Mathematika 61 (2015) 370–384 © 2015 University College London
doi:10.1112/S0025579314000400

WEIGHTED ESTIMATES ON FRACTAL DOMAINS

RAFFAELA CAPITANELLI AND MARIA AGOSTINA VIVALDI

Dedicated to Professor V. G. Maz'ya

Abstract. The aim of the paper is to establish estimates in weighted Sobolev spaces for the solutions of the Dirichlet problems on snowflake domains, as well as uniform estimates for the solutions of the Dirichlet problems on pre-fractal approximating domains.

§1. *Introduction.* In a previous paper [**5**], we established uniform estimates in weighted Sobolev spaces for the solutions of the Dirichlet problems on polygonal domains approximating the snowflake domain. In the present article, we deduce from the aforementioned estimates a regularity result for the Dirichlet problem on snowflake domain Ω_3. More precisely, we prove that the second derivatives of the solution belong to the weighted space $L^2(\Omega_3, \delta^{\mu})$, $\mu > \mu^* = \frac{2}{3} + \frac{1}{6}\log_3 4$, where by $\delta = \delta(x)$ we denote the distance of the point x to the boundary $\partial\Omega_3$ of Ω_3 (see Theorem 3.1). These results were presented by one of the authors at the meeting organized at the Department of Mathematical Sciences of the University of Liverpool in honour of Professor V. Maz'ya on the occasion of his 75th birthday. We are deeply grateful to Professor V. Maz'ya for having highlighted, in this meeting, Brennan's conjecture. Indeed, the regularity result of Theorem 3.1 could be greatly improved if Brennan's conjecture were shown to be true. As far as we know, Brennan's conjecture remains elusive, with only partial results having been established. In §4, we consider a larger class of fractal domains of the *snowflake* type constructed by means of the *generalized* Koch curves K_α, $\alpha \in (2, 4)$ (see §2) and we prove, using the upper bound found by Hedenmalm and Shimorin [**10**], that the first derivatives of the solution to the Dirichlet problem on snowflake domain Ω_α belong to the space $L^q(\Omega_\alpha)$, and the second derivatives belong to the weighted space $L^2(\Omega_\alpha, \delta^{\mu})$, $\mu > \mu_\alpha = (2/q) + ((q-2)/2q)\log_\alpha 4$, $q < q_0 = 3.752$ (see Theorem 4.1). We note that the exponent μ^* is strictly greater than μ_3 and the following inclusion holds for the weighted spaces $L^2(\Omega_\alpha, \delta^{\mu'}) \subset L^2(\Omega_\alpha, \delta^{\mu''})$ if $\mu' < \mu''$. Hence, Theorem 4.1 improves Theorem 3.1 also for the case $\alpha = 3$. In §5, by combining the tools and methods of the paper [**5**] with the results of §4, we establish uniform estimates that are more accurate than those established in [**5**], (compare Theorems 5.1 and 3.2). Lastly, in §6 we briefly discuss how we can extend the results of previous sections to the solutions of obstacle problems.

Received 28 March 2014, published online 3 February 2015.
MSC (2010): 28A80, 35D10, 35J08, 35J25, 35J85 (primary).

§2. *Snowflakes.* We recall the definition of the Koch curve with endpoints $A = (0,0)$ and $B = (1,0)$. We consider the family $\Psi^\alpha = \{\psi_1^\alpha, \ldots, \psi_4^\alpha\}$ of contractive similitudes $\psi_i^\alpha : \mathbb{C} \to \mathbb{C}$, $i = 1, \ldots, 4$ with contraction factor α^{-1}, $2 < \alpha < 4$:

$$\psi_1^\alpha(z) = \frac{z}{\alpha}, \qquad \psi_2^\alpha(z) = \frac{z}{\alpha} \mathrm{e}^{\mathrm{i}\theta(\alpha)} + \frac{1}{\alpha},$$
$$\psi_3^\alpha(z) = \frac{z}{\alpha} \mathrm{e}^{-\mathrm{i}\theta(\alpha)} + \frac{1}{2} + \mathrm{i}\sqrt{\frac{1}{\alpha} - \frac{1}{4}}, \qquad \psi_4^\alpha(z) = \frac{z-1}{\alpha} + 1,$$

where $\theta(\alpha) = \arcsin(\sqrt{\alpha(4-\alpha)}/2)$.

By the general theory of self-similar fractals (see [**12**]), there exists a unique closed bounded set K_α which is *invariant* with respect to Ψ^α, that is,

$$K_\alpha = \bigcup_{i=1}^{4} \psi_i^\alpha(K_\alpha). \tag{2.1}$$

Moreover, the Hausdorff dimension of the set K_α is $d_f = \ln_\alpha 4$. Let K^0 be the line segment of unit length that has as endpoints $A = (0,0)$ and $B = (1,0)$. We set, for each n in $\mathbb{N}$,

$$K_\alpha^1 = \bigcup_{i=1}^{4} \psi_i^\alpha(K^0), \qquad K_\alpha^2 = \bigcup_{i=1}^{4} \psi_i^\alpha(K_\alpha^1), \ldots, K_\alpha^{n+1} = \bigcup_{i=1}^{4} \psi_i^\alpha(K_\alpha^n); \tag{2.2}$$

K_α^n is the so-called nth pre-fractal curve. Moreover, the iterates K_α^n converge to the self-similar set K_α in the Hausdorff metric when n tends to infinity (see [**12**]). Let Ω^0 be the triangle with vertices $A = (0,0)$, $B = (1,0)$, and $C = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. We construct on the side with endpoints A and B the pre-fractal Koch curve defined before, which will be denoted by $K_{1,\alpha}^n$, and the Koch curve defined before, which will be denoted by $K_{1,\alpha}$. In a similar way, we construct on the other sides the analogous pre-fractal Koch curves (the Koch curves) denoting by $K_{2,\alpha}^n$ and $K_{3,\alpha}^n$ (by $K_{2,\alpha}$ and $K_{3,\alpha}$) the curves with endpoints B and C, and C and A, respectively. We denote by Ω_α^n the pre-fractal domain that is the set bounded by the pre-fractal Koch curves $K_{j,\alpha}^n$, $j = 1, 2, 3$. Moreover, we denote by Ω_α the snowflake that is the set bounded by the Koch curves $K_{j,\alpha}$, $j = 1, 2, 3$. We consider the homogeneous Dirichlet problem in the snowflake domain Ω_α as well as in the pre-fractal domains Ω_α^n:

$$\begin{cases} -\Delta u = f & \text{in } \Omega_\alpha, \\ u = 0 & \text{on } \partial\Omega_\alpha, \end{cases} \tag{2.3}$$

