \phi_i^* Y_i \otimes Y_i^*\psi_i = 0$, and this can now be checked by means of orthonormal base decomposition of ϕ_i, ψ_i.

8.4. Proposition *The natural maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$ are faithful and form a finite increasing sequence of inclusions of *–invariant subcoalgebras*

$$\mathbb{C} = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_m = \mathcal{C}$$

stabilizing to $\mathcal{C}$.

Proof The natural maps $\mathcal{C}_k \to \mathcal{C}_{k+1}$, $\mathcal{C}_k \to \mathcal{C}$ are faithful, by the previous lemma. On the other hand, there is an integer m such that every irreducible V_λ with $\lambda \in \Lambda_\ell$ is contained in $V^{\otimes n}$ for some $n \leq m$, hence $\mathcal{C}_k$ must stabilize to $\mathcal{C}_k = \mathcal{C}$ for $k \geq m$.

We next pass from $\mathcal{C}_k$ to a quotient $\widetilde{\mathcal{C}}_k$ and from inclusions $\mathcal{C}_k \to \mathcal{C}_{k+1}$ to (possibly non-faithful) linear maps $\widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{k+1}$. The advantage will be existence of natural, even associative, multiplication maps $\widetilde{\mathcal{C}}_h \otimes \widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{h+k}$.

More precisely, we observe that associativity failure of $\mathcal{D}$ can be described in terms of certain negligible intertwiners Z of the tilting category, as follows.

8.5. Lemma *For any triple $\alpha = \phi \otimes \psi$, $\beta = \phi' \otimes \psi'$, $\gamma = \phi'' \otimes \psi''$ of elements of $\mathcal{D}$, of grades m, n, r respectively,*

$$(\alpha\beta)\gamma - \alpha(\beta\gamma) =$$
$$(\phi\phi'\phi'')p_{m+n+r} \otimes Z(\psi\psi'\psi'') + (\phi\phi'\phi'') \circ Z^* \otimes p_{m+n+r}(\psi p_{n+r}(\psi'\psi'')),$$

where

$$Z = p_{m+n+r} \circ 1_m \otimes (1_{n+r} - p_{n+r}).$$

Consider the following spaces of negligible arrows of the tilting category,

$$\mathcal{Z}^{(k)} := \{p_{q+j+r} \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^r}; q + j + r \leq k\}$$

and then define

$$\widetilde{\mathcal{J}}_k := \text{l.s.}\{\mathcal{J}_k, \quad \phi \otimes Z \circ \psi', \quad \phi' \circ (Z')^* \otimes \psi\},$$

where Z, Z' vary in $\mathcal{Z}^{(k)}$, ψ, ϕ in the canonical truncated tensor powers of V, but ψ' and ϕ' belong to the full tensor powers.

8.6. Proposition *We have that* $\mathcal{D}_j\widetilde{\mathcal{J}}_k, \quad \widetilde{\mathcal{J}}_k\mathcal{D}_j \subset \widetilde{\mathcal{J}}_{j+k}$.

Proof Arguments similar to those of Prop. 8.1 a), but keeping track of the grades of homogeneous elements, show that $\mathcal{J}_k\mathcal{D}_j \subset \mathcal{J}_{j+k}$, and hence $\mathcal{J}_k\mathcal{D}_j$ is a subspace of $\widetilde{\mathcal{J}}_{j+k}$. Similar considerations hold for products $y\zeta$ with $y \in \widetilde{\mathcal{J}}_k$ of the form $y = \phi \otimes Z\psi'$ or $y = \phi \circ Z^* \otimes \psi$ and $\zeta := \xi \otimes \eta \in (V^h)^*p_h \otimes p_hV^h$,, $h \leq j$, noticing that in the first case for example

$$y\zeta = \phi\xi p_{n+h} \otimes p_{n+h} \circ Z \otimes 1_{V^h}(\psi'\eta)$$

and that the map $Z \to p_{n+h} \circ Z \otimes 1_h$ for $Z = p_r \circ 1_u \otimes (1 - p_s) \otimes 1_t$, $r = u + s + t \leq k$, takes $\mathcal{Z}^{(k)}$ to $\mathcal{Z}^{(k+j)}$. The left ideal property is more delicate to check, due to lack of associativity of the projections p_n. In order to check that $\zeta y \in \widetilde{\mathcal{J}}_{j+k}$ we compute

$$\zeta y = \xi\phi p_{h+n} \otimes p_{h+n} \circ 1_h \otimes Z(\eta\psi').$$

Write $p_{h+n} \circ 1_h \otimes Z$ in the form

$$p_{h+n} \circ 1_h \otimes Z = Z_1 - Z_2 \circ 1_{h+q} \otimes (1 - p_j) \otimes 1_r$$

with $Z_1 = p_{h+n} \circ 1_{h+q} \otimes (1 - p_j) \otimes 1_r$, $Z_2 = p_{h+n} \circ 1_h \otimes (1 - p_n)$ and notice that Z_i both lie in $\mathcal{Z}^{(j+k)}$ and this implies $\zeta y \in \widetilde{\mathcal{J}}_{j+k}$. We are left to show that $\zeta x \in \widetilde{\mathcal{J}}_{j+k}$, for all $\zeta \in \mathcal{D}_j$ as above and $x = \phi \otimes A \circ \psi - \phi \circ A \otimes \psi \in \mathcal{J}_k$. We claim that it suffices to the take $j = 1$ and $\zeta = v_{\xi,\eta}$ with $v_{\xi,\eta} := \xi \otimes \eta \in V^* \otimes V$. Indeed, we have already noticed that finite products $(\dots((v_{\xi_1,\eta_1} v_{\xi_2,\eta_2}) v_{\xi_3,\eta_3}) \dots v_{\xi_h,\eta_h})$, are total in $(V^h)^* p_h \otimes p_h V^h$, and multiplication of $\mathcal{D}$ is associative up to summing elements of $\widetilde{\mathcal{J}}_{j+k}$, by Lemma 8.5. We thus compute

$$v_{\xi,\eta} x = \xi\phi p_{1+n} \otimes [p_{1+n} \circ 1_V \otimes A(\eta\psi)] - [(\xi\phi) \circ 1_V \otimes A \circ p_{1+m}] \otimes p_{1+m}\eta\psi =$$

$$\xi\phi p_{1+n} \otimes (1\underline{\otimes}A)(p_{1+m}\eta\psi) - (\xi\phi p_{1+n})(1\underline{\otimes}A) \otimes p_{1+m}\eta\psi +$$

$$\xi\phi p_{1+n} \otimes Y_1(\eta\psi) - (\xi\phi) Y_2 \otimes p_{1+m}\eta\psi$$

where

$$Y_1 = p_{1+n} \circ 1_V \otimes A \circ (1 - p_{1+m}), \quad Y_2 = (1 - p_{1+n}) \circ 1_V \otimes A \circ p_{1+m}.$$

Hence $v_{\xi,\eta} x \in \widetilde{\mathcal{J}}_{k+1}$, provided we show that the last two terms in the sum vanish, or, equivalently, that $Y_1 \circ 1_V \otimes p_m = 0 = 1_V \otimes p_n \circ Y_2$, but this follows from Lemma 7.3, and the proof is complete.

8.7. Proposition *Each subspace $\widetilde{\mathcal{J}}_k$ is *–invariant.*

Proof As already noticed in the proof of Prop. 8.1, the proof of *-invariance of $\mathcal{J}_k$ is similar to that of the generic case. For $Z \in \mathcal{Z}^{(k)} \cap (V^n, V^{\underline{\otimes} n})$, $\phi \in V^n$, $\psi \in p_n V^n$,

$$(\psi^* \otimes Z\phi)^* = (\tau_n Z\phi)^* \otimes \tau_n^{-1}\psi =$$

$$[(\tau_n Z\phi)^* \otimes \tau_n^{-1}\psi - (\tau_n Z\phi)^* \tau_n^{-1} \otimes \psi] + \phi^* \circ Z^* \otimes \psi$$

which thus lies in $\widetilde{\mathcal{J}}_k$. One can similarly show that $((\psi \circ Z')^* \otimes \phi)^* \in \widetilde{\mathcal{J}}_k$ as well for $Z' \in (\mathcal{Z}^{(k)})^*$ and thus conclude that $\widetilde{\mathcal{J}}_k$ is *-invariant.

8.8. Proposition *We have that $\Delta(\widetilde{\mathcal{J}}_k) \subset \widetilde{\mathcal{J}}_k \otimes \mathcal{D}_k + \mathcal{D}_k \otimes \widetilde{\mathcal{J}}_k$.*

Proof This can be proved similarly to Prop. 6.5, keeping track of the grades of homogeneous elements.

We set $\widetilde{\mathcal{C}}_k := \mathcal{D}_k / \widetilde{\mathcal{J}}_k$ for $k \in \mathbb{N}$. The composition of the natural linear inclusion $\mathcal{D}_h \to \mathcal{D}_k$, $k > h$, with projection $\mathcal{D}_k \to \widetilde{\mathcal{C}}_k$ factors through a linear map

$$\widetilde{\mathcal{C}}_h \to \widetilde{\mathcal{C}}_k$$

and this is an inductive system. We have natural quotient maps

$$\mathcal{C}_k \to \widetilde{\mathcal{C}}_k$$

and we denote by $e^\lambda_{i,j} \in \widetilde{\mathcal{C}}_k$ the image of the matrix coefficient $v^\lambda_{i,j} \in \mathcal{C}_k$ corresponding to an orthonormal basis of V_λ. Let Λ^k_ℓ denote the set of $\lambda \in \Lambda_\ell$ for which V_λ is a summand of some $V^{\underline{\otimes} n}$ with $n \leq k$. In analogy with the properties of Prop. 8.1 for $\mathcal{C}$, we also summarise the results of the last two sections for $\widetilde{\mathcal{C}}_k$, which now take a stronger form.

8.9. Theorem *Assume that $\mathfrak{g} \neq E_8$ and let V be Wenzl's fundamental representation of $\mathfrak{g}$ ($V = V_{\kappa_1} \oplus V_{\kappa_2}$ in the type D case). Then*

a) *$\widetilde{\mathcal{C}}_k$ is a *–coalgebra linearly spanned by elements $e^\lambda_{i,j}$ labelling matrix units corresponding to V_λ, for $\lambda \in \Lambda^k_\ell$,*

b) *coproduct and involution satisfy*

$$\Delta(e^\lambda_{i,j}) = \sum_r e^\lambda_{i,r} \otimes e^\lambda_{r,j}$$

$$(e^\lambda_{i,j})^* = e^\lambda_{j,i},$$

in particular the involution is anticomultiplicative,

c) *there are associative multiplication maps* $\widetilde{\mathcal{C}}_h \otimes \widetilde{\mathcal{C}}_k \to \widetilde{\mathcal{C}}_{h+k}$ *and an element* $I \in \widetilde{\mathcal{C}}_0$ *acting as the identity. The involution is antimultiplicative and the coproduct is unital and multiplicative.*

9. Quasi-coassociative dual C^*–quantum groupoids $\widehat{\mathcal{C}(G, \ell)}$

The aim of the present and the next section is to show the main result of the paper, stating that if $G = \mathrm{SU}(N)$ then the dual groupoid $\widehat{\mathcal{C}(G, \ell)}$ can be made into a C^*–quantum groupoid, satisfying the axioms of a weak quasi-Hopf C^*-algebra in the sense of [31]. Furthermore the representation category $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ of $\widehat{\mathcal{C}}$ generated by the fundamental representation turns out to be a tensor C^*–category equivalent to the original fusion category $\mathcal{F}_\ell$.

We shall divide the proof in two parts. Throughout this section $\mathfrak{g}$ is general (but $\neq E_8$) and we assume to know that $\mathcal{C}(G, \ell)$ is cosemisimple with respect to the coalgebra structure introduced in Sect. 8. We then show that the above conclusions hold for $\widehat{\mathcal{C}(G, \ell)}$. More precisely, in Subsect. a) we upgrade $\mathcal{C}(G, \ell)$ to a non-associative bi-algebra with antipode associated to a fixed section of the quotient map $\mathcal{D}(V, \ell) \to \mathcal{C}(G, \ell)$, while in b) we pass to the dual groupoid $\widehat{\mathcal{C}(G, \ell)}$, in c) we construct the Drinfeld's associator of $\widehat{\mathcal{C}(G, \ell)}$ and discuss the main properties, in d) we explicit quasi-invertible R matrices for $\widehat{\mathcal{C}(G, \ell)}$, in e) we briefly discuss the relation between the groupoid structures associated to different sections, and finally in f) we show that $\mathrm{Rep}_V(\widehat{\mathcal{C}})$ is a tensor C^*–category equivalent to the fusion category $\mathcal{F}_\ell$. In the next section we verify cosemisimplicity in the type A case.

a) Algebra structure and antipode in $\mathcal{C}(G, \ell)$

Let V_λ be a copy of an irreducible representation of $U_q(\mathfrak{g})$ with highest weight $\lambda \in \Lambda_\ell$ and contained in some $V^{\underline{\otimes} n}$, and let M_λ denote the image of $V_\lambda^* \otimes V_\lambda$ in $\mathcal{C}$ under the quotient map $\mathcal{D} \to \mathcal{C}$, which we already know to be subcoalgebras independent of the choice of V_λ and spanning $\mathcal{C}$. We shall say that $\mathcal{C}(G, \ell)$ is *cosemisimple* if the subcoalgebras M_λ are independent matrix coalgebras in $\mathcal{C}$ as λ varies in Λ_ℓ, always understood of full dimension $\dim(V_\lambda)^2$.

