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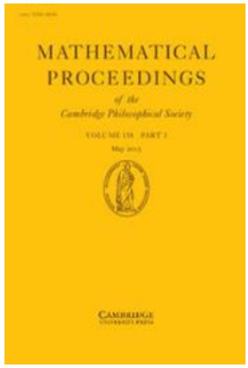




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# On maximal ideals in certain reduced twisted C*-crossed products

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*Abstract*

We consider a twisted action of a discrete group $G$ on a unital C$^*$-algebra $A$ and give conditions ensuring that there is a bijective correspondence between the maximal invariant ideals of $A$ and the maximal ideals in the associated reduced C*-crossed product.

## 1. *Introduction*

Let $A$ be a unital C$^*$-algebra and let $\mathcal{M}(A)$ denote the maximal ideal space of $A$, consisting of the maximal ideals of $A$. As is well known, a proper ideal of $A$ is maximal if and only if the associated quotient C$^*$-algebra is simple. Moreover, $\mathcal{M}(A)$ is a non-empty subset of the primitive ideal space Prim($A$) of $A$. In some cases, these spaces coincide (e.g. when $A$ is commutative or when $A$ is simple), and this corresponds to the fact that Prim($A$) is a T$_1$-space in the Jacobson topology. In general, computing Prim($A$) for a given $A$ is not an easy task. Determining $\mathcal{M}(A)$ still gives some valuable information: besides providing an invariant for $A$ in itself, it also gives a way to list all the simple quotients of $A$, and this might prospectively be useful if one aims to distinguish some given C$^*$-algebras by taking into account some of the invariants that have already been computed for several classes of simple C$^*$-algebras. Our main aim in this paper is to show how one can indeed determine the maximal ideal space of the reduced twisted C$^*$-crossed products associated with exact twisted actions of certain discrete groups on unital C$^*$-algebras. As all the groups in question belong to the class of C$^*$-simple groups, we first recall some relevant facts about the latter class.

Let $G$ denote a discrete group and let $C_r^*(G)$ denote its reduced group C$^*$-algebra, i.e., the C$^*$-algebra generated by the left regular representation of $G$ on $\ell^2(G)$. The group $G$ is then called C$^*$-simple [**1**] whenever $C_r^*(G)$ is simple. The class of C$^*$-simple groups is vast. It includes for example all Powers groups as defined by P. de la Harpe [**17**] (e.g. free nonabelian groups, as in Powers' original work [**29**], and free products of groups, with the exception of $\mathbb{Z}_2 * \mathbb{Z}_2$); all weak Powers groups, as introduced by F. Boca and V. Nitica [**6**] (e.g. direct products of Powers groups); the class of PH groups, as defined by S.D. Promislow [**31**] (e.g. extensions of weak Powers groups); the class of groups with property ($P_{\rm com}$), as defined by M. Bekka, M. Cowling and P. de la Harpe [**5**] (e.g. $PSL(n, \mathbb{Z})$ for every $n \geqslant 2$). We refer to [**17**] for a detailed overview of C*-simple groups and their properties. Some related articles written afterwards are [**7**, **19**, **22**, **25**, **27**, **30**, **34**].

In the very recent work [**7**], E. Breuillard, M. Kalantar, M. Kennedy and N. Ozawa show that if a C*-simple group $G$ acts on a unital C*-algebra $A$ in a minimal way (that is, the only invariant ideals of $A$ are $\{0\}$ and $A$), then the associated reduced C*-crossed product is simple. The case where $G$ is a Powers group was first established by P. de la Harpe and G. Skandalis [**18**]. Their result was later extended to cover weak Powers groups and twisted actions (see [**1**, **6**]), while the case where $G$ has property $(P_{\mathrm{com}})$ was handled by Bekka, Cowling and de la Harpe [**5**]. It is not clear to us that the result in [**7**] mentioned above holds in general for a twisted action of a C*-simple group $G$. Anyhow, as we show in this paper (cf. Corollary 3·10), this is certainly true when $G$ belongs to the class $\mathcal{P}$ consisting of all PH groups and all groups with the property $(P_{\mathrm{com}})$.

De la Harpe and Skandalis give in [**18**] an example of an action of a Powers group on a unital C*-algebra $A$ such that $A$ has exactly one nontrivial invariant ideal while the associated reduced C*-crossed product has infinitely many ideals. This could be taken as an indication that it is not possible to say something of interest about the lattice of ideals in a reduced C*-crossed product involving a non minimal action of a C*-simple group. Nevertheless, we will show (see Corollary 3·9) that if $G$ belongs to the class $\mathcal{P}$ introduced above, then one may describe the maximal ideal space of the reduced twisted C*-crossed product associated with an exact twisted action of $G$ on a unital C*-algebra. In the case where $G$ is a weak Powers group, this result was briefly discussed in [**4**, example 6·6].

As an important part of our work, we introduce a certain property for a twisted unital discrete C*-dynamical system $\Sigma = (A, G, \alpha, \sigma)$ that we call property (DP) (named after Dixmier and Powers). This property, which is weaker than the Dixmier property for the reduced crossed product $C_r^*(\Sigma)$, is always satisfied by the system $\Sigma$ whenever $G$ belongs to the class $\mathcal{P}$ (see Theorem 3·8 and Section 5). Moreover, we prove that if $\Sigma$ is exact [**4**, **33**] and has property (DP), then there is a one-to-one correspondence between the maximal ideal space of $C_r^*(\Sigma)$ and the set of maximal invariant ideals of $A$, and also a one-to-one correspondence between the set of all tracial states of $C_r^*(\Sigma)$ and the set of invariant tracial states of $A$ (see Theorem 3·7 and Proposition 3·4).

To illustrate the usefulness of our results, we describe in Section 4 the maximal ideal space of some C*-algebras that may be written as $C_r^*(\Sigma)$ for a suitably chosen system $\Sigma$. These examples include the reduced group C*-algebra of any discrete group $\Gamma$ such that the quotient of $\Gamma$ by its center is exact and belongs to $\mathcal{P}$, the reduced group C*-algebra of $\mathbb{Z}^3 \rtimes SL(3, \mathbb{Z})$ and the "twisted" Roe algebra $C_r^*(\ell^\infty(G), G, \mathrm{lt}, \sigma)$ associated to an exact group $G$ belonging to $\mathcal{P}$, the 2-cocycle $\sigma$ being then assumed to be scalar-valued.

We use standard notation. For instance, if $A$ is a unital C*-algebra, then $\mathcal{U}(A)$ denotes the unitary group of $A$ and $\mathrm{Aut}(A)$ denotes the group of all $*$-automorphisms of $A$. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ denotes the bounded linear operators on $\mathcal{H}$. By an ideal in a C*-algebra, we always mean a closed two-sided ideal, unless otherwise specified.

## 2. *Preliminaries*

Throughout this paper, we let $\Sigma = (A, G, \alpha, \sigma)$ denote a twisted, unital, discrete $C^*$-dynamical system (see for instance [**9**, **36**, **35**, **26**]). Thus, $A$ is a $C^*$-algebra with unit 1, $G$ is a discrete group with identity $e$ and $(\alpha, \sigma)$ is a twisted action of $G$ on $A$, that is, $\alpha$ is a map from $G$ into Aut$(A)$ and $\sigma$ is a map from $G \times G$ into $\mathcal{U}(A)$, satisfying

$$\alpha_g \circ \alpha_h = \mathrm{Ad}(\sigma(g,h)) \circ \alpha_{gh}$$
$$\sigma(g,h)\sigma(gh,k) = \alpha_g(\sigma(h,k))\sigma(g,hk)$$
$$\sigma(g,e) = \sigma(e,g) = 1,$$

for all $g, h, k \in G$. Of course, Ad$(v)$ denotes here the (inner) automorphism of $A$ implemented by some $v \in \mathcal{U}(A)$. One deduces easily that

$$\alpha_e = \mathrm{id},\ \sigma(g, g^{-1}) = \alpha_g(\sigma(g^{-1}, g))$$

and

$$\alpha_g^{-1} = \alpha_{g^{-1}} \circ \mathrm{Ad}(\sigma(g, g^{-1})^*) = \mathrm{Ad}(\sigma(g^{-1}, g)^*) \circ \alpha_{g^{-1}}.$$

Note that if $\sigma$ is trivial, that is, $\sigma(g,h) = 1$ for all $g, h \in G$, then $\Sigma$ is an ordinary $C^*$-dynamical system.

The reduced crossed product $C_r^*(\Sigma)$ associated with $\Sigma$ may (up to isomorphism) be characterised as follows [**3**, **36**]:

(i) $C_r^*(\Sigma)$ is generated (as a C*-algebra) by (a copy of) $A$ and a family $\{\lambda(g) \mid g \in G\}$ of unitaries satisfying

$$\alpha_g(a) = \lambda(g)\, a\, \lambda(g)^* \text{ and } \lambda(g)\,\lambda(h) = \sigma(g,h)\,\lambda(gh);$$

for all $g, h \in G$ and $a \in A$,

(ii) there exists a faithful conditional expectation $E : C_r^*(\Sigma) \to A$ such that $E(\lambda(g)) = 0$ for all $g \in G,\ g \neq e$.