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_\alpha^n, \\ u_n = 0 & \text{on } \partial\Omega_\alpha^n. \end{cases} \tag{2.4}$$

As is well known for any datum $f \in L^2(\Omega_\alpha)$, there exists a unique solution $u \in H_0^1(\Omega_\alpha)$ of (2.3) by Lax–Milgram theorem. Analogously for any $f \in L^2(\Omega_\alpha^n)$

there exists a unique solution $u_n \in H^1_0(\Omega^n_\alpha)$ of (2.4) and the estimate

$$\|u_n\||_{H^1_0(\Omega^n_\alpha)} \leqslant c\|f\|_{L^2(\Omega^n_\alpha)} \tag{2.5}$$

holds with the constant c independent of n. The previous estimate (2.5) follows from the following Poincaré-type inequality where the relevant fact is that the constant C_P is independent of n.

PROPOSITION 2.1. *There exists a constant C_P independent of n such that*

$$\|u\|_{L^2(\Omega^n_\alpha)} \leqslant C_P(\|\nabla u\|_{L^2(\Omega^n_\alpha)} + \|u\|_{L^2(\partial\Omega^n_\alpha)}) \tag{2.6}$$

for all $u \in H^1(\Omega^n_\alpha)$.

The proof can be achieved as in [**4**, Theorem 7.3] by making some natural changes.

In the following sections we establish regularity results in weighted Sobolev spaces for the Dirichlet problem on snowflake domain Ω_α and uniform estimates (in weighted Sobolev spaces) for the solutions of the Dirichlet problems on polygonal domains approximating the snowflake domain. From now on, by c we denote (possibly different) positive constants independent of n.

§3. *First regularity result.* We introduce the weighted Lebesgue space $L^2(\Omega_\alpha, \delta^\mu)$, where by $\delta = \delta(x)$ we denote the distance of the point x to the boundary $\partial\Omega_\alpha$ of Ω_α. The space $L^2(\Omega_\alpha, \delta^\mu)$ is the completion of the space $C^0(\overline{\Omega}_\alpha)$ with respect to the norm

$$\|u\|_{L^2(\Omega_\alpha,\delta^\mu)} = \left(\int_{\Omega_\alpha} |u|^2\delta^{2\mu}\,dx\right)^{1/2}. \tag{3.1}$$

In this section, we choose $\alpha = 3$ (and hence $\theta = \pi/3$); in this case, the self-similar fractal K_3 is the so-called equilateral Koch curve and we prove the following theorem.

THEOREM 3.1. *Let $u \in H^1_0(\Omega_3)$ be the solution of (2.3) with $f \in L^2(\Omega_3)$. Then, for every $\mu > \mu^* = \frac{2}{3} + \frac{1}{6}\log_3 4$,*

$$\sum_{|\beta|=2}\int_{\Omega_3} |D^\beta u|^2\delta^{2\mu}\,dx \leqslant c\|f\|^2_{L^2(\Omega_3)}. \tag{3.2}$$

In order to prove Theorem 3.1, we consider the solutions of the Dirichlet problems on polygonal domains Ω^n_3 approximating the snowflake domain Ω_3 and we use the uniform estimates in weighted Sobolev spaces established in [**5**]. We recall that as the domain Ω^n_3 is not convex, then the solution u_n of problem (2.4) does not belong to the space $H^2(\Omega^n_3)$. The second derivatives $|D^\beta u_n|$ ($\beta = (\beta_1, \beta_2)$, $|\beta| = 2$) actually belong to the weighted Lebesgue space $L^2(\Omega^n_3, \rho^\mu_n)$ for some positive exponent μ. Here $\rho_n = \rho_n(x)$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of re-entrant corners of Ω_3^n, and $L^2(\Omega_3^n, \rho_n^\mu)$ is the completion of the spaces $C^0(\overline{\Omega_3^n})$ with respect to the norm

$$\|u\|_{L^2(\Omega_3^n,\rho_n^\mu)} = \left(\int_{\Omega_3^n} |u|^2 \rho_n^{2\mu}\, dx\right)^{1/2}. \tag{3.3}$$

In our setting, the domains Ω_3^n are polygonal, non-convex, and with an increasing number of corners, where the amplitude of the re-entrant corners is equal to $\frac{4}{3}\pi$. Hence, the celebrated Kondratiev result implies that

$$\sum_{|\beta|=2} \int_{\Omega_3^n} |D^\beta u_n|^2 \rho_n^{2\mu}\, dx \leqslant c(n) \int_{\Omega_3^n} f^2 \rho_n^{2\mu}\, dx \tag{3.4}$$

with $\mu \in (\frac{1}{4}, 1)$, (see [**8**, **14**]).

As the boundaries are the union of an increasing number of graphs and develop at the limit a fractal geometry, then the sharp regularity result (3.4) involves constants that might diverge as the number of graphs becomes infinite. In [**5**] we proved that there exists a suitable value of μ^* depending on the structural parameter of the limit fractal domain Ω_3 for which uniform weighted estimates hold. More precisely, the following result holds.

THEOREM 3.2. *Let $u_n \in H_0^1(\Omega_3^n)$ be the solution of (2.4) with $f \in L^2(\Omega_3^n)$. Then, for every $\mu > \mu^* = \frac{2}{3} + \frac{1}{6}\log_3 4$,*

$$\sum_{|\beta|=2} \int_{\Omega_3^n} |D^\beta u_n|^2 \rho_n^{2\mu}\, dx \leqslant\ c\|f\|^2_{L^2(\Omega_3^n)} \tag{3.5}$$

with the constant c independent of n.