If we know that $\mathcal{C}(G, \ell)$ is cosemisimple, we can endow it both with an invertible antipode and with a non-associative algebra structure. Let's start with the antipode, which we introduce in a way similar to the generic case. Fix a complete set V_λ, $\lambda \in \Lambda_\ell$ of irreducibles contained in the various $V^{\underline{\otimes} n}$, and set, for $\phi^* \otimes \psi \in V_\lambda^* \otimes V_\lambda$ and $\lambda \in \Lambda_\ell$,

$$S(v^\lambda_{\phi,\psi}) = v^{\overline{\lambda}}_{j\psi, j\phi},$$

still independent of the choice of V_λ. It satisfies the relations

$$S(a^*) = S(a)^*, \qquad S^2(v^\lambda_{\phi,\psi}) = v^\lambda_{K\phi, K\psi},$$

$$\Delta \circ S = S \otimes S \circ \Delta^{\mathrm{op}}.$$

where $K = K_{-2\rho}$.

As regards the algebra structure, we pull back the product of $\mathcal{D}(V, \ell)$ via the choice of a section $s : \mathcal{C}(G, \ell) \to \mathcal{D}(V, \ell)$ of the quotient map $\mathcal{D}(V, \ell) \to \mathcal{C}(G, \ell)$. Correspondingly, we have a choice of irreducibles V_λ and s takes $v^\lambda_{\phi,\psi}$ to $\phi^* \otimes \psi$, for $\phi, \psi \in V_\lambda$. We thus set

$$v^\lambda_{\phi,\psi} v^\mu_{\xi,\eta} = [s(v^\lambda_{\phi,\psi}) s(v^\mu_{\xi,\eta})].$$

We always choose $V_0 = \mathbb{C}$, and $V_\kappa = V$ for the trivial and fundamental representation, respectively. In this way, denoting as before by $v_{\xi,\eta}$ the coefficients of V, products of the form $v^\lambda_{\phi,\psi}v_{\xi,\eta}$ encode fusion decomposition of the quotient category $\mathcal{F}_\ell$. However, the section is not unique, and the product of $\mathcal{C}(G,\ell)$ does depend on s (but cf. Subsect. e).)

9.1. Proposition *The product makes $\mathcal{C}(G,\ell)$ into a (non-associative) unital algebra and the coproduct of $\mathcal{C}(G,\ell)$ is a unital homomorphism, and furthermore the following relation holds,*

$$m \circ 1 \otimes S \circ \Delta = \varepsilon = m \circ S \otimes 1 \circ \Delta. \tag{9.1}$$

Proof The first statement follows from Prop. 7.4 b). The left hand side of (9.1) can be proved with computations similar those in the proof of Prop. 6.8, with the only variation that now $m \circ 1 \otimes S \circ \Delta(v^\mu_{\xi,\eta}) = \xi^*(j_\mu\eta)^* \circ p_{h+m} \otimes p_{h+m} \circ \overline{r}(1_\mathbb{C})$. It suffices to notice that $\overline{r}$ always lies in the range of the Wenzl projection p_{h+n}, where h and n correspond to the powers of V containing V_λ and $V_{\overline{\lambda}}$, respectively. More details on this argument can be found in the proof of the following Lemma 10.4.

Hence $\mathcal{C}(G,\ell)$ satisfies all the axioms of a Hopf algebra except associativity of the product and multiplicativity of the counit. The antipode is not antimultiplicative.

b) The dual quantum groupoid $\widehat{\mathcal{C}(G,\ell)}$

We next study the algebra structure associated to a fixed section $s : \mathcal{C}(G,\ell) \to \mathcal{D}(V,\ell)$. Notice that, as an algebra, $\mathcal{C}(G,\ell)$ is quite far from admitting an interpretation as a non commutative space, as compared, e.g., to the compact quantum groups of Woronowicz, in that it may lack some important properties, e.g. antimultiplicativity of the involution, or a C^*-norm, not to mention associativity. It is far more rewarding to pass to the dual $\widehat{\mathcal{C}(G,\ell)}$, and correspondingly consider its *–representations. In this subsection we show that $\widehat{\mathcal{C}(G,\ell)}$ satisfies most properties of the weak quasi-Hopf C^*–algebras of [31].

We identify elements of tensor power algebras $\widehat{\mathcal{C}}^{\otimes n}$ with functionals on $\mathcal{C}^{\otimes n}$. We shall need various elements of these algebras, and we start with $P \in \widehat{\mathcal{C}}^{\otimes 2}$, defined as follows

$$P(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, p_{h+k}\psi \otimes \eta)_{p,2},$$

where h, k are such that V_λ, V_μ are summands of $V^{\underline{\otimes} h}$, $V^{\underline{\otimes} k}$, respectively. Notice that these integers are specified by the section. Furthermore, the forms defining P is understood with respect to the product form of $V^{\underline{\otimes} h} \otimes V^{\underline{\otimes} k}$, each factor in turn endowed with Kirillov-Wenzl inner product.

9.2. Theorem *Endowed with dual bi-algebra structure, antipode and involution, $\widehat{\mathcal{C}} = \widehat{\mathcal{C}(G,\ell)}$ is isomorphic, as a *–algebra, to*

$$\widehat{\mathcal{C}} \simeq \bigoplus_{\lambda\in\Lambda_\ell} \mathcal{L}(V_\lambda),$$

hence it is a C^*-algebra. Furthermore, $(\widehat{\mathcal{C}}, \Delta, \varepsilon, S)$ satisfies all the axioms of a Hopf algebra except coassociativity and unitality of Δ. In particular, the following compatibility properties hold, for $\omega, \tau \in \widehat{\mathcal{C}}$,

$$\Delta(I) = P, \qquad \Delta(\omega\tau) = \Delta(\omega)\Delta(\tau), \tag{9.2}$$

$$\varepsilon(\omega\tau) = \varepsilon(\omega)\varepsilon(\tau), \qquad \varepsilon(\omega^*) = \overline{\varepsilon(\omega)}, \tag{9.3}$$

$$S(\omega\tau) = S(\tau)S(\omega), \qquad S(\omega^*) = S(\omega)^*, \tag{9.4}$$

Proof The first or second property stated in $(9.i)$ shall be referred to as $(9.i)_1$ or $(9.i)_2$ respectively. We show $(9.2)_1$. By duality, the identity I of $\widehat{\mathcal{C}}$ is the counit ε of $\mathcal{C}$, and $\Delta(I)$ is the two-variable functional given by

$$\Delta(I)(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = \varepsilon(v^\lambda_{\phi,\psi}v^\mu_{\xi,\eta}) = \varepsilon([\phi^*\xi^*p_{h+k} \otimes p_{h+k}\psi\eta]) =$$

$$\phi^*\xi^* p_{h+k}(p_{h+k}\psi\eta) = (\phi\xi, p_{h+k}\psi\eta)_{p,2} = P(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}).$$

The remaining properties follow from duality and corresponding properties of $\mathcal{C}(G,\ell)$. More precisely, the associative algebra structure and the non-coassociative but counital coalgebra structure of $\widehat{\mathcal{C}}$ obviously follow from the coassociative coalgebra and non-associative algebra structure in $\mathcal{C}$, respectively. Antimultiplicativity of the involution of $\widehat{\mathcal{C}}$ follows from anticomultiplicativity of that of $\mathcal{C}$. The antipode axioms $m\circ S\otimes 1\circ\Delta = 1 = m\circ 1\otimes S\circ\Delta$, $(9.2)_2$, and $(9.4)_2$ are selfdual and already shown in $\mathcal{C}$; $(9.3)_1$ and $(9.4)_1$ correspond respectively to unitality of the coproduct and anticomultiplicativity of the antipode in $\mathcal{C}$ (Prop. 9.1; Sect. 9a). Property $(9.3)_2$ is obvious.

The following Prop. 9.9 explicits a compatibility relation between Δ and *-involution in terms of the R-matrix. For completeness we remark that it may be $S\otimes S\circ\Delta \neq \Delta^{\mathrm{op}}\circ S$ in $\widehat{\mathcal{C}(G,\ell)}$, as this corresponds to antimultiplicativity failure of the antipode of $\mathcal{C}(G,\ell)$. However, this fact plays no role in the representation theory of $\widehat{\mathcal{C}(G,\ell)}$.

c) Drinfeld's associator

For a given weight $\lambda\in\Lambda_\ell$, let h_λ denote the truncated powers of V containing V_λ, as prescribed by the choice of a section s. It will be useful for later computations to have a multiplication rule for elements of $\widehat{\mathcal{C}}^{\otimes n}$ of the following form. Let $T = (T_n)$ be a sequence of linear maps $T_n : V^{\otimes n}\to V^{\otimes n}$ and associate the element ω_T of $\widehat{\mathcal{C}}^{\otimes n}$ defined by

$$\omega_T(v^{\lambda_1}_{\phi_1,\psi_1},\dots,v^{\lambda_n}_{\phi_n,\psi_n}) = (\phi_1\otimes\dots\otimes\phi_n, T_{h_{\lambda_1}+\dots+h_{\lambda_n}}\psi_1\otimes\dots\otimes\psi_n)_{p,n},$$

where the form is a tensor product form with n factors. The following lemma is a convenient formulation of the generalisation to tensor powers of the direct sum decomposition of the algebra $\widehat{\mathcal{C}}$ into its simple components.

9.3. Lemma *Given $S=(S_n)$ and $T=(T_n)$ as above, set $\omega=\omega_S\omega_T$. Then*

$$\omega(v^{\lambda_1}_{\phi_1,\psi_1},\dots,v^{\lambda_n}_{\phi_n,\psi_n}) = (\phi_1\dots\phi_n, S_{h_{\lambda_1}+\dots+h_{\lambda_n}}\circ p_{h_{\lambda_1}}\otimes\dots\otimes p_{h_{\lambda_n}}\circ T_{h_{\lambda_1}+\dots+h_{\lambda_n}}\psi_1\dots\psi_n)_{p,n}.$$

The following elements Φ, $\Psi\in\widehat{\mathcal{C}}^{\otimes 3}$ are important examples. Set

$$q_{h_\lambda,h_\mu,h_\nu} = \sum_{\gamma,i} 1_{h_\lambda}\otimes S_{\gamma,i}\circ p_{h_\lambda+h_\gamma}\circ 1_{h_\lambda}\otimes S^*_{\gamma,i},$$

where $S_{\gamma,i}\in(V_\gamma, V^{\underline{\otimes}(h_\mu+h_\nu)})$ are isometries of the fusion category satisfying $\sum_{\gamma,i}S_{\gamma,i}S^*_{\gamma,i} = p_{h_\mu+h_\nu}$. Observe that $q_{h_\lambda,h_\mu,h_\nu}$ does not depend on the choice of the isometries. We set

$$\Phi(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) = (\phi\otimes\xi\otimes\chi, q_{h_\lambda,h_\mu,h_\nu}\circ p_{h_\lambda+h_\mu+h_\nu}\psi\otimes\eta\otimes\zeta)_{p,3},$$

$$\Psi(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) = (\phi\otimes\xi\otimes\chi, p_{h_\lambda+h_\mu+h_\nu}\circ q_{h_\lambda,h_\mu,h_\nu}\psi\otimes\eta\otimes\zeta)_{p,3}.$$

The lemma then shows that Φ, Ψ are idempotent but not selfadjoint elements of the corresponding tensor power *-algebra. The meaning of these elements and computation of the adjoints will soon be apparent.