One easily cheks that the expectation $E$ is equivariant, that is, we have

$$E(\lambda(g)\, x\, \lambda(g)^*) = \alpha_g(E(x)),$$

for all $g \in G,\ x \in C_r^*(\Sigma)$. As is well known, it follows that if $\varphi$ is a tracial state on $A$ which is invariant (i.e. $\varphi(\alpha_g(a)) = \varphi(a)$ for all $g \in G,\ a \in A$), then $\varphi \circ E$ is a tracial state on $C_r^*(\Sigma)$ extending $\varphi$.

Let $J$ denote an invariant ideal of $A$ and set $\Sigma/J = (A/J, G, \dot{\alpha}, \dot{\sigma})$, where $(\dot{\alpha}, \dot{\sigma})$ denotes the twisted action of $G$ on $A/J$ naturally associated with $(\alpha, \sigma)$.

We will let $\langle J \rangle$ denote the ideal of $C_r^*(\Sigma)$ generated by $J$. Any ideal of this form is called an *induced ideal* of $C_r^*(\Sigma)$. Moreover, we will let $\tilde{J}$ denote the kernel of the canonical *-homomorphism from $C_r^*(\Sigma)$ onto $C_r^*(\Sigma/J)$. It is elementary to check that we have $E(\langle J \rangle) = J$ and $\langle J \rangle \subset \tilde{J}$. Another useful fact is that

$$\tilde{J} = \left\{x \in C_r^*(\Sigma) \mid \widehat{x}(g) \in J \text{ for all } g \in G\right\},$$

where $\widehat{x}(g) = E(x\,\lambda(g)^*)$ for each $x \in C_r^*(\Sigma),\ g \in G$. This may for instance be deduced from the proof of [**13**, theorem 5·1] by considering $C_r^*(\Sigma)$ as topologically graded C*-algebra over $G$:

$$C_r^*(\Sigma) = \overline{\bigoplus_{g \in G} A_g}^{\|\cdot\|},$$

where $A_g = \left\{a\,\lambda(g) \mid a \in A\right\}$ for each $g \in G$.

Following [**4**, **33**], we will say that the system $\Sigma$ is *exact* whenever we have $\langle J \rangle = \tilde{J}$ for every invariant ideal $J$ of $A$. It is known [**12**] that $\Sigma$ is exact whenever $G$ is exact. It is also known [**4**] that $\Sigma$ is exact whenever there exists a Fourier summing net for $\Sigma$ preserving the invariant ideals of $A$. This latter condition is for instance satisfied when $\Sigma$ has Exel's approximation property [**11**], e.g. when the associated action of $G$ on the center $Z(A)$ of $A$, obtained by restricting $\alpha$ to $Z(A)$, is amenable (as being defined in [**8**]).

We include here two lemmas illustrating the impact of the exactness of $\Sigma$ on the lattice of ideals of $C_r^*(\Sigma)$.

LEMMA 2·1. *Let $\mathcal{J}$ be an ideal of $C_r^*(\Sigma)$ and set $J = \overline{E(\mathcal{J})}$. Then $J$ is an invariant ideal of $A$ such that $\mathcal{J} \subset \tilde{J}$. Hence, if $\Sigma$ is exact, we have $\mathcal{J} \subset \langle J \rangle$.*

*Proof.* As $E$ is a conditional expectation, it follows readily that $J$ is an ideal of $A$. The invariance of $J$ is an immediate consequence of the equivariance of $E$. Let now $x \in \mathcal{J}$. Then, for each $g \in G$, we have $x\,\lambda(g)^* \in \mathcal{J}$, so

$$\widehat{x}(g) = E\big(x\,\lambda(g)^*\big) \in E(\mathcal{J}) \subset J.$$

Hence, $x \in \tilde{J}$. This shows that $\mathcal{J} \subset \tilde{J}$. The last assertion follows then from the definition of exactness.

An ideal $\mathcal{J}$ of $C_r^*(\Sigma)$ is called *$E$-invariant* if $E(\mathcal{J}) \subset \mathcal{J}$. Equivalently, $\mathcal{J}$ is $E$-invariant whenever $E(\mathcal{J}) = \mathcal{J} \cap A$ (so $E(\mathcal{J})$ is necessarily closed in this case). Any induced ideal of $C_r^*(\Sigma)$ is easily seen to be $E$-invariant. The converse is true if $\Sigma$ is exact, as shown below. (When $G$ is exact, this is shown in [**13**]; see [**4**] for the case where there exists a Fourier summing net for $\Sigma$ preserving the invariant ideals of $A$.)

LEMMA 2·2. *Let $\mathcal{J}$ be an $E$-invariant ideal of $C_r^*(\Sigma)$. If $\Sigma$ is exact, then $\mathcal{J}$ is an induced ideal. Indeed, we have $\mathcal{J} = \langle E(\mathcal{J}) \rangle$ in this case.*

*Proof.* Note that since $E(\mathcal{J}) = \mathcal{J} \cap A$ is closed, it is an invariant ideal of $A$ (cf. Lemma 2·1). Assume that $\Sigma$ is exact. Then Lemma 2·1 gives that $\mathcal{J} \subset \langle E(\mathcal{J}) \rangle$. On the other hand, since $E(\mathcal{J}) \subset \mathcal{J}$, we have $\langle E(\mathcal{J}) \rangle \subset \mathcal{J}$. Hence, $\mathcal{J} = \langle E(\mathcal{J}) \rangle$, as asserted.

## 3. *On maximal ideals and reduced twisted $C^*$-crossed products*

We set $\mathcal{U}_\Sigma = \mathcal{U}(C_r^*(\Sigma))$. When $S$ is a subset of a (complex) vector space, we let $\mathrm{co}(S)$ denote the convex hull of $S$.

*Definition* 3·1. The system $\Sigma$ is said to have *property* $(DP)$ whenever we have

$$0 \in \overline{\mathrm{co}\{v\,y\,v^* \mid v \in \mathcal{U}_\Sigma\}}^{\|\cdot\|} \tag{3·1}$$

for every $y \in C_r^*(\Sigma)$ satisfying $y^* = y$ and $E(y) = 0$.

*Remark* 3·2. Let $\mathcal{U}_G$ be the subgroup of $\mathcal{U}_\Sigma$ generated by the $\lambda(g)$'s. The above definition might be strengthened by replacing $\mathcal{U}_\Sigma$ with $\mathcal{U}_G$, that is, by requiring that

$$0 \in \overline{\mathrm{co}\{v\,y\,v^* \mid v \in \mathcal{U}_G\}}^{\|\cdot\|} \tag{3·2}$$

for every $y \in C_r^*(\Sigma)$ satisfying $y^* = y$ and $E(y) = 0$. All the examples of systems we are going to describe satisfy this strong form of property (DP). It can be shown (see Proposition 5·9) that if $\Sigma$ has this strong property (DP), then (3·2) holds for every $y \in C_r^*(\Sigma)$ satisfying $E(y) = 0$. It is not clear to us that if $\Sigma$ has property (DP), then (3·1) holds for every such $y$.

*Remark* 3·3. We recall that a unital C*-algebra $B$ is said to have the *Dixmier property* if

$$\overline{\mathrm{co}\{u\, b\, u^* \mid u \in \mathcal{U}(B)\}}^{\|\cdot\|} \cap \mathbb{C} \cdot 1 \neq \varnothing,$$

for every $b \in B$. As shown by L. Zsido and U. Haagerup in [**16**], $B$ is simple with at most one tracial state if and only if $B$ has the Dixmier property. Using [**16**, corollaire, p. 175], it follows that if $C_r^*(\Sigma)$ has the Dixmier property, then $\Sigma$ has the property (DP) introduced above. Property (DP) may be seen as a kind of relative Dixmier property for the pair $(A, C_r^*(\Sigma))$, generalizing the property considered by R. Powers [**29**] in the case where $\Sigma = (\mathbb{C}, \mathbb{F}_2, \mathrm{id}, 1)$. It should not be confused with the notion of relative Dixmier property for inclusions of C*-algebras considered by S. Popa in [**28**].

A first consequence of property $(DP)$ is the following:

PROPOSITION 3·4. *Assume $\Sigma$ has property $(DP)$. Then the map $\varphi \to \varphi \circ E$ is a bijection between the set of invariant tracial states of $A$ and the set of tracial states of $C_r^*(\Sigma)$. Especially, $C_r^*(\Sigma)$ has a unique tracial state if and only if $A$ has a unique invariant tracial state.*

*Proof.* It is clear that this map is injective, so let us prove that it is surjective. Let therefore $\tau$ be a tracial state on $C_r^*(\Sigma)$ and let $\varphi$ denote the tracial state of $A$ obtained by restricting $\tau$ to $A$. It follows from the covariance relation that $\varphi$ is invariant. We will show that $\tau = \varphi \circ E$.

Let $x^* = x \in C_r^*(\Sigma)$ and $\varepsilon > 0$. Set $y = x - E(x)$. As $y^* = y$ and $E(y) = E(x - E(x)) = E(x) - E(x) = 0$, property $(DP)$ enables us to pick $v_1, \ldots, v_n \in \mathcal{U}_\Sigma$ and $t_1, \ldots, t_n \in [0, 1]$ satisfying $\Sigma_{i=1}^n t_i = 1$ such that

$$\left\| \sum_{i=1}^n t_i\, v_i\, y\, v_i^* \right\| < \varepsilon.$$

As $\tau$ is a tracial, we have

$$\tau\left(\sum_{i=1}^n t_i\, v_i\, y\, v_i^*\right) = \sum_{i=1}^n t_i\, \tau(y) = \tau(y),$$

so we get

$$|\tau(y)| = \left|\tau\left(\sum_{i=1}^n t_i\, v_i\, y\, v_i^*\right)\right| \leqslant \left\| \sum_{i=1}^n t_i\, v_i\, y\, v_i^* \right\| < \varepsilon.$$

Hence, we can conclude that $\tau(y) = 0$. This gives that

$$\tau(x) = \tau(E(x)) = (\varphi \circ E)(x).$$

So $\tau$ agrees with $\varphi \circ E$ on the self-adjoint part of $C_r^*(\Sigma)$, and therefore on the whole of $C_r^*(\Sigma)$ by linearity.