Here, as before, ρ_n denotes the distance function from the set $\mathcal{R}_n$ of the vertices of re-entrant corners of Ω_3^n. It is well known that the solution u_n of the Dirichlet problem (2.4) realizes the minimum of the following functional F^n in $L^2(\Omega_3)$:

$$F^n[u] = \begin{cases} \displaystyle\int_{\Omega_3^n} |\nabla u|^2\, dx - 2\int_{\Omega_3^n} f u\, dx & \text{if } u|_{\Omega_3^n} \in H_0^1(\Omega_3^n), \\ +\infty & \text{otherwise in } L^2(\Omega_3). \end{cases} \tag{3.6}$$

It is easy to prove that the sequence of the functionals F^n M-converges to the functional

$$F[u] = \begin{cases} \displaystyle\int_{\Omega_3} |\nabla u|^2\, dx - 2\int_{\Omega_3} f u\, dx & \text{if } u \in H_0^1(\Omega_3), \\ +\infty & \text{otherwise in } L^2(\Omega_3), \end{cases} \tag{3.7}$$

(see [**15**, **16**] for definition and properties).

From now on, we denote by the same symbol u_n the extensions of the functions u_n to zero outside Ω_3^n. As a consequence of M-convergence we deduce that the sequence of the function u_n strongly converges in $H_0^1(\Omega_3)$ to the function u that minimizes the functional $F[u]$ defined in (3.7), and u is the solution of (2.3).

We are now in a position to prove Theorem 3.1.

Proof. For any fixed N, we consider regular open sets G_m approximating Ω_3^N, that is, $\bar{G}_m \subset G_{m+1} \uparrow \Omega_3^N$, $\bar{G}_m \subset \Omega_3^N$. We have, for any $n \geqslant N$, $D^2 u_n \in L^2(G_m)$ and $D^2 u \in L^2(G_m)$. We start by proving that, for a fixed m, the sequence $D^2 u_n$ weakly converges to $D^2 u$ in $L^2(G_m)$. Indeed, for $x \in G_m$, we have that, for any $n \geqslant N$,

$$\rho_n(x) \geqslant \delta_{m,N}$$

where $\delta_{m,N}$ denotes the distance of the set G_m from the boundary $\partial\Omega_3^N$ of Ω_3^N. From (3.5) we obtain

$$\int_{G_m} |D^2 u_n|^2 \, dx \leqslant \delta_{m,N}^{-2\mu} \int_{\Omega_3^n} |D^2 u_n|^2 \rho_n^{2\mu} \, dx \leqslant c, \tag{3.8}$$

with c independent of n.

Then, there exists $w \in L^2(G_m)$ such that, up to passing to a subsequence, $D^2 u_n$ weakly converges to w in $L^2(G_m)$. Now we prove that

$$w = D^2 u \quad \text{almost everywhere in } G_m, \tag{3.9}$$

that is,

$$\int_{G_m} (w - D^2 u)\varphi \, dx = 0$$

for any $\varphi \in C_0^1(G_m)$. In fact,

$$\begin{aligned} &\int_{G_m} (w - D^2 u)\varphi \, dx \\ &\quad = \int_{G_m} w\varphi \, dx + \int_{G_m} Du D\varphi \, dx = \int_{G_m} w\varphi \, dx + \lim_n \int_{G_m} Du_n D\varphi \, dx \\ &\quad = \int_{G_m} w\varphi \, dx - \lim_n \int_{G_m} D^2 u_n \varphi \, dx = 0. \end{aligned}$$

From (3.9) we deduce that (for any fixed m) the sequence $D^2 u_n$ weakly converges to $D^2 u$ in $L^2(G_m, \delta^\mu)$, that is,

$$\int_{G_m} (D^2 u_n - D^2 u)\varphi \delta^{2\mu} \, dx \to 0$$

for any $\varphi \in L^2(G_m)$. We show that $D^2 u \in L^2(\Omega, \delta^\mu)$. We recall that $\delta = \delta(x)$ denotes the distance of the point x to the boundary $\partial\Omega_\alpha$ of Ω_α. We set, for

$n \geqslant N$, $w_m^n = \chi_{G_m} D^2 u_n$, where by χ_{G_m} we denote the indicatrix function of the set G_m. Then we have that $\lim_m w_m^n = D^2 u_n$ almost everywhere in Ω_3^N. Hence,

$$\begin{aligned}\int_{\Omega_3^N} |D^2 u|^2 \chi_{G_m} \delta^{2\mu}\, dx &= \int_{G_m} |D^2 u|^2 \delta^{2\mu}\, dx \\ &\leqslant \liminf \int_{G_m} |D^2 u_n|^2 \delta^{2\mu}\, dx \\ &\leqslant \liminf \int_{\Omega_3^n} |D^2 u_n|^2 \rho_n^{2\mu}\, dx \leqslant c.\end{aligned}$$

As $\lim_m |D^2 u|^2 \chi_{G_m} d^{2\mu} = |D^2 u| \delta^{2\mu}$ almost everywhere in Ω_3^N, we obtain, by Fatou's lemma,

$$\int_{\Omega_3^N} |D^2 u|^2 \delta^{2\mu}\, dx \leqslant \liminf \int_{\Omega_3^N} |D^2 u|^2 \chi_{G_m} \delta^{2\mu}\, dx \leqslant c.$$

Finally, as the sets Ω_3^N tend to Ω_3 analogously, we have

$$\int_{\Omega_3} |D^2 u|^2 \delta^{2\mu}\, dx \leqslant c.$$

This concludes the proof of Theorem 3.1, and §3. □

§4. *Brennan's conjecture.* We point out that estimate (3.2) (with the same exponent μ^*) was established by Nyström in the more general framework of the class Domain$(2, M, r_0, q)$. We recall that an open, connected, and bounded subset D of $\mathbb{R}^2$ belongs to the class Domain$(2, M, r_0, q)$ if D is a non-tangentially accessible (NTA) domain with parameters M and r_0, supporting the reverse Hölder inequality (see (4.1) below). We say that a set D supports a reverse Hölder inequality if for all $P \in \partial D$, $r < r_0$,

$$J(P, r, D, x_0, q) \leqslant C J(P, r, D, x_0, 1), \tag{4.1}$$

where

$$J(P, r, D, x_0, a) = \left(\frac{1}{|B(P, r) \cap D|} \int_{B(P,r)\cap D} \left| \frac{G(x)}{\text{distance}(x, \partial D)} \right|^a dx \right)^{1/a}$$

for $a \in [1, \infty)$. Here $B(P, r)$ denotes an open ball, centred at P and of radius r, and $G(x) = G(x, x_0)$ the Green function of D with fixed pole x_0, such that the quotient of distance$(x_0, \partial D)$ and diameter(D) is bounded from above and below by absolute constants. By $|B(P, r) \cap D|$ we denote the 2-dimensional Lebesgue measure of the set. We now recall the definition of NTA domains (see [**13**]).