9.4. Theorem *The idempotents Φ, Ψ satisfy the following relations,*

$$\Psi\Phi = \Delta\otimes 1\circ\Delta(I),\qquad \Phi\Psi = 1\otimes\Delta\circ\Delta(I), \tag{9.5}$$

$$\Psi\Phi\Psi = \Psi,\qquad \Phi\Psi\Phi = \Phi \tag{9.6}$$

$$\Phi\Delta\otimes 1\circ\Delta(\omega) = 1\otimes\Delta\circ\Delta(\omega)\Phi, \tag{9.7}$$

$$\Psi 1\otimes\Delta\circ\Delta(\omega) = \Delta\otimes 1\circ\Delta(\omega)\Psi. \tag{9.8}$$

Proof To show (9.7) and (9.8) we pass to the predual $\mathcal{C}$, and the corresponding properties, which, in Sweedler notation, respectively read as

$$\Phi(a_1, b_1, c_1)(a_2b_2)c_2 = a_1(b_1c_1)\Phi(a_2, b_2, c_2), \tag{9.9}$$

$$\Psi(a_1, b_1, c_1)a_2(b_2c_2) = (a_1b_1)c_1\Psi(a_2, b_2, c_2), \tag{9.10}$$

for a, b, $c \in \mathcal{C}$. Set $a = v^\lambda_{\phi,\psi}$, $b = v^\mu_{\xi,\eta}$, $c = v^\nu_{\chi,\zeta}$. We write,

$$\Phi(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) = (\phi\xi\chi, T\psi\eta\zeta)_{p,3}$$

and look for a solution T of (9.9). Consider orthonormal bases ψ_r, η_s, ζ_t of the appropriate truncated tensor powers of V. The left hand side becomes

$$(\phi\xi\chi, T\psi_r\eta_s\zeta_t)_{p,3}[\psi^*_r\eta^*_s\zeta^*_t p_3 \otimes p_3\psi\eta\zeta],$$

where we have written p_3 for $p_{h_\lambda+h_\mu+h_\nu}$. But

$$(\phi\xi\chi, T\psi_r\eta_s\zeta_t)_{p,3}\psi^*_r\eta^*_s\zeta^*_t = ((\psi_r\eta_s\zeta_t, T^+\phi\xi\chi)_{p,3}\psi_r\eta_s\zeta_t)^+ =$$
$$(p_1 \otimes p'_1 \otimes p''_1 \circ T^+\phi\xi\chi)^+ = \phi^*\xi^*\chi^* \circ T \circ p_1 \otimes p'_1 \otimes p''_1$$

where $p_1 = p_{h_\lambda}$, $p'_1 = p_{h_\mu}$, $p''_1 = p_{h_\nu}$ and $+$ is the adjoint operator with respect to the triple product form. Notice that on each factor we are using Kirillov-Wenzl inner product, hence p_1, p'_1, p''_1 are selfadjoint. Thus the left hand side of (9.9) is the class of

$$(\phi^*\xi^*\chi^* \circ T \circ p_1 \otimes p'_1 \otimes p''_1 \circ p_3) \otimes p_3\psi\eta\zeta. \tag{9.11}$$

On the other hand, $b_1c_1 = \sum_{\gamma,i}[\xi^*\chi^* \circ S_{\gamma,i} \otimes S^*_{\gamma,i}\eta_s\zeta_t]$, hence the right hand side gives

$$(\phi^*\xi^*\chi^* \circ 1 \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma}) \otimes (p_{h_\lambda+h_\gamma} \circ 1 \otimes S^*_{\gamma,i}\psi_r\eta_s\zeta_t)(\psi_r\eta_s\zeta_t, T\psi\eta\zeta)_{p,3} =$$
$$(\phi^*\xi^*\chi^* \circ 1 \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma}) \otimes (p_{h_\lambda+h_\gamma} \circ 1 \otimes S^*_{\gamma,i} \circ p_1 \otimes p'_1 \otimes p''_1 \circ T\psi\eta\zeta). \tag{9.12}$$

Hence (9.11) and (9.12) coincide for $T = q_{h_\lambda,h_\mu,h_\nu} \circ p_{h_\lambda+h_\mu+h_\nu}$ thanks to Lemma 5.3. In a similar way, (9.10) leads to the equation

$$(\phi^*\xi^*\chi^* \circ T' \circ p_1 \otimes p'_1 \otimes p''_1 \circ 1 \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma}) \otimes (p_{h_\lambda+h_\gamma} \circ 1 \otimes S^*_{\gamma,i}\psi\eta\zeta) =$$
$$\phi^*\xi^*\chi^* \circ p_3 \otimes (p_3 \circ p_1 \otimes p'_1 \otimes p''_1 \circ T'\psi\eta\zeta),$$

which is solved for $T' = p_{h_\lambda+h_\mu+h_\nu} \circ q_{h_\lambda,h_\mu,h_\nu}$. The previous lemma shows that $\Psi\Phi$ and $\Phi\Psi$ are the functionals induced respectively by

$$p_3 \circ q \circ p_1 \otimes p'_1 \otimes p''_1 \circ q \circ p_3 = p_3,$$
$$q \circ p_3 \circ p_1 \otimes p'_1 \otimes p''_1 \circ p_3 \circ q = q \circ p_3 \circ q,$$

and it is easy to check that these coincide with the right hand sides of (9.5). One similarly shows (9.6).

9.5. Theorem *The element $\Phi \in \widehat{\mathcal{C}}^{\otimes 3}$ is a counital 3-cocycle:*

$$1 \otimes 1 \otimes \Delta(\Phi)\Delta \otimes 1 \otimes 1(\Phi) = I \otimes \Phi 1 \otimes \Delta \otimes 1(\Phi)\Phi \otimes I, \tag{9.13}$$
$$\varepsilon \otimes 1 \otimes 1(\Phi) = 1 \otimes \varepsilon \otimes 1(\Phi) = 1 \otimes 1 \otimes \varepsilon(\Phi) = P.$$

Proof We compute the elements $T_1, T_2, T_3 \in \widehat{\mathcal{C}}^{\otimes 4}$ corresponding to $\Delta \otimes 1 \otimes 1(\Phi)$, $1 \otimes \Delta \otimes 1(\Phi)$, and $1 \otimes 1 \otimes \Delta(\Phi)$. We first write

$$\Phi(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) = \phi^*\xi^*\chi^*(1_{h_\lambda} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\gamma} \circ 1_{h_\lambda} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\mu+h_\nu}\psi\eta\zeta)$$

with $S_{\gamma,i} \in (V_\gamma, V^{\underline{\otimes}(h_\mu+h_\nu)})$. For T_1 we need to replace the first variable by a product in $\mathcal{C}$. Explicitly, for $S_{\sigma,j} \in (V_\sigma, V^{\underline{\otimes}(h_\lambda+h_\tau)})$,

$$\Delta \otimes 1 \otimes 1(\Phi)(v^\lambda_{\phi,\psi}, v^\tau_{\alpha,\beta}, v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) = \Phi((v^\lambda_{\phi,\psi}v^\tau_{\alpha,\beta}), v^\mu_{\xi,\eta}, v^\nu_{\chi,\zeta}) =$$
$$\phi^*\alpha^*\xi^*\chi^*(S_{\sigma,j} \otimes 1_{h_\mu+h_\nu} \circ 1_{h_\sigma} \otimes S_{\gamma,i} \circ p_{h_\sigma+h_\gamma} \circ 1_{h_\sigma} \otimes S^*_{\gamma,i} \circ p_{h_\sigma+h_\mu+h_\nu} \circ S^*_{\sigma,j} \otimes 1_{h_\mu+h_\nu}\psi\beta\eta\zeta).$$

Taking into account Lemma 5.1 a),

$$T_1 = 1_{h_\lambda+h_\tau} \otimes S_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\gamma} \circ 1_{h_\lambda+h_\tau} \otimes S^*_{\gamma,i} \circ p_{h_\lambda+h_\tau+h_\mu+h_\nu}.$$

One also finds

$$T_2 = 1_{h_\lambda} \otimes [S_{\sigma',j'} \otimes 1_{h_\nu} \circ S_{\gamma',i}] \circ p_{h_\lambda+h_{\gamma'}} \circ 1_{h_\lambda} \otimes S^*_{\gamma',i'} \circ p_{h_\lambda+h_{\sigma'}+h_\nu} \circ 1_{h_\lambda} \otimes S^*_{\sigma',j'} \otimes 1_{h_\nu},$$

where $S_{\gamma',i'} \in (V_{\gamma'}, V^{\underline{\otimes}(h_{\sigma'}+h_\nu)})$, $S_{\sigma',j'} \in (V_{\sigma'}, V^{\underline{\otimes}(h_\tau+h_\mu)})$, and

$$T_3 = 1_{h_\lambda+h_\tau} \otimes S_{\sigma'',j''} \circ 1_{h_\lambda} \otimes S_{\gamma'',i''} \circ p_{h_\lambda+h_{\gamma''}} \circ 1_{h_\lambda} \otimes S^*_{\gamma'',i''} \circ p_{h_\lambda+h_\tau+h_{\sigma''}} \circ 1_{h_\lambda+h_\tau} \otimes S^*_{\sigma'',j''},$$

with $S_{\gamma'',i''} \in (V_{\gamma''}, V^{\underline{\otimes}(h_\tau+h_{\sigma''})})$, $S_{\sigma'',j''} \in (V_{\sigma''}, V^{\underline{\otimes}(h_\mu+h_\nu)})$. Verification of the cocycle condition (9.13) involves rather long formulas. We shall thus ease notation dropping most indices and writing for example p_2 for expressions of the form $p_{h_\tau+h_\gamma}$.

We compare the matrices corresponding to both sides of (9.13) taking into account Lemma 9.3, Lemma 5.3 and Lemma 5.1 a) again. For the left hand side we have

$$T_3 \circ p_1 \otimes p_1 \otimes p_1 \otimes p_1 \circ T_1 = T_3T_1 =$$
$$1 \otimes 1 \otimes S_{\sigma''} \circ 1 \otimes S_{\gamma''} \circ p_2 \circ 1 \otimes S^*_{\gamma''} \circ p_3 \circ 1 \otimes 1 \otimes S^*_{\sigma''} \circ 1 \otimes 1 \otimes S_\gamma \circ p_3 \circ 1 \otimes 1 \otimes S^*_\gamma \circ p_4 =$$
$$1 \otimes 1 \otimes S_{\sigma''} \circ 1 \otimes S_{\gamma''} \circ p_2 \circ 1 \otimes S^*_{\gamma''} \circ 1 \otimes 1 \otimes S^*_{\sigma''} \circ p_4,$$

while the right hand side gives, if T is the matrix defining Φ,

$$(I \otimes T) \circ p_1^{\otimes 4} \circ T_2 \circ p_1^{\otimes 4} \circ (T \otimes I) =$$
$$1 \otimes 1 \otimes S_\gamma \circ 1 \otimes p_2 \circ 1 \otimes 1 \otimes S^*_\gamma \circ 1 \otimes p_3 \circ p_1^{\otimes 4} \circ 1 \otimes (S_{\sigma'} \otimes 1 \circ S_{\gamma'}) \circ p_2 \circ$$
$$1 \otimes S^*_{\gamma'} \circ p_3 \circ 1 \otimes S^*_{\sigma'} \otimes 1 \circ p_1^{\otimes 4} \circ 1 \otimes S_{\gamma'''} \otimes 1 \circ p_2 \otimes 1 \circ 1 \otimes S^*_{\gamma'''} \otimes 1 \circ p_3 \otimes 1 =$$
$$1 \otimes 1 \otimes S_\gamma \circ 1 \otimes p_2 \circ 1 \otimes 1 \otimes S^*_\gamma \circ 1 \otimes (S_{\sigma'} \otimes 1 \circ S_{\gamma'}) \circ p_2 \circ$$
$$1 \otimes S^*_{\gamma'} \circ [p_3 \circ (p_2 \circ 1 \otimes S^*_{\sigma'} \circ p_3) \otimes 1] \circ 1 \otimes 1 \otimes 1 \otimes p_1 =$$

in the last equality we have deleted the extra idempotents $1 \otimes p_3$, the first $p_1^{\otimes 4}$ altogether, and the first three factors $p_1^{\otimes 3}$ of the second copy of the same idempotent, and moved the fourth to the far right. This has also allowed to use $S^*_{\sigma'} \circ S_{\gamma'''} = \delta_{\sigma',\gamma'''}$. Next write the term in square brackets as $(p_2 \circ 1 \otimes S^*_{\sigma'} \circ p_3) \otimes 1 \circ p_4$ and after elimination of p_2 and p_3, the above term becomes

$$1 \otimes 1 \otimes S_\gamma \circ 1 \otimes (p_2 \circ 1 \otimes S^*_\gamma \circ S_{\sigma'} \otimes 1 \circ S_{\gamma'}) \circ p_2 \circ$$
$$1 \otimes S^*_{\gamma'} \circ 1 \otimes S^*_{\sigma'} \otimes 1 \circ p_4 \circ 1 \otimes 1 \otimes 1 \otimes p_1 =$$
$$1 \otimes 1 \otimes S_\gamma \circ 1 \otimes S'_{\gamma'} \circ p_2 \circ 1 \otimes {S'_{\gamma'}}^* \circ 1 \otimes 1 \otimes S^*_\gamma \circ p_4 \circ 1 \otimes 1 \otimes 1 \otimes p_1$$

where $S'_{\gamma'} = p_2 \circ 1 \otimes S^*_\gamma \circ S_{\sigma'} \otimes 1 \circ S_{\gamma'}$ is another orthonormal system of isometries. Hence the two matrices induce the same functional.

