THE ADJOINT REPRESENTATION INSIDE THE EXTERIOR ALGEBRA OF A SIMPLE LIE ALGEBRA

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ABSTRACT. For a simple complex Lie algebra \mathfrak{g} we study the space of invariants $A = (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}}$, which describes the isotypic component of type \mathfrak{g} in $\bigwedge \mathfrak{g}^*$, as a module over the algebra of invariants $(\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$. As main result we prove that A is a free module, of rank twice the rank of \mathfrak{g} , over the exterior algebra generated by all primitive invariants in $(\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$, with the exception of the one of highest degree.

1. Introduction

Let \mathfrak{g} be a simple Lie algebra (over \mathbb{C}) of dimension n and rank r. Fix a Cartan subalgebra \mathfrak{h} in \mathfrak{g} . Let Δ be the corresponding root system, W the Weyl group, Δ^+ a positive system. We use the Killing form (\cdot, \cdot) to identify \mathfrak{g} and \mathfrak{g}^* when convenient.

The exterior algebra $\bigwedge \mathfrak{g}$ has been extensively studied as representation of \mathfrak{g} : see e.g. [11], [12], [17], [1]. The invariant algebra $\Gamma = (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ is the cohomology of \mathfrak{g} and it is an exterior algebra $\bigwedge (P_1, \ldots, P_r)$ over primitive generators P_i of degree $2m_i + 1$, where the integers m_i , with $m_1 \leq \ldots \leq m_r$ are the *exponents* of Δ , [2].

Among the other isotypic components, of particular interest is the component of type \mathfrak{g} , which is completely described as:

$$(1.1) A := \left(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*\right)^{\mathfrak{g}} = \hom(\bigwedge \mathfrak{g}, \mathfrak{g}^*)^{\mathfrak{g}} = \hom(\mathfrak{g}, \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}.$$

So A is the space of multilinear alternating functions from \mathfrak{g} to \mathfrak{g}^* which are \mathfrak{g} -equivariant.

By the work of Kostant [12], it is known that $\dim(A) = 2^r r$. Notice that the $\mathbb{Z}/2\mathbb{Z}$ grading of $\bigwedge \mathfrak{g}^*$ allows to define a Lie superalgebra structure on A.

The Poincaré polynomial GM(q) describing the dimension of A in each degree is given by a formula conjectured by Joseph and proved by Bazlov [1]:

(1.2)
$$GM(q) = (1+q^{-1}) \prod_{i=1}^{r-1} (1+q^{2m_i+1}) \sum_{i=1}^r q^{2m_i}.$$

Clearly A is a (left or right) module over the exterior algebra $\bigwedge(P_1,\ldots,P_r)$. Writing the right hand side of (1.2) as $\prod_{i=1}^{r-1} (1+q^{2m_i+1}) \sum_{i=1}^r [q^{2m_i}+q^{2m_i-1}]$ suggests that there might exist elements

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 $f_i, u_i \in A$ of degrees $2m_i, 2m_i - 1$ which generate A as a free module over $\bigwedge(P_1, \ldots, P_{r-1})$. This is indeed our main result.

1.1. **Description of the results.** The Killing form of \mathfrak{g} induces on A a graded symmetric form which will be denoted e(a,b), with values in Γ (see (2.8)). With the elements $f_i, u_i \in A$ as in Definition 2.3 we have:

Theorem 1.1. A is a free module, with basis the elements $f_i, u_i, i = 1, ..., r$, over the exterior algebra $\bigwedge(P_1, ..., P_{r-1})$.

The main tool needed to prove this Theorem is a description of the form e(-,-) on the proposed generators f_i, u_i , which is given by the following result. Recall that the primitive invariants are uniquely defined up to a non zero constant if the exponents are distinct. The latter condition is always verified unless \mathfrak{g} is of type D_{2n} ; in this case $m_n = m_{n+1} = 2n - 1$.

Proposition 1.2. Assume that the exponents of \mathfrak{g} are distinct. Then the two submodules spanned by the f_i and by the u_j are isotropic for the form e. For each pair i, j there exists a non-zero rational constant $c_{i,j}$ such that

(1.3)
$$e(f_i, u_j) = e(f_j, u_i) = \begin{cases} c_{i,j} P_k & \text{if } m_i + m_j - 1 = m_k \text{ is an exponent,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.3. If \mathfrak{g} is of type D_{2n} , the statements of Proposition 1.2 hold true unless $m_i + m_j - 1 = 2n - 1$ or $\{i, j\} = \{n, n + 1\}$. In this case we can choose P_n and P_{n+1} in such a way that

(1.4)
$$e(f_i, u_j) = e(f_j, u_i) = \begin{cases} c_{i,j}(P_n + P_{n+1}) & \text{if } i, j \notin \{n, n+1\}, \\ c_{i,j}P_n & \text{if } i = n \text{ and } j = 1, \\ c_{i,j}P_{n+1} & \text{if } i = n+1 \text{ and } j = 1. \end{cases}$$

and
$$e(f_n, u_{n+1}) = c_{n,n+1}P_{2n}$$
, $e(f_{n+1}, u_n) = c_{n+1,n}P_{2n}$ while $e(f_n, u_n) = e(f_{n+1}, u_{n+1}) = 0$.

Finally, the full module structure of A under the algebra of invariants is completed by the formulas for the multiplication of our basis by the last primitive element P_r .

Theorem 1.4. Set $c_i := c_{i,r-i+1}$. The multiplication by P_r is self adjoint for the form e. It is given by the formulas

(1.5)
$$e(f_i, u_{r-i+1}) \wedge f_i = -\sum_{j=1, \ j \neq i}^r c_i c_j^{-1} e(f_i, u_{r-j+1}) f_j, \qquad i = 1, \dots, r,$$

(1.6)
$$e(f_i, u_{r-i+1}) \wedge u_i = -\sum_{j=1, j \neq i}^r c_i c_j^{-1} e(f_i, u_{r-j+1}) u_j, \qquad i = 1, \dots, r.$$

The constants $c_{i,j}$ depend on the choice of the primitive invariants. We will exhibit, in the course of the paper, suitable choices of the P_i as to normalize as much as possible these constants, which will be then explicitly computed.

Clearly our Theorem 1.1 implies formula (1.2), affording a proof very different from Bazlov's one.

Theorem 1.1 has a Clifford counterpart due to Kostant [12]: A is free over the whole algebra of invariants Γ , generated under Clifford multiplication by f_1, \ldots, f_r . Kostant's theorem is based on two main ingredients: the simplicity of the Clifford algebra and the machinery of Chevalley's transgression. In our situation we cannot use the former fact.

However Chevalley transgression allows us to reduce the computation of the bilinear form e to that of another $S(\mathfrak{g})^{\mathfrak{g}}$ -valued bilinear form on $S(\mathfrak{g})^{\mathfrak{g}}$, and in turn, via Chevalley's restriction Theorem, to a $\mathbb{C}[\mathfrak{h}]^W$ -valued bilinear form on $\mathbb{C}[\mathfrak{h}]^W$.

This latter form can be introduced in the following general framework. Let G be a compact Lie group and V an orthogonal representation. Given two invariants a, b of degrees h, k we obtain a new invariant $a \circ b$ of degree h + k - 2 by the formula $a \circ b := (da, db)$ where da, db are the differentials and (da, db) is computed via the given scalar product. In the case of the adjoint representation, this pairing has been studied by Givental in [10] and Saito [18] (we thank M. Rais and E. Vinberg for pointing out to us these references) and we are going to use Givental's results at least for the exceptional groups.

Let us denote by R the ring of G-invariant polynomial functions on V, and say that a homogeneous invariant a is a *generator* if it does not belong to $(R^+)^2$, i.e. it is not a product of invariants of positive degree. Notice furthermore that, by the Leibnitz rule, the pairing $a \circ b$ induces a similar composition in the vector space $M := R^+/(R^+)^2$. We will prove the following theorem.

Theorem 1.5. Let V be the reflection representation of an irreducible Weyl group. If a, b are homogeneous generators of the invariants of degrees i, j and in degree i + j - 2 there is a generator, then $a \circ b$ is a generator, except the special case of type D_{2n} in degree 2n. In this case the two generators a, b of degree 2n can be chosen in such a way that $a \circ a = b \circ b = 0$ modulo squares while $a \circ b = 0$ is a generator.

The above theorem will be proved in Proposition 2.9 if $i + j \ge h$, h being the Coxeter number of \mathfrak{g} . In fact this will suffice to deduce Theorem 1.1.

In the remaining cases the result will follow from a case by case analysis and will represent a key step in the proof of Theorem 1.4.