Definition 4.1. A bounded domain $D \subset \mathbb{R}^2$ is an NTA domain when there exist constants M and $r_0 > 0$ such that:

(i) *Corkscrew condition*. For any $P \in \partial D$, $r < r_0$, there exists $A = A_r(P) \in D$ such that $M^{-1}r < |A - P| < r$ and $\mathrm{dist}(A, \partial D) > M^{-1}r$;

(ii) D^c satisfies the corkscrew condition (where D^c denotes the complement of D); and

(iii) *Harnack chain condition.* If $\varepsilon > 0$, P_1 and P_2 belong to D, $\mathrm{dist}(P_j, \partial D) > \varepsilon$, and $|P_1 - P_2| < C\varepsilon$, then there exists a sequence of M-non-tangential balls ($B(A, r)$ is an M-non-tangential ball if $M^{-1}r < \mathrm{dist}(B(A, r), \partial D) < Mr$) such that the first ball contains P_1, the last contains P_2, and such that consecutive balls have a non-empty intersection, whose length depends on C, but not ε.

We stress the fact that estimate (3.2) could be greatly improved if Brennan's conjecture were shown to be true.

More precisely, Brennan (see [2, Theorem 1]) proved that

$$\int_D |\phi'|^q \, dx < +\infty \quad \text{for all } q, 4/3 < q < q_0 \tag{4.2}$$

where the upper bound q_0 is strictly larger than three. Here, D denotes a simply connected domain in $\mathbb{R}^2$ with at least two boundary points and ϕ is a conformal map to the open disk B. Moreover, Brennan postulated that $q_0 = 4$ was indeed the correct upper bound for all domains D.

In our setting, we choose $D = \Omega_\alpha$, where Ω_α is the fractal domain of the *snowflake* type constructed by means of the *generalized* Koch curves K_α, $\alpha \in (2, 4)$ (see §2). Hence, if Brennan's conjecture were shown to be true, we could prove that the second derivatives of the solution to the Dirichlet problem on snowflake domain Ω_α belong to the weighted space $L^2(\Omega_\alpha, d^{\mu_\alpha})$, $\mu_\alpha > \mu_{B,\alpha} = \frac{1}{2} + \frac{1}{4}\log_\alpha 4$. We note that the exponent μ^* is strictly greater than $\mu_{B,3}$ and the following inclusion holds for the weighted spaces $L^2(\Omega_\alpha, \delta^{\mu'}) \subset L^2(\Omega_\alpha, \delta^{\mu''})$ if $\mu' < \mu''$.

Unfortunately, as far as we know, Brennan's conjecture remains elusive, and there are only partial results (see e.g. [7, 11]). More precisely, the upper bound for which (4.2) is known to hold has been increased by Pommerenke [19] to $q_0 = 3.399$ and by Hedenmalm and Shimorin [10] to $q_0 = 3.752$. We use the result of Hedenmalm and Shimorin to improve the results of Theorem 3.1.

THEOREM 4.1. *Let $u \in H_0^1(\Omega_\alpha)$ be the solution of (2.3) with $f \in L^2(\Omega_\alpha)$. Then, for every $\mu_\alpha > \mu_{P,\alpha} = (2/q_0) + ((q_0 - 2)/2q_0)\log_\alpha 4$, $q_0 = 3.752$,*

$$\sum_{|\beta|=2} \int_{\Omega_\alpha} |D^\beta u|^2 \delta^{2\mu_\alpha} \, dx \leqslant c\|f\|^2_{L^2(\Omega_\alpha)}. \tag{4.3}$$

We note that the exponent μ^* is strictly greater than $\mu_{P,3}$, hence Theorem 4.1 improves Theorem 3.1 also for the case $\alpha = 3$.

In order to prove Theorem 4.1, we need to introduce notation and preliminaries. An important key tool is the Whitney decomposition W_{Ω_α} of the snowflake Ω_α by means of closed cubes whose sides are parallel to a fixed system of coordinate axes. By x_Q we mean the centre of the cube Q and by HQ, $H > 0$, the cube Q dilated with respect to x_Q by a factor H; by $\ell(Q)$ we mean the side length of Q (see, for instance, [20]). The decomposition W_{Ω_α} of Ω_α has the following properties:

(a) $\Omega_\alpha = \bigcup_{j=1}^{+\infty} Q_j$;

(b) $Q_j^\circ \cap Q_k^\circ = \emptyset$ if $j \neq k$; and

(c) there exist constants C_1 and C_2 such that

$$C_1\ell(Q_j) \leqslant \text{distance } (Q_j, \partial\Omega\alpha) \leqslant C_2\ell(Q_j). \tag{4.4}$$

Associated with the decomposition, we consider cut functions $\phi_j \in C^\infty(\mathbb{R}^2)$ such that:

(d) $\phi_j = 1$ on Q_j;

(e) $$\operatorname{supp}\phi_j \subset HQ_j \quad \text{with } 1 < H < C_1 + 1; \tag{4.5}$$

(f) $$|D^\beta\phi_j| \leqslant \frac{c(H)}{\ell(Q_j)^{|\beta|}}; \text{ and} \tag{4.6}$$

(g) *Locally finite covering condition.* For any fixed j, $(\operatorname{supp}\phi_j)^\circ \cap (\operatorname{supp}\phi_k)^\circ \neq \emptyset$ only for a number M_0 (depending on H but not on j) of indices k.