9.6. Remarks a) Notice that not only the associators Φ, Ψ are quasi-invertible elements of a rather special kind, in that they are idempotent, but also there is a simple relation between iterated coproducts associated so

$$\Delta^{(n)}_{\text{left}} := \Delta \otimes 1_{n-1} \circ \cdots \circ \Delta \otimes 1 \circ \Delta$$

and arbitrary iterated coproducts $\Delta^{(n)}$ of order n, i.e. coproducts that can be obtained as compositions $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}} \to \cdots \to \widehat{\mathcal{C}}^{\otimes n+1}$ where the connecting maps $\widehat{\mathcal{C}}^{\otimes j} \to \widehat{\mathcal{C}}^{\otimes j+1}$ can be an arbitrary translates $1_r \otimes \Delta \otimes 1_{j-r-1}$ of Δ. Set $P_n = \Delta^{(n)}_{\text{left}}(I)$, hence in particular $P_1 = P$, $P_2 = \Psi\Phi$. Then

$$\Delta^{(n)}_{\text{left}}(\omega) = P_n \Delta^{(n)}(\omega) P_n$$

for all possible choices of $\Delta^{(n)}$. This relation can be derived with computations similar to those appearing in the proof of Theorem 9.4, and also in the following Prop. 9.7. b) In a similar way, the associators are explicitly related to non-unitality of Δ by $\Phi = 1 \otimes \Delta \circ \Delta(I) \Delta \otimes 1 \circ \Delta(I)$, and similarly for Ψ.

d) R-matrices

We may introduce quasi-invertible R-matrices $\mathcal{R}, \mathcal{R}_1 \in \widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$ appealing to the braided symmetry of the quotient category $\mathcal{F}_\ell$. More precisely, set

$$\mathcal{R}(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, \Sigma_{k,h} p_{k+h} \Sigma_{h,k} R p_{h+k} \psi \otimes \eta)_{p,2},$$

$$\mathcal{R}_1(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = (\phi \otimes \xi, p_{k+h} R^{-1} \Sigma_{k,h} p_{k+h} \Sigma_{h,k} \psi \otimes \eta)_{p,2},$$

with $\Sigma_{h,k} : V^{\otimes h} \otimes V^{\otimes k} \to V^{\otimes k} \otimes V^{\otimes h}$ the transposition operator.

9.7. Proposition *The following relations hold,*

$$\mathcal{R}\Delta(I) = \mathcal{R} = \Delta^{\mathrm{op}}(I)\mathcal{R}, \qquad \mathcal{R}_1\Delta^{\mathrm{op}}(I) = \mathcal{R}_1 = \Delta(I)\mathcal{R}_1 \tag{9.14}$$

$$\mathcal{R}\Delta(\omega)\mathcal{R}_1 = \Delta^{\mathrm{op}}(\omega), \qquad \mathcal{R}_1\Delta^{\mathrm{op}}(\omega)\mathcal{R} = \Delta(\omega), \tag{9.15}$$

where Δ^{op} is the opposite coproduct of $\widehat{\mathcal{C}(G,\ell)}$.

Proof We check $(9.15)_1$. Let $T = (T_n)$ and $T' = (T'_n)$ define functionals $\omega_T, \omega_{T'}$ on $\widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$. Then a relation $\omega_T\Delta(\tau)\omega_{T'} = \Delta^{\mathrm{op}}(\tau)$, for all $\tau \in \widehat{\mathcal{C}}$ is equivalent to $\omega_T(a_1, b_1)a_2b_2\omega_{T'}(a_3, b_3) = ba$ for all $a, b \in \mathcal{C}$. For $a = v^\lambda_{\phi,\psi}$, $b = v^\mu_{\xi,\eta}$, this relation reads

$$(\phi^*\xi^* \circ T \circ p_h \otimes p_k \circ p_{h+k}) \otimes (p_{h+k} \circ p_h \otimes p_k \circ T'\psi\eta) = \xi^*\phi^* \circ p_{k+h} \otimes p_{k+h}\eta\psi.$$

as class elements of $\mathcal{C}$. For $T = \Sigma_{k,h}p_{k+h}\Sigma_{h,k}Rp_{h+k}$, $T' = p_{k+h}R^{-1}\Sigma_{k,h}p_{k+h}\Sigma_{h,k}$ the left hand side becomes

$$(\xi^*\phi^*p_{k+h}\Sigma_{h,k}Rp_{h+k}) \otimes (p_{h+k}(\Sigma_{h,k}R)^{-1}p_{k+h}\eta\psi).$$

But $p_{k+h}\Sigma_{h,k}Rp_{h+k}$ and $p_{h+k}(\Sigma_{h,k}R)^{-1}p_{k+h}$ are braiding arrows and inverses of one another in the quotient category, hence they cancel out.

We next briefly discuss a relation between the original quantum group $U_q(\mathfrak{g})$ and $\widehat{\mathcal{C}(G,\ell)}$. There is a natural map

$$\pi : U_q(\mathfrak{g}) \to \widehat{\mathcal{C}(G,\ell)}$$

taking an element $a \in U_q(\mathfrak{g})$ to the functional

$$\pi(a)(v^\lambda_{\phi,\psi}) = (\phi, a\psi), \quad v^\lambda_{\phi,\psi} \in M_\lambda, \quad \lambda \in \Lambda_\ell. \tag{9.16}$$

9.8. Proposition π *is a surjective homomorphism of *–algebras satisfying*

$$P\pi \otimes \pi(\Delta(a)) = \Delta(\pi(a)) = \pi \otimes \pi(\Delta(a))P, \quad a \in U_q(\mathfrak{g}). \tag{9.17}$$

Proof The map π corresponds to $\oplus_{\lambda\in\Lambda_\ell}V_\lambda$ under the identification $\widehat{\mathcal{C}} \simeq \oplus_{\lambda\in\Lambda_\ell}\mathcal{L}(V_\lambda)$, and this shows the homomorphism property. Surjectivity is a consequence of irreducibility of the V_λ. The last statement is a straightforward computation.

We use the above link to resume a compatibility relation between coproduct and adjoint of $\widehat{\mathcal{C}}$.

9.9. Proposition *We have*

$$\Delta(\omega)^* = \pi \otimes \pi(\overline{R})\Delta(\omega^*)\pi \otimes \pi(\overline{R})^{-1}, \qquad \omega \in \widehat{\mathcal{C}}. \tag{9.18}$$

Proof We first show the relation for $\omega = I$. Recalling that $\Delta(I) = P$, we have

$$P^*(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}) = \overline{P((v^\lambda_{\phi,\psi})^*, (v^\mu_{\xi,\eta})^*)} =$$

$$\overline{P(v^\lambda_{\psi,\phi}, v^\mu_{\eta,\xi})} = \overline{(\psi\eta, p_{h+k}\phi\xi)}_{p,2} =$$

$$(p_{h+k}\phi\xi, \psi\eta)_{p,2} = (\phi\xi, \overline{R}p_{h+k}\overline{R}^{-1}\phi\xi)_{p,2} =$$

$$\pi \otimes \pi(\overline{R})P\pi \otimes \pi(\overline{R})^{-1}(v^\lambda_{\phi,\psi}, v^\mu_{\xi,\eta}).$$

Taking into account the relations $\Delta^{\rm op}(a) = \overline{R}\Delta(a)\overline{R}^{-1}$ and $\Delta(a)^* = \Delta^{\rm op}(a^*)$ in $U_q(\mathfrak{g})$ for $a \in U_q(\mathfrak{g})$, it follows that

$$\Delta(\pi(a))^* = (\pi \otimes \pi(\Delta(a))P)^* =$$
$$\pi \otimes \pi(\overline{R})P\pi \otimes \pi(\overline{R})^{-1}\pi \otimes \pi(\overline{R}\Delta(a^*)\overline{R}^{-1}) =$$
$$\pi \otimes \pi(\overline{R})\Delta(\pi(a^*))\pi \otimes \pi(\overline{R})^{-1}.$$

e) Comparing the groupoids associated to different sections

Let $s : \mathcal{C}(G, \ell) \to \mathcal{D}(V, \ell)$ and $s' : \mathcal{C}(G, \ell) \to \mathcal{D}(V, \ell)$ be different sections of the quotient map $\mathcal{D}(V, \ell) \to \mathcal{C}(G, \ell)$. Correspondingly, we upgrade the algebra $\widehat{\mathcal{C}(G, \ell)}$ to quasi-coassociative quantum groupoids in two different ways,

$$(\Delta, \Phi, \Psi, \mathcal{R}, \mathcal{R}_1) \quad \text{and} \quad (\Delta', \Phi', \Psi', \mathcal{R}', \mathcal{R}_1'),$$

(we have noticed before that S is independent of the section.) We claim that these are related by a twisting procedure in the sense of Drinfeld induced, again, by special quasi-invertible elements $\widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$. We sketch the intertwining relation between the coproducts, but we refrain from giving details for the remaining relations.

9.10. Proposition *Set* $P = \Delta(I)$, $P' = \Delta'(I)$. *Then for* $\omega \in \widehat{\mathcal{C}}$,

$$P\Delta'(\omega)P = \Delta(\omega), \qquad P'\Delta(\omega)P' = \Delta'(\omega).$$

Proof To show for example the first relation, we first write P in a different way, as follows. Let s and s' be defined by two choices of irreducible summands V_λ and V'_λ both of highest weight λ in $V^{\underline{\otimes} h_\lambda}$ and $V^{\underline{\otimes} h'_\lambda}$, respectively. For each $\lambda \in \Lambda_\ell$ there is a unitary intertwiner $U_\lambda : V'_\lambda \to V_\lambda$, unique up to a scalar multiple by an element of $\mathbb{T}$. We associate the element $F \in \widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$ defined by

$$F(v^\lambda_{\phi', \psi'}, v^\mu_{\xi', \eta'}) = (U_\lambda\phi' \otimes U_\mu\xi',\, p_{h_\lambda + h_\mu} \circ U_\lambda\psi' \otimes U_\mu\eta')_{p,2},$$

for ϕ', $\psi' \in V'_\lambda$, ξ', $\eta' \in V'_\mu$. On one hand we may use the defining identifications in $\mathcal{C}(G, \ell)$ and write, for ϕ, $\psi \in V_\lambda$, $v^\lambda_{\phi,\psi} = v^\lambda_{U^*_\lambda\phi, U^*_\lambda\psi}$, and this shows that $F = P$. On the other, we can use the computations of the first part of the proof of Prop. 9.7 and derive the relation $F\Delta'(\omega)F = \Delta(\omega)$.

f) The tensor *–equivalence $\mathcal{F}_\ell \to$ Rep$_V\widehat{\mathcal{C}(G, \ell)}$.

In this subsection we show that if $\mathcal{C}(G, \ell)$ is cosemisimple then the smallest full tensor subcategory $\text{Rep}_V\widehat{\mathcal{C}(G, \ell)}$ of the representation category of $\widehat{\mathcal{C}(G, \ell)}$ containing the fundamental representation is a tensor C^*–category equivalent to the fusion category $\mathcal{F}_\ell$.