Finally, Theorem 1.5 has another interesting application. If we allow the operation $a \circ b$ in the construction of invariants, we shall see that in each case, besides the quadratic invariant it is enough to add either one (cases A_n, C_n, G_2) or two more generators. In this way we shall compute, using a computer, the invariants in the case of E_8 .

Remark 1.6. When all degrees of generators are distinct, fixing generators ψ_1, \ldots, ψ_r we have constants $d_{i,j}$ such that $\psi_i \circ \psi_j = d_{i,j} \psi_k \mod (R^+)^2$ when ψ_k has the same degree as $\psi_i \circ \psi_j$. Using transgression, we associate to the ψ_i primitive invariants P_i , which in turn define the constants c_{ij} via formula (1.3). We will show in (2.24) that indeed, for any i, j, we have

$$c_{i,j} = \frac{d_{i,j}}{m_i + m_j}.$$

These constants will be computed explicitly for the classical groups in §2.7.1 and for the exceptional groups in Tables 2,3,4.

Some ideas and techniques used in this paper have been developed in [7] and [8]: we refer the reader to Section 4 for a brief outline of these results.

2. The main construction

2.1. **Setup.** Let V be an n-dimensional vector space. Recall that on the exterior algebra $\bigwedge V$ we have an action of elements $x \in V^*$ as derivations, denoted by i(x), which extends the duality action on V:

$$i(x)(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^k (-1)^{i+1} x(v_i) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k,$$

 $v_1, \ldots, v_k \in V$. This formula extends to a contraction action of $\bigwedge V^*$ on $\bigwedge V$. Remark that, given $u \in \bigwedge V$, the map $x \mapsto i(x)u$ can be thought of as the element $\sum_{h=1}^n i(x^h)u \otimes x_h \in \bigwedge V \otimes V$, where $\{x_h\}, \{x^j\}$ are dual bases of V, V^* .

Given an invariant $p \in (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$, the map $x \mapsto i(x)p$, $x \in \mathfrak{g}$ is \mathfrak{g} equivariant and therefore defines an element of A, which is represented by the tensor $\sum_{h=1}^{n} i(x_h)p \otimes x^h$ (using dual bases $\{x_h\}, \{x^h\}$ for $\mathfrak{g}, \mathfrak{g}^*$).

Let $\theta(a)$, $\theta^*(a)$ denote the adjoint and coadjoint action of $a \in \mathfrak{g}$ on $\mathfrak{g}, \mathfrak{g}^*$, respectively. These actions extend to the exterior algebras as derivations of degree 0.

In $\bigwedge \mathfrak{g}^*$ we have the Koszul differential δ which makes it a differential graded algebra. In degree 1, the differential $\delta: \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ is dual to the bracket map. Explicitly

$$\delta(u) = \frac{1}{2} \sum_{i=1}^{n} x^{i} \wedge \theta^{*}(x_{i})u.$$

Remark 2.1. Recall that on $\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*$ is defined the standard Koszul differential, which is the sum of $-\delta \otimes 1$ and of $\psi \otimes a \mapsto \sum_{i=1}^n x^i \wedge \theta(x_i)(\psi \otimes a)$, where $\{x_i\}, \{x^j\}$ are as above dual bases of \mathfrak{g} , \mathfrak{g}^* respectively. If $\psi \otimes a$ is invariant this second term vanishes and the total differential is just $-\delta \otimes 1$.

Using the non degenerate Killing form one defines also the differential (in homology):

$$\partial := -\delta^t$$

The Killing form identifies $\bigwedge \mathfrak{g}^*$ with $\bigwedge \mathfrak{g}$, so we can apply, by duality, the two differentials δ and ∂ to $\bigwedge \mathfrak{g}$. Under these identifications the contraction i is just the adjoint of the wedge multiplication $\varepsilon(x): \bigwedge \mathfrak{g} \to \bigwedge \mathfrak{g}, \varepsilon(x)(y) = x \wedge y, x, y \in \bigwedge \mathfrak{g}$. We have (see [12, (89),(90)])

(2.1)
$$\delta = \frac{1}{2} \sum_{h=1}^{n} \varepsilon(x_h) \theta(x^h), \quad \partial = \frac{1}{2} \sum_{h=1}^{n} \theta(x^h) i(x_h).$$

For the space $\Gamma = (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}} = (\bigwedge \mathfrak{g})^{\mathfrak{g}}$ of invariant forms Kostant [12, Proposition 22] proves:

(2.2)
$$\Gamma = \ker \delta \cap \ker \partial = \ker L,$$

where

(2.3)
$$L = \delta \partial + \partial \delta = \frac{1}{2} \sum_{i=1}^{n} \theta(z_i)^2$$

is the Laplacian and $\{z_i\}$ is an orthonormal basis of \mathfrak{g} with respect to the Killing form. Recall the useful formulas

(2.4)
$$i(x) \, \delta + \delta \, i(x) = \theta(x), \quad x \in \mathfrak{g}.$$

(2.5)
$$i(\delta x) = -(\partial i(x) + i(x)\partial), \quad x \in \mathfrak{g}.$$

Formula (2.4) follows from [12, (92)]. To prove (2.5), take $u, v \in \bigwedge \mathfrak{g}$; then $(i(\delta(x))(u), v) = (u, \delta(x) \land v) = (u, \delta(x \land v)) + (u, x \land \delta(v)) = -(\partial(u), x \land v)) + (i(x)(u), \delta(v)) = -(i(x)(\partial(u)), v) - (\partial(i(x)(u)), v)$.

Lemma 2.2. If $p \in \Gamma$, then

- (1) $\delta p = \partial p = 0$.
- (2) $\delta i(x)p = 0$.
- (3) $\partial i(x)p = -i(\delta x)p$.
- (4) $\delta i(\delta x)p = -\frac{1}{2}i(x)p$.

Proof. Relation (1) follows at once from (2.2). To prove (2), compute using (1) and (2.4):

$$\delta i(x)p = (\delta i(x) + i(x)\delta)(p) = \theta(x)(p) = 0.$$

Part (3) follows directly from (1) and (2.5). Finally, to prove (4), use (1), (2.5) and (2.3):

$$\delta i(\delta x)p = -\delta(\partial i(x) + i(x)\partial)(p) = -\delta\partial i(x)(p) = -(\delta\partial + \partial\delta)i(x)(p) = -\frac{1}{2}i(x)(p).$$

2.2. The elements f_i, u_i . We now introduce a basic definition. Let $m : \bigwedge \mathfrak{g} \otimes \mathfrak{g} \to \bigwedge \mathfrak{g}$ be the multiplication map

$$(2.6) m(\alpha \otimes b) := \alpha \wedge b.$$

Choose primitive generators P_1, \ldots, P_r for Γ and dual bases $\{x_h\}, \{x^h\} \in \mathfrak{g}$ with respect to the Killing form.

Definition 2.3. Set, for i = 1, ..., r

$$(2.7) f_i := \frac{1}{\deg(P_i)} \sum_{h=1}^n i(x_h) P_i \otimes x^h, \quad u_i := 2(\partial \otimes 1) f_i = \frac{2}{\deg(P_i)} \sum_{h=1}^n \partial i(x_h) P_i \otimes x^h.$$

Lemma 2.4. We have $m(f_i) = P_i$, $(\delta \otimes 1)u_i = f_i$, $(\delta \otimes 1)f_i = 0$.

Proof. The first statement is a consequence of the chosen normalization:

$$m(f_i) = \frac{1}{\deg(P_i)} \sum_{h=1}^n i(x_h) P_i \wedge x^h = \frac{1}{\deg(P_i)} \sum_{h=1}^n \varepsilon(x^h) i(x_h) P_i = P_i.$$

(Recall that P_i is homogeneous of odd degree. This allows us to swap the exterior multiplication by x^h over to the left: the second to last equality follows).

To prove the second statement, compute using Lemma 2.2:

$$(\delta \otimes 1)u_i = \frac{2}{\deg(P_i)} \sum_{h=1}^n \delta \partial i(x_h) P_i \otimes x^h = -\frac{2}{\deg(P_i)} \sum_{h=1}^n \delta i(\delta x_h) P_i \otimes x^h$$
$$= \frac{1}{\deg(P_i)} \sum_{h=1}^n i(x_h) P_i \otimes x^h = f_i.$$

The last identity $(\delta \otimes 1)f_i = 0$ follows from $\delta^2 = 0$.

Since we have that dim $A = 2^r r$ (see [12, Corollary 50]), Theorem 1.1 will follow from

Proposition 2.5. The elements f_i, u_i are linearly independent over $\bigwedge(P_1, \ldots, P_{r-1})$.

In fact, Proposition 2.5 can be reduced to

Lemma 2.6. The elements f_i are linearly independent over $\bigwedge(P_1,\ldots,P_{r-1})$.