PROPOSITION 4.1. *Let $u \in H_0^1(\Omega_\alpha)$ be the solution of (2.3) with $f \in L^2(\Omega_\alpha, \delta^\mu)$. Then, for every $\mu > 0$,*

$$\sum_{|\beta|=2}\int_{\Omega_\alpha} |D^\beta u|^2\delta^{2\mu}\,dx \leqslant c\left(\int_{\Omega_\alpha} f^2\delta^{2\mu}\,dx + \int_{\Omega_\alpha}|\nabla u|^2\delta^{2\mu-2}\,dx + \int_{\Omega_\alpha} u^2\delta^{2\mu-4}\,dx\right). \tag{4.7}$$

Proof. We extend the functions u and f to zero outside Ω_α denoting them by the same symbols. By using the decomposition of the domain Ω_α we obtain

$$\int_{\Omega_\alpha}|D^\beta u|^2\delta^{2\mu}\,dx = \sum_{j=1}^{+\infty}\int_{Q_j}|D^\beta u|^2\delta^{2\mu}\,dx. \tag{4.8}$$

We consider $u_j = u\phi_j$; then we have

$$\begin{cases} -\Delta u_j = \chi_{HQ_j} f_j & \text{in } HQ_j,\\ u = 0 & \text{on } \partial HQ_j, \end{cases} \tag{4.9}$$

where $f_j = f\phi_j - 2\nabla u\nabla\phi_j - u\Delta\phi_j$. By the classical results [6, 20] and also [8],

$$\sum_{|\beta|=2}\int_{HQ_j}|D^\beta u_j|^2\,dx \leqslant c\int_{HQ_j}|\Delta u_j|^2\,dx,$$

we obtain with the properties (4.6) and (4.4)

$$\sum_{|\beta|=2}\int_{Q_j}|D^\beta u|^2\,dx=\sum_{|\beta|=2}\int_{Q_j}|D^\beta u_j|^2\,dx$$
$$\leqslant c\Big(\int_{HQ_j} f^2\,dx+\int_{HQ_j}|\nabla u|^2\delta^{-2}\,dx+\int_{HQ_j}u^2\delta^{-4}\,dx\Big). \tag{4.10}$$

By multiplying (4.10) by $\ell_j^{2\mu}$ where $\ell_j=\ell(Q_j)$ and by using (4.4) and (4.5) we obtain

$$\sum_{|\beta|=2}\int_{Q_j}|D^\beta u|^2\delta^{2\mu}\,dx$$
$$\leqslant (C_2+1)^{2\mu}\sum_{|\beta|=2}\int_{Q_j}|D^\beta u|^2\ell_j^{2\mu}\,dx$$
$$\leqslant c\Big(\int_{HQ_j} f^2\delta^{2\mu}\,dx+\int_{HQ_j}|\nabla u|^2\delta^{2\mu-2}\,dx+\int_{HQ_j}u^2\delta^{2\mu-4}\,dx\Big). \tag{4.11}$$

From (4.8), and (4.11), we obtain the inequality (4.7) for every $\mu>0$. □

We evaluate the last term in estimate (4.7) by means of the Hardy inequality and we refer to [**17**, Theorem 5.1] for the proof (see also [**22**]).

PROPOSITION 4.2. *Let $u\in H_0^1(\Omega_\alpha)$; then there exists a constant c such that, for $s<2-\log_\alpha 4$,*

$$\int_{\Omega_\alpha}u^2\delta^{s-2}\,dx\leqslant c\int_{\Omega_\alpha}|\nabla u|^2\delta^s\,dx.$$

We can now prove Theorem 4.1.

Proof. From Proposition 4.1, we have that

$$\sum_{|\beta|=2}\int_{\Omega_\alpha}|D^\beta u|^2\delta^{2\mu}\,dx$$
$$\leqslant c\Big(\int_{\Omega_\alpha} f^2\delta^{2\mu}\,dx+\int_{\Omega_\alpha}|\nabla u|^2\delta^{2\mu-2}\,dx+\int_{\Omega_\alpha}u^2\delta^{2\mu-4}\,dx\Big). \tag{4.12}$$

By Proposition 4.2 with $s=2\mu-2$ we obtain

$$\int_{\Omega_\alpha}u^2\delta^{2\mu-4}\,dx\leqslant c\int_{\Omega_\alpha}|\nabla u|^2\delta^{2\mu-2}\,dx.$$

Hence, we only need to evaluate the L^q-norm of the gradient of the Green potential. At this point, we combine the estimate of the upper bound in Brennan's conjecture due to Hedenmalm and Shimorin (see [**10**]) with consequences for the

Green potential shown by Hedenmalm (see [9, Theorem 4.2 and Corollary 4.5]) to obtain

$$\left(\int_{\Omega_\alpha} |\nabla Gf|^q \, dx\right)^{1/q} \leqslant C(q)\left(\int_{\Omega_\alpha} |f|^2 \, dx\right)^{1/2} \tag{4.13}$$

with $2 < q < q_0$, $q_0 = 3.752$.

We use estimate (4.13) and we have

$$\begin{aligned}\int_{\Omega_\alpha} |\nabla u|^2 \delta^{2\mu-2} \, dx &\leqslant \left(\int_{\Omega_\alpha} |\nabla u|^q \, dx\right)^{2/q} \left(\int_{\Omega_\alpha} \delta^{(q/(q-2))(2\mu-2)} \, dx\right)^{(q-2)/q} \\ &\leqslant c\left(\int_{\Omega_\alpha} f^2 \, dx\right)\left(\int_{\Omega_\alpha} \delta^{(q/(q-2))(2\mu-2)} \, dx\right)^{(q-2)/q}.\end{aligned}$$

Then $\int_{\Omega_\alpha} \delta^{(q/(q-2))(2\mu-2)} \, dx$ is bounded as $(q/(q-2))(2\mu-2) + 2 - d_f > 0$ where $d_f = \log_\alpha 4$ is the Hausdorff dimension of $\partial\Omega_\alpha$. This concludes the proof of Theorem 4.1 and §4. □

§5. *Uniform estimates.* In this section, we establish uniform estimates for solutions of the homogeneous Dirichlet problem in the approximating domains Ω_α^n that improve those established in [5] (compare Theorems 5.1 and 3.2). Our approach combines tools and methods of the paper [5] with the result of §4. In order to use the decomposition of the pre-fractal snowflake constructed in [5] we focus our attention only on the case $\alpha = 3$ (and hence on Ω_3^n). We could obviously construct a suitable decomposition for any pre-fractal snowflake Ω_α^n, but as doing this with appropriate details would require some extra work, we prefer to address this tool in a forthcoming article. Moreover, as long as we consider the same framework of the paper [5], we can skip the details and highlight the main differences, referring to [5] for the complete proofs.

More precisely, we state the following result.