There is a known way to associate a tensor category to any weak quasi-Hopf algebra [31], that we briefly recall for $\widehat{\mathcal{C}}$. Let $\text{Rep}(\widehat{\mathcal{C}})$ be the category of unital representations of $\widehat{\mathcal{C}}$ on f.d. vector spaces. We have already noticed that $P = \Delta(I)$ is an idempotent of $\widehat{\mathcal{C}} \otimes \widehat{\mathcal{C}}$ commuting with all elements in the image of Δ. Hence if u and v are objects of $\text{Rep}(\widehat{\mathcal{C}})$ then $\omega \in \widehat{\mathcal{C}} \to u \otimes v \circ \Delta(\omega)$ is a representation of the same algebra on the tensor product space with support the image of the idempotent $u \otimes v(P)$. We define the tensor product object, still denoted $u \otimes v$, to be the unital representation of $\widehat{\mathcal{C}}$ obtained restricting the operators $u \otimes v \circ \Delta(\omega)$ to that subspace. The map $\Phi_{u,v,w}$ given by the restriction of $u \otimes v \otimes w(\Phi)$ to the space of $(u \otimes v) \otimes w$ is an invertible intertwiner from $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$, by Theorem 9.4, which, together with Theorem 9.5, guarantees that $\text{Rep}(\widehat{\mathcal{C}})$ becomes a tensor category with these associativity morphisms. We next consider the full subcategory $\text{Rep}_h(\widehat{\mathcal{C}})$ of $\text{Rep}(\widehat{\mathcal{C}})$ with objects *–representations of $\widehat{\mathcal{C}}$ on non-degenerate Hermitian spaces. Following a procedure similar to that of the non-semisimple case, if u and v are two objects of $\text{Rep}_h(\widehat{\mathcal{C}})$ then $u \otimes v$ is still a *–homomorphism, and hence an object of $\text{Rep}_h(\widehat{\mathcal{C}})$, provided we endow the corresponding vector space with the restriction of Wenzl's Hermitian form induced by $\overline{R}$. This can be seen with Prop. 9.9. Hence $\text{Rep}_h(\widehat{\mathcal{C}})$ is a tensor *–category. Now if u and v are Hilbert space *–representations then we can infer that $u \otimes v$ is still a Hilbert space representation provided $u \otimes v(P)$ is a positive operator with respect to Wenzl's form, and this may not always be the case. But there are sufficient objects of $\text{Rep}_h(\widehat{\mathcal{C}})$ for which the positivity condition holds. Consider the map $\hat{V} : \widehat{\mathcal{C}} \to \mathcal{B}(V)$ defined by

$$(\xi, \hat{V}(\omega)\eta) = \omega([\xi^* \otimes \eta]), \qquad \omega \in \widehat{\mathcal{C}}, \quad \xi, \eta \in V.$$

It is easy to see that this is a unital representation, and in fact a Hilbert space *–representation. Let $\text{Rep}_V(\widehat{\mathcal{C}})$ be the the smallest full tensor subcategory of $\text{Rep}_h(\widehat{\mathcal{C}})$ containing $\hat{V}$. The objects of $\text{Rep}_V(\widehat{\mathcal{C}})$ are the trivial representation, $\hat{V}$ and representations of the form $\omega \to \hat{V} \otimes \cdots \otimes \hat{V} \circ \Delta^{(n)}(\omega)$, $n \geq 1$, where $\Delta^{(n)}$ denotes an iteration of n translates of the coproduct as in Remark 9.6.

9.11. Theorem *$\text{Rep}_V(\widehat{\mathcal{C}})$ is a tensor C^*–category.*

Proof The representation $\omega \to \hat{V} \otimes \cdots \otimes \hat{V} \circ \Delta^{(n)}(\omega)$ acts on the subspace of $V^{\otimes n}$ corresponding to the idempotent $\hat{V} \otimes \cdots \otimes \hat{V} \circ \Delta^{(n)}(I)$. We need to show that Wenzl's inner product of $V^{\otimes n}$ is positive on that subspace. We limit ourselves to computing these idempotents in the example $\Delta^{(2)} = 1 \otimes \Delta \circ \Delta$. In Sweedler notation, $\Delta(I) = \omega_1 \otimes \omega_2$, $\Delta^{(2)}(I) = \omega_1 \otimes \omega_{2,1} \otimes \omega_{2,2}$. Hence

$$(\phi_1\phi_2\phi_3, \hat{V} \otimes \hat{V} \otimes \hat{V} \circ \Delta^{(2)}(I)\psi_1\psi_2\psi_3)_p =$$

$$(\phi_1, \hat{V}(\omega_1)\psi_1)(\phi_2, \hat{V}(\omega_{2,1})\psi_2)(\phi_3, \hat{V}(\omega_{2,2})\psi_3) =$$

$$\omega_1([\phi_1^* \otimes \psi_1])\omega_{2,1}([\phi_2^* \otimes \psi_2])\omega_{2,2}([\phi_3^* \otimes \psi_3]) =$$

$$\omega_1([\phi_1^* \otimes \psi_1])\omega_2([\phi_2^*\phi_3^* \circ p_2 \otimes p_2\psi_2\psi_3]) =$$

$$\omega_1([\phi_1^* \otimes \psi_1])\omega_2([\phi_2^*\phi_3^* \circ S_\alpha \otimes S_\alpha^*\psi_2\psi_3]) =$$

$$\varepsilon([(\phi_1^*\phi_2^*\phi_3^* \circ 1 \otimes S_\alpha \circ p_{1+n_\alpha}) \otimes (p_{1+n_\alpha} \circ 1 \otimes S_\alpha^*\psi_1\psi_2\psi_3)]) =$$

$$(\phi_1\phi_2\phi_3, 1 \otimes S_\alpha \circ p_{1+n_\alpha} \circ 1 \otimes S_\alpha^*\psi_1\psi_2\psi_3)_p,$$

where $S_\alpha \in (V_{\lambda_\alpha}, V^{\underline{\otimes} 2})$ are the isometries describing decomposition of p_2 into irreducibles with respect to the chosen section s, and V_{λ_α} is a summand of $V^{\otimes h_{\lambda_\alpha}}$, as before. Hence $\hat{V} \otimes \hat{V} \otimes \hat{V} \circ \Delta^{(2)}(I) = \sum 1 \otimes S_\alpha \circ p_{1+h_{\lambda_\alpha}} \circ 1 \otimes S_\alpha^*$, and this is a positive operator w.r.t. Wenzl's inner product.

We finally discuss the relation between the original (strict) fusion category $\mathcal{F}_\ell$ and $\text{Rep}_V(\widehat{\mathcal{C}})$. Let $X = V^{\underline{\otimes} n}$ be regarded as an object of $\mathcal{F}_\ell$, meaning that X is the truncated $U_q(\mathfrak{g})$–submodule of $V^{\otimes n}$, endowed with Hilbert space structure, cf. Theorem 5.4. We associate to X the map

$$\hat{X} : \widehat{\mathcal{C}(G, \ell)} \to \mathcal{B}(X), \qquad (\phi, \hat{X}(\omega)\psi) = \omega([\phi^* \otimes \psi]),$$

for ϕ, $\psi \in V^{\underline{\otimes} n}$, $\omega \in \widehat{\mathcal{C}(G, \ell)}$. This formula extends the previously introduced $\hat{V}$ to all objects of $\mathcal{F}_\ell$, and, as before, it is easily seen that $\hat{X}$ is a unital *–representation of $\widehat{\mathcal{C}(G, \ell)}$ on $V^{\underline{\otimes} n}$. One has $\hat{X} = \hat{V} \otimes \cdots \otimes \hat{V} \circ \Delta_{\text{left}}^{(n-1)}$, hence $\hat{X}$ is an object of $\text{Rep}_V(\widehat{\mathcal{C}})$.

9.12. Theorem *The map $X \to \hat{X}$ and acting identically on arrows is a tensor *–equivalence*

$$\mathcal{E} : \mathcal{F}_\ell \to \textit{Rep}_V(\widehat{\mathcal{C}}).$$

Proof An arrow $T \in (X, X')$ in $\mathcal{F}_\ell$ is an intertwiner of the corresponding modules of $U_q(\mathfrak{g})$. But it also lies in the arrow space $(\hat{X}, \hat{X}')$ of Rep$\widehat{\mathcal{C}(G,\ell)}$, as

$$(\phi, T\hat{X}(\omega)\psi) = (T^*\phi, \hat{X}(\omega)\psi) = \omega([(T^*\phi)^* \otimes \psi]) =$$

$$\omega([\phi^* \circ T \otimes \psi]) = \omega([\phi^* \otimes T\psi]) = (\phi, \hat{X}'(\omega)T\psi),$$

hence $\mathcal{E}$ is a functor between the stated categories, and it is easy to see that it is actually a *–functor. We verify that $\mathcal{E}$ has full image. Routine computations show that formula (9.16) defining the surjection $\pi : U_q(\mathfrak{g}) \to \widehat{\mathcal{C}(G,\ell)}$ extends to all representations, in that $\pi(a)([\phi^* \otimes \psi]) = (\phi, a\psi)$ holds for all X. Now if $T \in (\hat{X}, \hat{X}')$ then $T\hat{X}(\pi(a)) = \hat{X}'(\pi(a))T$, and expliciting this relation yields T as an intertwiner of the corresponding $U_q(\mathfrak{g})$-representations. Since it is obviously essentially surjective, we are left to construct a tensor structure for $\mathcal{E}$ making it into a tensor functor. Explicitly, since $\mathcal{F}_\ell$ is strict while Rep$_V(\widehat{\mathcal{C}})$ is not, we look for natural isomorphisms $\mathcal{E}_{X,Y} \in (\hat{X} \otimes \hat{Y}, \widehat{X \otimes Y})$ such that

$$\mathcal{E}_{W,X\otimes Y} \circ 1_{\hat{W}} \otimes \mathcal{E}_{X,Y} \circ \Phi_{\hat{W},\hat{X},\hat{Y}} = \mathcal{E}_{W\otimes X,Y} \circ \mathcal{E}_{W,X} \otimes 1_{\hat{Y}}. \tag{9.19}$$

For $W = V^{\underline{\otimes} m}$, $X = V^{\underline{\otimes} n}$, $Y = V^{\underline{\otimes} r}$, we write

$$\mathcal{E}_{X,Y} = \hat{V}^{\otimes(n+r)}(Q_{n,r})$$

and we are reduced to look for a quasi-invertible $Q_{n,r} \in \widehat{\mathcal{C}}^{\otimes n+r}$ satisfying the intertwining relation

$$Q_{n,r}\Delta_{\text{left}}^{(n-1)} \otimes \Delta_{\text{left}}^{(r-1)}(\Delta(\omega)) = \Delta_{\text{left}}^{(n+r-1)}(\omega)Q_{n,r}.$$

It is solved by

$$Q_{n,r} = \Delta_{\text{left}}^{(n+r-1)}(I)\Delta_{\text{left}}^{(n-1)} \otimes \Delta_{\text{left}}^{(r-1)}(\Delta(I))$$

and naturality easily follows. The tensorial structure (9.19) corresponds to

$$Q_{m,n+r}I_m \otimes Q_{n,r}\Delta_{\text{left}}^{(m-1)} \otimes \Delta_{\text{left}}^{(n-1)} \otimes \Delta_{\text{left}}^{(r-1)}(\Phi) = Q_{m+n,r}Q_{m,n} \otimes I_r. \tag{9.20}$$

Computations analogous to those of Theorem 9.5 give, in the same short notation, $Q_{n,r} = \omega_{T_{n,r}}$ with

$$T_{n,r} = p_{n+r} \circ 1_n \otimes S_\gamma \circ p_{n+1} \circ 1_n \otimes S_\gamma^*$$

and this expression shows that $Q_{n,r}$ is quasi-invertible. Furthermore $\Delta_{\text{left}}^{(m-1)}\otimes\Delta_{\text{left}}^{(n-1)}\otimes\Delta_{\text{left}}^{(r-1)}(\Phi)$ corresponds to

$$S_\sigma \otimes S_{\sigma'} \otimes S_{\sigma''} \circ 1 \otimes S_\gamma \circ p_2 \circ 1 \otimes S_\gamma^* \circ p_3 \circ S_\sigma^* \otimes S_{\sigma'}^* \otimes S_{\sigma''}^*$$

and we next check validity of the desired relation (9.20). At the left hand side we obtain

$$\begin{aligned} p_{m+n+r} \circ 1_m \otimes S_\gamma \circ p_{m+h_\gamma} \circ 1_m \otimes S_\gamma^* \circ p_m \otimes 1_{n+r} \circ 1_m \otimes p_{n+r} \circ \\ \circ 1_{m+n} \otimes S_\delta \circ 1_m \otimes p_{n+h_\delta} \circ 1_{m+n} \otimes S_\delta^* \circ S_\sigma \otimes S_{\sigma'} \otimes S_{\sigma''} \circ \\ \circ 1_m \otimes S_\alpha \circ p_{h_\sigma+h_\alpha} \circ 1 \otimes S_\alpha^* \circ p_{h_\sigma+h_{\sigma'}+h_{\sigma''}} \circ S_\sigma^* \otimes S_{\sigma'}^* \otimes S_{\sigma''}^* \end{aligned}$$

which equals

$$p_{m+n+r} \circ 1_m \otimes S_{\sigma'} \otimes S_{\sigma''} \circ p_{m+h_{\sigma'}+h_{\sigma''}} \circ 1_m \otimes S_{\sigma'}^* \otimes S_{\sigma''}^*$$

by repeated use of Lemma 5.1 and Lemma 5.3. The right hand side becomes

$$\begin{aligned} p_{m+n+r} \circ 1_{m+n} \otimes S_{\sigma''} \circ p_{m+n+h_{\sigma''}} \circ 1_{m+n} \otimes S_{\sigma''}^* \circ p_{m+n} \otimes p_r \circ \\ \circ 1_m \otimes S_{\sigma'} \otimes 1_r \circ p_{m+h_{\sigma'}} \otimes 1_r \circ 1_m \otimes S_{\sigma'}^* \otimes 1_r \end{aligned}$$

which in turn equals

$$p_{m+n+r} \circ 1_{m+n} \otimes S_{\sigma''} \circ [p_{m+n+h_{\sigma''}} \circ (p_{m+n} \circ 1_m \otimes S_{\sigma'} \circ p_{m+h_{\sigma'}}) \otimes 1_{h_{\sigma''}}] \circ 1_m \otimes S_{\sigma'}^* \otimes S_{\sigma''}^*.$$

It is easy to see that the expression in the square brackets can be rewritten as

$$(p_{m+n} \circ 1_m \otimes S_{\sigma'} \circ p_{m+h_{\sigma'}}) \otimes 1_{h_{\sigma''}} \circ p_{m+h_{\sigma'}+h_{\sigma''}}.$$

Substituting it in the right hand side formula and using the usual properties of projections we get the desired identity.