Taking this lemma for granted (the proof is at the end of this section), we can prove Proposition 2.5.

Proof of Proposition 2.5. Suppose that we have a relation $\sum_{i=1}^{r} \lambda_i u_i + \sum_{j=1}^{r} \mu_j f_j = 0$. Then apply

 $\delta \otimes 1$ and get $\sum_{i=1}^{r} \lambda_i f_i = 0$, so if we assume that the f_i are linearly independent, we get $\lambda_i = 0$ for all i and in turn that also all the μ_i are 0.

Notice that the only property we have used of the u_i is the fact that $(\delta \otimes 1)u_i = f_i$.

2.3. **The main form.** The Killing form on \mathfrak{g} induces an invariant graded symmetric bilinear form on $\bigwedge \mathfrak{g} \otimes \mathfrak{g}$ with values in $\bigwedge \mathfrak{g}$, given by

$$(2.8) e(a \otimes x, b \otimes y) = (x, y)a \wedge b$$

for $x, y \in \mathfrak{g}$, $a, b \in \bigwedge \mathfrak{g}$. In particular, by invariance, this form induces a form on A with values in $\Gamma = \bigwedge(P_1, \ldots, P_r)$.

2.3.1. Some pairings. Recall the following properties of the exponents, (cf. [2]): $m_1 = 1$ while $m_r = h - 1$, where h is the Coxeter number. Also, the m_i come naturally into pairs adding to h. We say that two indices $i, j = 1, \ldots, r$ are complementary if the corresponding exponents sum to the Coxeter number. Since we have ordered the exponents increasingly, the indices i, r - i + 1 are complementary. The exponents are all distinct except in the case of D_{2n} where the exponent 2n - 1 has multiplicity two.

Consider the matrix of scalar products $e(f_i, u_j)$; we will prove that, for complementary indices i, r - i + 1 we have

(2.9)
$$e(f_i, u_{r-i+1}) = c_i P_r, \quad 0 \neq c_i \in \mathbb{Q}$$

if \mathfrak{g} is not of type D_{2n} . In this case we have $m_n = m_{n+1} = 2n - 1$, and we can choose the primitive invariants in such a way that (2.9) holds and furthermore:

$$(2.10) e(f_n, u_n) = e(f_{n+1}, u_{n+1}) = 0.$$

Formulas (2.9), (2.10) will be proved in §2.7.2. Assuming these formulas we can finish the proof of Lemma 2.6, hence that of Theorem 1.1.

Proof of Lemma 2.6. Remark that, if there is a non trivial relation $\sum_{j=1}^{r} \mu_j f_j = 0$, we may assume that it is homogeneous, that is all terms have the same degree. Moreover, given an index j, multiplying by a suitable element of $\bigwedge(P_1, \ldots, P_{r-1})$ we can reduce ourselves to the case in which $\mu_j = P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1}$.

Notice now that the coefficient μ_h of the terms $\mu_h f_h$ for which $m_h < m_j$ has degree higher than the maximum allowed degree, hence it is zero. Thus, if we choose for j the maximum for which $\mu_j \neq 0$ and if the exponent m_j has multiplicity 1, we are reduced to prove that

$$(2.11) P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} f_i \neq 0.$$

The previous condition is satisfied for all exponents unless we are in the case of D_{2n} . In this case the exponent 2n-1 appears twice. The corresponding elements f_n, f_{n+1} have the same degree 4n-2, so that we have to show that $P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} f_n$ and $P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} f_{n+1}$ are linearly independent.

Unless we are in the case D_{2n} for the exponent of multiplicity two, we are reduced to prove (2.11). By (2.9) we have

$$e(P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1}f_j, u_{r-j+1}) = c_j \ P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} \wedge P_r \neq 0.$$

In the remaining case, linear independence of $P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} f_n$ and $P_1 \wedge P_2 \wedge \ldots \wedge P_{r-1} f_{n+1}$ follows in the same way using also formula (2.10).

- 2.4. **Proof of Formulas** (2.9) and (2.10). The proof of formulas (2.9) and (2.10) will be based on a computation in the symmetric algebra, which in turn, by the Chevalley restriction theorem, is deduced from the computations performed in §2.5.
- 2.5. Invariants of the Weyl group. The study of invariants both in the symmetric and the exterior algebra of \mathfrak{g}^* depends on special properties of the reflection representation of the Weyl group, in particular of the action of a Coxeter element. We refer to [2] for the details. Let c be a Coxeter element of W (i.e., a product of all simple reflections). Recall that its order h is the Coxeter number. Set $\zeta = e^{\frac{2\pi i}{h}}$. The eigenvalues of c in its reflection representation \mathfrak{h}^* are h-th roots of 1, ζ^{m_i} , $1 \leq m_i < h$ and the m_i are by definition the exponents. Since the reflection representation is real, together with any eigenvalue η also appears its conjugate $\bar{\eta} = \eta^{-1}$.

Fix a basis $\{X_1, \ldots, X_r\}$ of \mathfrak{h}^* made of eigenvectors relative to these eigenvalues. In particular, X_1 is relative to the eigenvalue ζ . We pair conjugate eigenvalues, and the Killing form on the space spanned by two conjugate eigenvectors X_i , X_{r-i+1} is (after normalization) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the case of the eigenvalue -1, which is conjugate of itself, we either have a single eigenvector, which we may assume to be of norm 1, or two eigenvectors in type D_{2n} . Then in [2, 6.2, Proposition 2] it is shown that to each exponent m_i we may associate a W-invariant polynomial on \mathfrak{h} such that its leading term in the variable X_1 is $X_1^{m_i}X_{r-i+1}$.

2.6. Symmetric algebra. Let $S = S(\mathfrak{g}^*)$ be the algebra of polynomial functions on \mathfrak{g} , and $R = S^{\mathfrak{g}}$ be the ring of invariants. We have that R is a polynomial algebra in generators ψ_1, \ldots, ψ_r of degrees $m_i + 1$. We observe that these invariants are so that when we restrict ψ_i to \mathfrak{h} the leading term in X_1 is a nonzero multiple $X_1^{m_i}X_{r-i+1}$, as we have seen in §2.5.

Let d be the usual differential of functions so that, when $f \in R$, we have that $df \in (S \otimes \mathfrak{g}^*)^{\mathfrak{g}}$.

Definition 2.7. The Bezoutiante matrix, is the matrix (with entries in R^+) of the scalar products $(d\psi_i, d\psi_i)$.

We first point out that the construction of the Bezoutiante is compatible with Chevalley restriction.

Lemma 2.8. Given two invariants $p, q \in R$ let \bar{p}, \bar{q} be their restriction to \mathfrak{h} . Then

$$\overline{(dp, dq)} = (d\bar{p}, d\bar{q}).$$

Proof. Use the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{u}^+ \oplus \mathfrak{u}^-)$, which is orthogonal with respect to the Killing form. Write a polynomial on \mathfrak{g} in the corresponding coordinates associated to root vectors, if F is such a polynomial we divide it as $F = \bar{F} + f$ where \bar{F} collects only the monomials in the coordinates of \mathfrak{h} while every monomial in f contains at least one coordinate x_{α} associated to a root. Now if F is invariant in particular each monomial must have weight 0 for \mathfrak{h} , hence if some variables x_{α} appear for a positive root, there must be other variables for negative roots. This means that f vanishes of order at least 2 on \mathfrak{h} , hence also its differential vanishes and we have the claim by the orthogonality relation.

Proposition 2.9. The entries of the matrix $(d\psi_i, d\psi_j)$ below the antidiagonal lie in $(R^+)^2$. On the antidiagonal they are non zero multiples of ψ_r modulo squares.

Proof. The first statement follows by degree considerations. As for the second, by Lemma 2.8 we can compute the Bezoutiante matrix on the W-invariant functions $\bar{\psi}_i$ on \mathfrak{h} .

The form in the basis X_1, \ldots, X_r introduced above, equals $\sum_{(i,\sigma(i))} X_i X_{\sigma(i)}$ (where σ is the involution pairing complementary indices, the sum is over the orbits of σ which are made by two or one element). It follows that the invariant $(d\bar{\psi}_i, d\bar{\psi}_j)$ has leading term in X_1 proportional to X_1^h if i,j are complementary indices, arising from

$$\frac{\partial X_1^{m_i} X_j}{\partial X_i} \frac{\partial X_1^{m_j} X_i}{\partial X_i} = X_1^h.$$

Hence, if i + j = r + 1, by the previous relation we have $(d\bar{\psi}_i, d\bar{\psi}_j) = c_i\bar{\psi}_r \mod (R^+)^2$ for some nonzero c_i .