THEOREM 5.1. *Let $u_n \in H_0^1(\Omega_3^n)$ be the solution of (2.4) with $f \in L^2(\Omega_3^n)$. Then, for every $\mu > \mu_{P,3} = (2/q_0) + ((q_0-2)/2q_0)\log_3 4$, $q_0 = 3.752$,*

$$\sum_{|\beta|=2} \int_{\Omega_3^n} |D^\beta u_n|^2 \rho_n^{2\mu} \, dx \leqslant c\|f\|^2_{L^2(\Omega_3^n)} \tag{5.1}$$

where the constant c is independent of n.

We recall that $\rho_n = \rho_n(x)$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of re-entrant corners of Ω_3^n.

Before proving Theorem 5.1, some remarks on the results of §4 are needed.

Remark 5.1. For a domain D in the class of self-similar domains SF(q) (see [18, Definition 12.1]) the validity of estimate (4.13) is equivalent to boundedness of the term

$$I_q(D) = \sum_{j\geqslant 4} 2^{j(q-2)} \sum_{Q\in W_j} G(x_Q)^q \tag{5.2}$$

where $W_j := \{Q \in W_D, \ell(Q) = 2^{-j}\}$, $G(x) = G(x, x_0)$, denotes the Green function and distance$(x_0, \partial D) \sim$ diameter(D). We recall that x_Q means the centre of the cube Q, $\ell(Q)$ means the side length of Q, and W_D denotes the Whitney decomposition of D mentioned in §4. Moreover, for a domain $D \in SF(q)$ it is possible to rephrase the reverse Hölder inequality condition (4.1) in terms of the sum over the Whitney cubes (5.2) (see [**18**, §12]).

In conclusion, as any domain Ω_3 belongs to the class $SF(q)$ for any $q > 1$ (see [**18**, §12]), then from (4.13) we deduce that any domain Ω_α supports the reverse Hölder inequality for any exponent $q < q_0 = 3.752$.

Remark 5.2. We stress the fact that any domain Ω_α is an NTA domain (see e.g. [**18**]) and from Remark 5.1 we deduce that any domain Ω_α belongs to the class Domain$(2, M, r_0, q)$ for any exponent $q < q_0 = 3.752$.

Remark 5.3. A peculiar property of the NTA domains is that if D is an NTA domain with parameters M, r_0, then $D \in$ Domain$(2, M, r_0, 1+1/(1-\beta))$ where $\beta = \beta(M) > 0$ is a constant describing the boundary behaviour of the Green function (see [**18**]). More precisely, the constant β is the constant appearing in the inequality (see [**18**])

$$G(x, y) \leqslant C(M)\frac{\text{dist}(y, \partial D)^\beta}{r^\beta} w(x, B(Q, r) \cap \partial D, D) \tag{5.3}$$

where $x \in D\backslash B(Q_0, Cr)$, $Q_0 \in \partial D$, $Cr < r_0$, and $y \in B(Q_0, r) \cap D$. Here $w(x, F, D)$ is the harmonic measure of $F \subset \partial D$ relative to D at $x \in D$. From Remark 5.2, we deduce that $\Omega_\alpha \in$ Domain$(2, M, r_0, 1 + 1/(1 - \beta))$ with $\beta = (q-2)/(q-1)$. Moreover, the value of β in (5.3) is the same as the one that appears in the following inequality (see [**13**, Lemma 4.1]):

$$u(x) < c(M)\left(\frac{|x-P|}{r}\right)^\beta C(u), \tag{5.4}$$

for all $P \in \partial D$, $r < r_0$, and for every positive harmonic function u in D such that u vanishes continuously on $B(P, r) \cap \partial D$. Here $x \in B(Q, r) \cap D$ and $C(u) := \sup\{u(y) : y \in \partial B(Q, r) \cap D\}$.

We are now in a position to prove Theorem 5.1.

Proof. By proceeding as in the proof of [**5**, Theorem 4.1], we establish the estimate for every $\mu > \frac{1}{4}$:

$$\sum_{|\beta|=2} \int_{\Omega_3^n} |D^\beta u|^2 \rho_n^{2\mu}\, dx \leqslant c\left(\int_{\Omega_3^n} f^2 \rho_n^{2\mu}\, dx + \int_{\Omega_3^n} |\nabla u|^2 \rho_n^{2\mu-2}\, dx + \int_{\Omega_3^n} u^2 \rho_n^{2\mu-4}\, dx\right) \tag{5.5}$$

with the constant c independent of n. We evaluate the last term in estimate (5.5) by means of the Hardy-type inequality established in [5, Theorem 5.1]:

$$\int_{\Omega_3^n} u^2 \rho_n^{s-2}\, dx \leqslant c \int_{\Omega_3^n} |\nabla u|^2 \rho_n^s \, dx$$

for any $u \in H_0^1(\Omega_3^n)$, $s < 2$, with the constant c independent of n and of u. Hence, we only need to establish uniform L^q-estimates for the gradient of the Green potential and we achieve this aim by using estimate (4.13) in §4, the geometry of the approximating domains Ω_3^n, a monotonicity argument, and the peculiar properties of a particular class of NTA domains mentioned in the previous remark (see also [13, 18]). Indeed, Nyström proves that (see [18]) for any $D \in \text{Domain}(2, M, r_0, q)$ where $q > 2$ there exists a constant $C = C = C(M, r_0, q)$ such that if $1/q = 1/p - 1/2$ then the following inequality is valid for all $f \in L^p(D)$:

$$\left(\int_D |\nabla G f|^q \, dx\right)^{1/q} \leqslant C(M, r_0, q) \left(\int_D |f|^p \, dx\right)^{1/p}. \tag{5.6}$$