10. COSEMISIMPLICITY IN TYPE A CASE

As already mentioned, in this section we formulate a sufficient condition for cosemisimplicity of $\mathcal{C}(G,\ell)$, that we next verify for $G=\mathrm{SU}(N)$. Our condition appeals to the existence of a Haar functional on $\mathcal{C}(G,\ell)$, but more is needed. In particular, we shall use the associative filtration $\widetilde{\mathcal{C}}_k$ constructed in Sect. 8. We should remark that space limit has prevented us from studying cosemisimplicity of $\mathcal{C}(G,\ell)$ for other Lie types. A positive answer would obviously suffice to extend our main result to other types provided $\mathfrak{g}\neq E_8$.

a) A sufficient condition for cosemisimplicity

In this subsection G is still general. It turns out useful to tackle the cosemisimplicity problem at the level of the filtration $\widetilde{\mathcal{C}}_k$, as this is a better behaved structure, in that it is provided with a multiplication. Of course, this involves the question of non-triviality of this filtration, or, more precisely, whether the image $\widetilde{M}_\lambda^k$ of M_λ in $\widetilde{\mathcal{C}}_k$ under the quotient map $\mathcal{C}_k\to\widetilde{\mathcal{C}}_k$ is a matrix coalgebra for some $k\geq n$ and sufficiently many $\lambda\in\Lambda_\ell$.

Linear independence can be easily settled.

10.1. Proposition

a) *The subcoalgebras M_λ, are linearly independent in $\mathcal{C}$ as λ varies in Λ_ℓ,*
b) *similarly, $\widetilde{M}_\lambda$ are linearly independent in $\widetilde{\mathcal{C}}_k$ as $\lambda\in\Lambda_\ell^k$, for all k.*

Proof We show b), the proof of a) is easier and can be done along similar lines. Let $V_{\lambda,n}$ denote the isotypic submodule of $p_n(V^{\otimes n})$ of type V_λ, with orthogonal complement $V_{\lambda,n}^\perp$. The subspaces $V_{\lambda,n}^*\otimes V_{\lambda,n}^\perp$, $(V_{\lambda,n}^\perp)^*\otimes V_{\lambda,n}$ and $V_{\lambda,n}^*\otimes V_{\lambda,n}$, for $n=0,\dots,k$, are linearly independent in $\mathcal{D}_k$. Let W_λ denote their span. Consider the projection $E_\lambda:\mathcal{D}_k\to W_\lambda$ with complement $\oplus_{n=0}^k({V_{\lambda,n}}^\perp)^*\otimes V_{\lambda,n}^\perp$. The main point is that $\widetilde{\mathcal{I}}_k$ is stable under E_λ. This can be seen noticing that $\widetilde{\mathcal{I}}_k$ is linearly spanned by $(V_{\lambda,n}^\perp)^*\otimes V_{\lambda,n}$, $V_{\lambda,n}^*\otimes V_{\lambda,n}^\perp$, $\phi^*\otimes A\psi-\phi^*A\otimes\psi$, with $\psi\in V_{\lambda,n}$, and $\phi\in V_{\lambda,m}$ or $\psi\in {V_{\lambda,n}}^\perp$ and $\phi\in {V_{\lambda,m}}^\perp$; and also by $V_{\lambda,n}^*\otimes ZV^{\otimes n}$, $({V_{\lambda,n}}^\perp)^*\otimes ZV^{\otimes n}$, $(V^{\otimes n})^*Z^*\otimes {V_{\lambda,n}}^\perp$, $(V^{\otimes n})^*Z^*\otimes V_{\lambda,n}$. Hence if a sum $T=\sum T_\mu$ of elements $T_\mu\in V_\mu^*\otimes V_\mu$ with $\mu\in\Lambda_\ell^k$ lies in $\widetilde{\mathcal{I}}_k$ then each $E_\lambda(T)=T_\lambda$ does as well.

Cosemisimplicity in the generic case was studied by means of the Haar functional. We look for a generalisation of that approach to the present setting. Notice that if one can establish that $\widetilde{M}_\lambda^k$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$ then M_λ is a matrix coalgebra in $\mathcal{C}(G,\ell)$ as well, by dimension count. We shall verify cosemisimplicity in this stronger form.

10.2. Definition A linear functional h on $\widetilde{\mathcal{C}}_k$ is said to be a Haar functional if $h(I)=1$ and h annihilates the subcoalgebras $\widetilde{M}_\lambda^k$ for $\lambda\in\Lambda_\ell^k\backslash\{0\}$.

Obviously a Haar functional on $\widetilde{\mathcal{C}}_k$ is unique. Furthermore, if $\widetilde{\mathcal{C}}_k$ admits a Haar functional then so does $\widetilde{\mathcal{C}}_h$ for $h<k$.

Notation For a given $\lambda\in\Lambda_\ell$, let $\deg(\lambda)$ denote the smallest integer k such that $\lambda\in\Lambda_\ell^k$, or, in other words, such that V_λ is a summand of $V^{\otimes k}$. Furthermore, set

$$m(\mathfrak{g},\ell):=\max\{\deg(\lambda),\lambda\in\Lambda_\ell\}.$$

10.3. Definition We will say that the pair $(\mathfrak{g},\ell)$ satisfies the cosemisimplicity condition if

1) there is $\tilde{m}\geq m(\mathfrak{g},\ell)$ such that $\widetilde{\mathcal{C}}_{\tilde{m}}$ admits a Haar functional,

2) every $\lambda \in \Lambda_\ell$ has a conjugate $\overline{\lambda} \in \Lambda_\ell$ satisfying $\deg(\lambda) + \deg(\overline{\lambda}) \leq \tilde{m}$.

10.4. Lemma *Let $\lambda \in \Lambda_\ell^k$ have a conjugate $\overline{\lambda} \in \Lambda_\ell^h$ such that $\widetilde{\mathcal{C}}_{h+k}$ admits a Haar functional. Then $\widetilde{M}_\lambda^k$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$.*

Proof Let V_λ be a summand of $V^{\otimes n}$, $n \leq k$. Let $r : \mathbb{C} \to V_{\overline{\lambda}} \otimes V_\lambda$ be defined as in the proof of Theorem 6.7. The composed arrow $r^* r : \mathbb{C} \to V_{\overline{\lambda}} \otimes V_\lambda \to \mathbb{C}$ is nonzero since $\lambda \in \Lambda_\ell$ [3, 49]. In particular, the trivial submodule defined by r is a summand of $V_{\overline{\lambda}} \otimes V_\lambda$. But $(1 - p_{h+n}) V_{\overline{\lambda}} \otimes V_\lambda$ can not contain a trivial submodule, as otherwise it would be a summand, by multiplicity count. Hence $(1 - p_{h+n}) \circ r = 0$. This shows that $r \in p_{h+n} V^{\otimes h+n}$. If a linear combination $x = \sum_{i,j} \mu_{i,j} e_{i,j}^\lambda = 0$ vanishes in $\widetilde{\mathcal{C}}_k$ then $h(ax) = 0$ for all $a \in \widetilde{\mathcal{C}}_h$, where h is a Haar functional for $\widetilde{\mathcal{C}}_{h+k}$. Now computations analogous to those of the generic case show that $\mu_{i,j} = 0$.

10.5. Theorem *If $(\mathfrak{g}, \ell)$ satisfies the cosemisimplicity condition 10.3 then*

a) *$\widetilde{M}_\lambda^k$ is a matrix coalgebra in $\widetilde{\mathcal{C}}_k$ for $k = \deg(\lambda)$,*
b) *M_λ is a matrix coalgebra in $\mathcal{C}(G, \ell)$, for all $\lambda \in \Lambda_\ell$, hence $\mathcal{C}(G, \ell)$ is cosemisimple:*

$$\mathcal{C}(G, \ell) = \bigoplus_{\lambda \in \Lambda_\ell} M_\lambda.$$

Proof The proof follows from Prop. 10.1 and Lemma 10.4.

b) The case $G = \mathrm{SU}(N)$

The rest of the section is dedicated to the proof of the following theorem, which concludes the main result of the paper.

10.6. Theorem *If $\mathfrak{g} = \mathfrak{sl}_N$ then $(\mathfrak{g}, \ell)$, satisfies the cosemisimplicity condition 10.3 for all $N \geq 2$ and $\ell \geq N + 1$ with $m(\mathfrak{g}, \ell) = (N-1)(\ell - N)$ and $\tilde{m} := m(\mathfrak{g}, \ell) + \ell - 1$.*

We start fixing notation of type A_{N-1} root systems [21]. Consider $\mathbb{R}^N$ with the usual euclidean inner product, and let $e_1, \dots, e_N$ be the canonical orthonormal basis. Consider the subspace $E \subset \mathbb{R}^N$ of elements $\mu_1 e_1 + \cdots + \mu_N e_N$ such that $\mu_1 + \cdots + \mu_N = 0$. The A_{N-1} root system is $\Phi = \{e_i - e_j, i \neq j\}$, the simple roots are $\alpha_i = e_i - e_{i+1}$, and the fundamental weights are $\omega_i = e_1 + \cdots + e_i - \frac{i}{N} e$, where $e := e_1 + \cdots + e_N$ and $i = 1, \dots, N-1$. The weight lattice and the dominant Weyl chamber of (E, Φ) are respectively

$$\Lambda = \{\lambda = \lambda_1 e_1 + \cdots + \lambda_{N-1} e_{N-1} - \frac{\lambda_1 + \cdots + \lambda_{N-1}}{N} e, \quad \lambda_i \in \mathbb{Z}\},$$

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq 0\}.$$

The highest root is $\theta = e_1 - e_N$. Direct computations show that

$$\Lambda_\ell = \{\lambda \in \Lambda^+ : \lambda_1 \leq \ell - N\}, \quad \overline{\Lambda_\ell} = \{\lambda \in \Lambda^+ : \lambda_1 \leq \ell - N + 1\}.$$

We have $V = V_{\omega_1}$, the vector representation, its weights are

$$\gamma_1 = \omega_1, \quad \gamma_i = e_i - \frac{1}{N} e, \quad i = 2, \dots, N.$$

In the next lemmas we shall make use of the decomposition into irreducibles

$$V_\lambda \otimes V \simeq \oplus V_{\lambda + \gamma_i}, \quad \lambda \in \Lambda_\ell,$$

in the category $\mathcal{T}_\ell$ of tilting modules, where the sum is extended to all i such that $\lambda + \gamma_i$ is dominant, (Theorem 4.4, c).) We derive two simple consequences.

10.7. Lemma *For any $\lambda \in \Lambda_\ell$, the negligible submodule N_λ of $V_\lambda \otimes V$ is non-zero if and only if $\lambda_1 = \ell - N$, and one has $N_\lambda \simeq V_{\lambda + \omega_1}$.*

Proof The indicated submodule is isomorphic to the sum corresponding to $\lambda + \gamma_i \in \overline{\Lambda_\ell} \setminus \Lambda_\ell$, which is realised if and only if $i = 1$ and $\lambda_1 = \ell - N$.

10.8. Lemma $m(\mathfrak{sl}_N, \ell) = (N-1)(\ell - N)$.

Proof Let $\lambda \in \Lambda_\ell$ be determined by non negative integers $\lambda_1, \dots, \lambda_{N-1}$ as above, and let us identify λ with $(\lambda_1, \dots, \lambda_{N-1})$. The dominant weight with coordinates all equal to $\ell - N$ lies in Λ_ℓ, it is a summand of $V^{\otimes(N-1)(\ell-N)}$ and this is the smallest possible power. We need to show that every module V_λ with $\lambda \in \Lambda_\ell$ is a summand of some $p_n V^n$ with $n \leq (N-1)(\ell-N)$. Notice that $(1, 0, \dots, 0), \dots, (\lambda_1, 0, \dots, 0), (\lambda_1, 1, 0, \dots, 0), \dots, (\lambda_1, \lambda_2, \dots, 0), \dots, (\lambda_1, \dots, \lambda_{N-1})$ is a sequence of $\lambda_1 + \lambda_2 + \dots + \lambda_{N-1}$ dominant weights of Λ_ℓ starting with ω_1 and obtained from one another by adding a weight of V. The fusion rules then show that V_λ is a summand of $p_n V^n$, where $n = \lambda_1 + \lambda_2 + \dots + \lambda_{N-1} \leq (N-1)(\ell - N)$.