2.7. **Transgression.** Consider the map $\delta: \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$. Since the elements in its image commute with respect to exterior multiplication, δ admits a unique extension to an algebra homomorphism

$$s: S(\mathfrak{g}^*) \to \bigwedge {}^{even}\mathfrak{g}^*.$$

Observe that s takes as values coboundaries, in particular $\delta s = 0$. Using the multiplication map m defined by (2.6) we set, for $a \in S(\mathfrak{g}^*)$:

$$(2.12) t(a) := m((s \otimes 1)da).$$

It follows from the explicit description of the transgression map, which is obtained in [5] (and in formula (231) of [12, Theorem 64]), that on homogeneous invariants of positive degree d our map t coincides up to the constant $(d!)^2/(2d+1)!$ with the classical transgression map. See also [15, Remark 6.11] for a discussion on terminology. We prefer to eliminate this constant since it is cumbersome in the formulas. The key result on the transgression map [4] can be reformulated as follows.

Theorem 2.10. The map t vanishes on $(R^+)^2$ and induces an isomorphism of $R^+/(R^+)^2$ with the space of primitive elements of Γ .

Thus, given generators ψ_i of degree $m_i + 1$ for R, one has that the elements $P_i := t(\psi_i)$ are primitive generators of Γ , of degrees $2m_i + 1$.

In particular, choose a set $\{\psi_1, \ldots, \psi_r\}$ of homogeneous generators of the ring R of symmetric invariants. We obtain a set of primitive generators $P_i := t(\psi_i)$ of Γ and the corresponding elements u_i , f_i defined by formula (2.3). To proceed further, we need some general formulas.

Lemma 2.11. For all homogeneous elements $a, b \in S(\mathfrak{g}^*)$ we have

$$(2.13) t(ab) = t(a) \wedge s(b) + s(a) \wedge t(b),$$

(2.14)
$$\delta t(a) = \deg(a)s(a).$$

Proof.

$$t(ab) = m((s \otimes 1)d(ab)) = m((s \otimes 1)(bd(a) + ad(b)))$$

= $m(s(b) \wedge (s \otimes 1)d(a)) + m(s(a) \wedge (s \otimes 1)(d(b))$
= $s(b) \wedge m((s \otimes 1)d(a)) + s(a) \wedge m((s \otimes 1)(d(b))) = t(a) \wedge s(b) + s(a) \wedge t(b).$

This proves (2.13). To prove (2.14), let $da = \sum_{h=1}^{n} i_S(x_h) a \otimes x^h$ (where $i_S(x_h)$ is the directional

derivative w.r.t. x_h), so that $\deg(a)a = \sum_{h=1}^n i_S(x_h)x^h$ and $t(a) = \sum_{h=1}^n s(i_S(x_h)) \wedge x^h$. Then

$$\delta t(a) = \sum_{h=1}^{n} \delta s(i_S(x_h)) \wedge x^h + \sum_{h=1}^{n} s(i_S(x_h)) \wedge \delta x^h = \sum_{h=1}^{n} s(i_S(x_h)) \wedge \delta x^h$$
$$= \sum_{h=1}^{n} s(i_S(x_h)) \wedge s(x^h) = s(\sum_{h=1}^{n} i_S(x_h)x^h) = \deg(a)s(a).$$

Definition 2.12. Define for any homogeneous element $a \in S(\mathfrak{g}^*)$:

$$(2.15) f(a) := (s \otimes 1)da, \quad u(a) := (t \otimes 1)da.$$

Note that, since $da = \sum_{h=1}^{n} i_S(x_h)a \otimes x^h$, we have $\deg(i_S(x_h)a) = \deg(a) - 1$, (2.14) implies

$$(2.16) \qquad (\delta \otimes 1)u(a) = (\deg(a) - 1)f(a).$$

Lemma 2.13. For every i = 1, ..., r, take f_i as in formula (2.7). Then $f_i = f(\psi_i)$.

Proof. Set for $a \in R$ homogeneous:

$$f_a := \frac{1}{2 \deg(a) + 1} \sum_{h=1}^{n} i(x_h) t(a) \otimes x^h,$$

so that $f_i = f_{\psi_i}$ by (2.7).

By [12], Theorem 73, one has that $i(x)t(a) = cs(i_S(x)a)$ for any x and a invariant, where the constant c depends on the normalizations we choose for t. Thus, since $(s \otimes 1)(da) = \sum_{h=1}^{n} s(i_S(x_h)a) \otimes x^h$, we have:

$$f(a) = (s \otimes 1)(da) = \sum_{h=1}^{n} s(i_S(x_h)a) \otimes x^h = \sum_{h=1}^{n} c \ i(x_h)t(a) \otimes x^h,$$

so f_i and $f(\psi_i)$ are proportional. To determine the proportionality constant c it is enough to apply the multiplication map m. f_i has been normalized so that $m(f_i) = P_i$, while $m(f(\psi_i)) = t(\psi_i) = P_i$ by definition. Thus $c = 1/(2 \deg(a) + 1)$ and everything follows.

Proposition 2.14. Given two elements $a, b \in S(\mathfrak{g}^*)$ we have

$$(2.17) e(f(a), f(b)) = s((da, db)), e(u(a), f(b)) + e(f(a), u(b)) = t((da, db)).$$

Proof. By construction $f(a) = (s \otimes 1)d(a)$, $f(b) = (s \otimes 1)d(b)$ hence

$$e(f(a), f(b)) = s((da, db)).$$

To prove the second relation, write $da = \sum_{j=1}^{n} \alpha_j \otimes z_j$, $db = \sum_{j=1}^{n} \beta_j \otimes z_j$ where $\{z_j\}$ is an

orthonormal basis. Then $(da, db) = \sum_{j=1}^{n} \alpha_j \beta_j$, hence by (2.13) and (2.15) we have

$$t((da, db)) = \sum_{j=1}^{n} t(\alpha_j) \wedge s(\beta_j) + \sum_{j=1}^{n} s(\alpha_j) \wedge t(\beta_j)$$

= $e((t \otimes 1)(da), (s \otimes 1)(db)) + e((s \otimes 1)(da), (t \otimes 1)(db))$
= $e(u(a), f(b)) + e(f(a), u(b)),$

which is the required relation.

Lemma 2.15. For $a, b \in \bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*$, we have that

(2.18)
$$\partial e(a,b) - e(\partial \otimes 1(a),b) + (-1)^{\deg(a)} e(a,\partial \otimes 1(b))$$

is orthogonal (for the Killing form) to all invariant elements, hence zero if it is an invariant.

Proof. Write, in an orthonormal basis $\{z_i\}$, $a = \sum_{i=1}^n a_i \otimes z_i$, $b = \sum_{i=1}^n b_i \otimes z_i$, $e(a,b) = \sum_{i=1}^n a_i \wedge b_i$. By

formula (4.7) of [13] we have that $\sum_{i=1}^{n} (\partial(a_i \wedge b_i) - \partial(a_i) \wedge b_i + (-1)^{\deg(a_i)} a_i \wedge \partial(b_i))$ is orthogonal

(for the scalar Killing form) to all invariant elements. Since $\partial \otimes 1(a) = \sum_{i=1}^{n} \partial a_i \otimes z_i$, $\partial \otimes 1(b) =$

$$\sum_{i=1}^{n} \partial b_i \otimes z_i$$
, the claim follows.

We now set for $a \in S(\mathfrak{g}^*)$ a homogeneous invariant

$$\bar{u}(a) := 2(\partial \otimes 1) f(a)$$

Proposition 2.16. If $a, b \in S(\mathfrak{g}^*)$ are homogeneous invariants, then

(2.19)
$$e(f(a), f(b)) = 0,$$

(2.20)
$$e(\bar{u}(a), \bar{u}(b)) = 0,$$

(2.21)
$$e(\bar{u}(a), f(b)) = e(f(a), \bar{u}(b)) = \frac{t((da, db))}{\deg(a) + \deg(b) - 2}.$$

Proof. From (2.17) and the definition of s we have that e(f(a), f(b)) is the value of s on an invariant. Now remark that elements in the image of s are coboundaries, and an invariant which is also a coboundary is 0. This proves (2.19).

In order to prove (2.20), we compute (2.18) for $\bar{u}(a)$, f(b). Using Lemma 2.15 we get

$$0 = \partial e(\bar{u}(a), f(b)) - e((\partial \otimes 1)(\bar{u}(a)), f(b)) + (-1)^{\deg(\bar{u}(a))} e(\bar{u}(a), (\partial \otimes 1)(f(b)))$$
$$= (-1)^{\deg(\bar{u}(a))} \frac{1}{2} e(\bar{u}(a), \bar{u}(b)),$$

since both $\partial e(\bar{u}(a), f(b)), e((\partial \otimes 1)(\bar{u}(a)), f(b))$ are clearly 0.