Next, we have:

PROPOSITION 3·5. *Assume that $\Sigma$ has property $(DP)$ and let $\mathcal{J}$ be a proper ideal of $C_r^*(\Sigma)$. Set $J = \overline{E(\mathcal{J})}$. Then $J$ is a proper invariant ideal of $A$.*

*Proof.* We know from Lemma 2·1 that $J$ is an invariant ideal of $A$. Assume that $J$ is not proper, i.e., $\overline{E(\mathcal{J})} = A$. Since $A$ is unital, we have $E(\mathcal{J}) = A$. So we may pick $x \in \mathcal{J}$ such that $E(x) = 1$.

Set $z = x^*x \in \mathcal{J}^+$. Using the Schwarz inequality for complete positive maps [**8**], we get

$$E(z) = E(x^*x) \geqslant E(x)^*E(x) = 1.$$

Now, set $y = z - E(z)$, so $y^* = y \in C_r^*(\Sigma)$ and $E(y) = 0$. Since $\Sigma$ has property $(DP)$, we can find $v_1, \ldots, v_n \in \mathcal{U}_\Sigma$ and $t_1, \ldots, t_n \in [0, 1]$ satisfying $\Sigma_{i=1}^n t_i = 1$ such that

$$(*) \quad \Big\| \sum_{i=1}^n t_i\, v_i\, z\, v_i^* - \sum_{i=1}^n t_i\, v_i\, E(z)\, v_i^* \Big\| = \Big\| \sum_{i=1}^n t_i\, v_i\, y\, v_i^* \Big\| < \frac{1}{2}.$$

Setting $z' = \Sigma_{i=1}^n t_i\, v_i\, z\, v_i^*$ , we have $z' \in \mathcal{J}^+$. Since $E(z) \geqslant 1$, we also have

$$\sum_{i=1}^n t_i\, v_i\, E(z)\, v_i^* \geqslant 1.$$

Hence, it follows from $(*)$ that $z'$ is invertible. So we must have $\mathcal{J} = C_r^*(\Sigma)$, which contradicts the properness of $\mathcal{J}$. This shows that $J$ is proper.

COROLLARY 3·6. *Assume $\Sigma$ has property $(DP)$ and is minimal (that is, $\{0\}$ is the only proper invariant ideal of $A$). Then $C_r^*(\Sigma)$ is simple.*

*Proof.* Since $E$ is faithful, this follows immediately from Proposition 3·5.

If $\Sigma$ is exact and has property $(DP)$, we can in fact characterize the maximal ideals of $C_r^*(\Sigma)$. We therefore set

$$\mathcal{MI}(A) = \big\{J \subset A \mid J \text{ is a maximal invariant ideal of } A\big\},$$
$$\mathcal{M}\big(C_r^*(\Sigma)\big) = \big\{\mathcal{J} \subset C_r^*(\Sigma) \mid \mathcal{J} \text{ is a maximal ideal of } C_r^*(\Sigma)\big\}.$$

It follows from Zorn's lemma that both these sets are non-empty.

THEOREM 3·7. *Assume $\Sigma$ is exact and has property $(DP)$.*
*Then the map $J \to \langle J\rangle$ is a bijection between $\mathcal{MI}(A)$ and $\mathcal{M}(C_r^*(\Sigma))$.*
*Thus, the family of all simple quotients of $C_r^*(\Sigma)$ is given by*

$$\Big\{C_r^*\big(\Sigma/J\big)\Big\}_{J\in\mathcal{MI}(A)}.$$

*Proof.* Let $J \in \mathcal{MI}(A)$. We have to show that $\langle J\rangle \in \mathcal{M}(C_r^*(\Sigma))$. We first note that $\langle J\rangle$ is a proper ideal of $C_r^*(\Sigma)$; otherwise, we would have $J = E(\langle J\rangle) = A$, contradicting that $J$ is a proper ideal of $A$.

Next, let $\mathcal{K}$ be a proper ideal of $C_r^*(\Sigma)$ containing $\langle J\rangle$, and set $K = \overline{E(\mathcal{K})}$. Since $\Sigma$ has property $(DP)$, Proposition 3·5 gives that $K$ is a proper invariant ideal of $A$. Moreover, we have $J = E(\langle J\rangle) \subset E(\mathcal{K}) \subset K$. By maximality of $J$, we get $J = K$, which gives

$$E(\mathcal{K}) = K = J \subset \langle J\rangle \subset \mathcal{K}.$$

Thus, $\mathcal{K}$ is $E$-invariant. Since $\Sigma$ is exact, we get from Lemma 2·2 that $\mathcal{K} = \langle K\rangle$. As $J = K$, we conclude that $\mathcal{K} = \langle J\rangle$. Thus, we have shown that $\langle J\rangle$ is maximal among the proper ideals of $C_r^*(\Sigma)$, as desired.

This means that the map $J \to \langle J\rangle$ maps $\mathcal{MI}(A)$ into $\mathcal{M}(C_r^*(\Sigma))$. This map is clearly injective (since $E(\langle J\rangle) = J$ for every invariant ideal $J$ of $A$).

To show that it is surjective, let $\mathcal{J} \in \mathcal{M}(C_r^*(\Sigma))$ and set $J = \overline{E(\mathcal{J})}$. We will show that $J \in \mathcal{MI}(A)$ and $\mathcal{J} = \langle J\rangle$.

Since $\Sigma$ has property $(DP)$ and $\mathcal{J}$ is a proper ideal of $C_r^*(\Sigma)$, Proposition 3·5 gives that $J$ is a proper invariant ideal of $A$. Further, since $\Sigma$ is exact, Lemma 2·1 gives that $\mathcal{J} \subset \langle J \rangle$. As $\mathcal{J}$ is maximal, we get $\mathcal{J} = \langle J \rangle$.

Finally, $J$ is maximal among the proper invariant ideals of $A$. Indeed, let $K$ be a proper invariant ideal of $A$ containing $J$. Then we have $\mathcal{J} = \langle J \rangle \subset \langle K \rangle$. By maximality of $\mathcal{J}$, we get $\langle J \rangle = \langle K \rangle$. This implies that $J = E(\langle J \rangle) = E(\langle K \rangle) = K$. Hence, we have shown that $J \in \mathcal{MI}(A)$.

To give examples of systems satisfying property $(DP)$, we let $\mathcal{P}$ denote the class of discrete groups consisting of PH groups [**31**] and of groups satisfying the property $(P_{\text{com}})$ introduced in [**5**]. The class $\mathcal{P}$, which is a subclass of the class of discrete C*-simple groups, contains a huge variety of groups, including for instance many amalgamated free products, HNN-extensions, hyperbolic groups, Coxeter groups, and lattices in semisimple Lie groups. For a more precise description, we refer to [**17**] (see also [**19**]). The following result may be seen as a generalization of results in [**1**, **5**, **6**, **18**, **31**]. For the convenience of the reader, we will give a proof in Section 5.

THEOREM 3·8. *Let $G \in \mathcal{P}$. Then $\Sigma$ has property $(DP)$.*

Thus, we get:

COROLLARY 3·9. *Let $G \in \mathcal{P}$. Then the map $\varphi \to \varphi \circ E$ is a bijection between the set of invariant tracial states of $A$ and the set of tracial states of $C_r^*(\Sigma)$.*

*Moreover, assume $\Sigma$ is exact. Then the map $J \to \langle J \rangle$ is a bijection between $\mathcal{MI}(A)$ and $\mathcal{M}(C_r^*(\Sigma))$. Thus, the family of all simple quotients of $C_r^*(\Sigma)$ is given by*

$$\left\{C_r^*(\Sigma/J)\right\}_{J \in \mathcal{MI}(A)}.$$

*Proof.* Since $G \in \mathcal{P}$, we know from Theorem 3·8 that $\Sigma$ has property $(DP)$. The result follows therefore from Proposition 3·4 and Theorem 3·7.

COROLLARY 3·10. *Assume $G \in \mathcal{P}$. If $A$ has a unique invariant tracial state, then $C_r^*(\Sigma)$ has a unique tracial state. If $\Sigma$ is minimal, then $C_r^*(\Sigma)$ is simple.*

*Proof.* This follows from Proposition 3·4, Corollary 3·6 and Theorem 3·8.

COROLLARY 3·11. *Let $G \in \mathcal{P}$ and let $\omega \in Z^2(G, \mathbb{T})$. Then $C_r^*(G, \omega)$ is simple with a unique tracial state.*

In fact, proceeding as in the proof of [**1**, corollary 4·10] and [**2**, corollary 4], one sees that Corollary 3·11 holds whenever $G$ is a *ultra-$\mathcal{P}$* group, meaning that $G$ has a normal subgroup belonging to $\mathcal{P}$ with trivial centralizer in $G$. Moreover, in the same way, one easily deduces that [**1**, corollaries 4·8 − 4·12] and [**2**, corollaries 5 and 6] still hold if one replaces *weak Powers group* by *group in the class $\mathcal{P}$*, and *ultraweak Powers group* by *ultra-$\mathcal{P}$ group* in the statement of these results.