A peculiar fact is that the constant in (5.6) depends only on the parameters M, r_0, and q (appearing in the definition of the class Domain(2, M, r_0, q)). Therefore, in order to prove uniform bounds, we only have to show that all domains Ω_3^n belong to the class Domain(2, M, r_0, q) with the same values M, r_0 and q. In [3, Lemma 2.3], it is proved that all domains Ω_3^n satisfy the Ahlfors three-point condition with the same Ahlfors constant A (independent of n). From [1, Lemma 2.5], we deduce that all domains Ω_3^n are NTA domains with the same constants M and r_0 and from Remark 5.3, $\Omega_3^n \in \text{Domain}(2, M, r_0, 1+1/(1-\beta))$ where $\beta = \beta(M) > 0$ is the constant appearing in (5.4). We note that the value of β in (5.4) can be expressed in terms of the capacity of the set $B(P, r) \cap (\Omega_3^n)^c$ (where $(\Omega_3^n)^c$ denotes the complement of Ω_3^n). In our setting, the sequence of domains Ω_3^n increases and converges to the snowflake Ω_3; moreover, the boundary of Ω_3 contains all the vertices of Ω_3^n. Then $B(P, r) \cap (\Omega_3^n)^c \supset B(P, r) \cap (\Omega_3)^c$ for any P vertex of Ω_3^n and for any point P^* of $\partial\Omega_3^n \setminus \mathcal{R}_n$ the capacity of the set $B(P, r) \cap (\Omega_3^n)^c$ is greater than or equal to the capacity of the set $B(P, r) \cap (\Omega_3^n)^c$ for some $P \in \mathcal{R}_n$. Hence, the value $\beta = (q_0 - 2)/(q_0 - 1)$, $q_0 = 3.752$ being a lower bound for the snowflake Ω_3 (see Remarks 5.2 and 5.3), provides a uniform lower bound for Ω_3^n. In conclusion,

$$\begin{aligned}\int_{\Omega_3^n} |\nabla u|^2 \rho_n^{2\mu-2}\, dx &\leqslant \left(\int_{\Omega_3^n} |\nabla u|^q \, dx\right)^{2/q} \left(\int_{\Omega_3^n} \rho_n^{(q/(q-2))(2\mu-2)} \, dx\right)^{(q-2)/q} \\ &\leqslant \left(\int_{\Omega_3^n} f^2 \, dx\right) \left(\int_{\Omega_3^n} \rho_n^{(q/(q-2))(2\mu-2)} \, dx\right)^{(q-2)/q}.\end{aligned}$$

Finally, to show the uniform boundedness of the term $\int_{\Omega_3^n} \rho_n^{(q/(q-2))(2\mu-2)}\, dx$, we note that for any $x \in \Omega_3^n$, we have that $\rho_n(x) \geqslant \delta(x)$ where $\delta(x)$ denotes the distance of the point x from the boundary of Ω_3. Then

$$\int_{\Omega_3^n} \rho_n^{(q/(q-2))(2\mu-2)}\,dx \leqslant \int_{\Omega_3^n} \delta^{(q/(q-2))(2\mu-2)}\,dx \leqslant \int_{\Omega_3} \delta^{(q/(q-2))(2\mu-2)}\,dx < +\infty$$

as $(q/(q-2))(2\mu-2)+2-d_f>0$ where $d_f=\log_3 4$ is the Hausdorff dimension of $\partial\Omega_3$. This completes the proof of Theorem 5.1 and §5. □

§6. *Obstacle problems.* In this section, we briefly discuss how we can extend the results of the previous sections to the solutions of obstacle problems. In the notation of the previous section, we consider the problem: find a function $u \in \mathcal{K}$ such that

$$\int_{\Omega_\alpha} \nabla u(\nabla u - \nabla v)\,dx \leqslant \int_{\Omega_\alpha} g(u-v)\,dx \quad \text{for all } v \in \mathcal{K} \tag{6.1}$$

where $\mathcal{K}=\{v \in H_0^1(\Omega_\alpha) : \varphi_1 \leqslant u \leqslant \varphi_2\}$, $\varphi_i \in H^1(\Omega_\alpha), i=1,2$, $\Delta\varphi_i \in L^2(\Omega_\alpha), i=1,2$, $\varphi_1 \leqslant \varphi_2$ in Ω_α, and $\varphi_1 \leqslant 0 \leqslant \varphi_2$ on $\partial\Omega_\alpha$.

Moreover, we consider the sequence of obstacle problems in the sets Ω_α^n: find a function $u \in \mathcal{K}^n$ such that

$$\int_{\Omega_\alpha^n} \nabla u(\nabla u - \nabla v)\,dx \leqslant \int_{\Omega_\alpha^n} g(u-v)\,dx \quad \text{for all } v \in \mathcal{K}^n \tag{6.2}$$

where $\mathcal{K}^n=\{v \in H_0^1(\Omega_\alpha^n) : \varphi_1^n \leqslant u \leqslant \varphi_2^n\}$, $\varphi_i^n \in H^2(\Omega_\alpha^n)$, $\varphi_1 \leqslant \varphi_2$ in Ω_α^n, and $\varphi_1^n \leqslant 0 \leqslant \varphi_2^n$ on $\partial\Omega_\alpha^n$.

As is well known for any datum $g \in L^2(\Omega_\alpha)$, there exists a unique solution $u \in H_0^1(\Omega_\alpha)$ of (6.1) by the Lax–Milgram theorem. Moreover, for any $g \in L^2(\Omega_\alpha^n)$ there exists a unique solution $u_n \in H_0^1(\Omega_\alpha^n)$ of (6.2) by the Lax–Milgram theorem and the following estimate holds:

$$\|u_n\||_{H_0^1(\Omega_\alpha^n)} \leqslant c(\|g\|_{L^2(\Omega_\alpha^n)} + \|\Delta\varphi_1^n\|_{L^2(\Omega_\alpha^n)} + \|\Delta\varphi_2^n\|_{L^2(\Omega_\alpha^n)}) \tag{6.3}$$

with the constant c independent of n.

We state the following results.

THEOREM 6.1. *Using the previous notation and assumptions, let u be the solution of (6.1) with $g \in L^2(\Omega_\alpha)$. Then, for every $\mu_\alpha > \mu_{P,\alpha} = (2/q_0) + ((q_0-2)/2q_0)\log_\alpha 4$, $q_0 = 3.752$,*

$$\sum_{|\beta|=2} \int_{\Omega_\alpha} |D^\beta u|^2 \delta^{2\mu_\alpha}\,dx \leqslant c(\|g\|^2_{L^2(\Omega_\alpha)} + \|\Delta\varphi_1\|^2_{L^2(\Omega_\alpha)} + \|\Delta\varphi_2\|^2_{L^2(\Omega_\alpha)}). \tag{6.4}$$

THEOREM 6.2. *Using the previous notation and assumptions, let u_n be the solution of (6.2) with $g \in L^2(\Omega_3)$ Then, for every $\mu > \mu_{P,3} = (2/q_0) + ((q_0-2)/2q_0)\log_3 4$, $q_0 = 3.752$,*

$$\sum_{|\beta|=2}\int_{\Omega_3^n} |D^\beta u_n|^2 \rho_n^{2\mu}\,dx \leqslant c(\|g\|^2_{L^2(\Omega_3^n)} + \|\triangle\varphi_1^n\|^2_{L^2(\Omega_3^n)} + \|\triangle\varphi_2^n\|^2_{L^2(\Omega_3^n)}) \quad (6.5)$$

where the constant c is independent of n.