We pass now to the last identity. We have for any homogeneous element $w \in A$,

$$0 = \delta(e(w, \bar{u}(b))) = e((\delta \otimes 1)(w), \bar{u}(b)) + (-1)^{\deg(w)} e(w, (\delta \otimes 1)(\bar{u}(b)))$$

= $e((\delta \otimes 1)(w), \bar{u}(b)) + (-1)^{\deg(w)} e(w, f(b)).$

It follows that

$$e(w, f(b)) = -(-1)^{\deg(w)} e((\delta \otimes 1)(w), \bar{u}(b)).$$

Since $f(a) = 1/(\deg(a) - 1)\delta \otimes 1u(a) = \delta \otimes 1\bar{u}(a)$, we deduce

$$e(u(a), f(b)) = (\deg(a) - 1)e(\bar{u}(a), f(b)).$$

We have $\delta e(\bar{u}(a), \bar{u}(b)) = 0$. Computing and using the fact that $\bar{u}(a)$ has odd degree, we get

$$0 = \delta e(\bar{u}(a), \bar{u}(b)) = e((\delta \otimes 1)(\bar{u}(a)), \bar{u}(b)) - e(\bar{u}(a), (\delta \otimes 1)(\bar{u}(b))) = e(f(a), \bar{u}(b)) - e(\bar{u}(a), f(b)),$$

that is $e(f(a), \bar{u}(b)) = e(\bar{u}(a), f(b))$. Formula (2.21) then follows immediately from (2.17). \Box

Corollary 2.17.

$$(2.22) e(f_i, f_j) = 0,$$

$$(2.23) e(u_i, u_i) = 0,$$

(2.24)
$$e(u_i, f_j) = e(f_i, u_j) = \frac{t((d\psi_i, d\psi_j))}{\deg(\psi_i) + \deg(\psi_j) - 2}.$$

2.7.1. Examples in the classical cases. For elements of R we introduce the following notation: $A \cong B$ if the two invariants are congruent modulo $(R^+)^2$. By abuse of notation we denote by the same symbol the invariant on \mathfrak{g} or its restriction to \mathfrak{h} .

Type A_n . Consider as generating invariants the normalized Newton polynomials

(2.25)
$$q_k := \frac{1}{k} \sum_{i=1}^{n+1} x_i^k, \ k = 2, \dots, n+1$$

(so that, in our general notation, $\psi_k = q_{k+1}, k = 1, \ldots, n$). We have

(2.26)
$$(dq_k, dq_g) = \sum_{i=1}^{n+1} x_i^{k+g-2} = (k+g-2)q_{k+g-2}.$$

We associate, via transgression, to these symmetric invariants the primitive invariants $P_i := t(\psi_i) = m((s \otimes 1)d\psi_i)$. We have the corresponding f_i, u_i and, plugging (2.26) into (2.21), we obtain:

(2.27)
$$e(f_i, u_j) = \frac{t((i+j)q_{i+j})}{i+j} = \begin{cases} P_{i+j-1} & \text{if } i+j \le n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Type C_n . Here we can choose as generators the normalized Newton functions $\psi_i = q_{2i}$, $i = 1, \ldots, n$. The formulas are the same as before:

$$e(f_i, u_j) = t(q_{2i+2j-2}) = \begin{cases} P_{i+j-1} & \text{if } 2i + 2j - 2 \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

which is indeed (2.27).

Type D_{2n+1} ,. Here we can choose as generators the normalized Newton functions $\psi_i = q_{2i}$, $i = 1, \ldots, n$, the function $\psi_{n+1} = \mathcal{P} := \sqrt{-2n} \prod_{i=1}^{2n+1} x_i$ and the functions $\psi_i = q_{2i-2}$ for $i = n+2, \ldots, 2n+1$. The formulas are the same as before for the Newton functions. As for those involving \mathcal{P} , we have

$$(2.28) (d\mathcal{P}, dq_2) = (2n+1)\mathcal{P}.$$

$$(2.29) (d\mathcal{P}, dq_{2k}) = 2(k-1)q_{2k-2} \cdot \mathcal{P} \in (R^+)^2, \ \forall k > 1,$$

$$(2.30) (d\mathcal{P}, d\mathcal{P}) \cong 4n \, q_{4n}.$$

Proof of (2.28), (2.29), (2.30). Formulas (2.28), (2.29) are straightforward. As for (2.30) we have

(2.31)

$$d\mathcal{P} = \sqrt{-2n} \sum_{i=1}^{2n+1} (\prod_{j \neq i} x_j) dx_i \implies (d\mathcal{P}, d\mathcal{P}) = -2n \sum_{i=1}^{2n+1} \prod_{j \neq i} x_j^2 = -2n \, s_{2n}(x_1^2, \dots, x_{2n+1}^2)$$

where by s_i we denote the i^{th} elementary symmetric function. From the formulas expressing elementary symmetric function in terms of power sum we see that, for each $i \leq 2n + 1$, we have $s_i(x_1, \ldots, x_{2n+1}) \cong (-1)^{i+1}q_i$ modulo squares, hence

$$(2.32) -2n s_{2n}(x_1^2, \dots, x_{2n+1}^2) \cong 2n q_{2n}(x_1^2, \dots, x_{2n+1}^2) = 4n q_{4n}(x_1, \dots, x_{2n+1}).$$

Combining (2.31) and (2.32) we get (2.30).

Now we can compute the scalar products $e(f_i, u_i)$; if f_i, u_i are constructed from Newton functions we still have formulas (2.27). As for f_{n+1}, u_{n+1} , the elements associated to \mathcal{P} , we have, by formulas (2.28), (2.29) and (2.30)

(2.33)
$$e(f_{n+1}, u_j) = 0, \ \forall j \neq n+1, \quad e(f_{n+1}, u_{n+1}) = \frac{t(4n \, q_{4n})}{4n} = P_{2n+1}.$$

Type D_{2n} , . Here we can choose as generators the normalized Newton functions $\psi_i=q_{2i},\ i=1,\ldots,n$ $1, \ldots, n-1$, the functions

$$\psi_n = q_{2n} + \sqrt{-1}\mathcal{P}, \quad \psi_{n+1} = q_{2n} - \sqrt{-1}\mathcal{P},$$

with $\mathcal{P} := \sqrt{2n-1} \prod_{i=1}^{2n} x_i$ and the functions $\psi_i = q_{2i-2}$ for $1 = n+2, \dots, 2n$. Notice that ψ_i has degree 2i for $i = 1, \dots, n-1$, has degree 2n for i = n, n+1 and has degree 2i-2 for $i=n+2,\ldots,2n$. The formulas are the same as before for the Newton functions. As for those involving \mathcal{P} , we have the formulas

$$(2.34) (d\mathcal{P}, dq_2) = 2n\mathcal{P},$$

$$(2.35) (d\mathcal{P}, dq_{2k}) = 2(k-1)q_{2k-2} \cdot \mathcal{P} \in (R^+)^2, \ \forall k > 1,$$

$$(2.36) (d\mathcal{P}, d\mathcal{P}) \cong (4n-2) q_{4n-2}.$$

whose proof is identical to that of formulas (2.28), (2.29), (2.30).

Reasoning as before it immediate to verify that, up to the case in which both i and j do not belong to $\{n, n+1\}$, we have

(2.37)
$$e(f_i, u_j) = \begin{cases} P_h & \text{if } m_i + m_j - 1 = m_h, \ h \neq n, n+1, \\ \frac{1}{2}(P_n + P_{n+1}) & \text{if } m_i + m_j - 1 = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand we have

$$(2.38) e(f_n, u_j) = \begin{cases} P_n & j = 1 \\ P_{n+j} & 2 \le j \le n - 1 \\ 0 & j = n \\ 2P_{2n} & j = n + 1 \\ 0 & j > n + 1 \end{cases} e(f_{n+1}, u_j) = \begin{cases} P_{n+1} & j = 1 \\ P_{n+j} & 2 \le j \le n - 1 \\ 2P_{2n} & j = n \\ 0 & j > n \end{cases}$$

2.7.2. Conclusion of the proof.

Proof of Theorem 1.1. Recall that we have reduced the proof of the theorem to the proof of formulas (2.9), (2.10). The former follows from Proposition 2.9 and formula (2.24). The latter, which is relative to D_{2n} , is obtained using also (2.33).

Proof of Theorem 1.5. Given an element $a \in \mathbb{R}^+$ set \tilde{a} equal to its image in $M := \mathbb{R}^+/(\mathbb{R}^+)^2$. Take the usual set of homogeneous generating invariants $\psi_1, \dots \psi_r \in \mathbb{R}^+$ of degrees $d_i = m_i + 1$. We have already seen that $\tilde{\psi}_i \circ \tilde{\psi}_j := \widetilde{\psi}_i \circ \psi_j$ is well defined. Assume \mathfrak{g} not of type D_{2n} . Then our claim will follow if we show that $\tilde{\psi}_i \circ \tilde{\psi}_j \neq 0$ if and only if $m_i + m_j - 1$ is an exponent.