We next derive information on the negligible summands of $V^{\otimes n}$, including the non-canonical ones, for the bounded values of n. The following Lemma plays a crucial role for the Haar functional.

10.9. Lemma *No negligible summand of $V^{\otimes n}$, with n up to $\tilde{m} = m(\mathfrak{sl}_N, \ell) + \ell - 1$ contains the trivial module among the successive factors of its Weyl filtrations.*

Proof A negligible summand of $V^{\otimes n}$ is isomorphic to a summand of $N_n = (1 - p_n)V^{\otimes n}$. Furthermore the inductive procedure described in Sect. 5 shows that N_n is in turn spanned by the summands

$$N(p_r V^{\otimes r} \otimes V) \otimes V^{\otimes(n-r-1)}, \quad r = 1, \dots, n-1, \tag{10.1}$$

where $N(p_r V^{\otimes r} \otimes V)$ is the canonical negligible summand of $p_r V^{\otimes r} \otimes V$, hence we are reduced to show the statement for these modules.

Now on one hand $N(p_r V^{\otimes r} \otimes V)$ is completely reducible and the dominant weights of the irreducible components are of the form $\lambda + \omega_1 = (\ell - N + 1, \lambda_2, \dots, \lambda_{N-1})$ by Lemma 10.7.

On the other, the dominant weights appearing in the Weyl filtrations of (10.1) are the same as those appearing in the irreducible decomposition of the corresponding module at the level of the semisimple category Rep($\mathfrak{g}$), see Prop. 3 and Remark 2 in [45].

Hence we are reduced to show that the smallest integer t such that $V_{\lambda+\omega_1} \otimes V^{\otimes t}$ contains the trivial module in Rep($\mathfrak{g}$) satisfies

$$t + r + 1 > (N-1)(\ell - N) + \ell - 1.$$

We compute t. For a general dominant weight $\mu = (\mu_1, \dots, \mu_{N-1})$, the shortest path to the trivial module is obtained as follows. If $\mu_{N-1} > 0$ we consider the path

$$\mu + \gamma_N, \quad \mu + 2\gamma_N, \quad \dots, \quad \mu + \mu_{N-1}\gamma_N$$

which lowers μ to

$$\mu' = (\mu_1 - \mu_{N-1}, \dots, \mu_{N-2} - \mu_{N-1}, 0),$$

and we have thus used μ_{N-1} powers of V. We need no such power if $\mu_{N-1} = 0$. We proceed in the same way for the $N-2$ coordinate and the new weight μ', but we now need to follow a longer path, due to vanishing of the last coordinate, and the shortest is

$$\mu' + \gamma_{N-1}, \quad \mu' + \gamma_{N-1} + \gamma_N, \quad \mu' + 2\gamma_{N-1} + \gamma_N, \quad \mu' + 2\gamma_{N-1} + 2\gamma_N, \dots$$

using $2(\mu_{N-2} - \mu_{N-1})$ more powers of V. Continuing in this way, we find

$$t = \mu_{N-1} + 2(\mu_{N-2} - \mu_{N-1}) + 3(\mu_{N-3} - \mu_{N-2}) + \dots + (N-1)(\mu_1 - \mu_2).$$

Taking into account the fact that in Rep($\mathfrak{sl}_N$), the dominant weights μ appearing in $V^{\otimes r+1}$ satisfy $\mu_1 + \dots + \mu_{N-1} \leq r + 1$, we easily get, for $\mu = \lambda + \omega_1$,

$$t + r + 1 \geq t + \mu_1 + \dots + \mu_{N-1} = N\mu_1 = (N-1)(\ell - N) + \ell > (N-1)(\ell - N) + \ell - 1.$$

which finally gives the desired estimate.

10.10. Remark The proof also shows that $N(p_r V^{\otimes r} \otimes V) = 0$ for $r < \ell - N$.

10.11. Corollary *Let* $n \leq \tilde{m}$.

a) e_n *is a central element of* $(V^{\otimes n}, V^{\otimes n})$,
b) $e_n \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u} = 0$, $q + j + u = n$.

Proof a) The multiplicity of the trivial representation in $p_n V^n$ is the same as that of the classical case, by the previous lemma, hence e_n is the specialisation of a central intertwiner of the generic case. b) $e_n \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u} \circ p_n = 0$ by Lemma 5.3, hence

$$e_n \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u} = e_n \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u} \circ (1 - p_n) =$$

$$1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u} \circ e_n \circ (1 - p_n) = 0$$

by a).

We next consider the $\tilde{m}$-th term, $\widetilde{\mathcal{C}}_{\tilde{m}}$, of the associative filtration $\widetilde{\mathcal{C}}_k$ corresponding to $\mathcal{C}(\mathrm{SU}(N), \ell)$. For convenience we recall that $\widetilde{\mathcal{C}}_{\tilde{m}} = \mathcal{D}_{\tilde{m}}/\widetilde{\mathcal{J}}_{\tilde{m}}$, where $\widetilde{\mathcal{J}}_{\tilde{m}}$ is spanned by elements of the form $\phi \otimes A \circ \psi - \phi \circ A \otimes \psi$, together with $\phi \otimes Z \circ \psi'$, and $\phi' \circ Z' \otimes \psi$, where

$$A \in (V^{\underline{\otimes} m}, V^{\underline{\otimes} n}), \qquad Z,\, Z'^{*} = p_{q+j+u} \circ 1_{V^q} \otimes (1 - p_j) \otimes 1_{V^u}$$

with m, n, $q + j + u \leq \tilde{m}$.

We define the linear functional

$$h : \mathcal{D}_{\tilde{m}} \to \mathbb{C},$$

setting

$$h(\phi \otimes \psi) = \phi(e_n \psi), \quad \phi \otimes \psi \in (V^n)^* p_n \otimes p_n V^n, \quad n \leq \tilde{m}.$$

where $e_n \in (p_n V^n, p_n V^n)$ is the orthogonal projection onto the isotypical component of the trivial representation.

10.12. Theorem *The functional* h *annihilates* $\widetilde{\mathcal{J}}_{\tilde{m}}$. *Hence it gives rise to a Haar functional on* $\widetilde{\mathcal{C}}_{\tilde{m}}$.

Proof The functional h obviously annihilates elements $\phi \otimes A\psi - \phi A \otimes \psi$. Furthermore it also annihilates elements of the form $\phi \otimes Z\psi'$, $\phi' Z' \otimes \psi \in \widetilde{\mathcal{J}}_{\tilde{m}}$, by Corollary 10.11, b). The rest is now clear.

We finally verify the needed upper bound for $\deg(\lambda) + \deg(\overline{\lambda})$ for all $\lambda \in \Lambda_\ell$. We are interested in property c) of the following proposition.

10.13. Proposition *If* $\lambda = (\lambda_1, \dots, \lambda_{N-1}) \in \Lambda^+$ *then*

a) *deg*$(\lambda) = \lambda_1 + \cdots + \lambda_{N-1}$,
b) $\overline{\lambda} = (\lambda_1, \lambda_1 - \lambda_{N-1}, \lambda_1 - \lambda_{N-2}, \dots, \lambda_1 - \lambda_2)$,
c) *deg*(λ) + *deg*$(\overline{\lambda}) = N\lambda_1 \leq \tilde{m}$ *for* $\lambda \in \Lambda_\ell$.

Proof Properties a) and b) are classical, and c) follows from an easy computation, since $\lambda_1 \leq (\ell - N)$ for $\lambda \in \Lambda_\ell$ and $\tilde{m} = (N-1)(\ell - N) + \ell - 1$. We sketch a proof for completeness. a) follows from the fusion rules for the powers of V, recalled at the beginning of this section. b) Let w_0 be longest element of the Weyl group. For $\mathfrak{sl}_N$, this is the permutation group $\mathbb{P}_N$ and w_0 is the permutation reversing the order of $(e_1, \dots, e_N)$. Then

$$\overline{\lambda} = -w_0 \lambda =$$

$$-(\lambda_1 e_N + \lambda_2 e_{N-1} + \cdots + \lambda_{N-1} e_2) + \frac{\lambda_1 + \cdots + \lambda_{N-1}}{N} e =$$

$$(\lambda_1, \lambda_1 - \lambda_{N-1}, \lambda_1 - \lambda_{N-2}, \dots, \lambda_1 - \lambda_2).$$

APPENDIX A.
ON THE FUSION C^*-CATEGORY OF TYPE A

In this appendix we illustrate a concrete realisation of the fusion C^*–category $\mathcal{F}_\ell$ associated to $\mathfrak{sl}_N$ in the sense of Theorem 5.4 and we identify a single generator, the quantum determinant element S. Most of the results are due to [23], but we have not been able to find a reference where S and the conjugation structure, Prop. A.5, are made explicit. The main aspect is the use of the Kirillov-Wenzl inner product.

The fusion rules seen in the previous section show that the trivial object of $\mathcal{F}_\ell$ is contained with multiplicity 1 in $V^{\underline{\otimes} N}$, hence there is a nonzero intertwiner, the quantum determinant element,

$$S \in (\iota, V^{\underline{\otimes} N})$$

unique up to a scalar multiple.

A.1. Theorem *The quotient category $\mathcal{F}_\ell$ of type A_{N-1} is generated by the quantum determinant S as a tensor *–category.*

Proof We have that $(V^{\underline{\otimes} s}, V^{\underline{\otimes} t}) \neq 0 \quad \Rightarrow s = t + hN$ for some $h \in \mathbb{Z}$, by the fusion rules. If for example $h > 0$ we can write an element $B \in (V^{\underline{\otimes} s}, V^{\underline{\otimes} t})$ in the form

$$B = S^* \otimes 1_t \circ S^* \otimes 1_{t+N} \circ \cdots \circ S^* \otimes 1_{t+hN} \circ Y, \quad \text{with} \quad Y \in (V^{\underline{\otimes} s}, V^{\underline{\otimes} s}).$$

Indeed, we may choose $Y = S \otimes 1_{t+hN} \circ \cdots \circ S \otimes 1_t \circ B$ which lies in the claimed space as

$$Y = S \otimes 1_{t+hN} \circ \cdots \circ S \otimes 1_t \circ p_t \circ B = p_s \circ S \otimes 1_{t+hN} \circ \cdots \circ S \otimes 1_t \circ p_t \circ B.$$

by Lemma 5.1. On the other hand, by Schur-Weyl duality at roots of unity [14] the intertwining space $(V^{\underline{\otimes} s}, V^{\underline{\otimes} s})$ is linearly spanned by the representation $\underline{\varepsilon}(b) = p_n \varepsilon(b) p_n$ of the braid group $\mathbb{B}_s$ obtained reducing the translates $1_w \otimes \varepsilon \otimes 1_{w'}$ of the basic braiding operator $\varepsilon \in (V^{\otimes 2}, V^{\otimes 2})$ by p_n. Since the p_n themselves lie in the image of $\mathbb{C}\mathbb{B}_s$, we are reduced to show that the basic braiding operator derived from the R-matrix acting on $(V^{\otimes 2}, V^{\otimes 2})$ lies in the tensor *–subcategory generated by S, and this will follow from Prop. A.5.

If for example $\ell \geq 2N$ then tensor powers $V^{\otimes k}$ with k up to N are completely reducible, with multiplicities given as in the classical case. In particular, the k-th fundamental representation V_{ω_k}, $k = 1, \dots, N-1$, is contained in $V_{\omega_{k-1}} \otimes V$, and hence in $V^{\otimes k}$, with multiplicity 1.

Set

$$\varepsilon = \Sigma R \in (V^{\otimes 2}, V^{\otimes 2}), \qquad g = q^{1/N} \varepsilon.$$

We recall that g gives rise to a representation of the Hecke algebra $H_n(q)$ of type A on $V^{\otimes n}$. Indeed g has the following spectral decomposition on $V^{\otimes 2}$,

$$g = q[2\omega_1] - q^{-1}[\omega_2],$$

hence the translates $g_i = 1_{i-1} \otimes g \otimes 1_{n-i-1}$ satisfy the defining relations $(g_i - q)(g_i + \frac{1}{q}) = 0$ of $H_n(q)$. In particular, ε is determined by the single projection $[\omega_2]$. We thus need to show that $[\omega_2]$ is in turn an arrow of the tensor *–category generated by S. Showing this claim, as we next see, involves determining the conjugate equations for the submodule V_{ω_2} of $V^{\otimes 2}$. Since it will not be more effort, we shall identify the conjugate equations for all the fundamental representations V_{ω_n}.