It may also be worth mentioning explicitely the following result:

COROLLARY 3·12. *Let $G \in \mathcal{P}$ and assume $A$ is abelian, so $A = C(X)$ for some compact Hausdorff space $X$. Then there is a one-to-one correspondence between the set of Borel probability measures on $X$ and the set of tracial states of $C_r^*(\Sigma)$ given by $\mu \to \int_X E(\cdot)\, d\mu$.*

*Moreover, assume $\Sigma$ is exact. Then there is a one-to-one correspondence between the set $\mathcal{Y}$ of minimal closed invariant subsets of $X$ and $\mathcal{M}(C_r^*(\Sigma))$ given by $Y \to \langle C_0(X \setminus Y)\rangle$. Moreover, the family of all simple quotients of $C_r^*(\Sigma)$ is given by*

$$\Big\{C_r^*\big(C(Y), G, \alpha_Y, \sigma_Y\big)\Big\}_{Y\in\mathcal{Y}}$$

*where $(\alpha_Y, \sigma_Y)$ denotes the twisted quotient action of $G$ on $C(Y)$ associated with $(\alpha, \sigma)$.*

*Proof.* This follows immediately from Theorem 3·9 and Gelfand theory.

When $\alpha$ is trivial, $\sigma$ is just some 2-cocycle on $G$ with values in $\mathcal{U}(Z(A))$, so $C_r^*(\Sigma)$ is a kind of "twisted" tensor product of $A$ with $C_r^*(G)$. In this case, we don't have to restrict our attention to maximal ideals of $C_r^*(\Sigma)$:

PROPOSITION 3·13. *Assume $\alpha$ is trivial, $\Sigma$ is exact and $G \in \mathcal{P}$. Then the map $J \to \langle J\rangle$ is a bijection between the set of ideals of $A$ and the set of ideals of $C_r^*(\Sigma)$.*

*Proof.* Since $\alpha$ is trivial and $\Sigma$ is exact, it follows immediately from Lemma 2·2 that the map $J \to \langle J\rangle$ is a bijection between the set of ideals of $A$ and the set of $E$-invariant ideals of $B = C_r^*(\Sigma)$. Hence, it suffices to show that any ideal of $B$ is $E$-invariant.

Let $\mathcal{J}$ be an ideal of $B$, $y^* = y \in \mathcal{J}$ and $\varepsilon > 0$. Set $x = y - E(y)$. Then $x^* = x \in B$ and $E(x) = 0$. Since $G \in \mathcal{P}$, it follows from the proof of Theorem 3·8 given in Section 5 that there exists a $G$-averaging process $\psi$ on $B$ (as defined in Section 5) such that $\|\psi(x)\| < \varepsilon$. Now, since $\alpha$ is trivial, any $G$-averaging process on $B$ restricts to the identity map on $A$. Thus, we get $\psi(x) = \psi(y) - \psi(E(y)) = \psi(y) - E(y)$, so

$$\|\psi(y) - E(y)\| < \varepsilon.$$

As any $G$-averaging process on $B$ preserves ideals, we have $\psi(y) \in \mathcal{J}$. Hence, we get $E(y) \in \overline{\mathcal{J}} = \mathcal{J}$. It clearly follows that $\mathcal{J}$ is $E$-invariant, as desired.

## 4. *Examples*

This section is devoted to the discussion of some concrete examples.

4·1. As a warm-up, we consider the simple, but instructive case of an action of a group $G$ on a non-empty finite (discrete) set $X$ with $n$ elements. Let $\alpha$ denote the associated action of $G$ on $A = C(X) \simeq \mathbb{C}^n$ and $\sigma \in Z^2(G, \mathbb{T})$.

We may then pick $x_1, \ldots, x_m \in X$ such that $X$ is the disjoint union of the orbits $O_j = \{g \cdot x_j \mid g \in G\}$ for $j = 1, \ldots, m$. Clearly, the $O_j$'s are the minimal (closed) invariant subsets of $X$. Hence, if $G$ is an exact group in the class $\mathcal{P}$, we get from Corollary 3·12 that the simple quotients of $B = C_r^*(C(X), G, \alpha, \sigma)$ are given by

$$B_j = C_r^*\big(C(O_j), G, \alpha_j, \sigma\big), \quad j = 1, \ldots, m,$$

where $\alpha_j$ is the action on $C(O_j)$ obtained by restricting $\alpha$ for each $j$.

The assumption above that $G$ is exact is in fact not necessary. Indeed, one easily sees that $B$ is the direct sum of the $B_j$'s. So if $G$ belongs to $\mathcal{P}$, then Corollary 3·10 gives that all the $B_j$'s are simple, and the same assertion as above follows readily.

Finally, assume that $\sigma = 1$. Then this characterisation of the simple quotients of $B$ still holds whenever $G$ is a C*-simple group. Indeed, letting $G_{x_j}$ denotes the isotropy group of $x_j$ in $G$ and identifying $O_j$ with $G/G_{x_j}$, one gets from [**9**, example 6·6] (see also [**23**, **32**]) that each $B_j$ is Morita equivalent to $C_r^*(G_{x_j})$. Now, if $G$ is C*-simple, then each $C_r^*(G_{x_j})$ is simple (i.e. $G_{x_j}$ is C*-simple) because $G_{x_j}$ has finite index in $G$ (cf. [**17**] and [**28**]), so the $B_j$'s are the simple quotients of $B$.

4·2. Consider the canonical action lt of a group $G$ by left translation on $\ell^\infty(G)$, in other words, the action associated with the natural left action of $G$ on its Stone-Čech compactification $\beta G$ [**10**, **21**], and let $\sigma \in Z^2(G, \mathbb{T})$.

It is known that $\beta G$ has $2^{2^{|G|}}$ minimal closed invariant subsets (see for instance [**20**, theorem 1·4] and [**21**, lemma 19·6]). Moreover, all these subsets are $G$-equivariantly homeomorphic to each other (this follows from [**21**, theorem 19·8]). Hence, letting $X_G$ denote one of these minimal closed invariant subsets, we get from Corollary 3·12 that if $G$ is exact and belongs to $\mathcal{P}$, then the simple quotients of the "twisted" Roe algebra $C_r^*(\ell^\infty(G), G, \mathrm{lt}, \sigma)$ are all isomorphic to $C_r^*(C(X_G), G, \mathrm{lt}, \sigma)$.

In general, if $G$ is exact and we assume that $\sigma = 1$, one may in fact deduce that there is a one-to-correspondence between the set of all invariant closed subsets of $\beta G$ and the ideals of the Roe algebra $C_r^*(\ell^\infty(G), G, \mathrm{lt})$; indeed, since the action of $G$ on $\beta G$ is known to be free [**10**, proposition 8·14], this follows from [**33**, theorem 1·20].

4·3. Let $\Gamma = \mathbb{Z}^3 \rtimes SL(3, \mathbb{Z})$ be the semidirect product of $\mathbb{Z}^3$ by the canonical action of $SL(3, \mathbb{Z})$. Since $\mathbb{Z}^3$ is a normal nontrivial amenable subgroup of $\Gamma$, it is well known that $\Gamma$ is not C*-simple. In aim to describe the maximal ideals of $C_r^*(\Gamma)$, we decompose

$$C_r^*(\Gamma) \simeq C_r^*\big(C_r^*(\mathbb{Z}^3), SL(3, \mathbb{Z}), \alpha\big) \simeq C_r^*\big(C(\mathbb{T}^3), SL(3, \mathbb{Z}), \tilde{\alpha}\big),$$

where $\alpha$ (resp. $\tilde{\alpha}$) denotes the associated action of $SL(3, \mathbb{Z})$ on $C_r^*(\mathbb{Z}^3)$ (resp. $C(\mathbb{T}^3)$). Now, $SL(3, \mathbb{Z})$ is exact [**8**] and belongs to $\mathcal{P}$ (since it has property $(P_{\mathrm{com}})$ [**5**]). Hence, appealing to Corollary 3·12, the maximal ideals of $C_r^*(\Gamma)$ are in a one-to-one correspondence with the minimal closed invariant subsets of $\mathbb{T}^3$. The orbits of the action of $SL(3, \mathbb{Z})$ on $\mathbb{T}^3$ are either finite or dense (see for instance [**15**, **24**]), hence the minimal closed invariant subsets of $\mathbb{T}^3$ are the orbits of rational points in $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$.

Let $x \in \mathbb{Q}^3/\mathbb{Z}^3 \subset \mathbb{T}^3$ and let $G_x$ denote the isotropy group of $x$ in $G = SL(3, \mathbb{Z})$. Then identifying the (finite) orbit $O_x$ of $x$ in $\mathbb{T}^3$ with $G/G_x$, we get that the simple quotient $B_x$ of $C_r^*(\Gamma)$ corresponding to $O_x$ is given by the reduced crossed product

$$B_x = C_r^*(C(O_x), G, \alpha^x) \simeq C_r^*(C(G/G_x), G, \beta^x)$$

where $\alpha^x$ is implemented by the action of $G$ on $O_x$ and $\beta^x$ is implemented by the canonical left action of $G$ on $G/G_x$. We note that $B_x$ has a unique tracial state since $G$ belongs to $\mathcal{P}$ and there is obviously only one invariant state on $C(O_x)$. Moreover, it follows from [**9**, example 6·6] (see also [**23**, **32**]) that $B_x$ is Morita equivalent to $C_r^*(G_x)$. This implies that $G_x$ is C*-simple, a fact that may also be deduced from [**17**] (see also [**28**]) since $G_x$ has finite index in $G$.