The fact that $\tilde{\psi}_i \circ \tilde{\psi}_j = 0$ if $m_i + m_j - 1$ is not an exponent is clear since M does not contain elements of degree $m_i + m_j = \deg(\tilde{\psi}_i \circ \tilde{\psi}_j)$.

When $m_i + m_j - 1$ is an exponent, the explicit analysis of the classical cases performed above shows that $\tilde{\psi}_i \circ \tilde{\psi}_j$ is actually non zero, hence a multiple of the corresponding generator of degree $m_i + m_j$. The statement of Theorem 1.5 in type D_{2n} follows by using formulas (2.34)-(2.36).

As for the exceptional cases, remark that the cases of G_2 and F_4 are an immediate consequence of Proposition 2.9.

It remains to discuss the case of algebras of type E. Let us now consider in each of the types E_6, E_7, E_8 , the equation p(w, x, y) of the corresponding simple surface singularities (see [19]). For convenience of the reader let us recall that they are

$$w^2 + x^3 + y^4$$
 for type E_6 ,
 $w^2 + x(x^2 + y^3)$ for type E_7 ,
 $w^2 + x^3 + y^5$ for type E_8 .

Take the algebra $Q = \mathbb{C}[x,y]/(p_x,p_y)$.

Let us first point out a few simple facts about Q which are readily verified. Q is a local algebra with maximal ideal m=(x,y). By suitably choosing the degrees of the variables w,x,y,p becomes in each case, a homogeneous polynomial of degree equal to the Coxeter number h. It follows that Q is graded. An easy computation shows that Q has Poincaré polynomial given by $\sum_{i=1}^{r} t^{m_i-1}$.

Consider now the operator S on Q which, for any homogeneous element $q \in Q$, is given by

$$Sq = \frac{\deg q + 2}{h}q.$$

By Corollary 2 in [10] there is an isomorphism $\Psi: M \to Q$ such that

(1) If $\tilde{a} \in M$ is homogeneous then

$$deg(\Psi(\tilde{a})) = deg(\tilde{a}) - 2.$$

(2) For any $\tilde{a}, \tilde{b} \in M$,

$$\Psi(\tilde{a} \circ \tilde{b}) = S(\Psi(\tilde{a})\Psi(\tilde{b})).$$

From this our claim follows by a straightforward case by case analysis.

Proof of Propositions 1.2, 1.3. Formulas (2.22) and (2.23) show that the u's and the f's generate isotropic subspaces.

Let us now pass to the determination of $e(f_i, u_j)$. Combining Theorem 2.10 and formula (2.22) we obtain that everything follows from Theorem 1.5.

2.8. Classical groups. If G is a classical group one can take a different approach. Let V be the defining representation of G, so that $\mathfrak{g} = Lie(G)$ is a subalgebra of End(V), which decomposes as $End(V) = \mathfrak{g} \oplus \mathfrak{p}$ where in case A_n the space \mathfrak{p} is the 1-dimensional trivial representation, while in the other cases it is the space of symmetric matrices for the corresponding involution, in all cases an irreducible representation. It is convenient to study the

associative invariant algebra, i.e. the algebra of G-equivariant maps

$$(2.39) A_G = (\bigwedge End(V)^* \otimes End(V))^G.$$

Then one can analyze inside A_G the super Lie subalgebra $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g})^G$. This among other topics is discussed in [3].

In the natural basis $\{e_{ij}\}$ of matrices with coordinates x_{ij} consider the element $X \in A_{GL(N)}$ (cf. (2.39)), which is the generic Grassmann matrix $X = \sum_{h,k} x_{hk} e_{hk}$. Its power $X^a = X^{\wedge a}$ equals the standard polynomial S_a computed in End(V), hence in this language the Amitsur–Levitzki Theorem, see [16], is the single identity $X^{2n} = 0$. In [3] the authors prove, among other results, the following

Theorem 2.18. The algebra $A_{GL(n)}$ is generated by X and the elements $tr(X^{2i-1})$, $i = 1, \ldots, n$. All these elements anticommute.

 $A_{GL(n)}$ is a free module with basis X^i , i = 0, ..., 2n - 1 over the Grassmann algebra in the elements $tr(X^{2i-1})$, i = 1, ..., n-1 and we have the two defining identities

(2.40)
$$X^{2n} = 0, \quad tr(X^{2n-1}) = -\sum_{i=1}^{n-1} X^{2i} \wedge tr(X^{2(n-i)-1}) + nX^{2n-1}.$$

As for the connection with $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}} \cong (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g})^{\mathfrak{g}}$ where $\mathfrak{g} = sl(n)$ we see immediately that

$$A_{GL(n)} = (\bigwedge \mathfrak{g}^* \otimes \mathfrak{g})^{\mathfrak{g}} \oplus \Gamma.$$

Write $X = Y + \frac{tr(X)}{n}$ where tr(Y) = 0 and see that for a > 1 we have $X^{2a} = Y^{2a}$. Hence $X^{2a} = Y^{2a}$ is in $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g})^{\mathfrak{g}}$ while in the odd case we have that $X^{2a+1} - \frac{X^{2a}tr(X)}{n}$ is in $(\bigwedge \mathfrak{g}^* \otimes \mathfrak{g})^{\mathfrak{g}}$. These are up no normalizations the elements f_i, u_i defined in §2.2 in the present case. Clearly the defining identities (2.40) which describe the associative algebra also allow us to compute all the relations for the Lie algebra case.

The other classical groups can be treated in a similar way: an investigation in this direction has been started in [8] (cf. Section 4).

3. Proof of Theorem 1.4

3.1. **The relations.** In order to complete the description of A as module over Γ we only need to express the elements $P_r \wedge u_i$ and $P_r \wedge f_i$ in our given basis.

Consider the relation for u_i ; we have

(3.1)
$$P_r \wedge u_i = \sum_{j=1}^r H_j \wedge u_j + \sum_{j=1}^r K_j \wedge f_j,$$

where $H_j, K_j \in \bigwedge(P_1, \dots, P_{r-1})$. Applying the differential $\delta \otimes 1$ we get

$$(3.2) P_r \wedge f_i = \sum_{j=1}^r H_j \wedge f_j.$$

Thus the relation for f_i involves only f_j 's. Also we have that the relation is homogeneous.

For each j, taking the scalar product with u_{r-j+1} , we have

$$P_r \wedge e(f_i, u_{r-j+1}) = H_j \wedge e(f_j, u_{r-j+1}) + \sum_{h \neq j} H_h \wedge e(f_h, u_{r-j+1})$$
$$= H_j \wedge c_j P_r + \sum_{h \neq j} H_h \wedge e(f_h, u_{r-j+1}).$$

Since the terms $\sum_{h\neq j} H_h \wedge e(f_h, u_{r-j+1})$ do not involve P_r , we must have

(3.3)
$$\sum_{h \neq j} H_h \wedge e(f_h, u_{r-j+1}) = 0,$$
$$-e(f_i, u_{r-j+1}) \wedge P_r = H_j \wedge c_j P_r$$

If $i \neq j$, we have that $e(f_i, u_{r-j+1})$ is not a multiple of P_r and we deduce that

$$e(f_i, u_{r-j+1}) = -c_j H_j.$$

If i = j, we deduce $H_j = 0$, so finally (3.2) becomes

(3.4)
$$P_r \wedge f_i + \sum_{i \neq j} c_j^{-1} e(f_i, u_{r-j+1}) \wedge f_j = 0.$$

Since $e(f_i, u_{r-i+1}) = c_i P_i$, formula (3.4) is indeed formula (1.5), as required. We go back to formula (3.1), which we now write:

(3.5)
$$P_r \wedge u_i = -\sum_{j=1}^r c_j^{-1} e(f_i, u_{r-j+1}) \wedge u_j + \sum_{j=1}^r K_j \wedge f_j.$$

Take the scalar product of both sides of (3.5) with u_{r-j+1} . We get

$$P_r \wedge e(u_i, u_{r-j+1}) = -\sum_{j=1}^r c_j^{-1} e(f_i, u_{r-j+1}) \wedge e(u_j, u_{r-j+1}) + \sum_{j=1}^r K_j \wedge e(f_j, u_{r-j+1}).$$

Since $e(u_h, u_k) = 0$, we deduce that

$$K_j \wedge e(f_j, u_{r-j+1}) + \sum_{i, i \neq j} K_i \wedge e(f_i, u_{r-j+1}) = 0$$

which in turn, by (2.9), becomes

$$c_j K_j \wedge P_r + \sum_{i, i \neq j} K_i \wedge e(f_i, u_{r-j+1}) = 0.$$

We claim that all K_j are zero. In fact, the only product containing P_r is $c_jK_j \wedge P_r$. On the other hand, it is clear that each element of Γ can be written in a unique way in the form $a + b \wedge P_r$ with $a, b \in \bigwedge(P_1, \dots, P_{r-1})$. We deduce that $K_j = 0$ for each j, and the proof of Theorem 1.4 is completed.