Consider the following sequence of central elements of $\alpha_{-n} \in H_n(q)$ inductively defined by $\alpha_{-1} = I$ and

$$\alpha_{-n-1} := \sigma(\alpha_{-n}) - q^{-1} g_1 \sigma(\alpha_{-n}) + \cdots + (-1)^n q^{-n} g_n \dots g_1 \sigma(\alpha_{-n}),$$

with $\sigma : H_n(q) \to H_{n+1}(q)$ the homomorphism taking g_i to g_{i+1}. In particular, α_{-2} is a scalar multiple of $[\omega_2]$. The next two lemmas recall certain known facts [23, 49], see also [37].

A.2. Lemma

a) $g_i \alpha_{-n} = \alpha_{-n} g_i = -q^{-1} \alpha_n$, $i = 1, \dots, n-1$,
b) $\alpha_{-n}^2 = \lambda_n \alpha_{-n}$, *with* $\lambda_n := q^{-n(n-1)/2} [n]_q!$.

Hence if $\ell > n$, we may consider the idempotent $E_{-n} := \lambda_n^{-1}\alpha_{-n}$. We next identify its range. The Hecke algebra generator $g = q^{1/N}\varepsilon$ acts on a suitable orthonormal basis ψ_i, $i = 1, \dots, N$ of the Hilbert space of $V = V_{\omega_1}$ as

$$g\psi_i \otimes \psi_i = q\psi_i \otimes \psi_i,$$
$$g\psi_i \otimes \psi_j = \psi_j \otimes \psi_i, \quad i > j$$
$$g\psi_i \otimes \psi_j = \psi_j \otimes \psi_i + (q - \frac{1}{q})\psi_i \otimes \psi_j, \quad i < j.$$

The quantum determinant $S \in (\iota, V^{\otimes N})$ is given by

$$S = \sum_{p \in \mathbb{P}_N} (-q)^{i(p)} \psi_{p(1)} \otimes \dots \otimes \psi_{p(N)}.$$

More generally, we also introduce the following antisymmetrized elements in $V^{\otimes n}$ for $n \leq N$ spanning the submodule V_{ω_n} of $V^{\otimes n}$. Consider indices $i_1 < i_2 < \dots < i_n$ with $i_j \in \{1, \dots, N\}$ and set, for $\underline{i} = (i_1, \dots, i_n)$,

$$S_{\underline{i}} = \sum_{p \in \mathbb{P}_n} (-q)^{i(p)} \psi_{i_{p(1)}} \otimes \dots \otimes \psi_{i_{p(n)}}.$$

It will be useful to introduce also corresponding antisymmetrized elements over q^{-1},

$$\tilde{S}_{\underline{i}} = \sum_{p \in \mathbb{P}_n} (-q^{-1})^{i(p)} \psi_{i_{p(1)}} \otimes \dots \otimes \psi_{i_{p(n)}}.$$

However, some caution is needed, as the $\tilde{S}_{\underline{i}}$ do *not* span a submodule of $V^{\otimes n}$.

A.3. Lemma *We have that*

a) $\alpha_{-n} = 0$ *for* $n > N$,
b) E_{-n} *is the idempotent of* $(V^{\otimes n}, V^{\otimes n})$ *corresponding to* V_{ω_n} *for* $n \leq N - 1$ *and to the trivial submodule for* $n = N$.

Proof This follows from $\alpha_{-n}\psi_{i_1} \otimes \dots \otimes \psi_{i_n} = 0$ if two indices repeat, while $q^{n(n-1)}\alpha_{-n}\psi_{i_{p^{-1}(1)}} \otimes \dots \otimes \psi_{i_{p^{-1}(n)}} = (-q)^{i(p)} S_{\underline{i}}$ for any $\underline{i} = i_1 < \dots < i_n$ and $p \in \mathbb{P}_n$.

A main aspect for the values $q \in \mathbb{T}$ is that the *–structure relies on the Kirillov-Wenzl inner product on tensor product spaces. We next identify an orthonormal basis of V_{ω_n} in the correct Hilbert space.

A.4. Proposition *Consider indices* $\underline{i} = i_1 < \dots < i_n$, $\underline{j} = j_1 < \dots < j_n$. *Then*

a) E_{-n} *is a selfadjoint projection of* $(V^{\otimes n}, V^{\otimes n})$ *for* $n \leq N$,
b) $\overline{R}^{(n)} S_{\underline{i}} = q^{n(n-1)/2} \tilde{S}_{\underline{i}}$.
c) $(S_{\underline{i}}, S_{\underline{j}}) = 0$ *for* $\underline{i} \neq \underline{j}$,
d) $\|S_{\underline{i}}\|^2 = [n]_q!$.

Proof a) Selfadjointness of E_{-n} was shown in [37] for the involution introduced in [48], which, however, does coincide with that induced by Kirillov-Wenzl inner product, as they both make the spectral idempotent $E_{-2} = [\omega_2]$ selfadjoint. b) Recall from Sect. 2 that

$$\overline{R}^{(n)} = R^{(n)}\Theta^{(n)}$$

and that

$$R^{(n)} = R^{(n-1)} \otimes 1 \circ \Delta^{(n-2)} \otimes 1(R) = R^{(n-1)} \otimes 1 \circ (R_{1\,n} \dots R_{n-1\,n}).$$

Hence

$$R^{(n)} = R_{12} \circ \dots \circ (R_{1\,n-1} \dots R_{n-2\,n-1}) \circ (R_{1\,n} \dots R_{n-1\,n})$$

It turns out convenient to decompose the permutation Σ_n as a product of transpositions

$$\Sigma_n = \Sigma_{n-1\,n} \circ \dots \circ (\Sigma_{23} \dots \Sigma_{n-1\,n}) \circ (\Sigma_{12}\Sigma_{23} \dots \Sigma_{n-1\,n}).$$

Distributing these transpositions among the $R_{i\,j}$ gives

$$R^{(n)} = \Sigma_n \circ \varepsilon_{n-1} \circ \dots (\varepsilon_2 \dots \varepsilon_{n-1}) \circ (\varepsilon_1 \dots \varepsilon_{n-1})$$

An easy computation shows that

$$g_i S_{\underline{i}} = -q^{-1} S_{\underline{i}}, \quad i = 1, \dots n-1$$

and this implies

$$q^{n(n-1)/2N} R^{(n)} S_{\underline{i}} = \Sigma_n \circ g_{n-1} \circ \dots (g_2 \dots g_{n-1}) \circ (g_1 \dots g_{n-1}) S_{\underline{i}} =$$
$$(-q^{-1})^{n(n-1)/2} \Sigma_n S_{\underline{i}} =$$
$$(-q^{-1})^{n(n-1)/2} \sum_{p \in \mathbb{P}_n} (-q)^{i(p)} \psi_{i_{p(n)}} \otimes \dots \otimes \psi_{i_{p(1)}} =$$
$$\sum_{p \in \mathbb{P}_n} (-q^{-1})^{i(p)} \psi_{i_{p(1)}} \otimes \dots \otimes \psi_{i_{p(n)}} = \tilde{S}_{\underline{i}}.$$

On the other hand, recall from Sect. 3 that $\Theta^{(n)}$ acts as scalar multiplication by

$$q^{n(\omega_1 + 2\rho, \omega_1) - (\omega_n + 2\rho, \omega_n)/2}$$

on the submodule V_{ω_n} of $V^{\otimes n}$. A straightforward computation gives $(\omega_n, \omega_n + 2\rho) = n + nN - n^2 - \frac{n^2}{N}$ and the conclusion follows. c) follows from b) and d) from explicit computation of the Kirillov-Wenzl norm,

$$\|S_{\underline{i}}\|^2 = (S_{\underline{i}}, \overline{R}^{(n)} S_{\underline{i}})_p =$$
$$q^{n(n-1)/2} \sum_{p, p' \in \mathbb{P}_n} ((-q)^{i(p)} \psi_{i_{p(1)}} \otimes \dots \otimes \psi_{i_{p(n)}}, (-q^{-1})^{i(p')} \psi_{i_{p'(1)}} \otimes \dots \otimes \psi_{i_{p'(n)}})_p =$$
$$q^{n(n-1)/2} \sum_{p \in \mathbb{P}_n} (q^{-2})^{i(p)} = [n]_q!$$

where inner products are assumed antilinear in the first variable.

Consider a partition of $N = m + n$, and fix indices $\underline{i} = i_1 < \dots < i_n$ in $\{1, \dots, N\}$. We denote by $\underline{j} = j_1 < \dots < j_m$ the indices obtained from $\{1, \dots, N\}$ after removing $i_1, \dots, i_n$. We shall refer to $\underline{j}$ as conjugate to $\underline{i}$. Obviously, $\underline{i}$ is conjugate to $\underline{j}$ as well. Fix permutations $s \in \mathbb{P}_m$ and $r \in \mathbb{P}_n$ and define $p \in \mathbb{P}_N$ by

$$p(1) = j_{s(1)}, \dots, p(m) = j_{s(m)},$$
$$p(m+1) = i_{r(1)}, \dots, p(m+n) = i_{r(n)}.$$

Every $p \in \mathbb{P}_N$ can be uniquely written in this form. Denoting by $N(\underline{i})$ the number of pairs (j_h, i_k) with $j_h > i_k$, we have $i(p) = i(r) + i(s) + N(\underline{i})$. We thus have

$$S = \sum_{\underline{i}} (-q)^{N(\underline{i})} S_{\underline{j}}^{(m)} \otimes S_{\underline{i}}^{(n)},$$

where we have introduced upper indices clarifying the corresponding tensor power of V. Exchanging the roles of m and n, we can also write

$$S = \sum_{\underline{h}} (-q)^{N(\underline{h})} S_{\underline{k}}^{(n)} \otimes S_{\underline{h}}^{(m)},$$

where $\underline{h}$ and $\underline{k}$ are conjugate indices as well. Similar relations hold for $\tilde{S}$, where q is replaced by q^{-1}. These decompositions turn out useful for the next result.

A.5. Proposition *The following conjugate equations hold in the tilting tensor *–category $\mathcal{T}_\ell$ associated to $\mathfrak{sl}_N$,*

$$S^* \otimes 1_{V^{\otimes m}} \circ 1_{V^{\otimes m}} \otimes S = (-1)^{mn} [m]_q! [n]_q! \, E_{-m}$$
$$S^* \otimes 1_{V^{\otimes n}} \circ 1_{V^{\otimes n}} \otimes S = (-1)^{mn} [m]_q! [n]_q! \, E_{-n},$$

where $m + n = N$. Hence V_{ω_m} and V_{ω_n} are conjugate of each other.

Proof It suffices to show the first equation. Evaluating the left hand side on a vector $\psi \in V^{\otimes m}$ gives

$$\sum_{\underline{h}}(-q)^{N(\underline{h})}(S, \psi \otimes S^{(n)}_{\underline{k}})S^{(m)}_{\underline{h}} =$$
$$\sum_{\underline{h}}(-q)^{N(\underline{h})}(\overline{R}^{(N)} S, \psi \otimes S^{(n)}_{\underline{k}})_p\, S^{(m)}_{\underline{h}} =$$
$$q^{-N(N-1)/2}\sum_{\underline{h}}(-q)^{N(\underline{h})}(\tilde{S}, \psi \otimes S^{(n)}_{\underline{k}})_p\, S^{(m)}_{\underline{h}} =$$
$$q^{-N(N-1)/2}\sum_{\underline{h},\underline{i}}(-q)^{N(\underline{h})}(-q)^{N(\underline{i})}(\tilde{S}^{(m)}_{\underline{j}} \otimes \tilde{S}^{(n)}_{\underline{i}}, \psi \otimes S^{(n)}_{\underline{k}})_p\, S^{(m)}_{\underline{h}} =$$
$$q^{-N(N-1)/2}\sum_{\underline{h}}(-q)^{N(\underline{h})}(-q)^{N(\underline{k})}(\tilde{S}^{(m)}_{\underline{h}}, \psi)_p(\tilde{S}^{(n)}_{\underline{k}}, S^{(n)}_{\underline{k}})_p\, S^{(m)}_{\underline{h}} =$$
$$(-1)^{mn}[n]_q!\sum_{\underline{h}}(S^{(m)}_{\underline{h}}, \psi)S^{(m)}_{\underline{h}} =$$
$$(-1)^{mn}[n]_q![m]_q!\, E_{-n}\psi.$$

where we have successively used selfadjointness of $\overline{R}^{(N)}$ with respect to the tensor product form, Prop. 3.6, $N(\underline{h}) + N(\underline{k}) = mn$, and the fact that $\{([m]_q!)^{-1/2}\, S^{(m)}_{\underline{h}}\}$ is an orthonormal basis of V_{ω_m} with respect to the Kirillov-Wenzl inner product.

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