4·4. Let $\Gamma$ be an exact discrete group such that $G = \Gamma/Z$ belongs to the class $\mathcal{P}$, where $Z = Z(\Gamma)$ denotes the center of $\Gamma$. We can then easily deduce that the ideals of $C_r^*(\Gamma)$ are in a one-to-one correspondence with the open (resp. closed) subsets of the dual group $\widehat{Z}$. Indeed, using [**1**, theorem 2·1], we can decompose

$$C_r^*(\Gamma) \simeq C_r^*\big(C_r^*(Z), G, \mathrm{id}, \omega\big) \simeq C_r^*\big(C(\widehat{Z}), G, \mathrm{id}, \widehat{\omega}\big)$$

where $\omega : G \times G \to \mathcal{U}(C_r^*(Z))$ is given by

$$\omega(g, h) = \lambda_Z\big(n(g)n(h)n(gh)^{-1}\big), \quad (g, h \in G),$$

for some section $n : G \to \Gamma$ of the canonical homomorphism $q : \Gamma \to G$ such that $n(e_G) = e_\Gamma$, while the second isomorphism is implemented by Fourier transform. So the assertion follows from Gelfand theory and Proposition 3·13.

Some specific examples are as follows:

(i) consider $\Gamma = SL(2n, \mathbb{Z})$ for some $n \in \mathbb{N}$. Then $Z = Z(\Gamma) \simeq \mathbb{Z}_2$. Also, $G = \Gamma/Z = PSL(2n, \mathbb{Z})$ is exact (cf. [**8**, Section 5.4]) and belongs to $\mathcal{P}$ (cf. [**5**]). Hence, we get that $C_r^*(SL(2n, \mathbb{Z}))$ has two nontrivial ideals;

(ii) consider the pure braid group $\Gamma = P_n$ on $n$ strands for some $n \geqslant 3$. Then $Z_n := Z(P_n) \simeq \mathbb{Z}$ and $G = P_n/Z_n$ is a weak Powers group (cf. [**14**] and [**6**]). Moreover $P_n$ is exact; this follows by induction on $n$, using the exact sequence

$$1 \longrightarrow \mathbb{F}_{n-1} \longrightarrow P_n/Z_n \longrightarrow P_{n-1}/Z_{n-1} \longrightarrow 1$$

(cf. [**14**, proposition 6], where $P_2 = Z_2 = 2\mathbb{Z}$) and the fact that extension of exact groups are exact (cf. [**8**, proposition 5·11]). Hence, we obtain that the ideals of $C_r^*(P_n)$ are in a one-to-one correspondence with the open (resp. closed) subsets of $\mathbb{T}$;

(iii) consider the braid group $\Gamma = B_3$ (i.e. the trefoil knot group). Then, $Z = Z(\Gamma) \simeq \mathbb{Z}$, and $G = \Gamma/Z \simeq \mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL(2, \mathbb{Z})$ belongs to $\mathcal{P}$. As, by definition of $P_3$, we have an exact sequence $1 \to P_3 \to B_3 \to S_3 \to 1$, where $S_3$ denotes the symmetric group on three symbols, it follows that $B_3$ is exact. (This also follows from the fact that braid groups are known to be linear groups.) Hence, we get that the ideals of $C_r^*(B_3)$ are in a one-to-one correspondence with the open (resp. closed) subsets of $\mathbb{T}$.

If one considers the braid group $B_n$ on $n$ strands for $n \geqslant 4$, then we believe that one should arrive at the same result as the one for $B_3$, but we don't know for the moment whether $B_n/Z_n$ belongs to the class $\mathcal{P}$. The group $B_n/Z_n$ is known to be a ultraweak Powers group (cf. [**1**, p. 536]), and Promislow has a result indicating that ultraweak Powers groups might be PH groups (see [**31**, theorem 8·1]), but this is open in general.

## 5. *Proof of Theorem* 3·8

We start by representing $B = C_r^*(\Sigma)$ faithfully on a Hilbert space. Without loss of generality, we may assume that $A$ acts faithfully on a Hilbert space $\mathcal{H}$, and let $(\pi, \lambda)$ be any regular covariant representation of $\Sigma$ on the Hilbert space $\ell^2(G, \mathcal{H})$; as in [**1**], we will work with the one defined by

$$\big(\pi(a)\xi\big)(h) = \alpha_{h^{-1}}(a)\,\xi(h),$$

$$\big(\lambda(g)\xi\,\big)(h) = \sigma(h^{-1},\, g)\,\xi(g^{-1}h),$$

for $a \in A,\ \xi \in \ell^2(G, \mathcal{H}),\ h, g \in G$.

We may then identify $B$ with $C^*(\pi(A), \lambda(G))$. The canonical conditional expectation from $B$ onto $\pi(A)$ will still be denoted by $E$. When $x \in B$, we set $\operatorname{supp}(x) = \{g \in G \mid \widehat{x}(g) \neq 0\}$, where $\widehat{x}(g) = E(x\,\lambda(g)^*)$. We will let $B_0$ denote the dense $*$-subalgebra of $B$ generated by $\pi(A)$ and $\lambda(G)$. So if $x \in B_0$, we have

$$x = \sum_{g \in \operatorname{supp}(x)} \widehat{x}(g)\,\lambda(g) \quad \text{(finite sum).}$$

If $D \subset G$, we let $P_D$ denote the orthogonal projection from $\ell^2(G, \mathcal{H})$ to $\ell^2(D, \mathcal{H})$ (identified as a closed subspace of $\ell^2(G, \mathcal{H})$).

Moreover, if $F \in \ell^\infty(G, \mathcal{B}(\mathcal{H}))$, that is, $F : G \to \mathcal{B}(\mathcal{H})$ is a map satisfying $\|F\|_\infty := \sup_{h\in G} \|F(h)\| < \infty$, we let $M_F \in \mathcal{B}(\ell^2(G, \mathcal{H}))$ be defined by

$$(M_F\, \xi)(h) = F(h)\, \xi(h)\,, \quad \xi \in \ell^2(G, \mathcal{H}),\ h \in G,$$

noting that $\|M_F\| = \|F\|_\infty < \infty$.

We remark that if $a \in A$ and we let $\pi_a : G \to \mathcal{B}(\mathcal{H})$ be defined by $\pi_a(h) = \alpha_{h^{-1}}(a)$ for each $h \in H$, then $\pi_a \in \ell^\infty(G, \mathcal{B}(\mathcal{H}))$ and $M_{\pi_a} = \pi(a)$.

Straightforward computations give that for $F \in \ell^\infty(G, \mathcal{B}(\mathcal{H}))$, $D \subset G$ and $g \in G$, we have

$$M_F\, P_D = P_D\, M_F, \quad \lambda(g)\, P_D = P_{gD}\, \lambda(g). \tag{5·1}$$

In passing, we remark that we also have $\lambda(g)\, M_F\, \lambda(g)^* = M_{F_g}$, where

$$F_g(h) = \sigma(h^{-1}, g)\, F(g^{-1}h)\, \sigma(h^{-1}, g)^*.$$

As a sample, we check that the second equation in (5·1) holds. Let $\xi \in \ell^2(G, \mathcal{H})$ and $h \in G$. Then we have

$$[(\lambda(g)\, P_D)\xi](h) = \sigma(h^{-1}, g)(P_D\xi)(g^{-1}h) = \begin{cases} \sigma(h^{-1}, g)\, \xi(g^{-1}h) & \text{if } g^{-1}h \in D, \\ 0 & \text{if } g^{-1}h \notin D \end{cases}$$

$$= \begin{cases} \sigma(h^{-1}, g)\, \xi(g^{-1}h) \text{ if } h \in gD, \\ 0 \qquad \text{if } h \notin gD \end{cases} = \begin{cases} (\lambda(g)\xi)(h) \text{ if } h \in gD, \\ 0 \qquad \text{if } h \notin gD \end{cases}$$

$= [(P_{gD}\, \lambda(g))\xi](h)$, as desired.

Let $H$ be a subgroup of $G$. By a *simple $H$-averaging process* on $B$, we will mean a linear map $\phi : B \to B$ such that there exist $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in H$ satisfying

$$\phi(x) = \frac{1}{n} \sum_{i=1}^{n} \lambda(h_i)\, x\, \lambda(h_i)^* \quad \text{for all } x \in B.$$

Moreover, an *$H$-averaging process on $B$* is a linear map $\psi : B \to B$ such that there exist $m \in \mathbb{N}$ and $\phi_1, \ldots, \phi_m$ simple $H$-averaging processes on $B$ with $\psi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$.

Let $\mathcal{U}_G$ denote the subgroup of $\mathcal{U}(B)$ generated by the $\lambda(g)$'s and let $\psi$ be a $G$-averaging process on $B$. Clearly, for all $x \in B$, we then have

$$\psi(x) \in \text{co}\big\{v\, x\, v^* \mid v \in \mathcal{U}_G\big\}.$$

Hence, to show that $\Sigma$ has (the strong) property (DP), it suffices to show that for every $x^* = x \in B$ satisfying $E(x) = 0$ and every $\varepsilon > 0$, there exists a $G$-averaging process $\psi$ on $B$ such that $\|\psi(x)\| < \varepsilon$.