4. Final Remarks

In view of Theorem 1.1, it is natural to ask whether, for some other irreducible representation L of \mathfrak{g} , the space of covariants of type L

$$A_L := \hom_{\mathfrak{g}}(L, \bigwedge \mathfrak{g}^*)$$

is free over $\bigwedge(p_1,\ldots,p_{r-1})$ of predictable rank. In [7] it is shown that the techniques and the outline of proof of Theorem 1.1 can be enhanced to prove that A_L is free over $\bigwedge(p_1,\ldots,p_{r-1})$ of rank $2\dim L_0$ (L_0 being the 0-weight space of L), in the following two cases:

- L is the little adjoint representation of \mathfrak{g} (i.e., the irreducible representation with highest weight the highest short root);
- \mathfrak{g} is of type A_{n-1} and $L = S^n(V)$ is the *n*-th symmetric power of the defining representation V or $L = S^n(V)^*$.

Another related question is the study of covariants of (indecomposable) infinitesimal symmetric spaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, more precisely of the spaces $(\mathfrak{g} \otimes \bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$, under the assumption that $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ is an exterior algebra (say on r generators). This can be successfully pursued in the classical cases using the associative superalgebra structure of (2.39) as in Subsection 2.8. The final outcome, proved by Dolce in [8] when \mathfrak{g} is simple, is that $(\mathfrak{g} \otimes \bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ is free of rank 4r over the exterior algebra generated by the r-1 invariants of degree less then the maximal one. Notice that our Theorem 1.1 fits into this picture by considering as \mathfrak{g} two copies of a simple Lie algebra switched by the flip involution.

5. Appendix

Formulas for invariants of exceptional groups can be found in several places, except case E_8 for which there are computer programs but no clear references. A possible method of computation that we partly followed appears in a paper of Lee [14]. We assume we are in a case in which the exponents are all distinct. Let \mathfrak{h} and W be as usual. For $a \in \mathfrak{h}^*$ and $m \in \mathbb{N}$ define

$$p_m(a) := \sum_{w \in W} w(a^m) = \sum_{w \in W} w(a)^m.$$

Lemma 5.1. Given an exponent m_i the set of a for which $p_{m_i+1}(a)$ is a generator is a non-empty open Zariski set.

Proof. The elements a^m generate linearly the symmetric power $S^m(\mathfrak{h}^*)$ and the operator $x \mapsto 1/|W| \sum_{w \in W} w(x)$ is the projection on the invariants therefore it is not possible that for all $a \in \mathfrak{h}^*$ the element $p_m(a) = \sum_{w \in W} w(a^m)$ is not a generator.

Let now m_i be an exponent and I_{m_1+1} , \bar{I}_{m_1+1} be the space of invariants of degree m_i+1 and the subspace of decomposable ones, respectively. Thus $\dim(I_{m_i+1}/\bar{I}_{m_i+1})=1$. Let $\pi:I_{m_1+1}\to I_{m_1+1}/\bar{I}_{m_1+1}$ be the projection. We thus have a map $a\mapsto \pi(p_{m_i+1}(a))$ which, trivializing the quotient, we think of as an invariant $Q_i(a)$.

Theorem 5.2. An element $a \in \mathfrak{h}^*$ is such that the set $\{p_{m_i+1}(a) \mid 1 \leq i \leq r\}$ is a system of generators if and only if $Q_i(a) \neq 0$ for every i.

Proof. The condition is tautological but it has an interesting interpretation. The fact that these elements are generators is equivalent to ask that the determinant of the associated Jacobian matrix J(a) is non-zero. This determinant, as function of a is non-zero by the previous lemma. By degree and symmetry considerations we deduce that it is a multiple, dependent on a, of the Weyl denominator $\prod_{\alpha \in \Delta^+} \alpha$. This multiple, as a function of a, is $\prod_{i=1}^r Q_i(a)$ (up to a constant). Hence

$$J(a) = \prod_{i=1}^{r} Q_i(a) \prod_{\alpha \in \Delta^+} \alpha.$$

In practice the previous Theorem is hard to use, so we will proceed in a different way. We compute only the first three invariants, choosing a in the simplest possible way and using the previous formula. Then compute the remaining ones by applying the construction $u \circ v$ which, by Theorem 1.5, will provide all the remaining generators (recall that the algebra Q is generated by the two homogeneous elements x, y in all cases).

In order to check that the three invariants we have found are indeed generators, it is enough to show that they are not decomposable. This can be verified by choosing a suitable monomial which never appears in the decomposable invariants of that degree and showing that the invariant we found has non–zero coefficient for that monomial. This is quite easy and we verified it for our choices.

5.1. Exceptional groups. Recall the exponents of exceptional groups.

Type	Exponents	h
E_6	1, 4, 5, 7, 8, 11	12
E_7	1, 5, 7, 9, 11, 13, 17	18
E_8	1, 7, 11, 13, 17, 19, 23, 29	30
F_4	1, 5, 7 ,11	12
G_2	1, 5	6

Table 1: exponents for the exceptional types.

In G_2 , F_4 there are no particular normalizations involved. For F_4 it is enough to define the invariant of degree 12 from the scalar product of those of degree 6, 8.

In this section we usually will write A_i for an invariant of the root system of degree i and implicitly for the corresponding invariant in $S(\mathfrak{g}^*)$. The invariant of degree 2 will always be taken to correspond to the Killing form. Then, by the definition of the composition $a \circ b =$

(da, db) between invariants, we have that

$$A_2 \circ A_i = 2iA_i, \forall i.$$

When we pass to the corresponding elements $P_i = t(A_i)$, the element $t(A_2)$ is denoted by P_1 and has degree 3; formula (2.21) implies $e(u_1, f_i) = \frac{t(A_2 \circ A_i)}{i} = 2P_i$. If we wish to have value 1 for the constants $c_{1,i}$ it is enough to let A_2 correspond to half the Killing form.

5.2. A sketch of the computations. In cases E_6, E_7 we are computing, on a space of dimension n=6,7 and coordinates x_1, \ldots, x_n , the scalar product of invariants written as polynomials in the power sums $p_i = \sum_{j=1}^n x_j^i$. Therefore the scalar product of two such invariants a,b is $\sum_{i,j} \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial p_j} (dp_i, dp_j)$, so it is necessary to compute a priori $(dp_i, dp_j) = \sum_{h,k} \frac{\partial p_i}{\partial x_h} \frac{\partial p_j}{\partial x_k} (dx_h, dx_k)$. In each case the quadratic function expressing the scalar product is of the form $ap_2 + bp_1^2$. This means that the matrix of the scalar products $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is $a1_n + bM_n$ where M_n is the $n \times n$ matrix with all entries equal to 1, hence $M_n^2 = nM_n$. The matrix of the scalar products (dx_i, dx_j) is

$$(a1_n + bM_n)^{-1} = a^{-1}(1_n - \frac{b}{a}(1 + \frac{bn}{a})^{-1}M_n)$$
$$(a1_n + bM_n)^{-1} = a^{-1}(1_n - b(a + bn)^{-1}M_n).$$

In the two cases we have

$$\frac{2}{3}(1-\frac{1}{9}M_6), \quad \frac{1}{2}(1-\frac{1}{9}M_7),$$

respectively, so that

$$p_i \circ p_j = \begin{cases} \frac{2ij}{3} (p_{i+j-2} - \frac{1}{9} p_{i-1} p_{j-1}) & \text{in type } E_6, \\ \\ \frac{ij}{2} (p_{i+j-2} - \frac{1}{9} p_{i-1} p_{j-1}) & \text{in type } E_7. \end{cases}$$

In case E_8 we have that the scalar product is $p_2 = \sum_{i=1}^8 x_i^2$. The invariants are polynomials in the power sums p_{2i} , i = 1, ..., 7 and the Pfaffian: $\mathcal{P} = \prod_{i=1}^8 x_i$, hence (cf. (2.31)):

$$p_i \circ p_j = ij \, p_{i+j-2}, \quad p_i \circ \mathcal{P} = i \, p_{i-1} \mathcal{P},$$

$$\mathcal{P} \circ \mathcal{P} = s_7(x_1^2, \dots, x_8^2) = \frac{1}{5040} p_2^7 - \frac{1}{240} p_2^5 p_4 + \frac{1}{48} p_2^3 p_4^2 - \frac{1}{48} p_2 p_4^3 + \frac{1}{72} p_2^4 p_6 - \frac{1}{12} p_2^2 p_4 p_6 + \frac{1}{24} p_4^2 p_6 + \frac{1}{18} p_2 p_6^2 - \frac{1}{24} p_2^3 p_8 + \frac{1}{8} p_2 p_4 p_8 - \frac{1}{12} p_6 p_8 + \frac{1}{10} p_2^2 p_{10} - \frac{1}{10} p_4 p_{10} - \frac{1}{6} p_2 p_{12} + \frac{1}{7} p_{14}.$$

With these formulas and the expressions of the basic invariants given below all computations can be easily reproduced.