We recall that $\rho_n = \rho_n(x)$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of re-entrant corners of Ω_3^n. The proofs follow from Theorems 4.1 and 5.1 by using the so-called *Lewy–Stampacchia inequality* (see [**21**, Theorem 4.35] and the reference quoted there). More precisely, we use the following.

PROPOSITION 6.1. *In the previous assumptions the solution u of (6.1) satisfies in Ω_α the inequality*

$$(-\triangle\varphi_2)\wedge g \leqslant (-\triangle u) \leqslant (-\triangle\varphi_1)\vee g. \quad (6.6)$$

Analogously, the solution u_n of (6.2) satisfies in Ω_α^n

$$(-\triangle\varphi_2^n)\wedge g \leqslant (-\triangle u_n) \leqslant (-\triangle\varphi_1^n)\vee g. \quad (6.7)$$

Proof of Theorem 6.1. By Proposition 6.1 the solution u of (6.1) is the solution of the Dirichlet problem (2.3) with datum $f \in L^2(\Omega_\alpha)$; in fact, from inequality (6.6) we obtain

$$(-\triangle\varphi_2)\wedge g \leqslant f \leqslant (-\triangle\varphi_1)\vee g.$$

Using Theorem 4.1 we thus conclude the proof. □

Proof of Theorem 6.2. By Proposition 6.1 the solution u_n of (6.2) is the solution of the Dirichlet problem (2.4) with datum $f \in L^2(\Omega_\alpha^n)$; in fact from inequality (6.7) we obtain

$$(-\triangle\varphi_2^n)\wedge g \leqslant f \leqslant (-\triangle\varphi_1^n)\vee g.$$

Using Theorem 5.1 we thus conclude the proof and §6. □

Acknowledgements. We wish to thank Professor Alexander Movchan for his kind invitation and Professor V. G. Maz'ya for his valuable suggestions.

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The authors are supported by Grant Ateneo "Sapienza" 2013.

References

1. B. Bennewitz and J. L. Lewis, On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.* **30**(2) (2005), 459–505.
2. J. E. Brennan, The integrability of the derivative in conformal mapping. *J. Lond. Math. Soc.* **18**(2) (1978), 261–272.
3. R. Capitanelli, Robin boundary condition on scale irregular fractals. *Commun. Pure Appl. Anal.* **9**(5) (2010), 1221–1234; doi:10.3934/cpaa.2010.9.1221.
4. R. Capitanelli and M. A. Vivaldi, On the Laplacean transfer across fractal mixtures. *Asymptot. Anal.* **83**(1–2) (2013), 1–33; doi:10.3233/ASY-2012-1149.
5. R. Capitanelli and M. A. Vivaldi, Uniform weighted estimates on pre-fractal domains. *Discrete Contin. Dyn. Syst. Ser. B* **19**(7) (2014), 1969–1985; doi:10.3934/dcdsb.2014.19.1969.
6. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (*Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **224**), 2nd edn edn., Springer (Berlin, 1983).
7. V. Gol'dshtein and A. Ukhlov, Brennan's Conjecture and universal Sobolev inequalities. *Bull. Sci. Math.* **138**(2) (2014), 253–269.
8. P. Grisvard, *Elliptic Problems in Non-Smooth Domains*, Pitman (Boston, 1985).
9. H. Hedenmalm, The dual of a Bergman space on simply connected domains. *J. Anal. Math.* **88** (2002), 311–335.
10. H. Hedenmalm and S. Shimorin, Weighted Bergman spaces and the integral means spectrum of conformal mappings. *Duke Math. J.* **127**(2) (2005), 341–393.
11. R. Hurri-Syrjänen and S. G. Staples, A quasiconformal analogue of Brennan's conjecture. *Complex Var. Elliptic Equ.* **35** (1998), 27–32.
12. J. E. Hutchinson, Fractals and selfsimilarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.
13. D. S. Jerison and C. E. Kenig, Boundary behaviour of harmonic functions in non-tangentially accessible domains. *Adv. Math.* **46** (1982), 80–147.
14. V. A. Kondratiev, Boundary value problems for elliptic equations in domains with conical or angular points. *Tr. Mosk. Mat. Obs.* **16** (1967), 209–292.
15. U. Mosco, Convergence of convex sets and of solutions of variational inequalities. *Adv. Math.* **3** (1969), 510–585.
16. U. Mosco, Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.* **123**(2) (1994), 368–421.
17. K. Nyström, Smoothness properties of Dirichlet problems in domains with a fractal boundary. *PhD Dissertation*, Umeà, 1994.
18. K. Nyström, Integrability of Green potentials in fractal domains. *Ark. Mat.* **34** (1996), 335–381.
19. Ch. Pommerenke, On the integral means of the derivative of a univalent function. *J. Lond. Math. Soc.* **32**(2) (1985), 254–258.
20. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (*Princeton Mathematical Series* **30**), Princeton University Press (Princeton, NJ, 1970).
21. G. M. Troianiello, *Elliptic Partial Differential Equations and Obstacle Problems* (*The University Series in Mathematics*), Plenum Press (New York, 1987).
22. A. Wannebo, Hardy inequalities. *Proc. Amer. Math. Soc.* **109**(1) (1990), 85–95.

Raffaela Capitanelli,
Dipartimento di Scienze di Base e
Applicate per l'Ingegneria,
"Sapienza Università di Roma",
Via A. Scarpa 16, 00161 Roma,
Italy
E-mail: raffaela.capitanelli@uniroma1.it

Maria Agostina Vivaldi,
Dipartimento di Scienze di Base e
Applicate per l'Ingegneria,
"Sapienza Università di Roma",
Via A. Scarpa 16, 00161 Roma,
Italy
E-mail: maria.vivaldi@sbai.uniroma1.it