In fact, it suffices to show the last claim for every $x^* = x \in B_0$ satisfying $E(x) = 0$ and every $\varepsilon > 0$. Indeed, assume that this holds and consider some $b^* = b \in B$ satisfying $E(b) = 0$ and $\varepsilon > 0$. Then pick $y^* = y \in B_0$ such that $\|b - y\| \leqslant \varepsilon/3$, and set $x = y - E(y)$. Then $x^* = x \in B_0$ and $E(x) = 0$, so we can find a $G$-averaging process on $B$ such that $\|\psi(x)\| < \varepsilon/3$. Since $\|E(y)\| = \|E(y - b)\| \leqslant \|y - b\| < \varepsilon/3$, we get

$$\begin{aligned} \|\psi(b)\| &\leqslant \|\psi(b - y)\| + \|\psi(y - E(y))\| + \|\psi(E(y))\| \\ &\leqslant \|b - y\| + \|\psi(x)\| + \|E(y)\| < \varepsilon, \end{aligned}$$

as desired.

5·1. In this subsection we will prove that Theorem 3·8 holds when $G$ is a PH group, as defined in [**31**]. We first recall the definition of a PH group.

If $g \in G$ and $A \subset G$, then set

$$< g >_A = \{aga^{-1} \mid a \in A\}.$$

Now, if $T \subset G$ and $\emptyset \neq M \subset G \setminus \{e\}$, then $T$ is said to be $M$-*large* (in $G$) if

$$m(G \setminus T) \subset T \quad \text{for all } m \in M.$$

Further, let $\emptyset \neq F \subset G \setminus \{e\}$ and $H \subset G$. Then $H$ is said to be a *Powers set for* $F$ if, for any $N \in \mathbb{N}$, there exist $h_1, \ldots, h_N \in H$ and pairwise disjoint subsets $T_1, \ldots, T_N$ of $G$ such that $T_j$ is $h_j F h_j^{-1}$-large for $j = 1, \ldots, N$. Moreover, if $g \in G \setminus \{e\}$, then $H$ is said to be a *c-Powers set for* $g$ if $H$ is a Powers set for $< g >_M$ for all finite, non-empty subsets $M$ of $H$.

If $G$ is a weak Powers group (see [**1**, **6**, **17**]), then $G$ is a c-Powers set for any $g \in G \setminus \{e\}$. More generally, $G$ is said to be a *PH group* if, given any finite non-empty subset $F$ of $G \setminus \{e\}$, one can write $F = \{f_1, f_2, \ldots, f_n\}$ and find a chain of subgroups $G_1 \subset G_2 \subset \cdots \subset G_n \subset G$ such that $G_j$ is a c-Powers set for $f_j$, $j = 1, \ldots, n$.

Note that in his definition of a PH group, Promislow just requires that one can find a chain of subsets $e \in G_1 \subset G_2 \subset \cdots \subset G_n$ of $G$ such that $G_j$ is a c-Powers set for $f_j$, $j = 1, \ldots, n$. Requiring these subsets to be subgroups of $G$ (or at least subsemigroups) seems necessary to us for the proof of his main result, [**31**, theorem 5·3], to go through. We will use the subsemigroup property in the proof of Lemma 5·3.

The class of PH groups has the interesting property that it closed under extensions [**31**, theorem 4·6].[1] For example, an extension of a weak Powers group by a weak Powers group is a PH group (but not necessarily a weak Powers group).

We will need a lemma of de la Harpe and Skandalis ([**18**, lemma 1]; see also [**1**, lemma 4·3]) in a slightly generalised form. For completeness, we include the proof, which is close to the one given in [**18**].

LEMMA 5·1. *Let $\mathcal{H}$ be a Hilbert space and $x^* = x \in \mathcal{B}(\mathcal{H})$. Assume that there exist orthogonal projections $p_1, p_2, p_3$ and unitary operators $u_1, u_2, u_3$ on $\mathcal{H}$ such that*

$$p_1\, x\, p_1 = p_2\, x\, p_2 = p_3\, x\, p_3 = 0$$

*and $u_1(1-p_1)u_1^*,\ u_2(1-p_2)u_2^*,\ u_3(1-p_3)u_3^*$ are pairwise orthogonal. Then we have*

$$\left\| \frac{1}{3} \sum_{j=1}^{3} u_j\, x\, u_j^* \right\| \leqslant \left( \frac{5}{6} + \frac{\sqrt{2}}{9} \right) \|x\| < 0.991\, \|x\|.$$

*Proof.* Without loss of generality, we may clearly assume that $\|x\| = 1$. Set $y = (1/3)\Sigma_{j=1}^{3} u_j\, x\, u_j^*$ and $q_j = u_j(1-p_j)u_j^*$, $j = 1, 2, 3$.

Let $\xi \in \mathcal{H}$, $\|\xi\| = 1$. Since the $q_j$'s are pairwise orthogonal, there exists an index $j$ such that $\|q_j\, \xi\|^2 \leqslant 1/3$. We may assume that $j = 1$, and set $\xi_1 = u_1^*\, \xi$.

As $\|(1-p_1)\, \xi_1\|^2 = \|q_1\, \xi\|^2 \leqslant 1/3$, one has

$$\|p_1 \xi_1\|^2 \geqslant 2/3 \quad \text{and} \quad \|p_1\, x\, (1-p_1)\, \xi_1\|^2 \leqslant 1/3\,.$$

[1] One easily checks that all the results in [**31**] are still true under our slightly more restrictive definition.

Now, since $p_1\, x\, p_1 = 0$ by assumption, we get

$$\|x\,\xi_1 - \xi_1\| \geqslant \|p_1\,\xi_1 - p_1\,x\,\xi_1\| = \|p_1\,\xi_1 - p_1\,x\,(1-p_1)\xi_1 - p_1\,x\,p_1\xi_1\|$$

$$\geqslant \big|\, \|p_1\,\xi_1\| - \|p_1\,x\,(1-p_1)\xi_1\|\,\big| \geqslant \frac{\sqrt{2}-1}{\sqrt{3}}.$$

As $\|x\,\xi_1 - \xi_1\|^2 \leqslant 2\,(1 - \langle x\,\xi_1, \xi_1\rangle)$, it follows that

$$\big\langle x\,\xi_1, \xi_1\big\rangle \leqslant 1 - \frac{1}{2}\|x\,\xi_1 - \xi_1\|^2 \leqslant 1 - \frac{1}{2}\left(\frac{\sqrt{2}-1}{\sqrt{3}}\right)^2 = \frac{3+2\sqrt{2}}{6}.$$

So, using the Cauchy–Schwarz inequality, we get

$$\big\langle y\,\xi, \xi\big\rangle \leqslant \frac{1}{3}\big\langle x\,\xi_1, \xi_1\big\rangle + \frac{2}{3} \leqslant \frac{1}{3}\left(\frac{3+2\sqrt{2}}{6} + 2\right) = \frac{5}{6} + \frac{\sqrt{2}}{9} < 0.991.$$

The same argument with $-x$ gives

$$\Big|\big\langle y\,\xi, \xi\big\rangle\Big| \leqslant \frac{5}{6} + \frac{\sqrt{2}}{9} < 0.991.$$

Since $y$ is self-adjoint, taking the supremum over all $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$, we obtain

$$\|y\| \leqslant \frac{5}{6} + \frac{\sqrt{2}}{9} < 0.991\,,$$

as desired.

LEMMA 5·2. *Let* $x^* = x \in B_0$ *satisfy* $E(x) = 0$. *Assume that* $\operatorname{supp}(x) \subset F \cup F^{-1}$ *for some finite non-empty subset* $F$ *of* $G \setminus \{e\}$ *and that there exists a subgroup* $H$ *of* $G$ *which is a Powers set for* $F$.

*Then there exists a simple* $H$*-averaging process* $\phi$ *on* $B$ *such that*

$$\|\phi(x)\| < 0.991\,\|x\|.$$

*Proof.* One easily sees that $H$ is also a Powers set for $S = F \cup F^{-1}$ (cf. [**31**, lemma 2·2]). We may therefore pick $h_1, h_2, h_3 \in H$ and pairwise disjoint subsets $T_1, T_2, T_3$ of $G$ such that $T_j$ is $h_j S h_j^{-1}$-large for $j = 1, 2, 3$.

For each $j = 1, 2, 3$, set $E_j = h_j^{-1} T_j$, $D_j = G \setminus E_j$ and let $p_j$ be the orthogonal projection from $\ell^2(G, \mathcal{H})$ onto $\ell^2(D_j, \mathcal{H})$. Then we have $p_j\, x\, p_j = 0$ for each $j$. Indeed, as is easily checked, $h_j S h_j^{-1}$-largeness of $T_j$ means that

$$s\, D_j \cap D_j = \varnothing \quad \text{for every } s \in S.$$

Thus, for $a \in A$ and $s \in S$, using the identities in (5·1), we get $p_j \pi(a)\lambda(s) p_j = \pi(a) p_j \lambda(s) p_j = \pi(a) P_{D_j} P_{sD_j} \lambda(s) = 0$. Since $\operatorname{supp}(x) \subset S$, the above assertion readily follows.