Formulas for E_6 . In this case the Weyl group contains the symmetric group S_6 as Weyl group of a root subsystem of type A_5 , and the restriction of the reflection representation to S_6 can be identified to the 6-dimensional permutation representation. Therefore we can express

the E_6 invariants as polynomials in symmetric functions; we will use the power sums p_i . In [9] the invariants are expressed through elementary symmetric functions. Changing variables we obtain for the invariants of degrees 2,5,6 the following polynomials

$$A_{2} = \frac{p_{1}^{2}}{2} + \frac{3p_{2}}{2}$$

$$A_{5} = \frac{11}{20}p_{1}^{5} - 6p_{1}^{3}p_{2} + \frac{27}{4}p_{1}p_{2}^{2} + \frac{27}{2}p_{1}^{2}p_{3} - \frac{27}{2}p_{2}p_{3} - \frac{27}{2}p_{1}p_{4} + \frac{81}{5}p_{5}$$

$$A_{6} = \frac{25}{8}p_{1}^{6} - \frac{99}{8}p_{1}^{4}p_{2} - \frac{297}{8}p_{1}^{2}p_{2}^{2} + \frac{243}{8}p_{2}^{3} + 270p_{1}p_{2}p_{3} - 135p_{3}^{2} + \frac{135}{2}p_{1}^{2}p_{4} - \frac{405}{2}p_{2}p_{4} - 324p_{1}p_{5} + 324p_{6}$$

Recall that write $A \cong B$ when two invariants are congruent modulo $(R^+)^2$. Computing the scalar products of these basic invariants we define

$$A_8 := A_5 \circ A_5, \quad A_9 := A_5 \circ A_6, \quad A_{12} := A_5 \circ A_9$$

and we have

$$8A_5 \circ A_9 \cong 9A_6 \circ A_8$$

The corresponding constants $d_{i,j}$ are given in in the following table.

	5	6	8	9
5	1	1	0	1
6	1	0	<u>8</u>	0
8	0	$\frac{8}{9}$	0	0
9	1	0	0	0

Table 2: coefficients d_{ij} for E_6 .

Formulas for E_7 . In this case the Weyl group contains the symmetric group S_7 as Weyl group of a root subsystem of type A_6 , and the restriction of the reflection representation to S_7 can be identified to the 7-dimensional permutation representation. Therefore we can express the E_6 invariants as polynomials in symmetric functions; we will use the power sums p_i . In [9] the invariants are expressed through elementary symmetric functions. We perform the change of variables obtaining for the invariants of degrees 2,6,8. One can normalize the two invariants of degrees 6,8 so that:

$$\begin{split} A_2 = & p_1^2 + 2p_2 \\ A_6 = & \frac{10}{3}(p_1^6 - 12p_1^4p_2 + 36p_1^2p_2^2 - 6p_2^3 + 40p_1^3p_3 - 120p_1p_2p_3 + 40p_3^2 - 60p_1^2p_4 + 60p_2p_4 + \\ & 144p_1p_5 - 96p_6) \\ A_8 = & \frac{10}{7}(p_1^8 + 224p_1^6p_2 - 1680p_1^4p_2^2 + 840p_1^2p_2^3 + 420p_2^4 - 1568p_1^5p_3 + 12320p_1^3p_2p_3 - \\ & 12320p_1^2p_3^2 - 4480p_2p_3^2 + 5040p_1^4p_4 - 18480p_1^2p_2p_4 - 3360p_2^2p_4 + 20160p_1p_3p_4 - \\ & 1680p_4^2 - 18816p_1^3p_5 + 12096p_1p_2p_5 + 2688p_3p_5 + 33600p_1^2p_6 + 6720p_2p_6 - 34560p_1p_7) \end{split}$$

We have normalized the invariants A_2 , A_6 , A_8 so that, computing the scalar products of these basic invariants, we get

$$A_{10} := A_6 \circ A_6, A_{12} := A_6 \circ A_8, A_{14} := A_6 \circ A_{10} \cong A_8 \circ A_8, A_{18} := A_6 \circ A_{14}$$

and we verify that

$$6A_{18} \cong 7A_8 \circ A_{12}, \ 5A_{18} \cong 7A_{10} \circ A_{10}$$

The constants $d_{i,j}$ are given in in the following table.

	6	8	10	12	14
6	1	1	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	0	1
8	1	1	0	$\frac{6}{7}$	0
10	1	0	$\frac{5}{7}$	0	0
12	0	$\frac{6}{7}$	0	0	0
14	1	0	0	0	0

Table 3: coefficients d_{ij} for E_7 .

Formulas for E_8 : The extended diagram of E_8 contains a subdiagram of type D_8 , so we can write the invariants of E_8 as polynomials in the invariants of D_8 , which are generated by p_{2i} , i = 1, ..., 7 and by the Pfaffian \mathcal{P} .

We have computed the invariants A_8 , A_{12} of degree 8 and 12 by the method of Lee [14] and the others by taking the scalar products (da, db) starting from these two invariants. The resulting invariants are generators by a simple inspection of their leading terms.

$$A_{2} = p_{2}$$

$$A_{8} = -10080\mathcal{P} - 105p_{2}^{2}p_{4} + 105p_{4}^{2} + 168p_{2}p_{6} - 180p_{8}$$

$$A_{12} = -103950\mathcal{P}p_{2}^{2} + \frac{10395}{64}p_{2}^{6} + 41580\mathcal{P}p_{4} - \frac{51975}{32}p_{2}^{4}p_{4} + \frac{51975}{16}p_{2}^{2}p_{4}^{2} - \frac{5775}{8}p_{4}^{3} + 3465p_{2}^{3}p_{6} - 6930p_{2}p_{4}p_{6} + 2772p_{6}^{2} - \frac{25245}{4}p_{2}^{2}p_{8} + \frac{10395}{2}p_{4}p_{8} + 8316p_{2}p_{10} - 7560p_{12}$$

We have the following relations:

$$A_8 \circ (A_8 \circ A_{12}) \cong \frac{9}{7} (A_8 \circ A_8) \circ A_{12}$$

 $(A_8 \circ A_8) \circ (A_8 \circ A_{12}) \cong \frac{3}{4} A_8 \circ ((A_8 \circ A_8) \circ A_{12}), \ A_{12} \circ (A_8 \circ (A_8 \circ A_8)) \cong \frac{5}{6} A_8 \circ ((A_8 \circ A_8) \circ A_{12}),$ that is the invariant of order 24 computed in two different ways gives two different values (modulo squares) as well as the invariant of order 30 computed in three different ways.

Set

$$\begin{split} A_{14} &:= A_8 \circ A_8, \\ A_{18} &:= A_8 \circ A_{12}, \\ A_{20} &:= A_8 \circ (A_8 \circ A_8) = A_8 \circ A_{14}, \\ A_{24} &:= (A_8 \circ A_8) \circ A_{12} \cong A_{12} \circ A_{14}, \\ A_{30} &:= A_8 \circ ((A_8 \circ A_8) \circ A_{12}) \end{split}$$

so that

$$A_{14} \circ A_{18} \cong \frac{3}{4}A_{30}, \quad A_{20} \circ A_{12} \cong \frac{5}{6}A_{30}.$$

We deduce the matrix of the constants $d_{i,j}$, which is displayed in the following table.

	8	12	14	18	20	24
8	1	$ \begin{array}{c} 12 \\ 1 \\ 0 \\ \hline 1 \\ 0 \\ \hline 6 \\ 0 \end{array} $	1	$\frac{9}{7}$	0	1
12	1	0	1	0	$\frac{5}{6}$	0
14	1	1	0	$\frac{3}{4}$	0	0
18	$\frac{9}{7}$	0	$\frac{3}{4}$	0	0	0
20	0	$\frac{5}{6}$	0	0	0	0
24	1	0	0	0	0	0

Table 4: coefficients d_{ij} for E_8 .

A full list of the invariants and the code to compute them is available in [6].

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