Moreover, for each $j = 1, 2, 3$, set $q_j = \lambda(h_j)(1 - p_j)\lambda(h_j)^*$. Then $q_j$ is the orthogonal projection from $\ell^2(G, \mathcal{H})$ onto $\ell^2(h_j E_j, \mathcal{H}) = \ell^2(T_j, \mathcal{H})$. Since the $T_j$'s are pairwise disjoint, the $q_j$'s are pairwise orthogonal. Thus, we can apply Lemma 5·1 and conclude that

$$\Big\| \frac{1}{3} \sum_{j=1}^{3} \lambda(h_j)\, x\, \lambda(h_j)^* \Big\| < 0.991\,\|x\|$$

which shows the assertion.

LEMMA 5·3. *Let $\delta > 0$, $g \in G \setminus \{e\}$ and assume that there exists a subgroup $H$ of $G$ which is a c-Powers set for $g$. Let $x^* = x \in B_0$ satisfy*

$$\text{supp}\,(x) \subset\ <g>_M \cup <g^{-1}>_M$$

*for some finite non-empty subset $M$ of $H$.*

*Then there exists an $H$-averaging process $\psi$ on $B$ such that $\|\psi(x)\| < \delta$.*

*Proof.* By assumption, $H$ is a Powers set for $<g>_M$. Applying Lemma 5·2 (with $F = \ <g>_M$), we get that there exists a simple $H$-averaging process $\phi_1$ on $B$ such that $\|\phi_1(x)\| < d\,\|x\|$, where $d = 0.991$. Now, one easily checks (cf. [**1**, lemma 4·4]) that

$$\text{supp}\,\big(\phi_1(x)\big) \subset\ <g>_{M_1} \cup <g^{-1}>_{M_1},$$

where $M_1$ is a finite non-empty subset of $H$ (since $H$ is closed under multiplication, being a subgroup). Moreover, $\phi_1(x)$ is a selfadjoint element of $B_0$ satisfying $E(\phi_1(x)) = 0$. Hence we can apply Lemma 5·2 (with $F =<g>_{M_1}$) and get that there exists a simple $H$-averaging process $\phi_2$ on $B$ such that

$$\|\phi_2(\phi_1(x))\| < d\,\|\phi_1(x)\| < d^2\,\|x\|.$$

Iterating this process, we get that for each $k \in \mathbb{N}$, there exist simple $H$-averaging processes $\phi_1, \ldots, \phi_k$ on $B$ such that

$$\|(\phi_k \circ \cdots \circ \phi_1)(x)\| < d^k\,\|x\|.$$

Choosing $k$ such that $d^k < \delta$ gives the result.

THEOREM 5·4. *Assume $G$ is a PH group. Then $\Sigma$ has property (DP).*

*Proof.* Let $x^* = x \in B_0$ satisfy $E(x) = 0$, and let $\varepsilon > 0$. Write $S = \text{supp}\,(x)$ as a disjoint union $S = R \cup F \cup F^{-1}$ where $R = \{s \in S \mid s^2 = e\}$.

Consider $R \cup F \subset G \setminus \{e\}$. Since $G$ is a PH group, we can write $R \cup F = \{s_1, s_2, \ldots, s_n\}$ and find a chain of subgroups $G_1 \subset G_2 \subset \cdots \subset G_n \subset G$ such that $G_j$ is a c-Powers set for $s_j$, $j = 1, \ldots, n$. Thus, each $G_j$ is a Powers set for $<s_j>_M$, for all finite subsets $M$ of $G_j$.

Write $x = \Sigma_{j=1}^n x_j$, where $x_j^* = x_j \in B_0$ and $\text{supp}\,(x_j) = \{s_j\} \cup \{s_j^{-1}\}$ for each $j$. (Note that if $s_j \in R$, we have $s_j^{-1} = s_j$, so $\text{supp}\,(x_j) = \{s_j\}$ in this case.)

Since $\text{supp}\,(x_1) = \ <s_1>_M \cup <s_1^{-1}>_M$, with $M = \{e\} \subset G_1$, and $G_1$ is a c-Powers set for $s_1$, Lemma 5·3 applies and gives that there exists a $G_1$-averaging process $\psi_1$ on $B$ such that $\|\psi_1(x_1)\| < \varepsilon/n$.

Now, consider $\tilde{x}_2 = \psi_1(x_2)$. Then $\text{supp}\,(\tilde{x}_2) \subset\ <s_2>_M \cup <s_2^{-1}>_M$ for some finite subset $M$ of $G_1$. Since $G_1$ is contained in $G_2$, and $G_2$ is a c-Powers set for $s_2$, Lemma 5·3 applies again and gives that there exists a $G_2$-averaging process $\psi_2$ on $B$ such that $\|\psi_2(\tilde{x}_2)\| < \varepsilon/n$, that is, $\|(\psi_2 \circ \psi_1)(x_2)\| < \varepsilon/n$.

Proceeding inductively, let $1 \leqslant k \leqslant n-1$ and assume that for each $j = 1, \ldots, k$, we have constructed a $G_j$-averaging process $\psi_j$ on $B$, such that $\|(\psi_j \circ \cdots \circ \psi_1)(x_j)\| < \varepsilon/n$ for $j = 1, \ldots, k$. Then consider $\tilde{x}_{k+1} = (\psi_k \circ \cdots \circ \psi_1)(x_{k+1})$. Then $\text{supp}\,(\tilde{x}_{k+1}) \subset\ <s_{k+1}>_M \cup <s_{k+1}^{-1}>_M$ for some finite subset $M$ of $G_k$. Since $G_k$ is contained in $G_{k+1}$, and $G_{k+1}$ is a c-Powers set for $s_{k+1}$, Lemma 5·3 applies and gives that there exists a $G_{k+1}$-averaging process $\psi_{k+1}$ on $B$ such that $\|\psi_{k+1}(\tilde{x}_{k+1})\| < \varepsilon/n$, that is, $\|(\psi_{k+1} \circ \cdots \circ \psi_1)(x_{k+1})\| < \varepsilon/n$.

Repeating this until $k = n-1$, we obtain, for each $1 \leqslant j \leqslant n$, a $G_j$-averaging process $\psi_j$ on $B$ such that $\|(\psi_j \circ \cdots \circ \psi_1)(x_j)\| < \varepsilon/n$. Set $\psi = \psi_n \circ \cdots \circ \psi_1$. Then $\psi$ is a $G$-averaging process on $B$ and, for each $1 \leqslant j \leqslant n$, we have

$$\|\psi(x_j)\| = \|(\psi_n \circ \cdots \circ \psi_{j+1} \circ \psi_j \circ \cdots \circ \psi_1)(x_j)\| \leqslant \|(\psi_j \circ \cdots \circ \psi_1)(x_j)\| < \varepsilon/n,$$

so we get

$$\|\psi(x)\| \leqslant \sum_{j=1}^{n} \|\psi(x_j)\| < \varepsilon.$$

This shows that $\Sigma$ satisfies (the strong) property DP.

5·2. We now turn to the proof that $\Sigma$ has property (DP) when $G$ satisfies property $(P_{\text{com}})$. We will adapt the arguments given in [**5**] to cover the twisted case. We recall from [**5**] that $G$ *is said to have property* $(P_{\text{com}})$ when the following holds given any non-empty finite subset $F \subset G \setminus \{e\}$, there exist $n \in \mathbb{N}$, $g_0 \in G$ and subsets $U, D_1, \ldots, D_n$ of $G$ such that:

(i) $G \setminus U \subset D_1 \cup \cdots \cup D_n$ ;

(ii) $g\,U \cap U = \emptyset$ for all $g \in F$ ;

(iii) $g_0^{-j} D_k \cap D_k = \emptyset$ for all $j \in \mathbb{N}$ and $k = 1, \ldots, n$ .

LEMMA 5·5. (*cf.* [**5**]). *Let* $g \in G \setminus \{e\}$ *and assume there exist* $n \in \mathbb{N}$ *and subsets* $U, D_1, \ldots, D_n$ *of* $G$ *such that*

$$G \setminus U \subset D_1 \cup \cdots \cup D_n \quad \text{and} \quad g\,U \cap U = \emptyset.$$

*Let* $F \in \ell^\infty\big(G, \mathcal{B}(\mathcal{H})\big)$ *and* $\xi, \eta \in \ell^2(G, \mathcal{H})$. *Then we have*

$$|\langle M_F\lambda(g)\xi,\ \eta\rangle| \leqslant \sum_{j=1}^{n} \big(\|M_F\lambda(g)\xi\|\,\|P_{D_j}\eta\| + \|P_{D_j}\xi\|\,\|M_F^*\eta\|\big). \tag{5·2}$$

*Proof.* We set $V = G \setminus U$, and note that $P_U\,P_{gU} = P_{U\cap gU} = 0$. Thus, making use of (5·1), we get

$$\begin{aligned}\langle M_F\lambda(g)\xi,\ \eta\rangle &= \langle M_F\lambda(g)P_U\,\xi,\ \eta\rangle + \langle M_F\lambda(g)P_V\,\xi,\ \eta\rangle \\ &= \langle P_{gU}M_F\lambda(g)\,\xi,\ (P_U + P_V)\,\eta\rangle + \langle \lambda(g)P_V\,\xi,\ M_F^*\,\eta\rangle \\ &= \langle P_{gU}M_F\lambda(g)\,\xi,\ P_V\,\eta\rangle + \langle \lambda(g)P_V\,\xi,\ M_F^*\eta\rangle.\end{aligned}$$

Thus, the triangle inequality and the Cauchy–Schwarz inequality give

$$\begin{aligned}|\langle M_F\lambda(g)\,\xi,\ \eta\rangle| &\leqslant |\langle P_{gU}M_F\lambda(g)\,\xi,\ P_V\,\eta\rangle| + |\langle \lambda(g)P_V\,\xi,\ M_F^*\eta\rangle| \\ &\leqslant \|M_F\lambda(g)\,\xi\|\,\|P_V\,\eta\| + \|P_V\,\xi\|\,\|M_F^*\eta\| \\ &\leqslant \sum_{j=1}^{n} \big(\|M_F\lambda(g)\,\xi\|\,\|P_{D_j}\eta\| + \|P_{D_j}\xi\|\,\|M_F^*\eta\|\big)\end{aligned}$$

since $\|P_V\zeta\| \leqslant \Sigma_{j=1}^{n}\|P_{D_j}\zeta\|$ for any $\zeta \in \ell^2(G, \mathcal{H})$, as is easily checked, using that $V \subset D_1 \cup \cdots \cup D_n$.

LEMMA 5·6. *Let $D \subset G$, $\zeta \in \ell^2(G, \mathcal{H})$ and assume there exist $N \in \mathbb{N}$ and $g_1, \ldots, g_N \in G$ such that $g_1 D, \ldots, g_N D$ are pairwise disjoint. Then we have*

$$\sum_{j=1}^{N} \| P_{g_j D}\, \zeta \| \leqslant \sqrt{N}\, \|\zeta\|.$$

*Proof.* The Cauchy–Schwarz inequality and the assumption give

$$\begin{aligned}\sum_{j=1}^{N} \| P_{g_j D}\, \zeta \| &\leqslant \sqrt{N} \Big[ \sum_{j=1}^{N} \| P_{g_j D}\, \zeta \|^2 \Big]^{1/2} \\ &= \sqrt{N} \Big[ \sum_{h \in g_1 D \cup \cdots \cup g_N D} \|\zeta(h)\|^2 \Big]^{1/2} \\ &\leqslant \sqrt{N}\, \|\zeta\|.\end{aligned}$$

LEMMA 5·7. *Assume that $G$ has property $(P_{\mathrm{com}})$.*
*Let $F$ be a finite non-empty subset of $G \setminus \{e\}$, $a_g \in A$ for each $g \in F$, and set $y_0 = \Sigma_{g \in F}\, \pi(a_g)\, \lambda(g) \in B$. Then we have*

$$0 \in \overline{co\{v\, y_0\, v^* \mid v \in \mathcal{U}_G\}}^{\,\|\cdot\|}.$$

*Proof.* Since $G$ has property $(P_{\mathrm{com}})$, we may pick $n \in \mathbb{N}$, $g_0 \in G$ and subsets $U, D_1, \ldots, D_n$ of $G$ so that (i), (ii) and (iii) in the definition of property $(P_{\mathrm{com}})$ hold with respect to the given $F$.

For each $j \in \mathbb{N}$, we set $g_j = g_0^{-j}$. Moreover, for each $N \in \mathbb{N}$, we set

$$y_N = \frac{1}{N} \sum_{j=1}^{N} \lambda(g_j)\, y_0\, \lambda(g_j)^* \in \mathrm{co}\{v\, y_0\, v^* \mid v \in \mathcal{U}_G\}.$$

We will show that

$$\|y_N\| \leqslant \frac{2n}{\sqrt{N}} \sum_{g \in F} \|a_g\|. \tag{5·3}$$

Thus, we will get that $\|y_N\| \to 0$ as $N \to \infty$, from which the assertion to be proven will clearly follow.

To prove (5·3), fix $N \in \mathbb{N}$. Since

$$y_N = \frac{1}{N} \sum_{g \in F} \sum_{j=1}^{N} \lambda(g_j)\, \pi(a_g)\, \lambda(g)\, \lambda(g_j)^*,$$

we have

$$\|y_N\| \leqslant \frac{1}{N} \sum_{g \in F} \|z_g\|, \tag{5·4}$$

where $z_g = \Sigma_{j=1}^{N}\, \lambda(g_j)\, \pi(a_g)\, \lambda(g)\, \lambda(g_j)^*$ for each $g \in F$.

Let $g \in F$ and $\xi, \eta \in \ell^2(G, \mathcal{H})$. As condition (iii) implies that for each $k \in \{1, 2, \ldots, n\}$, the sets $g_1 D_k, \ldots, g_N D_k$ are pairwise disjoint, Lemma 5·6 gives that

$$\sum_{j=1}^{N} \| P_{g_j D_k}\, \eta \| \leqslant \sqrt{N}\, \|\eta\| \quad \text{and} \quad \sum_{j=1}^{N} \| P_{g_j D_k}\, \xi \| \leqslant \sqrt{N}\, \|\xi\|. \tag{5·5}$$

Using Lemma 5·5 $N$ times (with $M_F = \pi(a_g)$) at the second step, we get

$$\begin{aligned}
|\langle z_g\,\xi,\ \eta\rangle| &\leqslant \sum_{j=1}^{N} \left|\left\langle \pi(a_g)\lambda(g)\,\lambda(g_j)^*\,\xi,\ \lambda(g_j)^*\,\eta\right\rangle\right| \\
&\leqslant \sum_{j=1}^{N}\sum_{k=1}^{n} \big(\|\pi(a_g)\lambda(g)\,\lambda(g_j)^*\,\xi\|\,\|P_{D_k}\lambda(g_j)^*\,\eta\| \\
&\quad + \|P_{D_k}\lambda(g_j)^*\,\xi\|\,\|\pi(a_g)^*\,\lambda(g_j)^*\,\eta\|\big) \\
&\leqslant \sum_{j=1}^{N}\sum_{k=1}^{n} \big(\|\pi(a_g)\|\,\|\xi\|\,\|P_{g_jD_k}\,\eta\| + \|P_{g_jD_k}\,\xi\|\,\|\pi(a_g)\|\,\|\eta\|\big) \\
&= \|a_g\| \sum_{k=1}^{n} \left( \|\xi\|\,\Big(\sum_{j=1}^{N}\|P_{g_jD_k}\,\eta\|\Big) \,+\, \|\eta\|\,\Big(\sum_{j=1}^{N}\|P_{g_jD_k}\,\xi\|\Big)\right) \\
&\leqslant \|a_g\|\,2\,n\,\sqrt{N}\,\|\xi\|\,\|\eta\|,
\end{aligned}$$

where we have used (5·5) to get the final inequality.

This implies that

$$\|z_g\| \;\leqslant\; 2\,n\,\sqrt{N}\,\|a_g\|.$$

Using (5·4), we therefore get

$$\|y_N\| \;\leqslant\; \frac{1}{N}\,2\,n\,\sqrt{N}\sum_{g\in F}\|a_g\| = \frac{2\,n}{\sqrt{N}}\sum_{g\in F}\|a_g\|\,,$$

that is, the inequality (5·3) holds, as desired.

THEOREM 5·8. *Assume that $G$ has property $(P_{\mathrm{com}})$. Then $\Sigma$ has property $(DP)$.*

*Proof.* Lemma 5·7 shows that if $x \in B_0$ satisfies $E(x) = 0$, and $\varepsilon > 0$, then there exists a $G$-averaging process on $B$ such that $\|\psi(x)\| < \varepsilon$. Hence, it follows that $\Sigma$ has (the strong) property (DP).

Note that the proof of Theorem 5·8 in fact implies that when $G$ has property $(P_{\mathrm{com}})$, then $\Sigma$ satisfies that

$$0 \;\in\; \overline{\mathrm{co}\{v\,y\,v^* \mid v \in \mathcal{U}_G\}}^{\;\|\cdot\|} \tag{5·6}$$

for every $y \in B$ satisfying $E(y) = 0$. As mentioned in Remark 3·2, this is true whenever $\Sigma$ satisfies the strong form of property (DP) (hence also when $G$ is a PH group):

PROPOSITION 5·9. *Assume that $\Sigma$ satisfies the strong form of property $(DP)$. Then* (5·6) *holds for every $y \in B$ satisfying $E(y) = 0$.*

*Proof.* Let $y \in B$ satisfy $E(y) = 0$ and $\varepsilon > 0$. Write $y = x_1 + i\,x_2$, where $x_1 = \mathrm{Re}(y)$, $x_2 = \mathrm{Im}(y)$. Note that $E(x_1) = (E(y) + E(y)^*)/2 = 0$, and, similarly, $E(x_2) = 0$. Using the assumption, we can find a $G$-averaging process $\psi_1$ on $B$ such that $\|\psi_1(x_1)\| < \varepsilon/2$. Now, set $\tilde{x}_2 = \psi_1(x_2)$. Then $\tilde{x}_2$ is self-adjoint, and, using the equivariance property of $E$, one deduces that $E(\tilde{x}_2) = 0$. Hence, we can find a $G$-averaging process $\psi_2$ on $B$ such that $\|\psi_2(\tilde{x}_2)\| < \varepsilon/2$. Set $\psi = \psi_2 \circ \psi_1$. Then we get

$$\|\psi(y)\| \leqslant \|\psi(x_1)\| + \|\psi(x_2)\| \leqslant \|\psi_1(x_1)\| + \|\psi_2(\tilde{x}_2)\| < \varepsilon,$$

and it follows that (5·6) holds.

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